Differential forms and the Hölder homeomorphism problem, after Gromov and Rumin

P. Pansu

February 1st, 2012

Question

(Gromov 1993). *Let M be an n-dimensional sub-Riemannian manifold. For which $\alpha \in (0,1)$ does there exist locally a homeomorphism $\mathbb{R}^n \longrightarrow M$ which is C^α-Hölder continuous ?*

Question

(Gromov 1993). *Let M be an n-dimensional sub-Riemannian manifold. For which $\alpha \in (0,1)$ does there exist locally a homeomorphism $\mathbb{R}^n \to M$ which is C^α-Hölder continuous ?*

Definition

Let $\alpha(M) = \sup\{\alpha \in (0,1) \,|\, \exists$ *locally a homeomorphism* $\mathbb{R}^n \to M\}$.

Question

(Gromov 1993). *Let M be an n-dimensional sub-Riemannian manifold. For which $\alpha \in (0,1)$ does there exist locally a homeomorphism $\mathbb{R}^n \to M$ which is C^α-Hölder continuous ?*

Definition

Let $\alpha(M) = \sup\{\alpha \in (0,1) \,|\, \exists$ *locally a homeomorphism* $\mathbb{R}^n \to M\}$.

Example

If G is a r-step Carnot group, the exponential map $\mathfrak{g} = Lie(G) \to G$ is locally $C^{1/r}$-Hölder continuous. Thus $\alpha(M) \geq 1/r$.

Question

(Gromov 1993). *Let M be an n-dimensional sub-Riemannian manifold. For which $\alpha \in (0,1)$ does there exist locally a homeomorphism $\mathbb{R}^n \to M$ which is C^α-Hölder continuous ?*

Definition

Let $\alpha(M) = \sup\{\alpha \in (0,1) \,|\, \exists$ *locally a homeomorphism* $\mathbb{R}^n \to M\}$.

Example

If G is a r-step Carnot group, the exponential map $\mathfrak{g} = Lie(G) \to G$ is locally $C^{1/r}$-Hölder continuous. Thus $\alpha(M) \geq 1/r$.

Theorem (Gromov 1993)

Let M be sub-Riemannian, with Hausdorff dimension Q. Then $\alpha(M) \leq \frac{n-1}{Q-1}$.
Let M be a $2m+1$-dimensional contact manifold. Then $\alpha(M) \leq \frac{m+1}{m+2}$ $(\leq \frac{2m}{2m+1})$.

Gromov's proof uses Hausdorff dimension of subset of given topological dimension: if all subsets of topological dimension k have Hausdorff dimension $\geq k'$, then $\alpha(M) \leq \frac{k}{k'}$.

Gromov's proof uses Hausdorff dimension of subset of given topological dimension: if all subsets of topological dimension k have Hausdorff dimension $\geq k'$, then $\alpha(M) \leq \frac{k}{k'}$.

To get lower bounds on Hausdorff dimension of subsets, Gromov constructs local foliations by horizontal submanifolds. If there are enough such dimension k foliations, all subsets of topological dimension $n-k$ have Hausdorff dimension $\geq Q-k$, therefore $\alpha(M) \leq \frac{n-k}{Q-k}$.

Gromov's proof uses Hausdorff dimension of subset of given topological dimension: if all subsets of topological dimension k have Hausdorff dimension $\geq k'$, then $\alpha(M) \leq \frac{k}{k'}$.

To get lower bounds on Hausdorff dimension of subsets, Gromov constructs local foliations by horizontal submanifolds. If there are enough such dimension k foliations, all subsets of topological dimension $n-k$ have Hausdorff dimension $\geq Q-k$, therefore $\alpha(M) \leq \frac{n-k}{Q-k}$.

Constructing horizontal curves amounts to solving a system of PDE's. If $k=1$, it is an ODE, the method applies to all (equiregular) sub-Riemannian manifolds. Gromov solves the relevant PDE for contact $2m+1$-manifolds and $k=m$, and, more generally, for generic h-dimensional distributions, and k such that $h-k \geq (n-h)k$.

Gromov's proof uses Hausdorff dimension of subset of given topological dimension: if all subsets of topological dimension k have Hausdorff dimension $\geq k'$, then $\alpha(M) \leq \frac{k}{k'}$.

To get lower bounds on Hausdorff dimension of subsets, Gromov constructs local foliations by horizontal submanifolds. If there are enough such dimension k foliations, all subsets of topological dimension $n-k$ have Hausdorff dimension $\geq Q-k$, therefore $\alpha(M) \leq \frac{n-k}{Q-k}$.

Constructing horizontal curves amounts to solving a system of PDE's. If $k=1$, it is an ODE, the method applies to all (equiregular) sub-Riemannian manifolds. Gromov solves the relevant PDE for contact $2m+1$-manifolds and $k=m$, and, more generally, for generic h-dimensional distributions, and k such that $h-k \geq (n-h)k$.

Today, I describe an alternative method, due again to Gromov, but based on Rumin's theory of differential forms on sub-Riemannian manifolds. A motivation to further study this theory in this seminar.

Definition

*On a metric space X, a (straight) q-*cochain of size ϵ *is a function c on $q+1$-uples of diameter $\leq \epsilon$. Its ϵ-absolute value is*

$$|c|_{\epsilon} = \sup\{c(\Delta)\,;\, diam(\Delta) \leq \epsilon\}.$$

In other words, straight cochains of size ϵ coincide with simplicial cochains on the simplicial complex whose vertices are points of X and a q-face joins $q+1$ vertices as soon as all pairwise distances are $\leq \epsilon$. Therefore, they form a complex $\mathcal{C}^{\cdot}_{\epsilon}$. There is a dual complex of chains $\mathcal{C}_{\cdot,\epsilon}$.

Lemma

Assume X is a manifold with boundary, or bi-Hölder homeomorphic to such, then the inductive limit complex $\varinjlim \mathcal{C}^{\cdot}_{\epsilon}$ computes cohomology.

Definition

Given a cohomology class κ and a number $\nu > 0$, one can define the ν-norm

$$\|\kappa\|_{\nu} = \liminf_{\epsilon \to 0} \epsilon^{-\nu} \inf\{|c|_{\epsilon} \mid \textit{ cochains } c \textit{ of size } \epsilon \textit{ representing } \kappa\}.$$

Definition

Let X be a metric space, let $q \in \mathbb{N}$. Define the metric weight *$MW_q(X)$ as the supremum of numbers ν such that there exist arbitrarily small open sets $U \subset M$ and nonzero straight cohomology classes $\kappa \in H^q(U, \mathbb{R})$ with finite ν-norm $\|\kappa\|_\nu < +\infty$.*

Proposition

In a Riemannian manifold with boundary, all straight cocycles c representing a nonzero class κ of degree q satisfy $|c|_\epsilon \geq const.(\kappa)\,\epsilon^q$. In other words, $\|\kappa\|_q > 0$.

Proof. Fix a cycle c' such that $\kappa(c') > 0$. Subdivide it as follows : fill simplices with geodesic singular simplices, subdivide them and keep only their vertices. This does not change the homology class. The number of simplices of size ϵ thus generated is $\leq \mathrm{const.}(c')\,\epsilon^{-q}$. For any representative c of size ϵ of κ,

$$\kappa(c') = c(c') \leq \mathrm{const.}\,\epsilon^{-q}|c|_\epsilon.\ q.e.d.$$

Corollary

Euclidean n-space has $MW_q \leq q$ for all $q = 1, \ldots, n-1$.

Proposition

Let $f : X \to Y$ be a C^α-Hölder continuous homeomorphism. Let $\kappa \in H^q(Y,\mathbb{R})$. Then

$$\|\kappa\|_\nu < +\infty \Rightarrow \|f^*\kappa\|_{\nu\alpha} < +\infty.$$

In particular, $MW_q(X) \geq \alpha MW_q(Y)$.

Proof. If σ is a straight simplex of size ϵ in X, $f(\sigma)$ has size $\epsilon' \leq \|f\|_{C^\alpha}\,\epsilon^\alpha$ in Y. If c is a representative of κ, f^*c is a representative of $f^*\kappa$, and

$$\begin{aligned}
\epsilon'^{-\nu}|c|_{\epsilon'} &\geq \epsilon'^{-\nu}|c(f(\sigma))| \\
&= \epsilon'^{-\nu}|f^*c(\sigma)| \\
&\geq \|f\|_{C^\alpha}^{-\nu}\,\epsilon^{-\nu\alpha}|f^*c(\sigma)|.
\end{aligned}$$

Therefore

$$\epsilon^{-\nu\alpha}|f^*c|_\epsilon \leq \|f\|_{C^\alpha}^{\nu}\,\epsilon'^{-\nu}|c|_{\epsilon'}.$$

This leads to

$$\|f^*\kappa\|_{\nu\alpha} \leq \|f\|_{C^\alpha}^{\nu}\,\|\kappa\|_\nu.\ q.e.d.$$

Let G be a Carnot group with Lie algebra $\mathfrak{g}$. Left-invariant differential forms on G split into homogeneous components under the dilations δ_ϵ,

$$\Lambda^*\mathfrak{g}^* = \bigoplus_w \Lambda^{*,w} \quad \text{where} \quad \Lambda^{*,w} = \{\alpha \,|\, \delta_\epsilon^*\alpha = \epsilon^w\alpha\}.$$

Therefore Lie algebra cohomology splits $H^q(\mathfrak{g}) = \bigoplus_w H^{q,w}(\mathfrak{g})$.

Example

If $G = Heis^{2m+1}$ is the Heisenberg group, for each degree $q \neq 0, 2m+1$,

$$\Lambda^q\mathcal{G}^* = \Lambda^{q,q} \oplus \Lambda^{q,q+1},$$

where $\Lambda^{q,q} = \Lambda^q(V^1)^*$ and $\Lambda^{q,q+1} = \Lambda^{q-1}(V^1)^* \otimes (V^2)^*$.

This gradation by weight depends on the group structure. What remains for general sub-Riemannian manifolds is a filtration.

Definition

Let (M,Δ) be a sub-Riemannian manifold, $m \in M$. Say a q-form α on T_mM has weight $\geq w$ if it vanishes on q-vectors of $\Delta^{i_1} \otimes \cdots \otimes \Delta^{i_q}$ whenever $i_1 + \cdots + i_q < w$. If (M,Δ) is equiregular, such forms constitute a subbundle $\Lambda^{q,\geq w}T^*M$. The space of its smooth sections is denoted by $\Omega^{*,\geq w}$.

Note that each $\Omega^{*,\geq w}$ is a differential ideal in Ω^*.

Proposition

Let M be an equiregular sub-Riemannian manifold. Let $U \subset M$ be a bounded open set with smooth boundary. Let ω be a closed differential form on U of weight $\geq w$. Then, for every ϵ small enough, the cohomology class $\kappa \in H^q(U, \mathbb{R})$ of ω can be represented by a straight cocycle c_ϵ (maybe defined on a slightly smaller homotopy equivalent open set) such that $|c_\epsilon|_\epsilon \leq const.\,\epsilon^w$. In other words, $\|\kappa\|_w < +\infty$.

Proof. In the case of a Carnot group G. Use the exponential map to push affine simplices in the Lie algebra to the group. Fill in all straight simplices in G of unit Carnot-Carathéodory size with such affine singular simplices. Apply δ_ϵ and obtain a filling σ_ϵ for each straight simplex σ in G of Carnot-Carathéodory size ϵ. Define a straight cochain c_ϵ of size ϵ on U by

$$c_\epsilon(\sigma) = \int_{\sigma_\epsilon} \omega.$$

Since ω is closed, Stokes theorem shows that c_ϵ is a cocycle. Its cohomology class in $H^q(U', \mathbb{R}) \simeq H^q(U, \mathbb{R})$ is the same as ω's. Furthermore,

$$|c_\epsilon(\sigma)| = \int_{\sigma_1} \delta_\epsilon^* \omega \leq V \, \|\delta_\epsilon^* \omega\|_\infty \leq \mathrm{const.}(\omega)\, \epsilon^w. \ q.e.d.$$

Definition

Let M be a sub-Riemannian manifold. Define the algebraic weight $AW_q(M)$ as the largest w such that there exists arbitrarily small open sets with smooth boundary $U \subset M$ and nonzero classes in $H^q(U,\mathbb{R})$ which can be represented by closed differential forms of weight $\geq w$.

Remark

Equiregular sub-Riemannian manifolds satisfy $MW_q \geq AW_q$.

Corollary

Let M be a sub-Riemannian manifold. Then for all $q = 1, \ldots, n-1$, $\alpha(M) \leq \frac{q}{AW_q}$.

So our goal now is to show that for certain sub-Riemannian manifolds, for certain degrees q, in every open set, every closed differential q-form is cohomologous to a form of high weight.

As a warm up, let us treat codimension 1 forms.

Proposition (Gromov)

Let (M,Δ) be a sub-Riemannian manifold of Hausdorff dimension Q. Every closed $n-1$-form is cohomologous to a form of weight $\geq Q-1$. Thus $AW_{n-1}(M)=Q-1$.

Proof. The filtration of $\Delta\subset\Delta^2\subset\cdots\subset TM$ induces a filtration $A^{n-1}=\mathcal{F}^1\subset\mathcal{F}^2\subset\cdots\subset\Omega^{n-1}$ as follows : $\alpha\in\mathcal{F}^j$ if and only if there exist an n-form ω and a vectorfield $Z\subset\Delta^j$ such that $\alpha=\iota_Z\omega$. We show that for all $j\geq 1$, $\mathcal{F}^{j+1}\subset\mathcal{F}^j+im(d)$.
Let ω be an n-form and X, Y be vectorfields such that $X\in\Delta$, $Y\in\Delta^j$. Then, using Lie derivatives, $\mathcal{L}_X(\iota_Y\omega)=\iota_{\mathcal{L}_X(Y)}\omega+\iota_Y(\mathcal{L}_X\omega)\iota_{[X,Y]}\omega \mod \mathcal{F}^j$. Thanks to Cartan's formula, $\mathcal{L}_X(\iota_Y\omega)=d(\iota_X\iota_Y\omega)+\iota_X d(\iota_Y\omega)\in im(d)+\mathcal{F}^j$. Therefore $\iota_{[X,Y]}\omega\in im(d)+\mathcal{F}^j$.
Let $\alpha\in\mathcal{F}^{j+1}$, $\alpha=\iota_Z\omega$ with $Z\in\Delta^{j+1}$. Write $Z=\sum_\ell a_\ell[X_\ell,Y_\ell]$ where a_ℓ are functions, X_ℓ, Y_ℓ are vectorfields, $X_\ell\in\Delta$, $Y_\ell\in\Delta^j$. Then $\alpha=\sum_\ell\iota_{[X_\ell,Y_\ell]}\omega_\ell$ (where $\omega_\ell=a_\ell\omega$), therefore $\alpha\in im(d)+\mathcal{F}^j$. This shows that $\mathcal{F}^{j+1}\subset\mathcal{F}^j+im(d)$.
The bracket generating assumption, $\Delta^r=TM$, implies that $\Omega^{n-1}=\mathcal{F}^r\subset im(d)+\mathcal{F}^1=im(d)+A^{n-1}$. Given a closed $n-1$-form ϕ, the equation

$$d\psi=-\phi \mod A^{n-1}$$

admits a smooth solution $\beta\in\Omega^{n-2}$. Then $\phi+d\psi\in A^{n-1}$ is a horizontal form. q.e.d.

Rumin's complex is a subcomplex of the de Rham complex, homotopic to it, consisting of differential forms of preferably high weights. The construction requires to invert the weight 0 component d_0 of d. d_0 identifies with the exterior differential on left-invariant forms on tangent Lie algebras $\mathfrak{g}_m$. So one needs that the cohomology $m \mapsto H^{q,w}(\mathfrak{g}_m)$ be constant, whence the word *equihomological*. It turns out that the obstruction for cohomologing q-forms towards weight $> w$ is $H^{q,w}(\mathfrak{g}_m)$.

Theorem (Rumin 2005)

Let M be a equihomological sub-Riemannian manifold. Assume that there exists a point $m \in M$ such that, in the cohomology of the tangent Lie algebra $\mathfrak{g}_m$, $H^{q,w'}(\mathfrak{g}_m) = 0$ for all $w' < w$. Then $AW_q(M) \geq w$.

On Carnot groups, the grading of cohomology is compatible with Poincaré duality, $H^{q,w}(\mathfrak{g}) = H^{n-q,Q-w}(\mathfrak{g})$. So

$$\exists m \; H^{n-q}(\mathfrak{g}_m) = H^{n-q,\leq Q-w}(\mathfrak{g}_m) \quad \Rightarrow \quad AW_q(M) \geq w.$$

Example

Degree $n-1$. On any Carnot Lie algebra $\mathfrak{g}$, closed 1-forms belong to $(V^1)^* = \Lambda^{1,1}$, so $H^1(\mathfrak{g}) = H^{1,1}(\mathfrak{g})$, and $AW_{n-1}(M) \geq Q-1$.

Example

Contact case. Closed m-forms belong to $\Lambda^{m,m}$. *Therefore* $H^m(\mathfrak{g}) = H^{m,m}(\mathfrak{g})$, *and* $AW_{m+1}(M) \geq m+2$.

Indeed, if $\omega \in \Lambda^{m,m+1}$, $\omega = \theta \wedge \phi$ where $\theta \in (V^2)^*$, $\phi \in \Lambda^{m-1,m-1}$, $(d\omega)^{m+1,m+1} = (d\theta) \wedge \phi \neq 0$ since $d\theta$ is symplectic on Δ.

Example

Contact case. Closed m-forms belong to $\Lambda^{m,m}$. Therefore $H^m(\mathfrak{g}) = H^{m,m}(\mathfrak{g})$, and $AW_{m+1}(M) \geq m+2$.

Indeed, if $\omega \in \Lambda^{m,m+1}$, $\omega = \theta \wedge \phi$ where $\theta \in (V^2)^*$, $\phi \in \Lambda^{m-1,m-1}$, $(d\omega)^{m+1,m+1} = (d\theta) \wedge \phi \neq 0$ since $d\theta$ is symplectic on Δ.

Example

Generic sub-Riemannian case. Let $h = dim(\Delta)$ and k be such that $h - k \geq (n-h)k$. Then $H^k(\mathfrak{g}_m) = H^{k,k}(\mathfrak{g}_m)$, thus $AW(M) \geq Q - k$.

Let θ be a $\mathbb{R}^{n-h}$-valued 1-form defining Δ. Say a k-plane $S \subset \Delta_m$ is *isotropic* if $d\theta_{|S} = 0$. Say S is *regular* if the map $\Delta_m \to Hom(S, \mathbb{R}^{n-h})$, $u \mapsto (\iota_u d\theta)_{|S}$ is onto. $h - k \geq (n-h)k$ is a necessary condition for existence of regular isotropic horizontal k-planes. It is generically sufficient. When it holds, closed left-invariant k-forms have to be of weight k, so $H^k(\mathfrak{g}_m) = H^{k,k}(\mathfrak{g}_m)$ for $m \in M$.

Remark

The method just exposed seems to cover all presently known results on the Hölder homeomorphism problem.