Singularities

How to keep away from them
after Y. Yomdin

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Plan

First talk

1. Motivation : Sard’s theorem
2. Elementary properties of semi-algebraic sets
3. Entropy
4. Multidimensional variations

Second talk

1. Quantitative Sard theorem
2. Proofs
3. Quantitative singularity theory

Sard's theorem

Theorem. Let $f : \mathbf{R}^n \to \mathbf{R}$ be of class C^k, $k \geq n$. Let $\Sigma(f)$ be the set of critical points and $\Delta(f)=f(\Sigma(f))$ be the set of critical values. Then $\Delta(f)$ has measure zero.

Remark. Differentiability assumption $k \geq n$ is sharp (Whitney).

Bézout's theorem

Theorem. Let f : $\mathbf{R}^n \rightarrow \mathbf{R}$ be a polynomial of degree d. Then the number of critical values of f is less than $(d-1)^n$.

Goal

Find an intermediate result between Bézout and Sard, giving, for a smooth function, a quantitative estimate on the size of $\Delta(f)$.

Method

- Prove a variant of Bezout’s theorem which is more stable under small perturbations.
- Approximate a function of class C^k with a polynomial of degree k.

Nearly critical values

Notation. Given $r > 0$ and $c > 0$, let

$$\Sigma(f,c,r)=\{x ; |x| < r \text{ and } |grad_x f| < c\}$$

and

$$\Delta(f,c,r)=f(\Sigma(f,c,r)).$$

Stable Bézout theorem

Theorem 1 (Y. Yomdin).

Let $f : \mathbf{R}^n \to \mathbf{R}$ be a polynomial of degree d. Then $\Delta(f,c,r)$ can be covered by N intervals of length cr, where N depends only on n and d.

Quantitative Sard theorem

Notation. If f is of class C^k, let

$$R_k(f) = r^k \sup \{|D^k f(x)| \; ; \; |x|<r\}.$$

Theorem 2 (Y. Yomdin). Let $\varepsilon < R_k(f)$. Then the set $\Delta(f,r)$ of critical values of f can be covered with

$$N \, (R_k(f)/\varepsilon)^{n/k}$$

intervals of length ε, where N depends only on n and k.

Consequences

Corollary. If $k \geq n$, the critical values of f have measure 0.

Corollary. Let $k > n$. If $0 \leq f \leq 1$ and $q^{1-n/k} > N(R_k(f))^{n/k}$, there is a non near-critical value of the form p/q, $0 \leq p \leq q$.

Proof of thm 2 from thm 1

Let $r' = r(\varepsilon/R_k(f))^{1/k}$.

Cover the r-ball in $\mathbf{R}^n$ with $(r/r')^n$ r'-balls B_j. On each of them, approximate f with its degree k Taylor polynomial g_j at the center. On B_j,

$|f - g_j| \le (r'/r)^k R_k(f) = \varepsilon$, $|\text{grad } f - \text{grad } g_j| \le \varepsilon/r'$.

Therefore critical points of f are c-critical points of g_j with $c = \varepsilon/r'$, and critical values of f are contained in the ε-neighborhood of $\cup_j \Delta(g_j,c,r)$, i.e. covered with $N(r/r')^n$ intervals of length 3ε.

Elementary properties of semi-algebraic sets

The proof of theorem 1 relies on three properties of semi-algebraic sets, i.e. sets defined by polynomial equations and inequations.

We measure the *complexity* of such a set with

- the number n of variables,
- the number of equations and inequations,
- the maximum degree of the polynomials

involved in its definition.

Connected components

Lemma 1 (Descartes,…).

The number of connected components of a semi-algebraic subset of $\mathbf{R}^n$ can be bounded in terms of its complexity only.

Curve selection lemma

Lemma 2.

Let $A \subset \mathbf{R}^n$ be semi-algebraic. Let x and y sit in the same connected component of A. Then there exists a semi-algebraic curve in A joining x to y, whose complexity depends only on that of A.

Length estimate

Lemma 3.

Let γ be a semi-algebraic curve contained in a ball of radius r in $\mathbf{R}^n$. Then

$$\text{length of } \gamma \leq K\, r$$

where K depends only on the complexity of γ.

Proof of theorem 1

Let f be a polynomial of degree d, $\Sigma(f,c,r)$ its c-critical points in a ball of radius r (its complexity is $\leq 2d$).

Goal : estimate the diameter of $f(\Sigma(f,c,r))$.

Lemma 1 : can assume $\Sigma(f,c,r)$ connected.

Lemma 2 : can replace $\Sigma(f,c,r)$ with a curve γ.

Lemma 3 : γ has length $\leq Kr$.

On γ, $|\mathrm{grad}\, f| \leq c$ so length $f(\gamma) \leq Kcr$.

Proofs of lemmas 1 and 2

Thom's lemma.

Let ¶ be a finite collection of polynomials on **R** which is stable under derivation. Then any subset of **R** of the form

$$\bigcap_{P \text{ in } ¶} \{P \geq \text{ or } \leq \text{ or } = 0\}$$

is connected.

Proof of Lemma 3

By lemma 1, for every hyperplane π, $\#(\gamma \cap \pi)$ is bounded by the complexity K of γ.

By Crofton's formula,

$$\text{length}(\gamma) = \int \#(\gamma \cap \pi)\, d\pi$$

$$\leq K \text{ vol}\{\pi \,;\, \pi \cap B(r) \neq \emptyset\}$$

$$\leq K r.$$

- **Goal :** collect tools for a generalization of the quantitative Sard theorem to maps $\mathbf{R}^n \to \mathbf{R}^m$.

- Keywords : entropy, multidimensional variation.

Entropy

Definition (Kolmogorov). Let $X \subset \mathbf{R}^n$, let $\varepsilon > 0$. Denote by

$$M(X, \varepsilon)$$

the minimal number of ε-balls needed to cover X.

The *entropy* of X is $\log M(X, \varepsilon)$.

Entropy versus volume

If X is a smooth k-manifold,

$$\mathrm{Vol}_k(X) = \mathrm{const}_n \lim_{\varepsilon\to 0} \varepsilon^k M(X, \varepsilon).$$

For X arbitrary, let X_ε = ε-neighborhood of X in $\mathbf{R}^n$.

Then

$$M(X, 2\varepsilon) \leq \mathrm{const}_n\, \varepsilon^{-n}\, \mathrm{vol}(X_\varepsilon).$$

Nevertheless, there is more information in M(X, ε) than in volume only. If M(X, ε) is small, X cannot contain a grid.

Multidimensional variations

Definition (Vitushkin). Let $X \subset \mathbf{R}^n$, $0 \leq k \leq n$. The *k-th variation* $V_k(X)$ of X is the integral of the number of connected components of the intersections of X with n-k planes.

Easy : $V_0(X)$ is the number of connected components of X, $V_n(X) = vol_n(X)$.

Case of C^2 submanifolds of dimension d :

$$V_k(X) < \infty \text{ for all } k,\ V_k(X) = 0 \text{ for } k > d.$$

$$\text{Crofton : } V_d(X) = vol_d(X).$$

Link with Lipschitz-Killing curvatures

The k-th Lipschitz-Killing curvature of a smooth submanifold X is

$$\Lambda_k(X) = \int_{\{n\text{-}k \text{ planes}\}} \chi(X \cap \pi)\, d\pi ,$$

where χ denotes Euler characteristic.

If X is a smooth *convex* set in $\mathbf{R}^3$, $V_1(X) = \Lambda_1(X)$ is the integral of mean curvature of the boundary.

If X is a closed surface, $V_1(X) \leq \text{const} \int_X |k_1|+|k_2|$

where k_1 and k_2 are principal curvatures.

Entropy/variations inequality

Weyl's formula. Let X be a smooth submanifold, X_ε its tubular neighborhood. Then for ε small enough,

$$\mathrm{Vol}_n(X_\varepsilon) = \sum_{k=0,..,n} \Lambda_{n-k}(X)\, \varepsilon^k .$$

This implies $M(X, \varepsilon) \leq \sum_{k=0,..,n} \Lambda_{n-k}(X)\, \varepsilon^{k-n}$.

Lemma 4 (Ivanov, Zerner). Let $X \subset \mathbf{R}^n$ be compact. Then for all $\varepsilon > 0$,

$$M(X, \varepsilon) \leq \mathrm{const}_n \sum_{k=0,..,n} V_k(X)\, \varepsilon^{-k} .$$

Towards higher dimensional quantitative Sard theorem

- Approximate C^k-smooth map with polynomial.
- Estimate the size of near critical values for a polynomial.
- Size = entropy $M(X,\varepsilon)$.
- **Lemma 4**. $M(X, \varepsilon) \le const_n \sum_{k=0,..,n} V_k(X)\, \varepsilon^{-k}$, where $V_k(X)$ is k-dimensional variation.

Behaviour of variations under polynomial maps

Lemma 5 (Yomdin). Let $A \subset \mathbf{R}^n$ be semi-algebraic, let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a polynomial. Assume that the k-th jacobian satisfies

$J_k(f) \leq c$ along A. Then

$$V_k(f(A)) \leq K\, c\, V_k(A)$$

where K depends on the complexity of A and the degree of f only.

Stable Bézout theorem

Theorem 3 (Yomdin). Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a polynomial. Given $\Lambda=\{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\}$, let $\sum(f,\Lambda,r)$ = points in a ball of radius r where the eigenvalues of $\sqrt{Df^T Df}$ are less than $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then

$$V_k(f(\sum(f,\Lambda,r))) \leq K \lambda_1 \lambda_2 \ldots \lambda_k r^k$$

where K depends on the degree of f only.

Quantitative Sard theorem

Theorem 4 (Yomdin). Let f : $\mathbf{R}^n \rightarrow \mathbf{R}^m$ be of class C^k. Let $R_k(f) = \sup |D_k f|$ on a ball of radius r.

If $\varepsilon \leq R_k(f)$, then

$$M(f(\Sigma(X,\Lambda,r)), \varepsilon) \leq K \sum_{i=0..\min(n,m)} \lambda_1 \lambda_2 \ldots \lambda_i \, (r/\varepsilon)^i \, (R_k(f)/\varepsilon)^{(n-i)/k} .$$

Proof of Lemma 4

- Let $X \subset \mathbf{R}^n$, let B be a unit ball. Let $V_k(X,B)$ denote the integral over n-k planes π of the number of components of $X \cap \pi$ which are contained in B.
- Show that if X contains the center of B,

$$\Sigma_{k=0..n} V_k(X,B) \geq 1.$$

- If M disjoint unit balls have their centers on X, then

$$M \leq \Sigma_{k=0..n} V_k(X) .$$

Proof of relative variation estimate (n=2)

- Let C be the connected component of X containing the center of B. If $C \subset B$, then

$$V_0(X,B) \geq 1.$$

- Otherwise, $\prod = \{\text{lines } \pi \text{ intersecting } B(1/2)\}$ has large measure. If for half the π in $\prod$ some connected component of $\pi \cap X$ intersecting $B(1/2)$ is contained in B, then $V_1(X,B) \geq 1$.
- Otherwise for half the π in $\prod$, $\text{length}(X \cap \pi) \geq 1/2$, therefore $V_2(X,B) = \text{area}(X \cap B) \geq 1$.

Proof of Lemma 5 (1/3)

1. $V_k(f(A)) \le \int V_0(A \cap f^{-1}(\pi))\, d\pi$.
2. $V_0(A \cap f^{-1}(\pi)) \le K$ depending on the complexity of A and the degree of f only.
3. $V_0(A \cap f^{-1}(\pi)) = 0$ unless f(A) intersects π.

Therefore

$$V_k(f(A)) \le K \int_{\text{k-planes thru origin}} vol_k((P_{\pi'} o f)(A)) d\pi'.$$

Proof of Lemma 5 (2/3)

Assume $k = \dim A = \dim (P_{\pi'} o f)(A)$.

Then $J_k f \le c$ implies

$$vol_k((P_{\pi'} o f)(A)) \le c\ vol_k(A),$$

thus

$$V_k(f(A)) \le K'\ c\ vol_k(A) = K'\ c\ V_k(A).$$

In general, replace A with $C \subset A$.

Proof of Lemma 5 (3/3)

Semi-algebraic axiom of choice. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a polynomial, let $A \subset \mathbf{R}^n$ be semi-algebraic. There exists a semi-algebraic set $C \subset A$ whose complexity is bounded by that of A, such that $\dim(C) = \dim(f(A))$ and $f(C) = f(A)$.

Lemma. Let $C \subset A$ be semi-algebraic. Then

$$V_k(C) \leq K\, V_k(A)$$

where K depends on the complexity of C only.

Quantitative singularity theory

Motto :

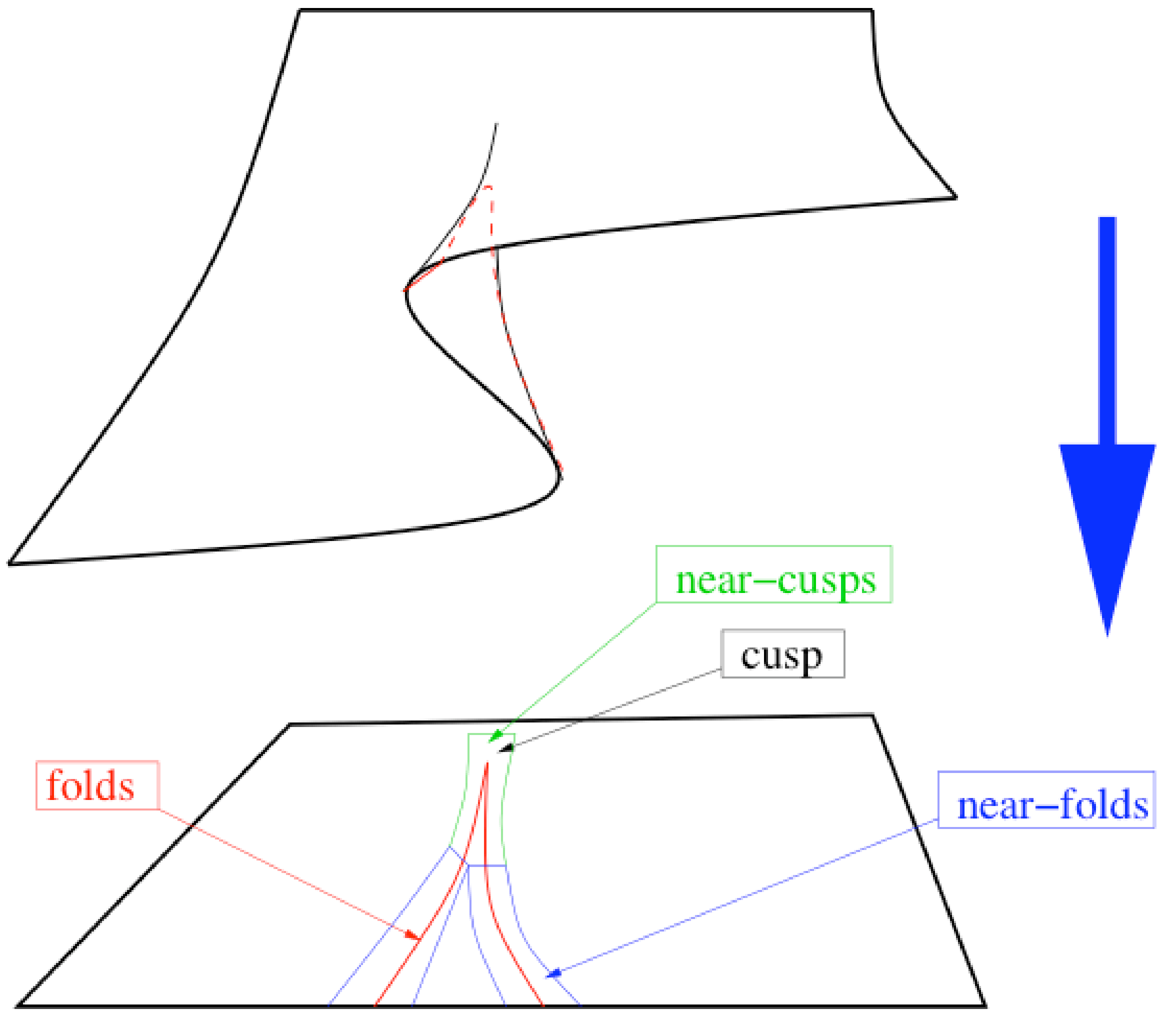
- Semi-algebraic algorithms are feasable far away from singularities.
- Normal forms near singularities are computable.

Example : axiom of choice

Problem : Given a smooth map f : $\mathbf{R}^2 \rightarrow \mathbf{R}^2$, find a triangulation such that f is a homeomorphism on each face.

Use Quantitative Sard Theorem to estimate the probability that f have only folds and cusps with large normal form neighborhoods.

Whitney's cusp



Inverting a cusp

Use standard algorithms away from folds and cusps.

Use normal forms

$(x,y) \rightarrow (x^2,y)$ near folds,

$(x,y) \rightarrow (x^3-3xy,y)$ near cusps.

Difficulty : given a near-singular point, choose the corresponding « center » singular point.

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