Large scale geometry of nilpotent Lie groups

Pierre Pansu

June 19-20th, 2023

Goal. The large scale behaviour of nilpotent Lie groups is governed by self-similarity:

- There is a subclass of nilpotent Lie groups, called Carnot groups, which admit (subFinsler) metrics, which are exactly self-similar.
- Every nilpotent Lie group is asymptotic to such a Carnot group.

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Plan of lecture

Lectures based on Enrico Le Donne's book project.
Ask him for updates.

We are interested in large scale invariants like

- Volume growth: $v(R)$ = number of unit balls needed to cover a ball of radius R.
- Isoperimetry: $I(v)$ = boundary volume needed to enclose volume at least v.
- More general filling inequalities: $Fill(L)$ = area needed to fill all loops of length at most L.

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$$\begin{aligned} dist(\delta_\epsilon(x), \delta_\epsilon(x')) &= \epsilon\, dist(x, x'), \\ vol(\delta_\epsilon(A)) &= \epsilon^n\, vol(A), \\ vol(\delta_\epsilon(\partial A)) &= \epsilon^{n-1}\, vol(\partial A), \\ length(\delta_\epsilon(\ell)) &= \epsilon\, length(\ell), \\ area(\delta_\epsilon(S)) &= \epsilon^2\, area(S) \end{aligned}$$

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Question. Are there other groups for which such scale invariance applies?

The 3-dimensional Heisenberg group is

$$\mathbb{H} = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \; ; x, y, z \in \mathbb{R} \}.$$

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If $M = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$, then $M^{-1} = \begin{pmatrix} 1 & -x & -z + xy \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}$ and

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gives us a basis of left invariant 1-forms. So a typical left-invariant metric is

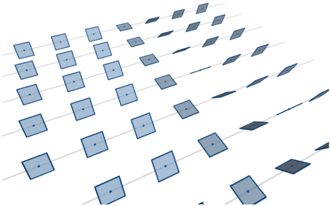
$$g_1 = dx^2 + dy^2 + (dz - xdy)^2.$$

Then

$$\delta_\epsilon^* g_1 = \epsilon^2 (dx^2 + dy^2 + \epsilon^2 (dz - xdy)^2).$$

Question. How does $g_\epsilon = dx^2 + dy^2 + \epsilon^2(dz - xdy)^2$ behave as $\epsilon \to \infty$?

The *horizontal plane field* $\mathrm{Ker}(dz - xdy)$ is generated by left-invariant vector fields $\xi = \frac{\partial}{\partial x}$ and $\eta = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$.



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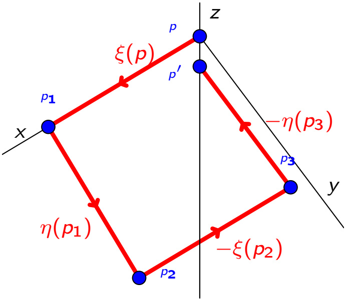
$$\zeta = [\xi, \eta] = \frac{\partial}{\partial z},$$

see the picture.



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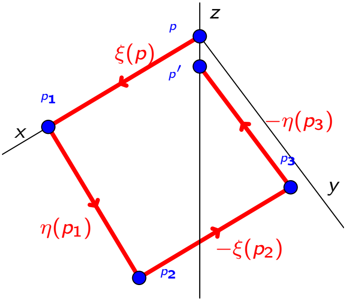

The family of distances d_ϵ defined by g_ϵ is increasing and bounded above by d_∞. Therefore it converges.

Proposition

$$\lim_{\epsilon \to \infty} d_\epsilon = d_\infty.$$

Proof given soon.

Lemma (π is a submetry)

The projection $\pi : \mathbb{H} \to \mathbb{R}^2$, $(x, y, z) \mapsto (x, y)$ *is a submetry, i.e. it maps* d_ϵ *balls to Euclidean balls of equal radii.*

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Indeed, $g_\epsilon \geq dx^2 + dy^2$, so π is 1-Lipschitz. Conversely, along horizontal curves, $g_\epsilon = dx^2 + dy^2$, so π preserves the length of horizontal curves. Since all plane curves have horizontal lifts (setting $z(t) = \int_0^t x(x)y'(s)ds$), the image of a ball is exactly a ball.

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The Carnot-Carathéodory distance satisfies

$$d_\infty((x,y,z),(0,0,0)) \leq \sqrt{x^2+y^2} + 4\sqrt{|z|}.$$

We already featured a horizontal path of length $4\sqrt{|z|}$ joining $(0,0,0)$ to $(0,0,z)$. Then the horizontal lift $t \mapsto (tx, ty, z + \frac{t^3}{3}xy)$ of the line segment from $(0,0)$ to (x,y) joins $(0,0,z)$ to (x,y,z) and has length $\sqrt{x^2+y^2}$.

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Remark. By compactness, there exists a constant $c > 0$ such that

$$c(\sqrt{x^2 + y^2} + 4\sqrt{|z|}) \leq d_\infty((x, y, z), (0, 0, 0)).$$

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There exists a constant C such that, on the d_∞ unit ball,

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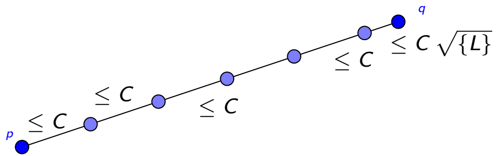
Indeed, if two points sit at d_1 distance L, cutting a d_1 geodesic in pieces of length 1, one sees that their d_∞-distance is at most $\lfloor L \rfloor C + C\sqrt{\{L\}}$.



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Corollary (Large scale geometry of d_1)

- *Volume growth:* $v(R) = O(R^4)$.
- *Isoperimetry:* $I(v) = O(v^{3/4})$.
- *Loop filling:* $Fill(L) = O(L^3)$.

Indeed, d_1 is quasiisometric to d_∞.

Theorem (Sharper - still unsharp - comparison between d_1 and d_∞)

$$d_\infty = d_1 + O(d_1^{1/2}).$$

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It is indeed sharper, since

$$\frac{d_\epsilon(e,p)}{d_\infty(e,p)} = \frac{d_1(e,\delta_\epsilon(p))}{d_\infty(e,\delta_\epsilon(p))}$$

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Proof of Theorem

Let $p \in \mathbb{H}$ at distance $L = d_1(e,p)$. Let γ be a minimizing geodesic from e to p, with constant speed L. Let γ_∞ be the horizontal lift from e of $\pi \circ \gamma$, let $\phi = \delta_{1/L} \circ \gamma$ and $\phi_\infty = \delta_{1/L} \circ \gamma_\infty$. Both have length 1. Then

$$\begin{aligned}\frac{1}{L} d_\infty(e,p) &= d_\infty(e,\phi(1)) \\ &\leq d_\infty(e,\phi_\infty(1)) + d_\infty(\phi_\infty(1),\phi(1)) \\ &\leq 1 + C\, d_1(\phi_\infty(1),\phi(1))^{1/2}.\end{aligned}$$

One views ϕ and ϕ_∞ as trajectories of left invariant time dependent vector fields X and Y: $X(q,t)$ is the left invariant vector field which equals $\phi'(t)$ at $\phi(t)$, idem for Y.

Lemma (Grönwall-type Lemma)

Let X and Y be time dependent vector fields on the unit ball of Euclidean space. Assume that for all points q, q' and all positive times t,

$$\|X(q,t) - Y(q',t)\| \leq C'(\epsilon + \|q - q'\|).$$

Then their trajectories ϕ and ψ from the origin satisfy

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Here, for all $t \in [0,1]$, $\gamma(t)$ and $\gamma_\infty(t)$ both lie in the d_1-ball of radius L, hence $\phi(t)$ and $\phi_\infty(t)$ both lie in a compact part of $\mathbb{H} = \mathbb{R}^3$ where g_1 upper bounded by a multiple of the Euclidean norm and where left invariant vector fields are Lipschitz.

$$\frac{1}{L} d_\infty(e,p) \leq 1 + C\, d_1(\phi_\infty(1), \phi(1))^{1/2}.$$

Lemma (Grönwall-type Lemma)

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Here, for all q, $Y(q,t)$ is the horizontal component of $X(q,t)=\phi'(t)=\delta_{1/L}(\gamma'(t))$ left-translated to q. Since $Z=\delta_L(Y(q,t)-X(q,t))$ is vertical,

$$\|Y(q,t)-X(q,t)\|_1=\|\delta_{1/L}(Z)\|_1=\frac{1}{L^2}\|Z\|_1\leq\frac{1}{L},$$

so a similar inequality holds for the Euclidean norm. Changing one of the q to q' adds a Lipschitz term $O(\|q-q'\|)$. The Lemma yields

$$d_1(\phi_\infty(1),\phi(1))\leq(e^{C'}-1)\frac{\text{const.}}{L},$$

and $\frac{1}{L}d_\infty(e,p)\leq 1+O(L^{-1/2})$, where $L=d_1(e,p)$.

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Remark. The sharper comparison does not directly yield asymptotics for the isoperimetric profile nor the filling function. Nevertheless, the strategy of approximating Riemannian by subRiemannian quantities should apply there as well.

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A subFinsler metric on (M, Δ) is the data of a continuous family on norms on Δ. Minimizing the lengths of horizontal curves yields a distance, the subFinsler distance.

Heisenberg group is a prototype in a broader family.

Definition (Carnot group)

*A **Carnot grading** on a Lie algebra $\mathfrak{g}$ is a gradation*

$$\mathfrak{g} = \bigoplus_{i=1}^{s} \mathfrak{g}_i,$$

*such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, and $\mathfrak{g}_1$ generates $\mathfrak{g}$. s is called the **step** of $\mathfrak{g}$.*
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The maps such that $\delta_\epsilon = \epsilon^i$ on $\mathfrak{g}_i$ are Lie algebra (and thus group via exp) automorphisms. A norm on $\mathfrak{g}_1$ defines a left invariant subFinsler metric on the group, which is selfsimilar: δ_ϵ multiplies it exactly by ϵ.

Theorem (Sublinear equivalence, Breuillard-Le Donne 2013, Giannella 2017)

Let G be a Carnot group of step s, let d_1 be a left invariant subFinsler metric on G. There exists a left invariant ***selfsimilar*** *subFinsler metric d_∞ on G such that*

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Take the norm on $\mathfrak{g}_1 \simeq G/[G,G]$ that makes the projection $G \to G/[G,G]$ a submetry. Then same proof as for Heisenberg group.

Definition

Let $\mathfrak{g}$ be a Lie algebra. Define recursively $\mathfrak{g}^1 = \mathfrak{g}$ and $\mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{g}]$. A Lie algebra is ***nilpotent*** *if there exists s such that $\mathfrak{g}^{s+1} = \{0\}$. The smallest such s is the* ***step*** *of $\mathfrak{g}$.*

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Notation

Let G be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. The number

$$Q = \sum_{i=1}^{s} i \dim(\mathfrak{g}^i/\mathfrak{g}^{i+1}) = \sum_{i=1}^{s} \dim(\mathfrak{g}^i)$$

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We shall see below that Q is the degree of polynomial volume growth of G.

Every Carnot Lie algebra is nilpotent. Not every nilpotent Lie algebra admits a Carnot grading.

Example

The Lie algebra $\mathfrak{n}$ with basis $X_1, \ldots, X_5$ and nonzero brackets $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$, $[X_1, X_4] = X_5 = [X_2, X_3]$ does not admit a Carnot grading.

To see this, we first define

Definition (The associated Carnot graded Lie algebra $Car(\mathfrak{g})$)

Let $\mathfrak{g}$ be a nilpotent Lie algebra. Let

$$Car(\mathfrak{g}) = \bigoplus_{i=1}^{s} \mathfrak{g}^i/\mathfrak{g}^{i+1},$$

with the induced brackets $\mathfrak{g}^i/\mathfrak{g}^{i+1} \times \mathfrak{g}^j/\mathfrak{g}^{j+1} \to \mathfrak{g}^{i+j}/\mathfrak{g}^{i+j+1}$. It is a Lie algebra, which comes with a tautological Carnot grading.

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If a Lie algebra $\mathfrak{g}$ admits a Carnot grading, then $\mathfrak{g}$ and $Car(\mathfrak{g})$ are isomorphic as Carnot Lie algebras. Here, $Car(\mathfrak{n})$ has one less nonzero bracket, $[X_2, X_3] = 0$, so it is not isomorphic to $\mathfrak{n}$.

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Definition (Associated Carnot group)

*Let G be a simply connected nilpotent Lie group. The **associated Carnot group** $Car(G)$ is the simply connected Lie group with Lie algebra $Car(\mathfrak{g})$.*

Theorem (Breuillard-Le Donne 2013, Gianella 2017)

Let G be a simply connected nilpotent Lie group of step s, let d_1 be a left invariant subFinsler metric on G. There exists a left invariant selfsimilar subFinsler metric d_∞ on $Car(G)$ and a diffeomorphism $f : G \to Car(G)$ such that

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The same method applies, with a change: the vector fields X and Y are now left invariant with respect to distinct group structures on the vector space $\mathfrak{g}$, the G and $Car(G)$ multiplications, transported to $\mathfrak{g}$ and $Car(\mathfrak{g})$ by exponentials, and a linear isomorphism $\mathfrak{g} \to Car(\mathfrak{g})$ that preserves filtrations by commutators (but not gradings).

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We now show that this change is insignificant.
Everything is carried to the normed vector space $\mathfrak{g}$. Recall that $p \in \mathfrak{g}$ lies at G-distance $L = d_1(e, p)$. Let γ be a d_1-minimizing geodesic from 0 to p, with constant speed L. Let γ_∞ be the $Car(G)$-horizontal lift from 0 of $\pi \circ \gamma$, let $\phi = \delta_{1/L} \circ \gamma$ and $\phi_\infty = \delta_{1/L} \circ \gamma_\infty$.

$X(q,t)$ is the G-left invariant vector field which equals $\phi'(t)$ at $\phi(t)$, and $Y(q,t)$ is the $Car(G)$-left invariant vector field which equals $\phi'_{\infty}(t)$ at $\phi_{\infty}(t)$.

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X and Y still have the same projection on $\mathfrak{g}/\mathfrak{g}^2$, so $X - Y$ takes values in $\mathfrak{g}^2$.

$$Z = \delta_L(Y(q,t) - X(q,t)) = \gamma'_\infty(t) - \gamma'(t) = O(L).$$

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Also, $\phi(t)$ and $\phi_\infty(t)$ stay bounded, so the restriction of the norm to $\mathfrak{g}^2$, left-translated to these points, is still multiplied by $\frac{1}{L^2}$ by $\delta_{1/L}$, up to a bounded multiplicative error.

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so Grönwall's Lemma applies and shows that

$$d_1(\phi_\infty(1),\phi(1)) \leq (e^{C'}-1)\frac{\text{const.}}{L}, \quad \text{and hence} \quad \frac{1}{L}d_\infty(0,p) \leq 1 + C\, d_1(0,p)^{-1/s}.$$

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Since the inequality $d_1\leq d_\infty$ need not hold, one must exchange G and $Car(G)$ and repeat the argument to get the opposite inequality, for $L'=d_\infty(0,p)$,

$$\frac{1}{L'}d_1(0,p)\leq 1+C\,d_\infty(0,p)^{-1/s}.$$

Definition (Gromov-Hausdorff convergence)

*Given two metric spaces X and Y, an ϵ-**approximation** is a map $\phi : X \to Y$ whose image is ϵ-dense and which preserves distances up to an additive error of ϵ:*

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Example

Let G be a Carnot group with selfsimilar left invariant subFinsler metric d. Assume G admits a discrete cocompact subgroup Γ. Then the family of subgroups $(\delta_\epsilon(\Gamma), d_{|\delta_\epsilon(\Gamma)})$, with the induced distances, converges to (G, d) as $\epsilon \to 0$.

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Example

The map which to a triple $([\cdot,\cdot], \Delta, \|\cdot\|)$ of a Lie algebra structure $[\cdot,\cdot]$ on $\mathbb{R}^n$, a generating linear subspace $\Delta \subset \mathbb{R}^n$ and a norm $\|\cdot\|$ on Δ associates the induced subFinsler metric on the corresponding simply connected Lie group is continuous.

Start from a metric space X and look at it from farther and farther away. Does the picture stabilize?

Definition (GH asymptotic cone)

Let (X, d, x_0) and (K, d_K, k_0) be based metric spaces. Say that K is a Gromov-Hausdorff asymptotic cone if there exists a sequence $\epsilon_j \to 0$ such that the sequence of based metric spaces $(X, \epsilon_j d, x_0)$ converges to K.

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Corollary

Let (G, d_1) be a simply connected nilpotent Lie group with a left invariant subFinsler metric. Then G admits a unique Gromov-Hausdorff asymptotic cone, which is isometric to $Car(G)$ equipped with a selfsimilar left invariant subFinsler metric.

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GH-asymptotic cones are a special case of the general notion of asymptotic cone of a based metric space. Asymptotic cones always exist, but they are GH-asymptotic cones and locally compact only for space of polynomial volume growth.

Corollary (Volume growth)

Let G be a simply connected nilpotent Lie group of homogeneous dimension Q and step s, let d_1 be a left invariant subFinsler metric on G. There exists a positive number ω such that the volume of balls satisfies

$$v(R) = \omega R^Q + O(R^{Q-\frac{1}{s}}).$$

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However filling functions of nilpotent groups are not fully determined by their asymptotic cones. The following general inequality holds (Gromov, Papasoglu): define the filling degree of a Lie group G as the infimal ν such that $Fill^G(L) = O(L^\nu)$ as $L \to +\infty$. Then

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However, it is not always an inequality.

Example (Llosa-Isenrich, Pallier, Tessera 2023)

There exist simply connected nilpotent Lie groups whose filling degree differs from that of their asymptotic cone.

Given a based metric space (X, d), the speed of convergence to the asymptotic cone K is measured by the function

$$R \mapsto \sigma(R) = dist_{GH}((B^X(R), \frac{1}{R}d), (B^K(1), d_K)).$$

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The following is a consequence of the asymptotic expansion of distances on nilpotent Lie groups.

Corollary

Let G be a simply connected nilpotent Lie group of step s, equipped with a left invariant subFinsler metric on G. Then

$$\sigma(R) = O(R^{-\frac{1}{s}}).$$

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Question. How sharp is this estimate?

Example (Breuillard-Le Donne 2013)

Let $G = \mathbb{R} \times \mathbb{H}$. This is a step 2 nilpotent group, which satisfies $\sigma(R) \sim R^{-\frac{1}{2}}$.

The authors relate this slow convergence phenomenon to the presence of abnormal curves ($\mathbb{R}$ factors) for the limiting subFinsler metric.

Abnormality of curves is defined in general for smooth distributions Δ. In the Carnot case, horizontal lifting defines a map from curves starting from 0 in $G/[G,G]$ to curves starting from e in G, and in particular an endpoint map $H^1([0,1],G/[G,G]) \to G$. A curve is *abnormal* if this map is not a submersion.

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Definition

Say a 2-step nilpotent Lie group is ***nonsingular*** *if in its Lie algebra $\mathfrak{g}$, for every $X \notin \mathfrak{g}^2$, $ad_X : \mathfrak{g} \to \mathfrak{g}^2$ is onto.*

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Theorem (Breuillard-Le Donne 2013, Tashiro 2021)

If G is a nonsingular 2-step Carnot group, for any left invariant subFinsler metric d_1 on G there exists a selfsimilar left invariant Finsler metric d_∞ on G such that $d_1 - d_\infty$ is bounded.

Abnormality of curves is defined in general for smooth distributions Δ. In the Carnot case, horizontal lifting defines a map from curves starting from 0 in $G/[G,G]$ to curves starting from e in G, and in particular an endpoint map $H^1([0,1],G/[G,G]) \to G$. A curve is *abnormal* if this map is not a submersion.

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The geodesics in question are merely horizontal 1-parameter groups, whence the algebraic characterization.

Theorem (Breuillard-Le Donne 2013, Tashiro 2021)

If G is a nonsingular 2-step Carnot group, for any left invariant subFinsler metric d_1 on G there exists a selfsimilar left invariant Finsler metric d_∞ on G such that $d_1 - d_\infty$ is bounded.

Goal. The large scale behaviour of nilpotent Lie groups is governed by self-similarity:

- There is a subclass of nilpotent Lie groups, called Carnot groups, which admit (subFinsler) metrics, which are exactly self-similar.
- Every nilpotent Lie group is asymptotic to such a Carnot group.

This behaviour is specific to nilpotent groups: self-similarity and isometric homogeneity characterize subFinsler Carnot groups.

Plan of lecture

1. *Large scale geometry of the Heisenberg group*
2. *Generalization to nilpotent groups*
3. *GH-asymptotic cones*
4. *Characterization of self-similar isometrically homogeneous spaces*
5. *Quasiisometries of nilpotent groups*

Lectures based on Enrico Le Donne's book project.
Ask him for updates.

Theorem (Berestovskii 2004, Le Donne 2015)

If a metric space is

1. *locally compact,*
2. *geodesic,*
3. *isometrically homogeneous,*
4. *self-similar, i.e. it admits a homeomorphism that multiplies distance by a number* > 1*,*

then it is isometric to a subFinsler Carnot group.

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The proof relies on

1. Gleason-Montgomery-Zippin's theorem providing sufficient topological and metric conditions for the isometry group of a general metric space to be a Lie group.

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1. Gleason-Montgomery-Zippin's theorem providing sufficient topological and metric conditions for the isometry group of a general metric space to be a Lie group.
2. Berestovskii's complement in the geodesic case.
3. Mitchell's description of metric tangents to equiregular subFinsler manifolds: they are Carnot groups equipped with selfsimilar subFinsler metrics.

Theorem (Gleason, Montgomery-Zippin 1952)

Let X be a metric space. Assume that

- *X is connected and locally connected,*
- *X is locally compact,*
- *X has finite topological dimension,*
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Then $\mathrm{Isom}(X)$ has the structure of a Lie group with finitely many connected components, and X has the structure of an analytic manifold.

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In our setting, finite dimensionality follows from self-similarity, via *doubling*.

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A metric space X is ***doubling*** *if there exists a N such that every r-ball can be covered with N $r/2$-balls.*

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self-similar + locally compact $\implies$ doubling $\implies$ finite Hausdorff dimension $\implies$ finite topological dimension.

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Indeed, use local coordinates near some point x_0 to define self-homeomorphisms δ_ϵ of space X. Consider rescaled small balls $\beta_r = \delta_{1/r}(B(x_0, r))$ and take the intersection $\beta = \bigcap_{r \to 0} \beta_r$.

One shows that β is a compact convex and centrally symmetric set. Thus it is a convex body in a linear subspace $L \subset T_{x_0}X$. It is invariant under the stabilizer G_{x_0}. Therefore, by translation, it defines a subFinsler metric which coincides with d.

The last step exploits self-similarity, using the fact that smooth subFinsler manifolds admit metric tangents.

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Definition (Metric tangents)

Let (X, d) be a metric space. A metric tangent at x_0 is a Gromov-Hausdorff limit of rescaled based spaces $(X, \epsilon d, x_0)$ as $\epsilon \to \infty$.

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A smooth distribution Δ is ***equiregular*** *if the subspaces of the tangent bundle spanned by 1st, 2nd... order brackets have constant rank.*

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Let X be a smooth equiregular subFinsler manifold. Then X admits at every point a unique metric tangent: it is a Carnot group equipped with a selfsimilar subFinsler metric.

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We see that tangents happen to be groups. Why?

Here is a heuristic suggested by Bellaïche and implemented by O. Mohsen.

Let X be a topological space. Then $X \times X$ is a pseudogroup, i.e. there exists a partially defined multiplication rule, here

$$(x_1, x_2),\ (x_2, x_3) \mapsto (x_1, x_3).$$

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Assuming that X admits a metric tangent T_x at each point, then there exist approximate isometries $\phi_\epsilon^x : (X, \epsilon d) \to T_x$. One can introduce a topology on the disjoint union

$$X \times X \times (0,1] \cup TX, \quad \text{where } TX := \bigcup_{x \in X} T_x.$$

where (x, x', ϵ) converges to (z, v) when $\epsilon \to 0$, $x \to z$ and $\phi_\epsilon^x(x') \to v$.

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Then the pseudogroup structure on $X \times X$ converges to a pseudogroup structure on TX, where elements (x, v) and (x', v') can be multiplied if and only if $x = x'$. Therefore each metric tangent T_x carries a multiplication. Inverses exist in $X \times X$, so they exist in T_x, therefore T_x is a group.

Theorem

If two nilpotent Lie groups G and G' are quasiisometric, then their associated Carnot groups $Car(G)$ and $Car(G')$ are isomorphic (as groups with gradings).

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Indeed, setting $d_\epsilon = \epsilon d_1$, $d'_\epsilon = \epsilon d'_1$,

$$-C + \frac{1}{L} d_1(x, x') \leq d'_1(f(x), f(x')) \leq L d_1(x, x') + C$$

$$\Longrightarrow \ -\epsilon C + \frac{1}{L} d_\epsilon(x, x') \leq d'_\epsilon(f(x), f(x')) \leq L d_\epsilon(x, x') + \epsilon C.$$

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Composing with ϵ-approximations yields maps $Car(G) \to Car(G')$ which are biLipschitz up to small additive error. Arzela-Ascoli $\implies$ a (nonunique) limiting biLipschitz map.

Step 2. From now on, G, G' are Carnot, $f : G \to G'$ is biLipschitz with respect to selfsimilar subFinsler metrics. We would like that $\delta_\epsilon \circ f \circ \delta_{1/\epsilon}$ converges as $\epsilon \to 0$. If so, the limit should be a selfsimilar, i.e. graded, group homomorphism. This suggests the

Definition (Differentiabilty)

*Say a map f between neighborhoods of e in Carnot groups is **differentiable** at e if $\delta_{1/\epsilon} \circ f \circ \delta_\epsilon$ converges uniformly on compact sets as $\epsilon \to 0$ to a graded group homomorphism. Using left translations, one defines differentiability at other points.*

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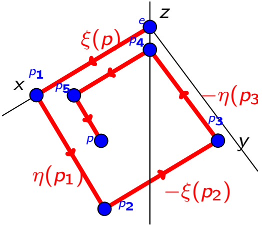
This proves the above quasiisometric rigidity theorem.

Case where $f : \mathbb{H} \to \mathbb{R}$ is Lipschitz.
A Lipschitz map admits horizontal partial derivatives almost everywhere. These derivatives are approximately continuous almost everywhere.

Write $e = p_0$, $p = p_6$, $\partial_i =$ successively $\xi, \eta, -\xi, -\eta, \xi, \eta,$



$$f(p) - f(e) = \sum_{i=1}^{6} f(p_i) - f(p_{i-1})$$

$$= \sqrt{|z|} \sum_{i=0}^{3} (\partial_i f)(p_i) + x(\xi f)(p_4) + y(\eta f)(p_5) + \epsilon$$

$$= \sqrt{|z|} \sum_{i=0}^{3} (\partial_i f)(e) + x(\xi f)(e) + y(\eta f)(e) + \epsilon$$

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where $\epsilon = o(|x| + |y| + \sqrt{|z|})$, provided e is a point of existence and approximate continuity of all partial derivatives, the Lipschitz bound allowing the needed flexibility in the choice of break points.

Case where $f : \mathbb{R} \to \mathbb{H}$ is Lipschitz.

Assume that $f(0) = e$, $f(t) = (x(t), y(t), z(t))$. One shows that f is horizontal, hence

$$z(t) - \frac{1}{2}x(t)y(t) = \int_0^t \frac{1}{2}(x(s)y'(s) - x'(s)y(s))\, ds,$$

is the area enclosed by the projection $s \mapsto (x(s), y(s))$ and the line segment from $(x(t), y(t))$ to $(0, 0)$.

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So, if x and y have derivatives at 0, $z(t) - \frac{1}{2}x(t)y(t) = o(t^2)$.

$$\delta_{1/\epsilon}(x(\epsilon), y(\epsilon), z(\epsilon)) = (\frac{1}{\epsilon}x(\epsilon), \frac{1}{\epsilon}y(\epsilon), \frac{1}{\epsilon^2}z(\epsilon))$$

tends to $(x'(0), y'(0), \frac{1}{2}x'(0)y'(0))$.

General case is a mixture.

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Absolute continuity. In order to show that differentials are bijective, one needs show that for a.e. $p \in G$, f^{-1} is differentiable at $f(p)$. I.e. that f^{-1} of the differentiability locus of f^{-1} has full measure. This is obvious for biLipschitz maps, but not for quasiconformal maps.

Lemma (Mostow 1970)

Quasiconformal homeomorphisms between open sets in Carnot groups are absolutely continuous on horizontal lines, and hence map null sets to null sets.

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The Rademacher-Stepanov type theorem and the absolute continuity theorem have been extended to smooth equiregular distributions by Margulis and Mostow: differentials become graded homomorphisms between metric cones. The theorems hold under weaker differentiability assumptions on distributions, see work by Vodopjanov and his school.
Recently, differentiability loci of Lipschitz mapping on Carnot groups have been investigated by De Philippis, Marchese, Merlo, Pinamonti and Rindler: the Rademacher-Stepanov theorem fails for all measure classes but Haar's.