Shapes of crystals

P. Pansu, Université Paris-Sud 11 and Ecole Normale Supérieure

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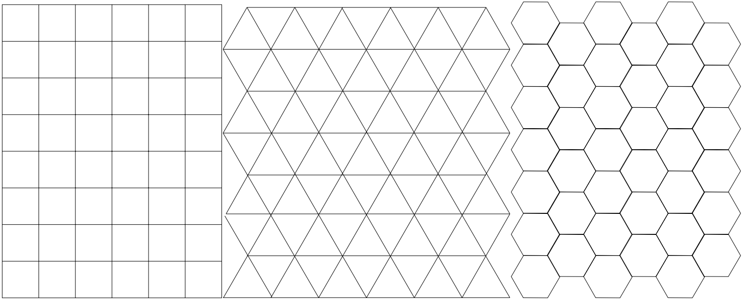
On a game board, form a shape using A identical pieces. The *perimeter* is the number of outer edges, i.e. edges which are not shared by two pieces. The number A is the *area* of the shape.



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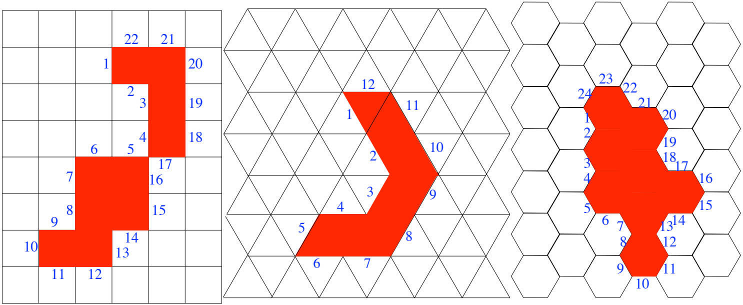

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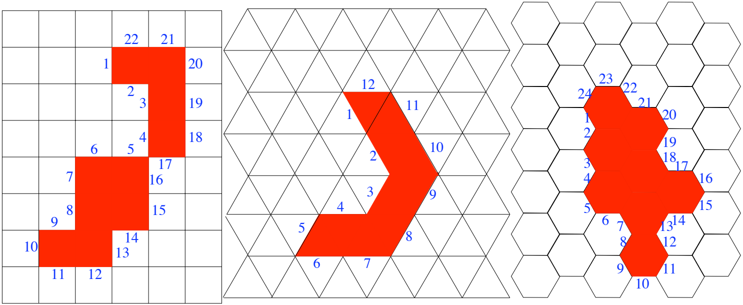

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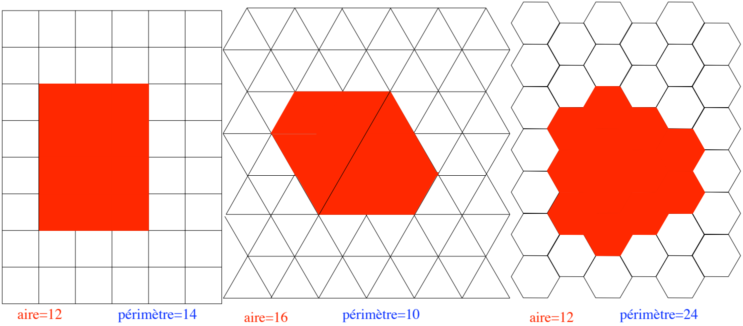
Aim of the game : find the shape which, for a given area, minimizes perimeter.

These are three *discrete* isoperimetric problems.

Here are a few solutions of isoperimetric games.



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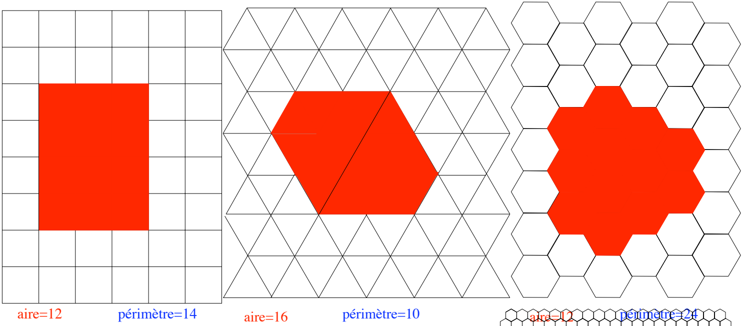

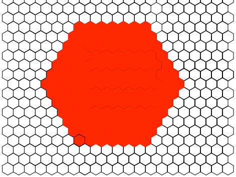

When area gets larger, solutions have the same rough shapes, approximately square or hexagonal.

Theorem

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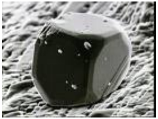

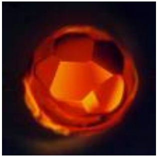

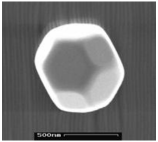

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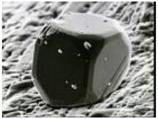
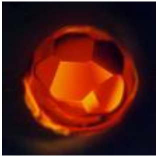
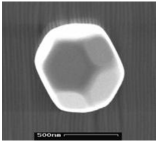
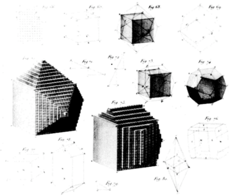


R.J. Haüy (1784) proposed that matter should be formed of tiny polyhedra glued together. This was naive, but the idea of microscopic periodicity was a good one, and finally was confirmed experimentally in 1912.



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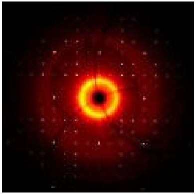
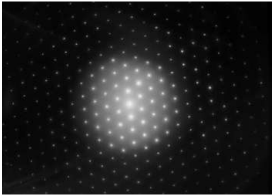
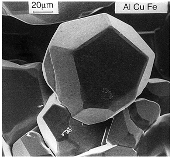
During 200 years, crystallographers have taken this rule for granted. It has been confirmed by *diffraction patterns* (obtained using X-rays), which reflect the crystal's symmetry.





How can one explain the 5-fold symmetry visible in the diffraction pattern below ? And the dodecahedral shape of a monocrystal of aluminium-copper-iron alloy ?



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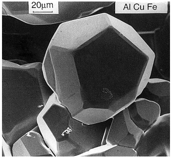

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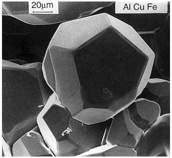

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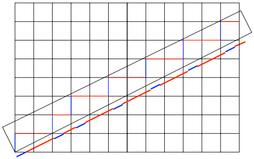
Quasiperiodicity can produce crystals. A phenomenon in 3-space is *quasiperiodic* if it is the trace, on a 3-dimensional vector subspace, of a higher dimensional periodic phenomenon.

To obtain a non periodic tiling of a line, one can consider an oblique slab in the plane tiled by squares, and project on a line the vertices contained in the slab.



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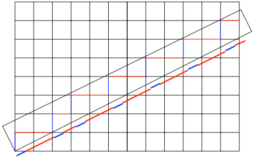

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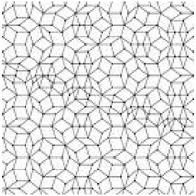

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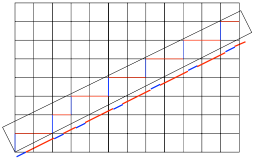

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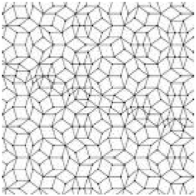

The same process applied in dimension 6 generates non periodic tilings of 3-dimensional space, whose diffraction patterns are made of spots, and which admit screw rotations with angle $2\pi/5$. These are good candidates for models of *quasicrystals*.