Quasiconformal classification of 3D contact Lie groups

Pierre Pansu

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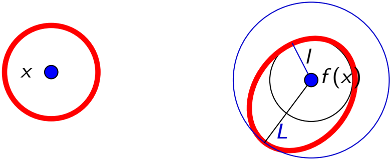
$$\limsup_{r\to 0}\frac{L(f,x,r)}{\ell(f,x,r)}\leq H.$$

We focus on contact subRiemannian spaces, i.e. smooth contact manifolds with subRiemannian metrics.



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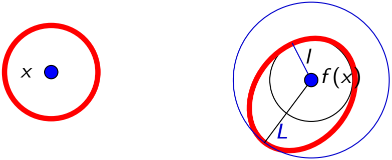

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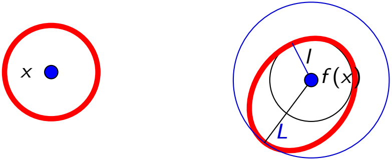

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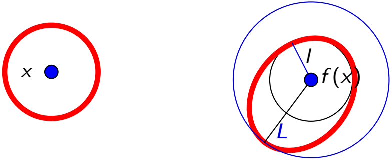

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We call a *contact Lie group* the data of a connected Lie group G and a left-invariant contact structure ξ on G.

Any two choices of left-invariant subFinsler metrics on ξ are biLipschitz equivalent, in particular quasiconformally equivalent. Therefore one can discuss wether two contact Lie groups are quasiconformally equivalent or not.

We are concerned first with the quasiconformal classification of 3-dimensional contact Lie groups.

Theorem

1. *The left invariant contact structures on* $U(1) \times Dil(\mathbb{R})$ *and* $PSl(2,\mathbb{R})$ *are quasiconformally equivalent, as are their coverings.*
2. *In all other cases, two contact 3-dimensional Lie groups are quasiconformally equivalent if and only if they are isomorphic as Lie groups.*

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All contact structures encountered (on $\mathbb{R}^3$, $\mathbb{R}^2 \times S^1$) are smoothly contactomorphic. So no interference with contact topology. Nevertheless, a trick is borrowed from contact topology.

The previous classification relies on the classical tools of quasiconformal geometry: the modulus of a curve family and the capacity of a condenser. They are 1-dimensional invariants, expressible in terms of lengths of curves or L^p norms of closed 1-forms.

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Theorem (Work in progress)

There exist (high-dimensional) pairs of contact subRiemannian manifolds (M, N) such that

1. *Both M and N admit cocompact isometry groups.*
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The use of higher degree differential forms in the quasiworld is not so frequent:

Donaldson-Sullivan, *Quasiconformal 4-manifolds*. Acta Math. (1989).

Iwaniec-Martin, *Quasiregular mappings in even dimensions*. Acta Math. (1993).

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Lemma (Parabolicity of 3D contact Lie groups)

Here is the list of conformally parabolic 3*-dimensional contact groups, grouped in topological classes:*

- $SU(2)$.
- $SO(3)$.
- *Proper quotients of* $\widetilde{Mot(\mathbb{R}^2)}$ *and* $Heis$.
- $\widetilde{Mot(\mathbb{R}^2)}$ *and* $Heis$ *themselves.*

Since qc homeos lift to qc homeos on coverings, it suffices to show that $\widetilde{Mot(\mathbb{R}^2)}$ and $Heis$ are not quasiconformally equivalent.

Theorem (Fässler-Koskela-Le Donne (2015))

$\widetilde{Mot(\mathbb{R}^2)}$ *and Heis are not quasiconformally equivalent.*

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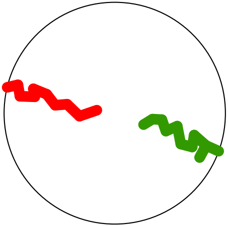
Their argument uses capacities of condensers of Teichmüller's type: formed by two unbounded curves.



Definition (Ferrand (1972))

Say that a noncompact subRiemannian manifold ***has a Teichmüller distance*** *if*

$$d_T(A,B) = (\inf \int |du|^{2n+2}\,;\ u = 0\ (\textit{resp. } 1)\ \textit{on an unbounded curve containing } A\ (\textit{resp. } B)\})^{-1/(2n+1)}$$

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Because volume growth in $\widetilde{Mot}(\mathbb{R}^2)$ is slow (cubic), there exist functions whose horizontal gradient has finite L^4 norm, but which take values 1 (resp. 0) along two geodesic rays. Therefore $\widetilde{Mot}(\mathbb{R}^2)$ has a Teichmüller distance.

However, by self-similarity, *Heis* cannot have a Teichmüller distance.

Proposition (qc $\implies$ qi principle)

Let M and N be bounded geometry $2n+1$-dimensional contact subRiemannian manifolds. If both have isoperimetric dimensions $> 2n+2$, then every quasiconformal mapping $M \to N$ is a quasiisometry.

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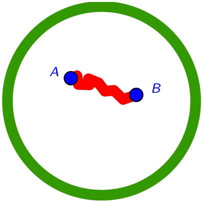
The prototype of such distorsion estimates is due to Grötzsch (1928). It is based on capacities of condensers formed by ∞ and a curve.



Parabolicity means that the capacities of all such condensers vanish.

Definition (Gal (1960), Vuorinen (1988))

On a nonparabolic contact subRiemannian manifold, the ***Grötzsch distance*** *is*

$$d_G(A,B)=\inf \int |du|^{2n+2}\,;\ u \text{ compactly supported, } u=1 \text{ on a curve joining } A \text{ to } B\}.$$

Isoperimetry (in fact, Sobolev inequality) implies that d_G grows at least like some function of the distance. Bounded geometry implies that d_G grows at most like some function of the distance. Whence the qi estimate on qc mappings.

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Remark. For 3-dimensional contact Lie groups,

$$\text{conformal parabolicity} \iff \mathit{IsopDim} \leq 4,$$

so the qc $\implies$ qi principle applies to all nonparabolic 3-dimensional contact Lie groups.

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5. One shows that all quasiisometric pairs are indeed quasiconformally equivalent.

For instance, $PSL_2(\mathbb{R})$ carries two $PSL_2(\mathbb{R})$-equivariantly nonequivalent left invariant contact structures, they turn out to be $Dil(\mathbb{R})$-equivariantly isomorphic (mirror + isotopy).

Cohomological interpretation of parabolicity

Let X be a Riemannian n-manifold. Fix a ball B. This creates a relative 1-cohomology class in $H^1_c(X, B; \mathbb{R})$, represented by differentials of smooth compactly supported functions that take value 1 on B. X is conformally parabolic $\iff$ the L^n norm of this cohomology class vanishes.

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Vanishing of Grötzsch distance (i.e. conformal parabolicity) is close to (but not exactly the same as) $L^{\infty,n}H^1$ not being Hausdorff. It implies that for every $q < \infty$, $L^{q,n}H^1$ is not Hausdorff.

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Vanishing of Teichmüller distance is close to (but not exactly the same as) $L^{\infty,n}H^1 \neq 0$. For every $q < \infty$, it follows from $L^{q,n}H^1 = 0$.

We want to use a subRiemannian avatar, in dimension $2n+1$, with $\frac{n}{k}$ replaced with $\frac{2n+2}{\alpha(k)}$, where $\alpha(k)$ is the minimal Hausdorff dimension of k-dimensional submanifolds,

$$\alpha(k) = \begin{cases} k & \text{if } k \leq n, \\ k+1 & \text{otherwise,} \end{cases}$$

and de Rham's complex replaced with *Rumin*'s complex. Let us call it *Rumin conformal cohomology*.

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Therefore quasiconformal mappings induce functorial chain maps between Rumin conformal complexes, hence isomorphisms between Rumin conformal cohomologies.

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The construction relies on facts about the $L^{q,p}$ cohomology of Carnot groups.

Fact (Pansu-Rumin 2018)

Let G be a Carnot group with homogeneous dimension Q.

1. $L^{q,p}H^1(G) = 0 \iff \frac{1}{p} - \frac{1}{q} \geq \frac{1}{Q}$.
2. *Let $[w_{min}, w_{max}]$ denote the range of weights occurring in the Lie algebra cohomology $H^2(\mathfrak{g})$. Then*
   1. $\frac{1}{p} - \frac{1}{q} \geq \frac{w_{max}-1}{Q} \implies L^{q,p}H^2(G) = 0.$
   2. $\frac{1}{p} - \frac{1}{q} < \frac{w_{min}-1}{Q} \implies L^{q,p}H^2(G) \neq 0.$

The basic bricks are pairs of step-3 Carnot Lie algebras $\mathfrak{f}$ and $\mathfrak{g}$ such that

1. $\dim(\mathfrak{f}) = \dim(\mathfrak{g}) = m$.
2. $Q(\mathfrak{f}) = Q(\mathfrak{g}) = 2n + 2$.
3. $w_{min}(\mathfrak{f}) \geq 3$, $w_{max}(\mathfrak{g}) = 2$.

Then one forms products with spheres $V_1 = F \times S^{n+1-m}$, $V_2 = G \times S^{n+1-m}$, and one takes projectivized cotangent bundles $M = P(T^* V_1)$, $N = P(T^* V_2)$, in order to get contact manifolds with cocompact F- (resp. G-) actions.

By quasiisometry invariance of $L^{q,p}$ Rumin cohomology (Baldi-Franchi-Pansu-Tripaldi (2021)), the cohomological properties, including parabolicity and vanishing of Teichmüller distance, pass from F, G to M, N.

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Lemma

A Carnot Lie algebra $\mathfrak{f}$ has $w_{min}(\mathfrak{f}) \geq 3$ if and only if its 2-step quotient $\mathfrak{f}/\mathfrak{f}^{(3)}$ is free.

Lemma

A 3-step Carnot Lie algebra $\mathfrak{g}$ is determined by the choices of

1. *A vectorspace E_1.*
2. *A subvectorspace E_2 of $\Lambda^2 E_1$.*
3. *A subvectorspace E_3 of the kernel of the tautological map*

$$E_1 \otimes E_2 \to E_1 \otimes \Lambda^2 E_1 \to \Lambda^3 E_1.$$

Then $w_{max}(\mathfrak{g}) = 2$ if and only if

1. $E_3 = \mathrm{Ker}(E_1 \otimes E_2 \to \Lambda^3 E_1)$.
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So E_3 is uniquely determined by E_2, and it is subject to an open condition, which holds for generic choices of E_2 in $\Lambda^2 E_1$ provided dimensions match:

$$\frac{(n_1 - 1)(n_1 - 2)}{6} \leq n_2 \leq \frac{2}{n_1 + 1}\binom{n_1}{3}. \tag{1}$$

For every integers p, and r such that

$$0 \leq r \leq \frac{p^3 - p}{6}, \tag{2}$$

there exists step-3 Lie algebras $\mathfrak{f}$ such that $\mathfrak{f}/\mathfrak{f}^{(3)}$ is free on p generators and $\dim(\mathfrak{f}^{(3)}) = r$.

The equations for $\dim(\mathfrak{g}) = \dim(\mathfrak{f})$ and $Q(\mathfrak{g}) = Q(\mathfrak{f})$ boil down to

$$\begin{cases} n_2 &= \frac{1}{2}p^2 + \frac{3}{2}p - 2n_1 \\ r &= (1+n_1)(\frac{1}{2}p^2 + \frac{3}{2}p - 2n_1) - \binom{n_1}{3} + n_1 - \frac{1}{2}p^2 - \frac{1}{2}p. \end{cases}$$

If one picks p and n_1 such that $\frac{n_1}{p}$ belongs to the interval $(x_0, \sqrt{3})$, where $x_0 = 1.53...$ is the largest root of equation $x^3 - 3x + 1 = 0$, conditions (1) and (2) hold. This yields the required Lie algebras.