Conformal Hölder exponents

P. Pansu

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Proposition

Let M have Hausdorff dimension Q. Then $\alpha(M) \leq \frac{n}{Q}$.

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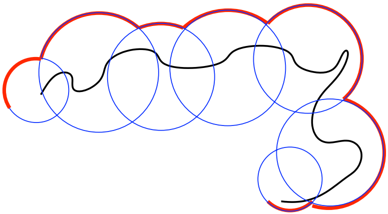
$$vol(D)^{Q-1/Q} \leq const.\, \mathcal{H}^{Q-1}(\partial D).$$

It follows that the boundary of any non smooth domain Ω has Hausdorff dimension at least $Q-1$. Indeed, cover $\partial\Omega$ with balls B_j and apply (*) to $\Omega \cup \bigcup B_j$. This gives a lower bound on $\mathcal{H}^{Q-1}(\partial(\bigcup B_j)) \leq \sum \mathcal{H}^{Q-1}(\partial B_j) \leq const. \sum \mathrm{diameter}(B_j)^{Q-1}$.



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(Gromov 1993). *Let* $\mathbb{H}^m$ *denote* $2m+1$*-dimensional Heisenberg group. Let* $V \subset \mathbb{H}^m$ *be a subset of topological dimension* $m+1$*. Then the Hausdorff dimension of* V *is at least* $m+2$*. It follows that* $\alpha(\mathbb{H}^m) \leq \frac{m+1}{m+2}$.

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Proof. According to topological dimension theory (Alexandrov), there exists an m-dimensional polyhedron P and a continuous map $f : P \to \mathbb{H}^m$ such that every map sufficiently C^0-close to f hits V.

Gromov approximates f with piecewise *horizontal* maps which sweep an open set U. This gives rise to a local projection $p : U \to \mathbb{R}^{m+1}$ such that for every ball B, the tube $p^{-1}(p(B))$ has volume $\leq$ const. $\mathrm{diameter}(B)^{m+2}$.

Cover V with balls B_j. The corresponding tubes $T_j = p^{-1}(p(B_j))$ cover U. Then the volume of U is less than $\sum \mathrm{diameter}(B_j)^{m+2}$, which shows that $dim_{Hau}(V) \geq m+2$.

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Theorem

(Gromov 1993). *Let M be a generic subRiemannian manifold of dimension n, Hausdorff dimension Q, with an h-dimensional distribution. Let $k \leq h$ be such that $h-k \geq (n-h)k$. Then $\alpha(M) \leq \frac{n-k}{Q-k}$.*

Curvature pinching

Definition

*Let M be a Riemannian manifold. Let $-1 \leq \delta < 0$. Say M is δ-*pinched *if sectional curvature ranges between -1 and δ. Define the* optimal pinching $\delta(M)$ *of M as the least $\delta \geq -1$ such that M is biLipschitz to a δ-pinched simply connected Riemannian manifold.*

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Example

Rank one symmetric spaces of noncompact type are hyperbolic spaces over the reals $H^n_{\mathbb{R}}$, the complex numbers $H^m_{\mathbb{C}}$, the quaternions $H^m_{\mathbb{H}}$, and the octonions $H^2_{\mathbb{O}}$.
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Question

What is the optimal pinching of $H^m_{\mathbb{C}}$?

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Say two geodesic rays in a Riemannian manifold are asymptotic *if their Hausdorff distance is finite. The* visual boundary *of a negatively curved manifold is the set of asymptoticity classes of geodesic rays.*

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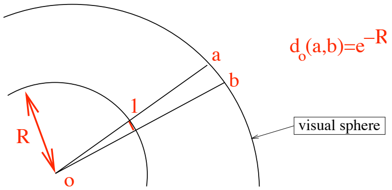
Facts.

- The visual boundary, seen from a point o, is a sphere (use polar coordinates).
- It carries a *visual metric* d_o.
- Different visual metrics d_o and $d_{o'}$ are equivalent.
- BiLipschitz maps between negatively curved Riemannian manifolds induce *quasisymmetric* maps between ideal boundaries.



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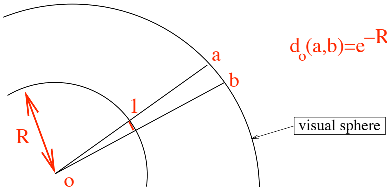

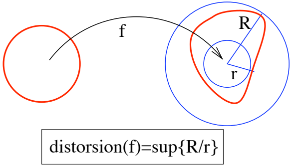

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Proposition

Let M be a simply connected δ-pinched Riemannian manifold. Equip the ideal boundary X of M with a visual metric. The natural homeomorphism $S^{n-1} \to X$ is C^{α} with $\alpha = \sqrt{-\delta}$, and its inverse is Lipschitz.

Indeed, geodesics from a unit ball to a point come together exponentially fast, with exponents ranging from $\sqrt{-\delta}$ to 1 (Rauch comparison theorem, 1950's).

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Question

Let X be a nonRiemannian subRiemannian manifold. Let $\alpha > 1/2$. Does there exist quasisymmetricly equivalent metrics on X which locally admit C^α homeomorphisms from Euclidean space ? With Lipschitz inverses ?

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Proposition

Assume (X, d) has topological dimension n and conformal dimension $Q > n$. Pick a quasisymmetricly equivalent metric d' on X. If $\alpha > \frac{n}{Q}$, there are no local C^{α} homeomorphisms $\mathbb{R}^n \to (X, d')$.

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Example

Carnot groups have conformal dimension equal to their Hausdorff dimension. Heisenberg group $\mathbb{H}^m$ has conformal dimension $2m + 2$. Even up to a quasisymmetric change of metric, no C^α homeomorphisms from $\mathbb{R}^{2m+1}$ if $\alpha > \frac{2m+1}{2m+2}$.

Note: no restriction on inverse mapping.

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Remark

Vertical dilation invariant subgroups often have conformal dimension $Q-1$.

Example

In $\mathbb{H}^1$, subgroup $\{y=0\}$ is a Rickman rug, *i.e. a product $(\mathbb{R}, d) \times (\mathbb{R}, d^{1/2})$, conformal dimension is $1+2=3$.*

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In $\mathbb{H}^m$, $m \geq 2$, subgroup $\{y_m=0\}$ is a Carnot group isometric to $\mathbb{H}^{m-1} \times \mathbb{R}$.

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In $\mathbb{H}^1$, *subgroup* $\{y=0\}$ *is a* Rickman rug, *i.e. a product* $(\mathbb{R}, d) \times (\mathbb{R}, d^{1/2})$, *conformal dimension is* $1+2=3$.

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In $\mathbb{H}^m$, $m \geq 2$, *subgroup* $\{y_m = 0\}$ *is a Carnot group isometric to* $\mathbb{H}^{m-1} \times \mathbb{R}$.

This is not sufficient: a Hölder map might pull back all of these to fractal subsets of $\mathbb{R}^n$.

Leading intuition: Given a function $u : X \to \mathbb{R}$, $\int |\nabla u|^Q$ is a conformal invariant. It may be written

$$\int |\nabla u|^Q = \int_{\mathbb{R}} \left(\int_{u^{-1}(t)} |\nabla u|^{Q-1} \right) dt.$$

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Notation

Given a set function ϕ, a subset $Y \subset X$, let

$$\Phi^{p;\epsilon}(Y) = \inf\{\sum_i \phi(B_i)^p \,;\, B_i \text{ countable covering of } Y \text{ by balls of diameters } \leq \epsilon\}.$$

Let

$$\Phi^p(Y) = \lim_{\epsilon \longrightarrow 0} \Phi^{p;\epsilon}(Y).$$

Denote by

$$osc_u(B) = \max_B u - \min_B u.$$

Lemma

On a Carnot group, given two open sets A and B with disjoint closures,
$cap_Q(A,B) := \inf\{OSC_u^Q(X)\,;\, u \text{ continuous},\ u(A)=0,\ u(B)=1\} > 0.$

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Observation: Let B_i be a countable covering of X.

$$\sum_i osc_u(B_i)^Q = \int_{\mathbb{R}} \left(\sum_{\{t\,;\,B\cap u^{-1}(t)\neq\emptyset\}} osc_u(B_i)^{Q-1} \right) dt.$$

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Lemma

Let $A \subset X$*, let* $u : A \to [0,1]$ *be continuous. Then*

$$OSC_u^Q(X) \geq \int_{\mathbb{R}} OSC_u^{Q-1}(u^{-1}(t))\,dt.$$

Given a homeomorphism $f:\mathbb{R}^n \to X$, let $u = v\circ f^{-1}$, v linear. Want to relate $OSC_u^{Q-1}(u^{-1}(t))$ to $\mathcal{H}^{\alpha(Q-1)}(v^{-1}(t))$.

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Definition

Let $f:X\to Y$ be a homeomorphism. The conformal Hölder exponent $CH(f)$ of f is the supremum of α's such that for all $\ell>0$, there exists $L>0$ such that for all x, x', x'' in X,

$$d(f(x),f(x''))\le \ell\, d(f(x),f(x'))\Rightarrow d(x,x'')\le L\, d(x,x')^{\alpha}.$$

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Say that f is a conformally Hölder homeomorphism if $CH(f) > 0$. Let $CH(X, Y)$ denote the supremum of α's such that there locally exist homeomorphisms $X \to Y$ with conformal Hölder exponents $\geq \alpha$.

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If $f : X \to Y$ is C^{α} and f^{-1} is C^{β}, then $CH(f) \geq \alpha\beta$.

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Example

If $f:X\to Y$ is C^α and f^{-1} is C^β, then $CH(f)\geq\alpha\beta$.

Lemma

Let $f:X\to Y$ and $g:Y\to Z$ be homeomorphisms. Assume that g is quasisymmetric. Then $CH(g\circ f)=CH(f)$.

Lemma

Let $f: X \to Y$ be a homeomorphism, $v: X \to \mathbb{R}$ a Lipschitz function. Let $\alpha < CH(f)$. Then

$$OSC_u^{Q-1}(u^{-1}(t)) \leq const.\, \mathcal{H}^{\alpha(Q-1)}(v^{-1}(t)).$$

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$$OSC_u^{Q-1}(u^{-1}(t)) \leq const.\,\mathcal{H}^{\alpha(Q-1)}(v^{-1}(t)).$$

Compare to previous inequalities:

$$\inf\{OSC_u^Q(X)\,;\, u \text{ continuous},\ u(A)=0,\ u(B)=1\} > 0.$$

$$OSC_u^Q(X) \geq \int_{\mathbb{R}} OSC_u^{Q-1}(u^{-1}(t))\,dt.$$

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