Aspects of geometric group theory: growth, isoperimetry, ℓ^p-cohomology

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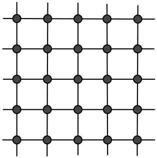
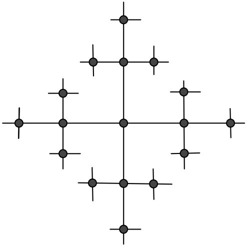
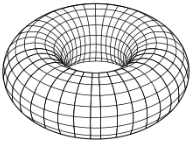
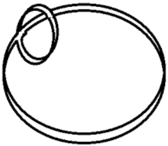


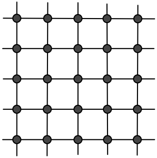


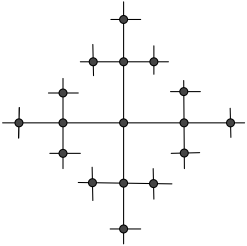
Free abelian group
$\langle a, b | aba^{-1}b^{-1} \rangle$
Fundamental group of 2-torus

Free group
$\langle a, b | \rangle$
Fund. group of punctured 2-torus

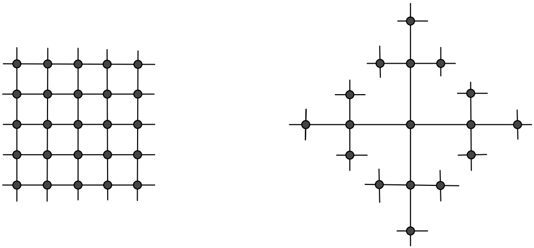








Motto: a finitely presented group $\langle S|R\rangle$ is a metric space.



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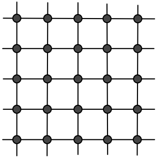

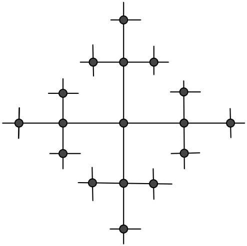

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Need to focus on rough, large scale features.

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Invariant up to $n \to Cn$.

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Finitely generated group G has polynomial growth $\iff$ G is virtually nilpotent.

Corollary. The class of finitely generated virtually nilpotent groups is *rigid*: a group quasiisometric to a group in the class belongs to the class.

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Theorem (Erschler-Zheng 2018)

The growth of Grigorchuk's example satisfies $\frac{\log \log v(n)}{\log n} \to \alpha_0$, $\alpha_0 = 0.7674\ldots$

Random walks are used as a tool.

Question: What are the possible growths for finitely generated groups?

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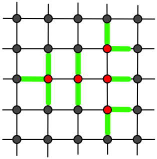
Using Lie algebras, Bartholdi and Grigorchuk (2000) solved the case of residually p groups.

Shalom and Tao's effective version of Gromov's theorem (2009) implies:

$$v(n) \leq n^{c(\log\log n)^c} \implies G \text{ is virtually nilpotent.}$$



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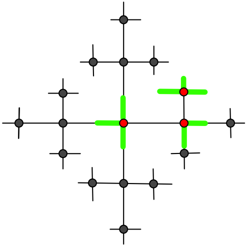

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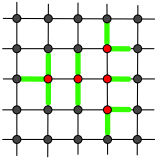

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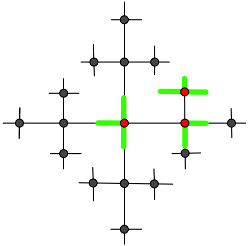

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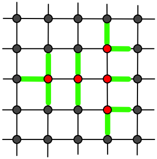
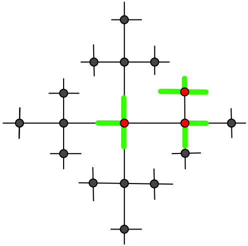
Følner function. Every finite set of vertices V such that $\frac{|\partial V|}{|V|} \leq \frac{1}{n}$ has at least $Føl(n)$ points.





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Free group
$F\o l(n) = +\infty$

Theorem (Følner 1955)

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Theorem (Erschler 2006)

There exist finitely generated amenable groups whose Følner functions grow arbitrarily fast.

Notation. For u finitely supported function on the vertices of a graph,

$$du(\text{edge } xy) = u(y) - u(x).$$

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Sobolev inequality. Say Sobolev inequality $Sob_{q,p}$ holds on a graph if there exists C such that for all finitely supported functions u,

$$\|u\|_q \leq C\,\|du\|_p.$$

Proposition (Faber, Krahn, circa 1923)

$F\o l(n) = n^Q \iff Sob_{Q/(Q-1),1} \implies Sob_{Qp/(Q-p),p}\ \forall p \in [1, Q)$.
$F\o l(n) = +\infty \iff Sob_{1,1} \implies Sob_{p,p}\ \forall p \in [1, \infty)$.

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Note: Sobolev inequality states that $d : \ell^q \to \ell^p$ has a closed range.

Definition ($\ell^{q,p}$ cohomology of simplicial complex)

$$\ell^{q,p}H^k = \{k - cocycles\ in\ \ell^p\}/d(\{k-1-cochains\ in\ \ell^q\}).$$

This is a quasiisometry invariant.

Negative curvature case

Negatively curved group = hyperbolic group. Such groups have $F\o l(n) = +\infty$ hence $Sob_{p,p}$ $\forall p \in [1, \infty)$ hence $\ell^{p,p}H^1(G)$ is Hausdorff if $p < \infty$.

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Note that $\ell^{\infty,\infty}H^1(G) \neq 0$ if G is infinite.

Theorem (Bourdon-Pajot 2002)

For every hyperbolic group G, $\exists p_0 \geq 1$ such that $\ell^{p,p}H^1(G) \neq 0$ for $p > p_0$.

The infimal such p_0 is a kind of Hausdorff dimension of the ideal boundary of G. $\ell^{p,p}H^1(G)$ identifies with a function space on it (e.g. Besov space for Fuchsian G).

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Theorem (Drutu-Mackay 2016)

Random groups in the density model, in the range of densities $\frac{1}{3} < d < \frac{1}{2}$, and at depth m, satisfy

$$\sqrt{\log m} \leq p_0 \leq \log m.$$

The lower bound is a special case of a property of linear representations of G.

Higher degrees?

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Question

For hyperbolic G, $\ell^{p,p}H^k(G) = 0$ *for all* $k \geq 2$ *and* p *large enough?*

Polynomial growth case

Theorem (Folklore, Bourgain-Brezis 2007 for $p = 1$)

$\ell^{q,p}H^k(\mathbb{Z}^d) = 0 \iff \frac{1}{p} - \frac{1}{q} \geq \frac{1}{d}$ *except if* $(p, k) = (1, d)$ *or* $(d, 1)$.

Proof: Calderon-Zygmund if $p > 1$, geometric if $p = 1$.

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Heisenberg group $\mathbb{H}^m$ has $F\o l(n) = n^Q$ with $Q = 2m + 2$.

Theorem (Baldi-Franchi-Pansu-Tripaldi 2019)

$\ell^{q,p}H^k(\mathbb{H}^m) = 0 \iff \frac{1}{p} - \frac{1}{q} \geq \frac{1}{Q}$ *(replace Q with $Q/2$ if $k = m + 1$), except if* $(p, k) = (1, 2m + 1)$ *or* $(Q, 1)$.