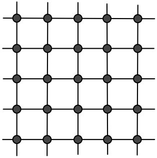
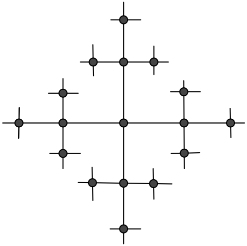
An invitation to geometric group theory

Pierre Pansu, Université Paris-Sud

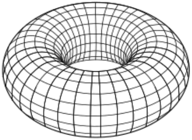
May 30th, 2019

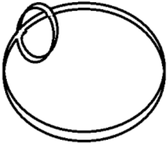


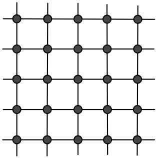
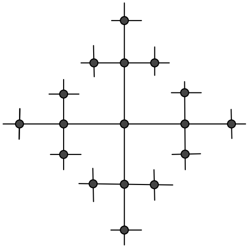


Free abelian group
$\langle a, b | aba^{-1}b^{-1} \rangle$
Fundamental group of 2-torus

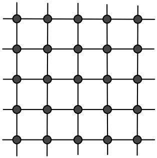
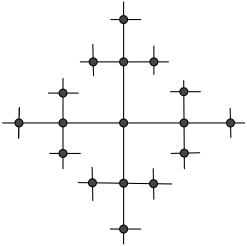
Free group
$\langle a, b | \rangle$
Fund. group of punctured 2-torus



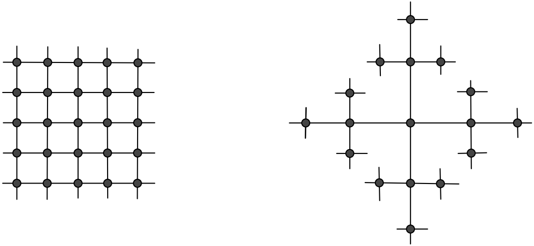


Motto: a finitely presented group $\langle S|R\rangle$ is a metric space.

Motto: a finitely presented group $\langle S|R\rangle$ is a metric space.
Caveat: up to quasiisometry (passing to an equivalent distance).



Motto: a finitely presented group $\langle S|R\rangle$ is a metric space.
Caveat: up to quasiisometry (passing to an equivalent distance).
Need to focus on rough, large scale features.

Growth: Count number $v(n)$ of vertices in n-ball.
Invariant up to $n \to Cn$.

Growth: Count number $v(n)$ of vertices in n-ball.
Invariant up to $n \to Cn$.

Free abelian group	Free group
$\langle a, b \mid aba^{-1}b^{-1}\rangle$	$\langle a, b \mid \rangle$
$v(n) = 2n^2 + 2n + 1.$	$v(n) = 4.3^{n-1}$

Growth: Count number $v(n)$ of vertices in n-ball.
Invariant up to $n \to Cn$.

Free abelian group	Free group
$\langle a, b \vert aba^{-1}b^{-1}\rangle$	$\langle a, b \vert \rangle$
$v(n) = 2n^2 + 2n + 1.$	$v(n) = 4.3^{n-1}$

Theorem (Misha Gromov 1982)

Finitely generated group G has polynomial growth $\iff$ G is virtually nilpotent.

Theorem (Rostislav Grigorchuk 1984)

There are finitely generated groups with growth between e^{n^α} and e^{n^β}, $0 < \alpha < \beta < 1$.

Growth: Count number $v(n)$ of vertices in n-ball.
Invariant up to $n \to Cn$.

Free abelian group	Free group
$\langle a, b \mid aba^{-1}b^{-1} \rangle$	$\langle a, b \mid \rangle$
$v(n) = 2n^2 + 2n + 1.$	$v(n) = 4.3^{n-1}$

Theorem (Misha Gromov 1982)

Finitely generated group G has polynomial growth $\iff$ G is virtually nilpotent.

Theorem (Rostislav Grigorchuk 1984)

There are finitely generated groups with growth between e^{n^α} and e^{n^β}, $0 < \alpha < \beta < 1$.

Theorem (Anna Erschler - Tianyi Zheng 2018)

The growth of Grigorchuk's example satisfies $\frac{\log \log v(n)}{\log n} \to \alpha_0$, $\alpha_0 = 0.7674\ldots$.

Random walks are used as a tool.

Question: What are the possible growths for finitely generated groups?

Conjecture (Grigorchuk)

If not polynomial, growth cannot be $< e^{\sqrt{n}}$.

Question: What are the possible growths for finitely generated groups?

Conjecture (Grigorchuk)

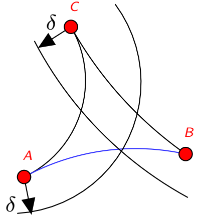
If not polynomial, growth cannot be $< e^{\sqrt{n}}$.

Using Lie algebras, Laurent Bartholdi and Grigorchuk (2000) solved the case of residually p groups.

Yehuda Shalom and Terry Tao's effective version of Gromov's theorem (2009) implies:

$$v(n) \leq n^{c(\log\log n)^c} \implies G \text{ is virtually nilpotent.}$$

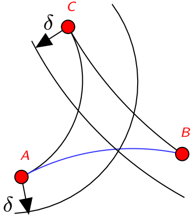
In 1986, Gromov extracted the following *hyperbolicity* axiom, a property satisfied by Lobatchevsky's nonEuclidean geometry.



Definition

Let $\delta \in \mathbb{R}_+$. Say a finitely generated group G is δ-hyperbolic if geodesic triangles in G are δ-thin: each side is contained in the δ-neighborhood of the two others.

In 1986, Gromov extracted the following *hyperbolicity* axiom, a property satisfied by Lobatchevsky's nonEuclidean geometry.

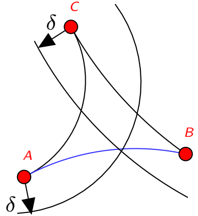


Definition

Let $\delta \in \mathbb{R}_+$. Say a finitely generated group G is δ-hyperbolic if geodesic triangles in G are δ-thin: each side is contained in the δ-neighborhood of the two others.

Hyperbolic groups arise in topology: free groups, fundamental groups of higher genus surfaces are hyperbolic.

In 1986, Gromov extracted the following *hyperbolicity* axiom, a property satisfied by Lobatchevsky's nonEuclidean geometry.



Definition

Let $\delta \in \mathbb{R}_+$. Say a finitely generated group G is δ-hyperbolic if geodesic triangles in G are δ-thin: each side is contained in the δ-neighborhood of the two others.

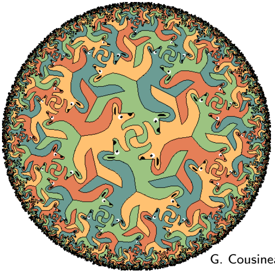
Hyperbolic groups arise in topology: free groups, fundamental groups of higher genus surfaces are hyperbolic.

Hyperbolic groups are plentiful:

- Random groups (picking large random relators) are hyperbolic.
- Adding large relators does not kill the group.
- Taking limits of hyperbolic groups produces heaps of examples and counterexamples of weird finitely generated groups.

Geometry of hyperbolic groups

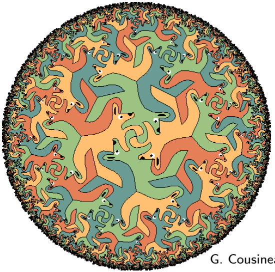
Symmetry groups of nonEuclidean tilings are hyperbolic



G. Cousineau

Geometry of hyperbolic groups

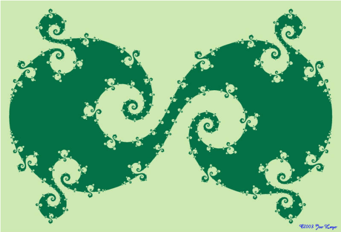
Symmetry groups of nonEuclidean tilings are hyperbolic



G. Cousineau

A hyperbolic group possesses an ideal boundary: a fractal space equipped with a *quasisymmetric structure* allowing a new kind of real analysis: *Analysis on Metric Spaces*.

Keywords: differentiability, function spaces.



Model theory
Circa 1945, Alfred Tarsky asked for a structure theory for algebraic varieties in free groups. A satisfactory solution was provided by Zlil Sela in 2001, based on properties of hyperbolic groups and their limits.

Model theory
Circa 1945, Alfred Tarsky asked for a structure theory for algebraic varieties in free groups. A satisfactory solution was provided by Zlil Sela in 2001, based on properties of hyperbolic groups and their limits.

Operator algebras
At the end of the 1960's, differential topology came to a halt, its methods came short when dealing with nonsimply connected manifolds. Sergei Novikov's conjecture on homotopy invariance of higher signatures (1970) was put into an ambitious operator algebraic framework by Paul Baum and Alain Connes in 1982. It involves analysis of singular integral operators on the fundamental group. Circa 2000, Vincent Lafforgue solved it for a subclass of hyperbolic groups (soon extended to all hyperbolic groups by Igor Mineyev and Guoliang Yu).

Model theory
Circa 1945, Alfred Tarsky asked for a structure theory for algebraic varieties in free groups. A satisfactory solution was provided by Zlil Sela in 2001, based on properties of hyperbolic groups and their limits.

Operator algebras
At the end of the 1960's, differential topology came to a halt, its methods came short when dealing with nonsimply connected manifolds. Sergei Novikov's conjecture on homotopy invariance of higher signatures (1970) was put into an ambitious operator algebraic framework by Paul Baum and Alain Connes in 1982. It involves analysis of singular integral operators on the fundamental group. Circa 2000, Vincent Lafforgue solved it for a subclass of hyperbolic groups (soon extended to all hyperbolic groups by Igor Mineyev and Guoliang Yu).

Counterexamples to a stronger form of Baum-Connes' conjecture were obtained as limits of hyperbolic groups (Nigel Higson-Lafforgue-Georges Skandalis 2002).

Model theory
Circa 1945, Alfred Tarsky asked for a structure theory for algebraic varieties in free groups. A satisfactory solution was provided by Zlil Sela in 2001, based on properties of hyperbolic groups and their limits.

Operator algebras
At the end of the 1960's, differential topology came to a halt, its methods came short when dealing with nonsimply connected manifolds. Sergei Novikov's conjecture on homotopy invariance of higher signatures (1970) was put into an ambitious operator algebraic framework by Paul Baum and Alain Connes in 1982. It involves analysis of singular integral operators on the fundamental group. Circa 2000, Vincent Lafforgue solved it for a subclass of hyperbolic groups (soon extended to all hyperbolic groups by Igor Mineyev and Guoliang Yu).

Counterexamples to a stronger form of Baum-Connes' conjecture were obtained as limits of hyperbolic groups (Nigel Higson-Lafforgue-Georges Skandalis 2002).

Lafforgue's method requires the group to act isometrically on a metric space which satisfy a slightly stronger assumption than hyperbolicity. The *Green metric* of a random walk does the job for hyperbolic groups,

$$d(x,y) = -\log \mathbb{P}(\text{random walk from } x \text{ eventually passes through } y).$$

Low dimensional topology
In 1968, 3-dimensional topology came short of analyzing 3-manifolds if they cannot be cut into simpler pieces by incompressible surfaces. Friedhelm Waldhausen conjectured that this never happens, up to taking finite covers.

Ian Agol's 2012 solution relies on the machinery of hyperbolic groups and their isometric actions on nonpostively curved cube complexes.

[1] SemiDefinite Programming

Low dimensional topology

In 1968, 3-dimensional topology came short of analyzing 3-manifolds if they cannot be cut into simpler pieces by incompressible surfaces. Friedhelm Waldhausen conjectured that this never happens, up to taking finite covers.

Ian Agol's 2012 solution relies on the machinery of hyperbolic groups and their isometric actions on nonpostively curved cube complexes.

Theoretical computer science

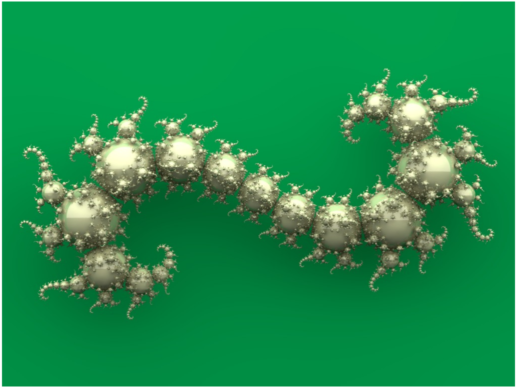
Michel Goemans and Nati Linial's SDP[1] relaxation of SPARSEST CUT raises the following question: what is the worst minimal distorsion of L^1-embeddings among n-point metric spaces which is square root embeddable in L^2? Assaf Naor and Robert Young's 2017 answer is $\sqrt{\log n}$.

The spaces which are hardest to embed in L^1 are balls in a finitely generated nilpotent group, the 5-dimensional Heisenberg group. The proof relies on fine geometric measure theory on subRiemannian Heisenberg Lie groups.

$\sqrt{\log n}$ is indeed an upper bound for the approximability ratio of SPARSEST CUT in polynomial time (Sanjeev Arora, James Lee and Naor, 2008).

[1] SemiDefinite Programming

Geometric group theory in United Kingdom: Bristol, Cambridge, Durham, Edinburgh, Glasgow, London, Southampton, Warwick. Main center: Oxford.



Geometric group theory in France: Caen, Grenoble, Lille, Lyon, Montpellier, Nantes, Nice, Orsay, Paris-Diderot, Rennes, Strasbourg, Vannes.