Flat compactness of normal Rumin currents on the Heisenberg group

Pierre Pansu, joint with A. Julia

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- 2005-2016: Franchi and Serapioni (often with Serra-Cassano) study the differentiability of intrinsic Lipschitz functions, guided by the idea of currents.

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Theorem (Representation of charges)

A (certain) dual of the space of normal Rumin currents with compact support is the space $C^0 + d_R C^0$.

This is an avatar of a result by T. De Pauw, L. Moonens and W. Pfeffer (2009) in Euclidean space.

Rumin forms on Heisenberg group can be viewed as a subset of differential forms, defined by the vanishing of certain components. M. Rumin has defined a differential d_R on Rumin forms that satisfies $d_R \circ d_R = 0$ and whose cohomology vanishes locally. d_R has order two in the middle dimension.

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The *mass* $M(T)$ of a Rumin current is

$$M(T) = \sup\{\langle T, \omega \rangle\, ;\, \|\omega\|_\infty \leq 1\}.$$

A Rumin current is *normal* if $N(T) := \max\{M(T), M(\partial_R T)\} < +\infty$.

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The *flat norm* of a Rumin current T is

$$F(T) := \min\{M(R) + M(S) \,;\, T = R + \partial_R S\}.$$

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Currents of integration. In $\mathbb{H}^1$, a smooth 2-submanifold V with boundary defines a Rumin current T_V of mass equal to its Hausdorff 3-dimensional measure. ∂_R differs from ∂: if ω is a smooth compactly supported 1-form,

$$\langle \partial_R T_V, \omega \rangle = \langle \partial T_V, \omega - \frac{d\omega}{d\theta} \theta \rangle.$$

Therefore $M(\partial_R T_V) < \infty$ only if ∂V is horizontal.

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By duality (the adjoint of a compact operator is compact), this follows from a corresponding statement for Rumin forms. Consider the Heisenberg nilmanifold $Z = \mathbb{H}^n / \mathbb{H}^n_{\mathbb{Z}}$.

Proposition

Let X be the space of Rumin forms on Z with vanishing averages, equipped with the norm

$$\|\omega\|_X := |\omega|_\infty + |\mathrm{d}_R \omega|_\infty,$$

and let Y be the space of Rumin forms on Z equipped with the norm

$$\|\omega\|_Y := \inf\{|\phi|_\infty + |\psi|_\infty, \, \omega = \psi + \mathrm{d}_c \phi\}.$$

Then the injection $X \hookrightarrow Y$ is compact.

Pick $p > 2n + 2$. Then every closed Rumin form $\omega \in L^q$ has a primitive $\phi \in W^{1,q}$.
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If (ω_j) is bounded in X,

$$\omega_j = S\omega_j + d(Q\omega_j) + Q(d\omega_j)$$

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Sobolev's embedding $W^{1,p} \hookrightarrow C^\alpha$, $\alpha = 1 - (2n+2)/p$, implies that each summand is bounded in C^α, hence compact in L^∞.

Thus (ω_j) is compact in Y.

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*A **Rumin charge** is a linear functional on the space of compactly supported normal Rumin currents, which is continuous on each $S(K,\nu)$.*

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Theorem (Representation of Rumin charges)

The space of Rumin charges is $C^0 + dC^0$, i.e. if ϕ is a continuous Rumin form on Heisenberg group, both ϕ and $d_R\phi$ define Rumin charges. Conversely, every Rumin charge is of the form $\phi + d_R\psi$, where ϕ and ψ are continuous Rumin forms on Heisenberg group.

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The Euclidean case is part of Laurent Moonens' PhD Thesis, UC Louvain, 2008.

Fix a compact set K, let $\mathcal{M}_K$ (resp. $\mathcal{N}_K$) be the space of currents of finite mass (resp. finite N norm) with support in K. Then $\mathcal{M}_K = C^0(K)'$,

$$\mathcal{N}_K = \mathcal{M}_K \cap d_R^{-1}(\mathcal{M}_K) = (C^0(K) + d_R C^0(K))'.$$

So we are claiming reflexivity of $C^0 + d_R C^0$, which fails!

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So we are claiming reflexivity of $C^0 + d_R C^0$, which fails! The point is the change of topology on $\mathcal{N}_K$, passing from the N norm to the flat norm. The keyword is *semireflexivity*.

Definition (Bourbaki)

A linear form on a topological vectorspace is ***strongly continuous*** *if it is bounded on bounded subsets.*
A locally convex topological vectorspace X is ***semireflexive*** *if every strongly continuous linear form on its topological dual X' arises from an element of X.*

Let X be a locally convex space, let $\mathcal{S}$ be a dilation stable family of convex subsets of X. There is a topology $\mathcal{T}_{\mathcal{S}}$ on X, inducing the initial topology on each $S \in \mathcal{S}$, such that a linear map $X \to Y$ is continuous if and only if all its restrictions to elements of $\mathcal{S}$ are continuous.

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Theorem (T. De Pauw, L. Moonens, W. Pfeffer 2009)

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If $\mathcal{N}$ is the space of compactly supported normal Rumin currents, endowed with the flat topology, and $\mathcal{S} = \{S_{K,\nu}\}$, then Rumin charges are precisely continuous linear functionals on $(\mathcal{N}, \mathcal{T}_S)$,

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The theorem applies. By semireflexivity, every strongly continuous linear functionals on the space of charges CH arises from a normal current of compact support,

$$(\mathcal{N}, \mathcal{T}_S) \simeq CH^*.$$

The second ingredient is the identification of bounded subsets of C^0.

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We want to show that $\Theta : C^0 \oplus C^0 \to CH$, $(\phi, \psi) \mapsto \phi + d_R\psi$ is onto. We proceed by showing that its adjoint $\Theta^* : \mathcal{N} = CH^* \to (C^0 \oplus C^0)^*$, given by

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is proper. If $S \subset \mathcal{N}$ and $\Theta^*(S)$ is bounded on all bounded subsets of $C^0 \oplus C^0$, then, for $T \in S$, $N(T)$ is bounded and $\mathrm{supp}(T)$ is in a common compact set.

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Hence closedness of the range of Θ^*. This shows that Θ is onto. Every charge can be written $\phi + d_R\psi$, for $\phi, \psi \in C^0$.