Density of Zarisky density for surface groups, II

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Definition

Let G be the group of real points of an algebraic group. Let Γ be a finitely generated group. Say a homomorphism $\phi : \Gamma \to G$ is

- fully flexible *if Zariski dense homomorphisms are dense in a neighborhood of ϕ in $Hom(\Gamma, G)$.*
- partially rigid *if there are no Zariski dense homomorphisms in a neighborhood of ϕ in $Hom(\Gamma, G)$.*

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Determine which homomorphisms of surface groups to reductive real algebraic groups are fully flexible, or partially rigid, or neither.

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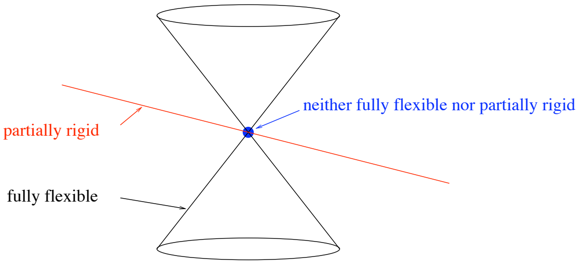
The story started with W. Goldman's flexibility result for non-maximal surface groups in $SU(2,1)$ (1985). The full flexibility of surface groups in $Sl(n,\mathbb{R})$ is due to O. Guichard (2007).

Theorem (No third option)

A homomorphism from a high genus surface group Γ to a reductive real algebraic group G is either fully flexible or partially rigid.



$$\begin{cases} x^2+y^2-tz-t^2 &= 0,\\ x^2+y^2+z^2-t^2 &= 0. \end{cases}$$

Let G be semi-simple, $\phi : \Gamma \to G$ a reductive homomorphism. Let $\mathfrak{c}$ be the center of the centralizer of $\phi(\Gamma)$. It splits the complexified Lie algebra of G into root spaces $\mathfrak{g}_\lambda$. When λ is pure imaginary, the sesquilinear Killing form $\bar{X} \cdot X$ on $\mathfrak{g}_\lambda$ is non-degenerate, giving rise to a representation ρ_λ in $U(p_\lambda, q_\lambda)$, and a Toledo invariant T_λ.

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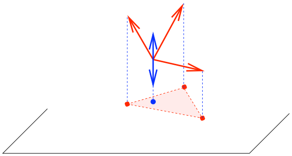
Definition

Among the above roots, let P *be the subset of pure imaginary roots* λ *such that* ρ_λ *is a maximal representation with* $T_\lambda > 0$ *and vanishing signature* ($p_\lambda = q_\lambda$)*. Say* $\mathfrak{c}$ *is balanced with respect to* ϕ *if 0 belongs to the interior of the sum of the convex hull of the imaginary parts of elements of* P *and the linear span of the real and imaginary parts of roots not in* $\pm P$*.*



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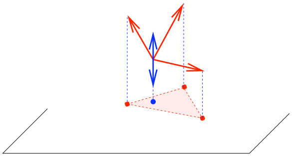

Theorem (Fully flexible $\Leftrightarrow$ balanced)

A homomorphism ϕ of a high genus surface group to a semisimple real algebraic group is fully flexible if and only if $\mathfrak{c}$ is balanced with respect to ϕ.

Theorem (Hernandez, Bradlow-García-Prada-Gothen, Burger-Iozzi-Wienhard)

Maximal representations in simple real algebraic groups whose symmetric space is Hermitian and not of tube type are partially rigid: they factor through tube type Hermitian subgroups.

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Theorem (Classification)

Converse holds for classical simple real algebraic groups (i.e. real forms of $Sl(n,\mathbb{C})$, $O(n,\mathbb{C})$ or $Sp(n,\mathbb{C})$).

In other words, flexibility is the rule unless $G = SU(p,q)$, $q > p$ and $\phi(\Gamma)$ is contained in and maximal in a conjugate of $S(U(p,p) \times U(q-p)) \subset SU(p,q)$, or $G = SO^*(2n)$, n odd, and $\phi(\Gamma)$ is contained in and maximal in a conjugate of $SO^*(2n-2) \times SO(2) \subset SO^*(2n)$.

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Remark

1. The restriction on genus is probably irrelevant.
2. Non constructive : deformations are not given by explicit formulae nor geometric constructions.

Theorem

(W. Goldman, 1985). *If* Γ *is a surface group and* ρ *is reductive, then, in a neighborhood of the conjugacy class of* ρ*,* $Hom(\Gamma, G)$ *is analytically equivalent to*

$$\{u \in H^1(\Gamma, \mathfrak{g}_{ad\circ\rho}) \,|\, [u \smile u] = 0\}.$$

Here, smiling bracket denotes cup-product : $H^1(\Gamma, \mathfrak{g}_{ad\circ\rho}) \to H^2(\Gamma, \mathfrak{g}_{ad\circ\rho})$.

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Cup-products can be computed thanks to

Theorem

(W. Meyer, 1972). *Let* (E, Ω) *be a flat symplectic vector bundle over* Σ*. The quadratic form* $Q(a) = \int_\Sigma \Omega(a \smile a)$ *on* $H^1(\Sigma, E)$ *is nondegenerate of signature* $4c_1(E, \Omega)$.

Proposition

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Proposition

Levi factors of centralizers are treated by explicitly deforming given cohomology classes u *such that* $[u \smile u] = 0$.

Such deformations v, satisfying $[v \smile u] = 0$, $[v \smile v] = 0$, are obtained as zeroes of sections of bundles on the Grassmannian of symplectic subspaces of $H^1(\Gamma, \mathbb{R})$. They exist due to nonvanishing of powers the top Chern class of the universal bundle. This requires a high dimensional Grassmannian and thus high genus.

Let $\mathfrak{c}$ denote the center of the centralizer of $\phi(\Gamma)$. $\mathfrak{g}\otimes\mathbb{C}$ splits under $\mathfrak{c}$ into root spaces $\mathfrak{g}_\lambda$. $H^1(\Gamma,\mathfrak{g})\otimes\mathbb{C}$ splits accordingly.

Lemma

$[\cdot\smile\cdot]$ *vanishes on each* $H^1(\Gamma,\mathfrak{g}_\lambda)$.
$H^1(\Gamma,\mathfrak{g}_\lambda)$ *and* $H^1(\Gamma,\mathfrak{g}_\mu)$ *are orthogonal with respect to* $[\cdot\smile\cdot]$ *unless* $\lambda+\mu=0$.

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On each $\mathfrak{g}_{\lambda,\mathbb{R}}=\mathfrak{g}\cap(\mathfrak{g}_\lambda\oplus\mathfrak{g}_{-\lambda})$, all ad_Z, $Z\in\mathfrak{c}$ are proportional. Therefore the corresponding alternating forms $(X,Y)\to Z\cdot[X,Y]$ are proportional to the symplectic form $\Omega_\lambda=\Im m(s_\lambda)$. On $H^1(\Gamma,\mathfrak{g}_{\lambda,\mathbb{R}})$, all $Z\cdot[\cdot\smile\cdot]$ are proportional to the quadratic form $Q_\lambda(u,u)=\int_\Sigma\Omega_\lambda(u\smile u)$.

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Lemma

If $\lambda\neq 0$, Q_λ is nondegenerate and its index is equal to $4T_\lambda$. Therefore

$$4|T_\lambda|\leq\dim(H^1(\Gamma,\mathfrak{g}_{\lambda,\mathbb{R}}))=-\chi(\Sigma)\dim(\mathfrak{g}_{\lambda,\mathbb{R}}).$$

In particular, Q_λ is definite if and only if ρ_λ is a maximal representation.

On $H^1(\Gamma,\mathfrak{g})$, $[\cdot \smile \cdot] = \sum_\lambda Q_\lambda \lambda$. Smooth points of $\chi(\Gamma, G)$ correspond to classes $u = \sum_\lambda u_\lambda$, $u_\lambda \in H^1(\Gamma, \mathfrak{g}_{\lambda,\mathbb{R}})$, such that $[u \smile u] = 0$ and $u_\lambda \neq 0$ for a spanning set of λ's. If u is not a smooth point, too many u_λ's vanish, perturb them to nonzero v_λ's. Indefinite quadratic forms Q_λ allow arbitrary moves in the direction of λ, but positive definite ones allow one to reach only points of a convex hull. If 0 belongs to it, nonzero v_λ's can be found such that $[v \smile v] = 0$.

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Real or $\mathbb{C}$-linear roots have vanishing Toledo invariant. Let λ be pure imaginary with Q_λ definite. Then Toledo invariant equals

$$T_\lambda=\frac{|\chi(\Gamma)|}{2}(p_\lambda+q_\lambda).$$

Milnor-Wood inequality implies that

$$|T_\lambda|\leq|\chi(\Gamma)|\mathrm{rank}(U(p_\lambda,q_\lambda))=|\chi(\Gamma)|\min\{p_\lambda,q_\lambda\}).$$

This implies $p_\lambda=q_\lambda$. Whence the definition of P.

When G is classical, one can determine centers of centralizers of reductive subgroups H, and the root space decomposition, to a large extent. Indeed, $G \subset Gl(V)$ is the full automorphism group of a bilinear (resp. sesquilinear) form on the standard representation V. If

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Theorem (Burger-Iozzi-Wienhard, 2007)

Let S be a semisimple Lie group whose symmetric space is Hermitian. Let $\rho : \Gamma \to S$ be a maximal representation of a surface group. Then ρ is tight. Its Zariski closure H is reductive of Hermitian type. The embedding $H \hookrightarrow S$ is tight. If S is of tube type, so is H.

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Example: $G = SU(p, q)$.
Let H be the Zariski closure of $\phi(\Gamma)$. A root $\ell - \ell'$ belongs to P iff the sesquilinear Γ action on $Hom(I_{\ell'}, I_\ell)$ is maximal and positive. Since this action factors through $U(I_{\ell'}) \times U(I_\ell)$, the morphism $U(I_{\ell'}) \times U(I_\ell) \to U(Hom(I_{\ell'}, I_\ell))$ is tight, this implies that either $U(I_{\ell'})$ or $U(I_\ell)$ is compact, the Γ action on the other factor is maximal. The noncompact factor must be of tube type and therefore its signature vanishes.

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Assume that $\mathfrak{c}$ is not balanced with respect to ϕ. Then one shows that there exists one root ℓ' such that all $\ell - \ell'$ belong to P. This implies that ϕ falls into $S(U(p, p) \times U(q - p))$. Also, Toledo invariants sum up with correct signs, showing that ϕ is maximal in $S(U(p, p) \times U(q - p))$.