L^p-cohomology of symmetric spaces

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August 2, 2006

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$$\begin{aligned} H^{k,p} &= \textit{closed } k\textit{-forms in } L^p / d((k-1)\textit{-forms in } L^p), \\ R^{k,p} &= \textit{closed } k\textit{-forms in } L^p / \textit{closure of } d((k-1)\textit{-forms in } L^p), \\ T^{k,p} &= \textit{closure of } d((k-1)\textit{-forms in } L^p) / d((k-1)\textit{-forms in } L^p). \end{aligned}$$

$R^{k,p}$ is called the reduced cohomology*. $T^{k,p}$ is called the* torsion*.*

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More generally, for $p > 1$, $T^{1,p} = 0$ and $H^{1,p}$ is equal to the Besov space $B^{1/p}_{p,p}(\mathbb{R}/2\pi\mathbb{Z})$ mod constants.

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$T^{1,p}$ is non zero and thus infinite dimensional.
Indeed, the 1-form $\frac{dt}{t}$ (cut off near the origin) is in L^p for all $p > 1$ but it is not the differential of a function in L^p.

What are our favourite spaces ?

- In talks by J. Rohlfs, L. Saper, B. Speh, S. Zucker : manifolds with thin ends. L^p-cohomology is related to the topology of a compactification.

- In this talk : manifolds with large ends, e.g. symmetric spaces themselves. L^p-cohomology is related to analytic features of a compactification (compare A. Koranyi's lectures).

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The obvious map $\mathbb{Z} \to \mathbb{R}$ is uniform. Any homomorphism between groups (with left invariant metrics) is uniform. The parametrization of a cusp by a punctured disk is not uniform.

Proposition

Among contractible Riemannian manifolds admitting a cocompact isometric group action, L^p-cohomology is natural under uniform maps.

L^p-cohomology can be discretized.

L^p-cohomology of discrete groups

L^p-cohomology can be discretized. It makes sense for discrete groups, and cannot see any difference between a cocompact lattice in a semi-simple Lie group G, the Lie group G itself or the Riemannian symmetric space G/K.

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In conclusion,

- L^p-cohomology is a tool to investigate discrete groups.
- It shares nearly all properties of usual cohomology.
- Nevertheless, it is not easy to calculate it.
- In the case of cocompact lattices in Lie groups, it can probably be computed by analytic means.

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In this talk, we explain 3 applications of L^p-cohomology to negatively curved Riemannian manifolds and groups.

1. Hopf's conjecture about Euler characteristic
2. Cannon conjecture on groups with boundary a 2-sphere
3. Curvature pinching

Outline

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Remark

- *Compact 2-dimensional negatively curved manifolds have negative Euler characteristic.*
- $2m$*-dimensional compact hyperbolic manifolds have Euler characteristic proportional to* $(-1)^m$.
- *This generalizes to all compact negatively curved locally symmetric spaces (Gauss-Bonnet).*

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(H. Hopf). If M is $2m$-dimensional compact negatively curved, then $(-1)^m\chi(M) > 0$.

Theorem

(M. Gromov, 1991). This is true provided M also admits a Kähler *metric.*

Role of the Kähler condition

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Corollary

Let M^{2m} be a complete Kähler manifold with Kähler form ω. Then wedging with ω maps L^2-harmonic forms to L^2-harmonic forms, and this induces an injection *in reduced L^2-cohomology $R^{k,2}(M) \to R^{k+2}(M)$ for all $k < m$.*

Role of negative curvature

Proposition

(M. Gromov). Let $\tilde{M}$ be a complete simply connected negatively curved Riemannian manifold. Let $k \geq 2$.

- *(Coning of cycles). Every $k-1$-cycle z spans a k-chain c with $vol(c) \leq const.\ vol(z)$.*

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$$\omega \wedge \alpha = d(b \wedge \alpha) \quad \text{and} \quad b \wedge \alpha \in L^2,$$

thus $\omega \wedge \alpha = 0$ in $R^{k+2,2}(\tilde{M})$. If α is harmonic, conclude that $\alpha = 0$ in $R^{k,2}(\tilde{M})$.

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Examples

(W. Lück). If M admits a tower of finite degree d_j normal coverings M_j such that $\bigcap_j \pi_1(M_j) = \{1\}$, then

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Let $\tilde{M}$ cover a compact manifold M. Then

$$\chi(M) = \sum_k (-1)^k b^{k,2}(M).$$

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Proof of Gromov's theorem. Assume M is compact and admits both a negatively curved metric and a Kähler metric. Then all $b^{k,2}(M)$ vanish except $b^{m,2}(M)$, which is nonzero, thus $(-1)^m\chi(M) = b^{m,2}(M) > 0$.

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- Lefschetz mechanism, L^2-Betti numbers.
- Vanishing of L^∞-cohomology.
- Cup-product $H^{k,2} \otimes H^{2,\infty} \to H^{k+2,\infty}$.

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Conjecture

(J. Cannon). Let Γ be a hyperbolic group whose ideal boundary is a 2-sphere. Then Γ is virtually a cocompact lattice in $PSL(2,\mathbb{C})$.

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Strategy (B. Kleiner). Prove that this infimum is achieved. Then prove that dimension minimizing metrics are Riemannian if boundary is 2-dimensional. Then apply a result of D. Sullivan (1978): *every uniformly quasiconformal group of the standard 2-sphere is conjugate to a subgroup of* $PSL(2, \mathbb{C})$.

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Theorem

(S. Keith-T. Laakso, M. Bonk-B. Kleiner 2005). Let Γ *be a hyperbolic group whose ideal boundary is a 2-sphere. If conformal dimension is achieved, then* Γ *is virtually a cocompact lattice in* $PSL(2,\mathbb{C})$.

Remark

For hyperbolic groups, $H^{1,p}$ is nonzero for p large, but zero for p small.

L^p-dimension

Remark

For hyperbolic groups, $H^{1,p}$ is nonzero for p large, but zero for p small. For instance, $H^{1,p}(SO(n,1)) = 0 \Leftrightarrow p \leq n-1$.

Definition

Define the L^p-dimension *of a group as the least $p > 1$ such that its $H^{1,p}$ is nonzero.*

L^p-dimension

Remark

For hyperbolic groups, $H^{1,p}$ is nonzero for p large, but zero for p small. For instance, $H^{1,p}(SO(n,1)) = 0 \Leftrightarrow p \leq n-1$.

Definition

Define the L^p-dimension *of a group as the least $p > 1$ such that its $H^{1,p}$ is nonzero.*

Theorem

(Same people + M. Bourdon-H. Pajot 2003). Let Γ be a hyperbolic group. Then L^p-dimension is less than or equal to conformal dimension.

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Examples

(M. Bourdon-H. Pajot). There exist hyperbolic groups for which conformal dimension $> 2 \geq L^p$-dimension. For such groups, conformal dimension cannot be achieved.

In conclusion, we have used

- Mayer-Vietoris and L^2-Betti numbers.
- Expression of $H^{1,p}$ as a function space on the ideal boundary.

Outline

1. Hopf's conjecture about Euler characteristic
2. Cannon conjecture on groups with boundary a 2-sphere
3. Curvature pinching

Remark

Rank one symmetric spaces are hyperbolic spaces over the reals $H^n_{\mathbb{R}}$, the complex numbers $H^m_{\mathbb{C}}$, the quaternions $H^m_{\mathbb{H}}$, and the octonions $H^2_{\mathbb{O}}$.
Real hyperbolic space has sectional curvature -1. Other rank one symmetric spaces are $-\frac{1}{4}$-pinched, *i.e. their sectional curvature ranges between -1 and $-\frac{1}{4}$.*

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Conjecture

The optimal pinching of $SU(m,1)$, $Sp(m,1)$ ($m \geq 2$) and F_4^{-20} is $-\frac{1}{4}$.

Theorem

If M^n is simply connected and δ-pinched for some $\delta \in [-1, 0)$, then

$$p < 1 + \frac{n-k}{k-1}\sqrt{-\delta} \quad \Rightarrow \quad T^{k,p}(M) = 0.$$

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This is sharp. For instance, consider the semidirect product $G = \mathbb{R}^3 \rtimes_\alpha \mathbb{R}$ where $\alpha = diag(1,1,2)$.

- It admits a $-\frac{1}{4}$-pinched left-invariant Riemannian metric, therefore $\delta(G) \leq -\frac{1}{4}$.
- It has $T^{2,p}(G) \neq 0$ for $2 < p \leq 4$. This implies that $\delta(G) = -\frac{1}{4}$.

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Remark

Complex hyperbolic plane $H^2_{\mathbb{C}}$ is isometric to $G' = Heis^3 \rtimes_\alpha \mathbb{R}$ where $\alpha = diag(1, 1, 2)$ and Heis denotes the Heisenberg group. Therefore it is very close to G.

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Theorem

$T^{2,p}(H^2_{\mathbb{C}}) = 0$ *for* $2 < p < 4$.

Proof of torsion comparison theorem

Use the gradient vectorfield ξ of a Busemann function and its flow ϕ_t, whose derivative is controlled by sectional curvature. For α a closed k-form in L^p,

$$\phi_t^*\alpha = \alpha + d\left(\int_0^t \phi_s^*\iota_\xi\alpha\, ds\right)$$

has a limit as $t \to +\infty$ under the assumptions of the theorem. This boundary value map injects $H^{k,p}$ into a function space of closed forms on the ideal boundary, showing that $H^{k,p}$ is Hausdorff.

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Proof of torsion vanishing for $H^2_{\mathbb{C}}$

For $p \notin \{4/3, 2, 4\}$, differential forms α on $H^2_{\mathbb{C}}$ split into components α_+ and α_+ which are contracted (resp. expanded) by ϕ_t. Then

$$B_t : \alpha \mapsto \int_0^t \phi_s^* \iota_\xi \alpha_+ \, ds - \int_{-t}^0 \phi_s^* \iota_\xi \alpha_- \, ds$$

converges as $t \to +\infty$ to a bounded operator B on L^p. $P = 1 - dB - Bd$ retracts the L^p de Rham complex onto a complex of differential forms on $Heis^3$ with missing components and weakly regular coefficients. If $2 < p < 4$, this complex is nonzero in degrees 1 and 2, but it is so small that its cohomology can be shown to be Hausdorff.

Non-vanishing of torsion

Use Poincaré duality. Let $p' = p/p-1$ denote the conjugate exponent. In order to prove that a closed k-form α is nonzero in cohomology, it suffices to construct a sequence ψ_j of $(n-k)$-forms such that $\| d\psi_j \|_{L^{p'}}$ tends to zero but $\int \alpha \wedge \psi_j$ does not tend to zero.

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In conclusion, we have used

- Poincaré duality.
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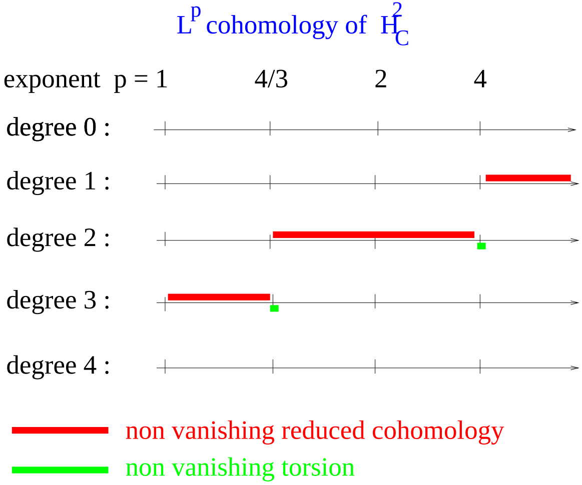
Conjecture

- *For rank 1 symmetric spaces, $T^{k,p} = 0$ except for at most 1 value of p in each degree.*
- *For higher rank symmetric spaces, $H^{k,p} = 0$ for $k < rank$, $T^{k,p} = 0$ for $k = rank$.*
- *For $k = rank$, $R^{k,p} \neq 0$ for p large, and $R^{k,p}$ is a function space on the maximal boundary.*
- *For each $p > 1$, there exists k such that $H^{k,p} \neq 0$.*

L^p-cohomology of $H^2_{\mathbb{C}}$



L^p-cohomology of G