Persistence in geometric group theory

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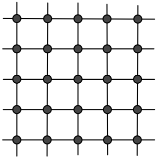
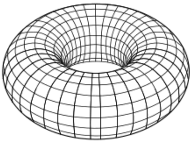
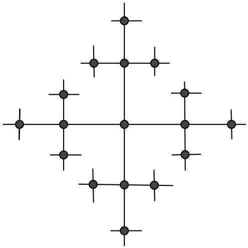


Free abelian group

$\langle a, b | aba^{-1}b^{-1} \rangle$

Fundamental group of 2-torus

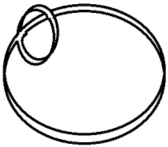


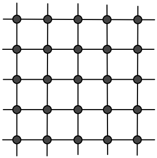


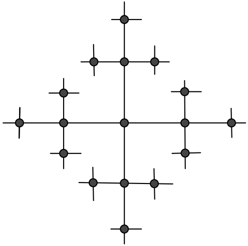
Free group

$\langle a, b | \rangle$

Fund. group of punctured 2-torus







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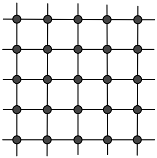

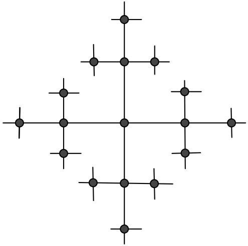

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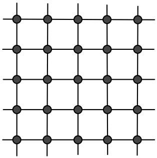

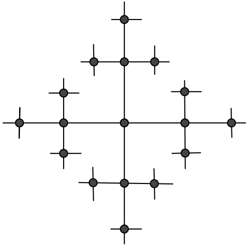

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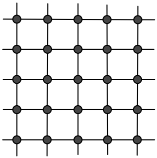

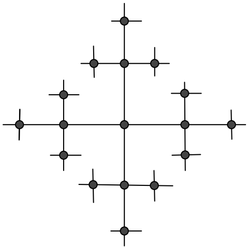

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Need to focus on rough, large scale features.
Allows to use continuous tools.

The word problem. Let $\langle S|R\rangle$ be a finite presentation of a group G. Find an algorithm that, given as input a (reduced) word w in the alphabet S, returns YES if and only if w is trivial in G, i.e. a product of conjugates of relators (elements of R).

Quantitative version. Estimate the minimal number $A(w)$ of relators in terms of the length $|w|$ of w, i.e. the function

$$Dehn^{\langle S|R\rangle} : n \mapsto \max\{A(w)\,;\,|w| \leq n\}.$$

Examples. In a free group, $A(w) = 0$. In a free abelian group, $A(w) \sim \frac{1}{16}|w|^2$.

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Theorem (P. Novikov 1952, W. Boone 1959)

There exist finitely presented groups with unsolvable word problem.

Geometric interpretation. In the cell complex of the presentation, a word w trivial in G corresponds to an edge-loop, $A(w)$ is the area of disk filling it.
This helps when showing that the growth type of $Dehn^{\langle S|R\rangle}$ depends only on the group G, not on the choice of S.

Question. Which functions arise as $Dehn^{\langle S|R\rangle}$ of some finite presentation of a group?

Theorem (Papasoglu, Bowditch)

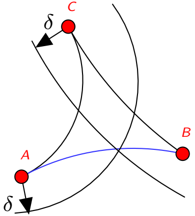
$Dehn^{\langle S|R\rangle}$ subquadratic $\implies$ $Dehn^{\langle S|R\rangle}$ at most linear.

Theorem (Birget, Olshanski, Rips, Sapir 2002)

Every at least quartic function that can be simulated by a Turing machine is $Dehn^{\langle S|R\rangle}$ of some finite presentation of a group.

Question. Fill the gap between quadratic and quartic.

For a finitely presented group, a linear filling inequality turns out to be equivalent to the following *hyperbolicity* axiom, a property satisfied by Lobatchevsky's nonEuclidean geometry (**M Gromov 1986**).



Definition

Let $\delta \in \mathbb{R}_+$. Say a finitely generated group G is δ-hyperbolic if geodesic triangles in G are δ-thin: each side is contained in the δ-neighborhood of the two others.

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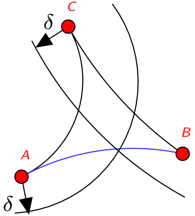

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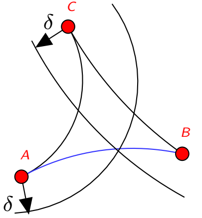

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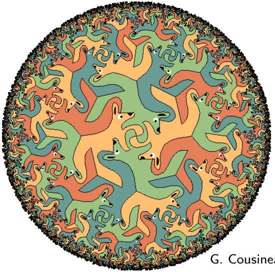
Hyperbolic groups are plentiful:

- Random groups (picking large random relators) are hyperbolic.
- Adding large relators does not kill the group.
- Taking limits of hyperbolic groups produces heaps of examples and counterexamples of weird finitely generated groups.

Geometry of hyperbolic groups

Symmetry groups of nonEuclidean tilings are hyperbolic



G. Cousineau

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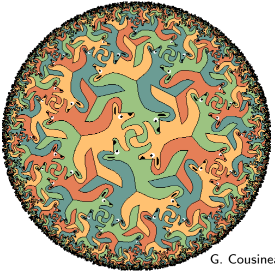

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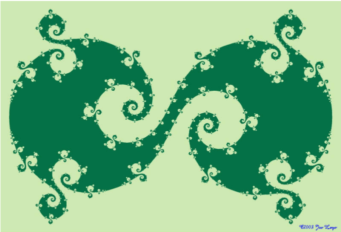
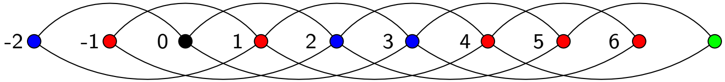
A hyperbolic group possesses an ideal boundary: a fractal space equipped with a *quasisymmetric structure* allowing a new kind of real analysis: *Analysis on Metric Spaces*.

Keywords: differentiability, function spaces.

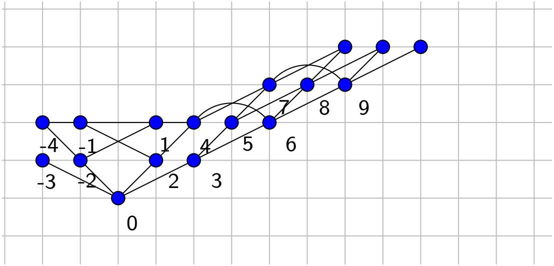


In Cayley graphs of groups, the distance $g \mapsto |g|$ to e sometimes achieves local maxima.

Example. $G = \mathbb{Z}$ with generating system $\{2, 3\}$.



Other view, showing distance to 0:



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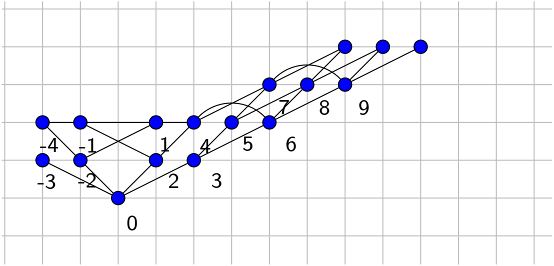
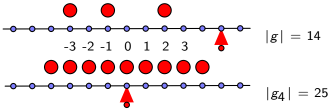


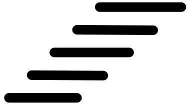
Define the deadend depth of an element g of G with respect to a generating set S as the distance of g to the complement of the ball $B(e, |g|)$.

Example (T. Riley and A. Warshall 2006). There exists a finitely presented group with unbounded deadend depth for some generating system, and bounded deadend depth for an other generating system.

Example. The *lamplighter group* is $\langle a, t | a^2, [a^{t^n}, a], \forall n \in \mathbb{Z}\rangle$.



The element g_n satisfies $|g_n| = 6n + 1$. $\forall h \in \partial B(g_n, n)$, $|h| \leq |g_n| - n$.
Hence a superlevel set H^0 barcode:



Say a graph has *very deep deadends* if there exists a vertex such that the distance barcode has arbitrarily long bars (p, q) with $p = o(q)$.
Question. Do there exist finitely generated/presented groups with very deep deadends?
This is a quasiisometry invariant property.

On the **space of loops** in a Cayley graph, the length function sometimes achieves deep local minima, leading to H^0 barcodes with long bars.

Define the *loop shortening function* of a presentation $\langle S|R\rangle$ as follows: let $lsf : \mathbb{N} \to \mathbb{N}$ be the smallest function such that every loop of length $\leq n$ can be reduced to the constant loop by applications of relators one at a time via loops of length $\leq lsf(N)$.

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Easy fact. For Novikov-Boone groups, lsf cannot be bounded above by a recursive function.

Question. Which functions arise as lsf of some finite presentation of a group?

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Easy fact. For Novikov-Boone groups, *lsf* cannot be bounded above by a recursive function.

Question. Which functions arise as *lsf* of some finite presentation of a group?

Question. What *lsf* arise for specific subclasses of groups?

Easy fact. For hyperbolic groups, *lsf* is linear.

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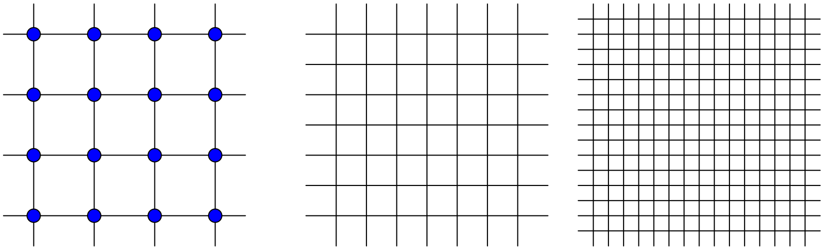

As a metric space, $\epsilon Cay(\mathbb{Z}^2)$ tends to $\mathbb{R}^2$ in ℓ^1 metric as $\epsilon \to 0$.

This extends to finitely generated nilpotent groups. Let G be a nilpotent group generated by finite S, $X = Cay(G, S)$. Then

$$\lim_{\epsilon \to 0} \epsilon X = G',$$

where G' is a selfsimilar nilpotent Lie group (G' admits a 1-parameter group of automorphisms homothetic for a leftinvariant geodesic metric).

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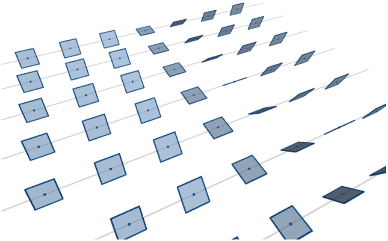
Example: Heisenberg group

$$G = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} ; x,y,z \in \mathbb{Z} \}.$$

$$\delta_t \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & tx & t^2z \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{pmatrix}.$$



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Dehn **: counterexample (C. Llosa-Isenrich, G. Pallier and R. Tessera 2019)**. There exists a finitely generated nilpotent group G such that $Dehn^G(n) \sim n^3$ and $Dehn^{G'}(n) \sim n^4$.

X metric space. An η-filling of a loop $c : \partial D \to X$ is an extension of c to the 1-skeleton of a triangulation of the disk D such that the images of the boundaries of triangles have length $\leq \eta$. Its area is the number of triangles.

Define $Area_\eta(c)$ as the the minimum area of η-fillings of c. Define a function $Fill_\eta^X : \mathbb{N} \to \mathbb{N}$ by

$$Fill_\eta^X(n) = \sup\{Area_\eta(c)\,;\, Length(c) \leq n\}.$$

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Since $Fill_\eta^{\epsilon X}(n) = Fill_{\eta/\epsilon}^X(n/\epsilon)$, understanding the discontinuity of $\epsilon \mapsto Dehn^{\epsilon G}$ is equivalent to understanding $Fill_\eta^X(n)$ as a 2-variable function.

Alternatively, $Fill_{\eta/\epsilon}^X(n/\epsilon) = Dehn^{S(\epsilon),R(\epsilon)}(n)$ where $S(\epsilon)$ is the ball of radius $1/\epsilon$ and $R(\epsilon)$ the set of trivial words of length η/ϵ.