Measures constrained by linear PDEs [after de Philippis and Rindler]

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A constant coefficient linear PDE on a vector-valued function $u:\Omega\subset\mathbb{R}^d\to\mathbb{R}^N$ takes the form

$$\mathcal{A}(u)=\sum_\alpha A_\alpha\partial^\alpha u=0,$$

where $A_\alpha\in Hom(\mathbb{R}^N,\mathbb{R}^n)$ are linear maps. Here $\partial^\alpha u=\partial_1^{\alpha_1}\cdots\partial_k^{\alpha_k}u$ denote partial derivatives.

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Example

Assume $u \in C^\infty(\Omega, (\mathbb{R}^d)^*)$ is a differential 1-form on open set Ω. Its exterior differential $du \in C^\infty(\Omega, \Lambda^2(\mathbb{R}^d)^*)$. Then $du = \mathcal{A}(u)$ where $\mathcal{A}$ has order 1. Putting all A_α together into a single linear map

$$\begin{aligned} A \in & \, Hom(\mathbb{R}^d, Hom((\mathbb{R}^d)^*, \Lambda^2(\mathbb{R}^d)^*)) \\ = & \, Hom((\mathbb{R}^d)^* \otimes (\mathbb{R}^d)^*, \Lambda^2(\mathbb{R}^d)^*), \end{aligned}$$

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Assume $u \in C^\infty(\Omega, Hom(\mathbb{R}^d, \mathbb{R}^\ell))$ is a $\mathbb{R}^\ell$-valued differential 1-form. Its exterior differential $du \in C^\infty(\Omega, Hom(\Lambda^2\mathbb{R}^d, \mathbb{R}^\ell))$. Then $du = \mathcal{A}(u)$ where $\mathcal{A}$ has order 1. Putting all A_α together into a single linear map

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Question. Assume a vector-valued measure μ satisfies $\mathcal{A}(\mu)=0$ in distributional sense. Ellipticity should imply restrictions on the singular part μ^s of μ. Which restrictions?

The principal symbol. In Fourier, if $\xi \in (\mathbb{R}^d)^*$,

$$\mathcal{F}(\mathcal{A}(u))(\xi) = A(2\pi i\xi)(\mathcal{F}(u)(\xi)),$$

where $A(\xi) = \sum_\alpha A_\alpha \xi^\alpha \in Hom(\mathbb{R}^N, \mathbb{R}^n)$. It is a sum of homogeneous terms, the term of highest degree $A^k(\xi)$ is the *principal symbol* of $\mathcal{A}$ evaluated at ξ.

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Assume $\mathcal{A}$ is *elliptic*, i.e. $A(\xi)$ is injective for all $\xi \neq 0$. Then $\mathcal{A}(u) = 0 \implies \mathrm{supp}(\mathcal{F}(u)) = \{0\}$, so u is a polynomial.

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Conversely, assume that $\mathcal{A}$ is homogeneous. If there exists nonzero $\xi \in (\mathbb{R}^d)^*$ and $\lambda \in \mathbb{R}^n$ such that $\lambda \in \ker(A(\xi))$, then for every distribution $h : \mathbb{R} \to \mathbb{R}$, the plane wave $u = (h \circ \xi)\lambda$ solves $\mathcal{A}(u) = 0$. Thus $\mathcal{A}$ has nonsmooth solutions.

Nonelliptic behaviours are concentrated in the *wave cone*.

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The wave cone of an order k operator $\mathcal{A}$ *is* $\Lambda_{\mathcal{A}} := \bigcup_{\xi \neq 0} \ker A^k(\xi) \subset \mathbb{R}^N$.

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For the exterior differential d, $\Lambda_{\mathcal{A}} = \bigcup_{\xi\neq 0} \xi\otimes\mathbb{R}^{\ell} \subset (\mathbb{R}^d)^*\otimes\mathbb{R}^{\ell} = Hom(\mathbb{R}^d,\mathbb{R}^{\ell})$ *consists of all rank* ≤ 1 *linear maps* $\mathbb{R}^d \to \mathbb{R}^{\ell}$.

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Indeed, the kernel of $\xi \wedge \cdot : \Lambda^1 \to \Lambda^2$ coincides with the image of $\xi \wedge \cdot : \Lambda^0 \to \Lambda^1$. This reflects (at the symbol level) the (local) exactness of the de Rham complex.

Theorem (de Philippis-Rindler 2018)

Let μ be an $\mathbb{R}^N$-valued measure which solves $\mathcal{A}(\mu) = 0$ on open set Ω. Then $\frac{d\mu}{d|\mu|}(x) \in \Lambda_{\mathcal{A}}$ for $|\mu|^s$-almost every point $x \in \Omega$.

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When $\mathcal{A}$ is the exterior differential on 1-forms, Hom$(\mathbb{R}^d, \mathbb{R}^\ell)$-valued measures solving $d\mu = 0$ are differentials of $\mathbb{R}^\ell$-valued BV functions.

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This is Alberti's Rank One Theorem (1993), solving a conjecture of Ambrosio-De Giorgi (1988).

I stick to degree 1 operators. Let μ be an $\mathbb{R}^N$-valued measure solving $\mathcal{A}(\mu) = 0$. Let $|\mu|$ denote its total variation. Consider the Radon-Nikodym decomposition

$$\mu = \frac{d\mu}{d|\mu|} d|\mu|.$$

Let $E = \{x \in \mathbb{R}^d \, ; \, \frac{d\mu}{d|\mu|}(x) \notin \Lambda_{\mathcal{A}}\}$. By contradiction, assume that $|\mu|^s(E) > 0$.

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One can pick $x_0 \in E$ and a subsequence r_k of radii which satisfy the following. Let μ_k denote the blowing up of μ at x_0, at scale r_k, normalized by $|\mu|(B(r_k))$.

1. $|\mu_k|$ converges vaguely to a measure ν.
2. $\lambda_0 = \frac{d\mu}{d|\mu|}(0) \notin \Lambda_{\mathcal{A}}$.
3. μ_k converges vaguely to the measure $\nu\lambda_0$.
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On the other hand, one shows that $|\mu_k|$ converges to ν in total variation. This is a contradiction, since $\||\mu_k| - \nu| \geq |\mu_k|^s(B(r_k))/|\mu|(B(r_k))$ which tends to 1.

First step : absolute continuity of tangent measures

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Since $\mathrm{supp}(\mathcal{F}(\nu)) = \{0\}$, ν is a (polynomial) function times Lebesgue measure.

Second step : from vague convergence to convergence in total variation

Let $\nu_k = |\mu_k|^s$.
Recall that $\frac{|\mu|^s(B(r_k))}{|\mu|(B(r_k))}$ tends to 1. One can furthermore assume that $\fint_{B(r_k)} |\frac{d\mu}{d|\mu|} - \lambda_0|\, d|\mu|^s$ tends to 0. It follows that $|\nu_k\lambda_0 - \mu_k|(B(1))$ tends to 0.

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Pick a smooth cut-off function χ. Then $|\chi\nu_k\lambda_0 - \chi\mu_k|(\mathbb{R}^d)$ tends to 0.

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Since $\mathcal{A}(\mu_k) = 0$,

$$\mathcal{A}(\chi\nu_k\lambda_0) = \mathcal{A}(\chi\nu_k\lambda_0 - \chi\mu_k) + A(d\chi)\mu_k.$$

We shall see that ellipticity allows to invert $\mathcal{A}$ with estimates. By density, one can replace measures with smooth functions and total variation with L^1 norm. The goal is to show that $\chi\nu_k$ is precompact in L^1. The estimates will stem from Hörmander-Mikhlin's multiplier theorem.

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Taking Fourier transforms,

$$\forall\xi, \quad \mathcal{F}(\chi\nu_k)A(\xi)\lambda_0 = A(\xi)\mathcal{F}(V_k)(\xi) + \mathcal{F}(R_k)(\xi),$$

where $V_k = \chi\nu_k\lambda_0 - \chi\mu_k$ tends to 0 in L^1 and $R_k = A(d\chi)\mu_k$ is bounded in L^1.

Inner multiplying with $\overline{A(\xi)\lambda_0}$, one gets

$$|A(\xi)\lambda_0|^2\mathcal{F}(\chi\nu_k) = A(\xi)\lambda_0 \cdot A(\xi)\mathcal{F}(V_k)(\xi) + A(\xi)\lambda_0 \cdot \mathcal{F}(R_k)(\xi),$$

and then

$$(1+|A(\xi)\lambda_0|^2)\mathcal{F}(\chi\nu_k) = A(\xi)\lambda_0 \cdot A(\xi)\mathcal{F}(V_k)(\xi) + A(\xi)\lambda_0 \cdot \mathcal{F}(R_k)(\xi) + \mathcal{F}(\chi\nu_k).$$

So $\mathcal{F}(\chi\nu_k) = T_0(V_k) + T_1((Id-\Delta)^{-1/2}R_k) + T_2((Id-\Delta)^{-1}\mathcal{F}(\chi\nu_k))$ is the sum of three terms, involving operators defined by Fourier multipliers

$$\begin{aligned} m_0(\xi) &= (1+|A(\xi)\lambda_0|^2)^{-1}\overline{A(\xi)\lambda_0} \cdot A(\xi), \\ m_1(\xi) &= (1+|A(\xi)\lambda_0|^2)^{-1}(1+4\pi^2|\xi|^2)^{1/2}\overline{A(\xi)\lambda_0}, \\ m_2(\xi) &= (1+|A(\xi)\lambda_0|^2)^{-1}(1+4\pi^2|\xi|^2). \end{aligned}$$

$(Id-\Delta)^{-1/2}$ is bounded and compact from L^1 to some L^q, $q>1$. T_1 and T_2 are bounded from L^q to L^q. Therefore $w_k := T_1((Id-\Delta)^{-1/2}R_k) + T_2((Id-\Delta)^{-1}\mathcal{F}(\chi\nu_k))$ is precompact in L^1.

Since $\lambda_0 \notin \Lambda_{\mathcal{A}}$, $|A(\xi)\lambda_0|/|\xi|$ is bounded away from 0. This implies that T_0 is bounded from L^1 to the Lorentz space $L^{1,\infty}$ (Hörmander-Mikhlin). Hence $u_k := T_0(V_k)$ tends to 0 in measure.

Since $u_k + w_k = \chi\nu_k \geq 0$, u_k^- is dominated by $|w_k|$. By Vitali's convergence theorem (strengthening of dominated convergence theorem where a.e. convergence is replaced with convergence in measure), u_k^- tends to 0 in L^1.

Since $V_k \to 0$ in distributional sense, so does $u_k = T_0(V_k)$. For every smooth cut-off function ϕ, $0 \leq \phi \leq 1$,

$$\int \phi|u_k| = \int \phi u_k + 2\int \phi u_k^- \leq \int \phi u_k + 2\int u_k^-$$

tends to 0, so u_k tends to 0 in L^1.

Hence $\chi\nu_k = u_k + w_k$ is precompact in L^1. Its vague limit is $\chi\nu$, so it subconverges in L^1 to $\chi\nu$. This shows that $|\mu_k|^s$ converges to ν in total variation.

This contradicts the fact that ν is absolutely continuous, and completes the proof of the theorem.