L^p-cohomology and curvature pinching

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Definition

A Carnot group *is a Lie group* G *admitting a one-parameter group* δ_t *of automorphisms, generated by a derivation whose bottom eigenspace generates the Lie algebra.* G *carries left invariant* subRiemannian *(or* Carnot-Caratheodory*) metrics* d *such that* $d \circ \delta_t = td$*.*

Question

(Gromov 1993). *Let* G *be an n-dimensional Carnot group. For which* $\alpha \in (0,1)$ *does there exist locally a homeomorphism* $\mathbb{R}^n \to G$ *which is* C^α*-Hölder continuous ? I.e. compute* $\alpha(G) = \sup\{\alpha \in (0,1) \,|\, \exists$ *locally a homeomorphism* $\mathbb{R}^n \to G\}$*.*

Example

If G *is a r-step Carnot group, the exponential map* $\mathfrak{g} = Lie(G) \to G$ *is locally* $C^{1/r}$*-Hölder continuous. Thus* $\alpha(G) \geq 1/r$*.*

Proposition

Let M *have Hausdorff dimension* Q*. Then* $\alpha(M) \leq \frac{n}{Q}$*.*

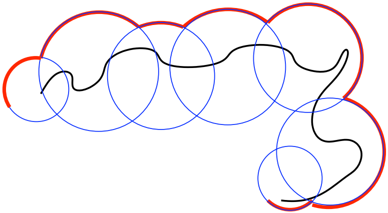
Proposition

Let G be a Carnot group of dimension n and Hausdorff dimension Q. Then $\alpha(M) \leq \frac{n-1}{Q-1}$.

Proof. Use the isoperimetric inequality for piecewise smooth domains $D \subset M$,

$$vol(D)^{Q-1/Q} \leq const.\, \mathcal{H}^{Q-1}(\partial D).$$

It follows that the boundary of any non smooth domain Ω has Hausdorff dimension at least $Q-1$. Indeed, cover $\partial\Omega$ with balls B_j and apply (*) to $\Omega \cup \bigcup B_j$. This gives a lower bound on $\mathcal{H}^{Q-1}(\partial(\bigcup B_j)) \leq \sum \mathcal{H}^{Q-1}(\partial B_j) \leq const. \sum \mathrm{diameter}(B_j)^{Q-1}$.



Proposition

(Gromov 1993). *Let $\mathbb{H}^m$ denote $2m+1$-dimensional Heisenberg group. Let $V \subset \mathbb{H}^m$ be a subset of topological dimension $m+1$. Then the Hausdorff dimension of V is at least $m+2$. It follows that $\alpha(\mathbb{H}^m) \leq \frac{m+1}{m+2}$.*

Proof. According to topological dimension theory (Alexandrov), there exists an m-dimensional polyhedron P and a continuous map $f : P \to \mathbb{H}^m$ such that every map sufficiently C^0-close to f hits V.

Gromov approximates f with piecewise *horizontal* maps which sweep an open set U. This gives rise to a local projection $p : U \to \mathbb{R}^{m+1}$ such that for every ball B, the tube $p^{-1}(p(B))$ has volume $\leq$ const. $\mathrm{diameter}(B)^{m+2}$.

Cover V with balls B_j. The corresponding tubes $T_j = p^{-1}(p(B_j))$ cover U. Then the volume of U is less than $\sum \mathrm{diameter}(B_j)^{m+2}$, which shows that $dim_{Hau}(V) \geq m+2$.

Theorem

(Gromov 1993). *Let M be a generic subRiemannian manifold of dimension n, Hausdorff dimension Q, with an h-dimensional distribution. Let $k \leq h$ be such that $h-k \geq (n-h)k$. Then $\alpha(M) \leq \frac{n-k}{Q-k}$.*

Curvature pinching

Definition

Let M be a Riemannian manifold. Let $-1 \leq \delta < 0$. Say M is δ-pinched *if sectional curvature ranges between -1 and δ. Define the* optimal pinching $\delta(M)$ *of M as the least $\delta \geq -1$ such that M is quasiisometric to a δ-pinched simply connected Riemannian manifold.*

Example

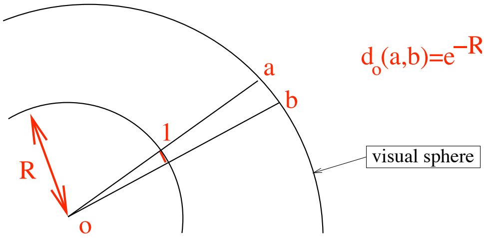
Rank one symmetric spaces of noncompact type are hyperbolic spaces over the reals $H^n_{\mathbb{R}}$, the complex numbers $H^m_{\mathbb{C}}$, the quaternions $H^m_{\mathbb{H}}$, and the octonions $H^2_{\mathbb{O}}$. Real hyperbolic space has sectional curvature -1. Other rank one symmetric spaces are $-\frac{1}{4}$-pinched.

Question

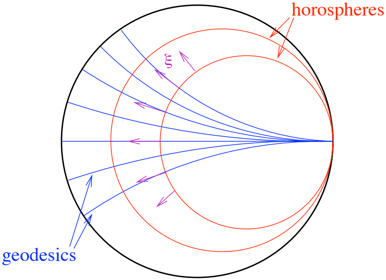
Is it true that the optimal pinching of $H^m_{\mathbb{C}}$, $H^m_{\mathbb{H}}$ $(m \geq 2)$ and $H^2_{\mathbb{O}}$ is $-\frac{1}{4}$?

Facts. Let M be a simply connected pinched Riemannian manifold.

- The ideal boundary, seen from a point o, is a sphere.
- It carries a *visual metric* d_o (see picture for U. Hamenstädt's definition).
- Different visual metrics d_o and $d_{o'}$ are equivalent.
- Quasiisometric maps between negatively curved Riemannian manifolds induce *quasisymmetric* maps between ideal boundaries.



Negatively curved symmetric spaces admit simply transitive groups of isometries of the form $\mathbb{R} \ltimes G$ where G is a Carnot group and $\mathbb{R}$ acts by dilations δ_t.



The eigenvalues $e^{t/2}$, e^t of δ_t reflect extrema of sectional curvature.

Corollary

If M is a rank one symmetric space, visual metrics on its ideal boundary are equivalent to subRiemannian metrics. If $M = H^m_{\mathbb{C}}$, $\partial M = \mathbb{H}^{m-1} \cup \{\infty\}$.

Proposition

Let M be a simply connected δ-pinched Riemannian manifold. Then visual metrics on the ideal boundary of M are C^{α}-Hölder equivalent to the round metric, with $\alpha = \sqrt{-\delta}$.

Indeed, geodesics from a unit ball to a point come together exponentially fast, with exponents ranging from $\sqrt{-\delta}$ to 1 (Rauch comparison theorem, 1950's).

Question

Let G be a nonabelian Carnot group. Do there exist quasisymmetricly equivalent distances on G which are locally C^{α}-Hölder equivalent to Riemannian metrics, with $\alpha > 1/2$?

$$\begin{tikzcd} & \boxed{\text{Hölder quasisymmetric}} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ \boxed{\text{Hölder}} & & \boxed{\text{pinching}} \end{tikzcd}$$

Theorem

The optimal pinching of $H^2_{\mathbb{C}}$ is equal to $-\frac{1}{4}$.

Scheme of proof

- *Define, in a quasiisometry-invariant manner, families $\mathcal{R}_p$, $p > 1$, of algebras of functions on 4-dimensional negatively curved manifolds M, and, given $u \in \mathcal{R}_p$, a vectorsubspace $\mathcal{S}_p(u) \subset \mathcal{R}_p$.*
- *If M is δ-pinched and $p < 2 + 4\sqrt{-\delta}$, then for every u, $\mathcal{S}_p(u)$ is a subalgebra of $\mathcal{R}_p$.*
- *If $M = H^2_{\mathbb{C}}$, for all $p \in (4, 8)$, there exists (locally) $u \in \mathcal{R}_p$ such that $\mathcal{S}_p(u)$ is not a subalgebra of $\mathcal{R}_p$.*

$\mathcal{R}_p$ can be viewed as a quasisymmetrically invariant function space on the visual boundary of M. However, $\mathcal{S}_p(u)$ does not seem to be definable directly in terms of the visual boundary only.

L^p-cohomology

Definition

Let M be a Riemannian manifold. Let $p>1$. L^p-cohomology of M is the cohomology of the complex of L^p-differential forms on M whose exterior differentials are L^p as well,

$$\begin{aligned}
H^{k,p} &= \textit{closed } k\textit{-forms in } L^p / d((k-1)\textit{-forms in } L^p),\\
R^{k,p} &= \textit{closed } k\textit{-forms in } L^p / \textit{closure of } d((k-1)\textit{-forms in } L^p),\\
T^{k,p} &= \textit{closure of } d((k-1)\textit{-forms in } L^p)/d((k-1)\textit{-forms in } L^p).
\end{aligned}$$

$R^{k,p}$ *is called the* reduced cohomology. $T^{k,p}$ *is called the* torsion.

Example

The real line $\mathbb{R}$.

$H^{0,p}=0$.

$R^{1,p}=0$, since every function in $L^p(\mathbb{R})$ can be approximated in L^p with derivatives of compactly supported functions. Therefore $H^{1,p}$ is only torsion.

$T^{1,p}$ *is non zero and thus infinite dimensional.* Indeed, the 1-form $\frac{dt}{t}$ (cut off near the origin) is in L^p for all $p>1$ but it is not the differential of a function in L^p.

Example : the real hyperbolic plane $H^2_{\mathbb{R}}$

Here $H^{0,p} = 0 = H^{2,p}$ for all p.
If $p = 2$, since the Laplacian on L^2 functions is bounded below, $T^{1,2} = 0$. Therefore

$$\begin{aligned} H^{1,2} &= R^{1,2} \\ &= \{L^2 \text{ harmonic 1-forms}\} \\ &= \{\text{harmonic functions } h \text{ on } H^2_{\mathbb{R}} \text{ with } \nabla h \in L^2\}/\mathbb{R}\,. \end{aligned}$$

Using conformal invariance, switch from hyperbolic metric to euclidean metric on the disk D.

$$\begin{aligned} H^{1,2} &= \{\text{harmonic functions } h \text{ on } D \text{ with } \nabla h \in L^2\}/\mathbb{R} \\ &= \{\text{Fourier series } \Sigma a_n e^{in\theta} \text{ with } a_0 = 0, \Sigma |n|\,|a_n|^2 < +\infty\}, \end{aligned}$$

which is Sobolev space $H^{1/2}(\mathbb{R}/2\pi\mathbb{Z})$ mod constants.

More generally, for $p > 1$, $T^{1,p} = 0$ and $H^{1,p}$ is equal to the Besov space $B^{1/p}_{p,p}(\mathbb{R}/2\pi\mathbb{Z})$ mod constants.

Proposition

Let M be a simply connected negatively curved Riemannian manifold. Functions u on M whose differential belongs to L^p have boundary values u_∞ on the visual boundary. The cohomology class $[du] \in H^{1,p}(M)$ vanishes if and only if u_∞ is constant.

Indeed, since volume in polar coordinates grows exponentially, and $L^p(e^t\,dt) \subset L^1(dt)$, the radial derivative belongs to L^1, so $u_\infty(\theta) = \lim_{t\to\infty} u(\theta, t)$ exists a.e. If $u_\infty = 0$, Sobolev inequality $\|u\|_{L^p} \leq \|du\|_{L^p}$ applies, and $[du] = 0$.

This suggests

Definition

(Bourdon-Pajot 2004). *For a negatively curved manifold M, define the* Royden algebra *$\mathcal{R}_p(M)$ as the space of L^∞ functions u on M such that $du \in L^p$, modulo $L^p \cap L^\infty$ functions.*

Then $\mathcal{R}_p(M)$ identifies with an algebra of functions on the visual boundary of M. If M is a symmetric space, $\mathcal{R}_p(M)$ is a (possibly anisotropic) Besov space.

Remark

L^p-cohomology is quasiisometry invariant. Wedge product $\alpha, \beta \mapsto \alpha \wedge \beta$ induces cup-product $[\alpha] \smile [\beta] : H^{k,p} \times H^{k',p} \to H^{k+k',p/2}$ in a quasiisometry invariant manner.

Definition

Let M be a simply connected negatively curved manifold, let $p > 2$, let $u \in \mathcal{R}_p(M)$. Define

$$\mathcal{S}_p(u) = \{v \in \mathcal{R}_p(M) \,|\, [dv] \smile [du] = 0 \in H^{2,p/2}(M)\}.$$

Remark: As a function space on the visual boundary, $\mathcal{R}_p$ is a quasisymmetric invariant. Not so clear for $\mathcal{S}_p(u)$.

Theorem

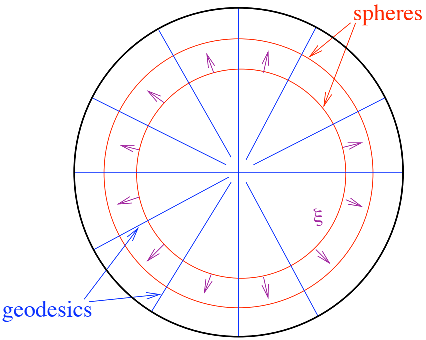
If $\dim(M) = 4$, M is δ-pinched and $p < 2 + 4\sqrt{-\delta}$, then for all $u \in \mathcal{R}_p(M)$, $\mathcal{S}_p(u)$ is a subalgebra of $\mathcal{R}_p(M)$.

Step 1. *For $q = p/2$ small, closed L^q 2-forms admit boundary values.*
Use the radial vectorfield $\xi = \frac{\partial}{\partial r}$ in polar coordinates and its flow ϕ_t, whose derivative is controlled by sectional curvature.

Use Poincaré's homotopy formula :
For α a closed 2-form in L^q,

$$\phi_t^* \alpha = \alpha + d \left(\int_0^t \phi_s^* \iota_\xi \alpha \, ds \right)$$

has a limit as $t \to +\infty$ under the assumptions of the theorem.



Step 2. *Boundary value determines cohomology class.*

Step 3. *This implies $\mathcal{S}_p(u)$ is a subalgebra.*
Let v, $v' \in \mathcal{S}_p(u)$. Then $[dv] \smile [du]$ vanishes if and only if its boundary value $dv_\infty \wedge du_\infty = 0$ a.e. Then $v'_\infty dv_\infty \wedge du_\infty + v_\infty dv'_\infty \wedge du_\infty = 0$ a.e., showing that $[d(vv')] \smile [du] = 0$, i.e. $vv' \in \mathcal{S}_p(u)$.

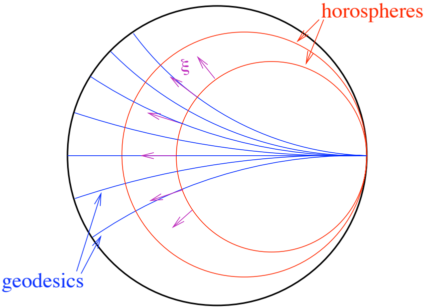
Now we compute $H^{2,q}(H^2_{\mathbb{C}})$ for $2 < q = p/2 < 4$.

Step 1. *Switch point of view. Use horospherical coordinates. View $H^2_{\mathbb{C}}$ as a product $\mathbb{H}^1 \times \mathbb{R}$. Prove a Künneth type theorem.*

For $q \notin \{4/3, 2, 4\}$, differential forms α on $H^2_{\mathbb{C}}$ split into components α_+ and α_+ which are contracted (resp. expanded) by ϕ_t. Then

$$h_t : \alpha \mapsto \int_0^t \phi_s^* \iota_\xi \alpha_+ \, ds - \int_{-t}^0 \phi_s^* \iota_\xi \alpha_- \, ds$$

converges as $t \to +\infty$ to a bounded operator h on L^q. $P = 1 - dh - hd$ retracts the L^q de Rham complex onto a complex $\mathcal{B}$ of differential forms on $\mathbb{H}^1$ with missing components and weakly regular coefficients.



Step 2. *If $2 < q < 4$, this complex is nonzero in degrees 1 and 2.*
$\mathcal{B}^1$ consists of 1-forms which are multiples of the left-invariant contact form τ on $\mathbb{H}^1$.

Step 3. *If $2 < q < 4$, vanishing of degree 2 cohomology classes is characterized by a differential equation.*
If $\alpha \in \mathcal{B}^2$ is a 2-form, then $\alpha \in d\mathcal{B}^1$ if and only if α satisfies the linear differential equation

$$\alpha = d(\frac{\tau \wedge \alpha}{\tau \wedge d\tau}\tau).$$

If $dv \wedge du$ is a solution, $d(v^2) \wedge du$ is not a solution, unless dv is proportional to du.

Failure of the subalgebra theorem for $H^2_{\mathbb{C}}$.
In coordinates (x, y, z) on $\mathbb{H}^1$, one can take (locally) $u = y$ and $v = x$. Then $dv \wedge du = -d\tau$ belongs to $d\mathcal{B}^1$, whereas $d(v^2) \wedge du$ does not. So for $4 < p = 2q < 8$, $\mathcal{S}_p(u)$ is not (locally) a subalgebra of $\mathcal{R}_p(H^2_{\mathbb{C}})$.

Other rank one symmetric spaces.
The comparison theorem works for all of them: in the definition of $\mathcal{S}_\kappa$, replace du by a cohomology class κ of degree 1, resp. 3 resp. 7. Steps 1 and 2 of the L^q computation in degree 2 resp. 4 resp. 8 are unchanged. It turns out that for all spaces but $H^2_{\mathbb{C}}$, the differential equation of Step 3 is a consequence of $d\alpha = 0$, so $\mathcal{S}_\kappa$ is an algebra in these cases.

Questions from the audience

- **Peter Haïssinski**: *doesn't conformal dimension provide a lower bound on pinching ?*
  Answer: It does. If M is n-dimensional and δ-pinched, then conformal dimension is $\leq (n-1)/\sqrt{-\delta}$. This yields the bound $\delta(H^2_{\mathbb{C}}) \leq -9/16$. Also, this gives a partial (non sharp) answer to the Hölder+quasisymmetric problem for Heisenberg group: if there is a map $\mathbb{R}^3 \to \mathbb{H}^1$ which is the composition of a C^α (local) homeomorphism and a quasisymmetric (local) homeomorphism, then $\alpha \geq 3/4$.
- **Mario Bonk**: *how do cup-products behave for Rickmann's rug and the corresponding negatively curved Riemannian homogeneous space ?*
  Answer: Consider the 3-dimensional rug leading to the solvable group $S = \mathbb{R} \ltimes \mathbb{R}^3$ with $\mathbb{R}$ acting on $\mathbb{R}^3$ via the diagonal matrix $diag(e^{t/2}, e^{t/2}, e^t)$. It is $-\frac{1}{4}$-pinched. For $2 < q < 4$, any closed form in $\mathcal{B}^2$ is exact in a neighborhood of a point of the ideal boundary. So the local version of $\mathcal{S}_u$ is an algebra. However, $H^{2,q}(S)$ has torsion, indicating that the condition for a closed form in $\mathcal{B}^2$ to be globally exact is subtle, so the global version of $\mathcal{S}_u$ is probably not an algebra. Nevertheless, the fact that $T^{2,q}(S) \neq 0$ alone suffices to prove that $\delta(S) = -\frac{1}{4}$.