Flexibility of surface groups in semi-simple Lie groups

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Definition

Let G be the group of real points of an algebraic group. Let Γ be a finitely generated group. Say a homomorphism $\phi : \Gamma \to G$ is flexible *if Zariski dense homomorphisms are dense in a neighborhood of ϕ in $Hom(\Gamma, G)$.*

Problem

Determine which homomorphisms of surface groups to almost simple Lie groups are flexible.

Plan of lecture

We survey (global) rigidity results concerning surface groups in semisimple Lie groups.

We complement them with new flexibility results.

Let X be a Hermitian symmetric space, with Kähler form Ω (the metric is normalized so that the minimal sectional curvature equals -1). Let Σ be a closed surface of negative Euler characteristic, let $\Gamma = \pi_1(\Sigma)$ act isometricly on X. Pick a smooth equivariant map $\tilde{f} : \tilde{\Sigma} \to X$.

Definition

Define the Toledo invariant of the action $\rho : \Gamma \to Isom(X)$ by

$$T_\rho = \frac{1}{2\pi} \int_\Sigma \tilde{f}^* \Omega.$$

Then

1. T_ρ depends continuously on ρ.
2. There exists $\ell_X \in \mathbb{Q}$ such that $T_\rho \in \ell_X \mathbb{Z}$.
3. $|T_\rho| \leq |\chi(\Sigma)||\mathrm{rank}(X)|$.

Example

When $X = H^1_{\mathbb{C}}$ is a disk, inequality $|T_\rho| \leq |\chi(\Sigma)|$ is due to J. Milnor (1958). Furthermore $\ell_X = 1$, T takes all integer values between $-|\chi(\Sigma)|$ and $|\chi(\Sigma)|$.

Theorem

(W. Goldman, 1980). *Let $X = H^1_{\mathbb{C}}$. The level sets of T coincide with the connected components of the character variety $\chi(\Gamma, PU(1,1))$. Furthermore $|T_\rho| = |\chi(\Sigma)|$ if and only if $\rho(\Gamma)$ is discrete and cocompact in $PU(1,1) = Isom(H^1_{\mathbb{C}})$.*

Note that all components of $\chi(\Gamma, PU(1,1))$ have the same dimension $3|\chi(\Sigma)|$.

Theorem

(D. Toledo, 1979, 1989). *Let $X = H^n_{\mathbb{C}}$ have rank 1. Then $|T_\rho| \leq |\chi(\Sigma)|$. Furthermore, $|T_\rho| = |\chi(\Sigma)|$ if and only if $\rho(\Gamma)$ stabilizes a complex geodesic $H^1_{\mathbb{C}}$ in X and acts cocompactly on it.*

It follows that, for $n \geq 2$, different components of $\chi(\Gamma, PU(n,1))$ can have different dimensions.

Definition

Actions ρ *such that* $|T_\rho| = |\chi(\Sigma)|\mathrm{rank}(X)$ *are called* maximal representations.

Examples

Pick cocompact actions $\rho_1, \ldots, \rho_r$ *of* Γ *on* $H^1_{\mathbb{C}}$. *Then the direct sum representation on the polydisk* $(H^1_{\mathbb{C}})^r$ *is maximal. When the polydisk is embedded in a larger symmetric space of rank* r*, it remains maximal. It follows that all Hermitian symmetric spaces admit maximal representations.*

Proposition

(Toledo, 1987). *In case* X *is Siegel's upper half space (i.e.* $Isom(X) = Sp(n, \mathbb{R})$*), such actions can be deformed to become Zariski dense.*

But this may fail for other Hermitian symmetric spaces.

Theorem

(L. Hernàndez Lamoneda, 1991, S. Bradlow, O. García-Prada, P. Gothen, 2003). *Maximal reductive representations of* Γ *to* $PU(p, q)$, $p \leq q$, *can be conjugated into* $P(U(p, p) \times U(q - p))$.

Tube type

Gromov, Thurston, Domic-Toledo : Surfaces admit ideal triangulations. T_ρ is the sum of Kähler areas of ideal triangles in an ideal triangulation.
Observation : in $H^n_{\mathbb{C}}$, the Kähler area of ideal triangles takes a full interval of values for $n > 1$, whereas only one value for $n = 1$. In a polydisc, finitely many values.

Theorem

(J.-L. Clerc, B. Ørsted 2003). *The Kähler area of an ideal triangle either takes a full interval of values (non tube type case) or finitely many values (tube type case), in both cases bounded by* $\frac{1}{2}\mathrm{rank}(X)$. *Equality holds iff the triangle is contained in a subsymmetric space of tube type.*

Definition

Say a Hermitian symmetric space is of tube type *if it can be realized as a domain in* $\mathbb{C}^n$ *of the form* $\mathbb{R}^n + iC$ *where* $C \subset \mathbb{R}^n$ *is a proper open cone.*

Examples

Siegel's upper half spaces and Grassmannians with isometry groups $PO(2, q)$ *are of tube type. The Grassmannian* $\mathcal{D}_{p,q}$, $p \leq q$, *with isometry group* $PU(p, q)$ *is of tube type iff* $p = q$. *Products of tube type spaces are of tube type, so polydisks are of tube type.*

Tube type rigidity

Theorem

(Burger, Iozzi, Wienhard, 2003). *Let Γ be a closed surface group and X a Hermitian symmetric space. Every maximal representation $\Gamma \to \mathrm{Isom}(X)$ stabilizes a tube type subsymmetric space Y and is Zariski dense in $\mathrm{Isom}(Y)$.*

In particular, maximal representations of surface groups in non tube type Hermitian symmetric spaces are globally rigid.

Examples

In case X is the n-ball $\mathcal{D}_{1,n}$ (resp. $\mathcal{D}_{p,q}$), one recovers Toledo's (resp. Barlow et al.) results.

Flexibility problem

Problem

Characterize flexible actions of closed surface groups on (non necessarily Hermitian) symmetric spaces.

All previously known examples of flexibility can be obtained by bending representations.

Examples

(Burger, Iozzi, Wienhard). *In a tube type Hermitian symmetric space, surface groups stabilizing a maximal polydisk and acting diagonally on it (Toledo's 1987 examples) are flexible.*

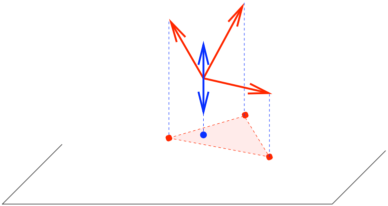
We shall give a somewhat computable flexibility criterion. When flexibility is proven, deformations do not look like bending.

Flexibility theorem

Let G be a semisimple Lie group. Let Γ be the fundamental group of a closed surface of high enough genus. Let $\phi : \Gamma \to G$ be a homomorphism with reductive Zariski closure. Let $\mathfrak{c}$ be the center of the centralizer of $\phi(\Gamma)$. It splits the Lie algebra of G into real root spaces $\mathfrak{g}_{\lambda,\mathbb{R}}$ which carry natural symplectic structures. In particular, to each pure imaginary root λ, there corresponds a symplectic representation ρ_λ and a Toledo invariant T_λ.

Definition

Among the above roots, let P be the subset of pure imaginary roots λ such that ρ_λ is a maximal representation with $T_\lambda > 0$. Say $\mathfrak{c}$ is balanced with respect to ϕ if 0 belongs to the interior of the sum of the convex hull of the imaginary parts of elements of P and the linear span of the real and imaginary parts of roots not in $\pm P$.



Theorem

ϕ is flexible if and only if $\mathfrak{c}$ is balanced with respect to ϕ.

Consequences

Corollary

If ϕ is not flexible, then one of the noncompact irreducible factors of the symmetric space of a Levi factor of the Zariski closure of $\phi(\Gamma)$ is Hermitian of tube type, and the action of Γ on this factor is a maximal representation.

In case G is of Hermitian type, this is close to a converse to the result by Burger et al. (up to the restriction on genus).

Corollary

If G has real rank one, then ϕ is flexible, unless G is $PU(m,1)$ and $\phi(\Gamma)$ is discrete, cocompact in a conjugate of $P(U(1,1)\times U(m-1))\subset PU(m,1)$.

This is an exact converse to Toledo's theorem (up to the restriction on genus).

Work in progress

Classify all non flexible homomorphisms. Based on classification of root systems.

Easy cases : complex Lie groups, $PSL(n,\mathbb{R})$, $PSL(n,\mathbb{H})$, in which cases $P=\emptyset$.

Remark

1. The restriction on genus is probably irrelevant.
2. Non constructive : deformations are not given by explicit formulae nor geometric constructions.

Theorem

(W. Goldman, 1985). *If Γ is a surface group and ρ is reductive, then, in a neighborhood of the conjugacy class of ρ, $Hom(\Gamma, G)/G$ is analytically equivalent to*

$$\{u \in H^1(\Gamma, \mathfrak{g}_{ad\circ\rho}) \,|\, [u \smile u] = 0\}/Z_G(\rho(\Gamma)).$$

Here, smiling bracket denotes cup-product : $H^1(\Gamma, \mathfrak{g}_{ad\circ\rho}) \to H^2(\Gamma, \mathfrak{g}_{ad\circ\rho})$.

Remark

This can prove flexibility without providing explicit deformations.

The dimension of $H^1(\Gamma, \mathfrak{g}_{ad\circ\rho})$ can be computed via Euler characteristic and Poincaré duality: $H^2(\Gamma, \mathfrak{g}) = (H^0(\Gamma, \mathfrak{g}^*))^*$.

Cup-products can be computed thanks to

Theorem

(W. Meyer, 1972). *Let (E, Ω) be a flat symplectic vector bundle over Σ. The quadratic form $Q(a) = \int_\Sigma \Omega(a \smile a)$ on $H^1(\Sigma, E)$ is nondegenerate of signature $4c_1(E, \Omega)$.*

Levi factors of centralizers

Notation

$\chi(\Gamma, G) = Hom(\Gamma, G)/G$.

Proposition

- *The dimension of* $\chi(\Gamma, G)$ *at points with trivial centralizers is* $|\chi(\Sigma)|\dim(G)$.
- *If the genus of* Σ *is large enough, non Zariski dense homomorphisms form a subset of* $\chi(\Gamma, G)$ *of dimension less than* $|\chi(\Sigma)|\dim(G)$.
- *Therefore it is sufficient to prove density of smooth points in neighborhoods of homomorphisms with nontrivial centralizers.*

Proposition

Levi factors of centralizers are treated by explicitly deforming given cohomology classes u *such that* $[u \smile u] = 0$.

Such deformations v, satisfying $[v \smile u] = 0$, $[v \smile v] = 0$, are obtained as zeroes of sections of bundles on the complex Grassmannian. They exist due to nonvanishing of powers the top Chern class of the universal bundle (requires high genus).

Let $\mathfrak{c}$ denote the center of the centralizer of $\phi(\Gamma)$. $\mathfrak{g}\otimes\mathbb{C}$ splits under $\mathfrak{c}$ into root spaces $\mathfrak{g}_\lambda$. $H^1(\Gamma,\mathfrak{g})\otimes\mathbb{C}$ splits accordingly.

Lemma

$[\cdot\smile\cdot]$ vanishes on each $H^1(\Gamma,\mathfrak{g}_\lambda)$.
$H^1(\Gamma,\mathfrak{g}_\lambda)$ and $H^1(\Gamma,\mathfrak{g}_\mu)$ are orthogonal with respect to $[\cdot\smile\cdot]$ unless $\lambda+\mu=0$.

On each $\mathfrak{g}_{\lambda,\mathbb{R}}=\mathfrak{g}\cap(\mathfrak{g}_\lambda\oplus\mathfrak{g}_{-\lambda})$, all ad_Z, $Z\in\mathfrak{c}$ are proportional. Therefore the corresponding alternating forms $(X,Y)\to Z\cdot[X,Y]$ are proportional to a single symplectic form Ω_λ. On $H^1(\Gamma,\mathfrak{g}_{\lambda,\mathbb{R}})$, all $Z\cdot[\cdot\smile\cdot]$ are proportional to the quadratic form $Q_\lambda(u,u)=\int_\Sigma\Omega_\lambda(u\smile u)$. Let $\rho_\lambda:\Gamma\to Sp(\mathfrak{g}_{\lambda,\mathbb{R}},\Omega_\lambda)$ denote the composed symplectic linear representation, and T_λ the corresponding Toledo invariant. Meyer's formula yields

Lemma

If $\lambda\neq 0$, Q_λ is nondegenerate and its index is equal to $4T_\lambda$. Therefore

$$4|T_\lambda|\leq\dim(H^1(\Gamma,\mathfrak{g}_{\lambda,\mathbb{R}}))=-\chi(\Sigma)\dim(\mathfrak{g}_{\lambda,\mathbb{R}}).$$

In particular, Q_λ is definite if and only if ρ_λ is a maximal representation.

End of proof

On $H^1(\Gamma, \mathfrak{g})$, $[\cdot \smile \cdot] = \sum_\lambda Q_\lambda \lambda$. Smooth points of $\chi(\Gamma, G)$ correspond to classes $u = \sum_\lambda u_\lambda$, $u_\lambda \in H^1(\Gamma, \mathfrak{g}_{\lambda,\mathbb{R}})$, such that $[u \smile u] = 0$ and $u_\lambda \neq 0$ for a spanning set of λ's. If u is not a smooth point, too many u_λ's vanish, perturb them to nonzero v_λ's. Indefinite quadratic forms Q_λ allow arbitrary moves in the direction of λ, but positive definite ones allow one to reach only points of a convex hull. If 0 belongs to it, nonzero v_λ's can be found such that $[v \smile v] = 0$.

The fact that the Zariski closure of $\phi(\Gamma)$ has to be Hermitian of tube type follows from the following theorem.

Theorem

(Burger, Iozzi, Wienhard, 2007). *Let S be a semisimple Lie group whose symmetric space is Hermitian. Let $\rho : \Gamma \to S$ be a maximal representation of a surface group. Then ρ is tight. Its Zariski closure H is reductive of Hermitian type. The embedding $H \hookrightarrow S$ is tight. If S is of tube type, so is H.*

Calculation of examples (e.g., rank one) relies on further obstructions for ρ_λ to be definite : ρ_λ is not only symplectic, it must be unitary with respect to a Hermitian form of vanishing signature.