L^1 analysis and related geometry

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A few milestones in Bruno Franchi's achievements

- 1983: Franchi and Lanconelli extend de Giorgi's Hölder regularity of solutions of linear PDEs. Ellipticity needs be measured with respect to a metric naturally associated with the operator. Birth of *Analysis on Metric Spaces*.
- 1999: Franchi, Hajlasz and Koskela define Sobolev spaces on metric spaces. Consolidation of Analysis on Metric Spaces.
- 2001: Franchi, Serapioni and Serra Cassano inaugurate *Geometric Measure Theory* on Carnot groups. Their first rectifiability result is crucial in Cheeger, Kleiner and Naor's work on the Goemans-Linial conjecture.
- 2011: Franchi, Serapioni and Serra Cassano introduce intrinsic Lipschitz graphs, as candidates for models of rectifiable sets. These play a key role in Naor and Young's final answer to Goemans-Linial's conjecture.
- 2009-2021: Franchi and coauthors extend the theory from functions to differential forms, guided by the idea of currents.

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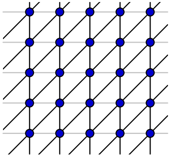
$(Sobol_{p,q})$ has a discrete version on $\mathbb{Z}^n$, $n \geq 2$: every finitely supported 1-cocycle c admits a primitive u such that

$$\|u\|_q \leq C'(n,p,q)\|c\|_p.$$



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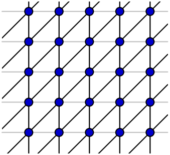

We study a higher dimensional generalization, which has a topological flavour.

Definition

X simplicial complex. Is every ℓ^p cocycle the coboundary of a ℓ^q cochain?

$$\ell^{q,p}H^k(X) = \{\ell^p \; k\text{-cocycles}\}/d\{\ell^q \; k-1\text{-cochains}\}.$$

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Example. $X =$ the line tiled with equal intervals. Then $\ell^{q,p}H^0(X) = 0$ except $\ell^{q,\infty}H^0(X) = \mathbb{R}$. In degree 1, $\ell^{\infty,1}H^1(X) = 0$, all other $\ell^{q,p}H^1(X) \neq 0$.

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Example. $X =$ plane tiled with equal triangles. Then $\ell^{q,p}H^1(X) = 0$ if $\frac{1}{p} - \frac{1}{q} \geq \frac{1}{2}$.
For finitely supported 1-cocycles, this is the discrete Sobolev inequality.

Two steps,

- one passes from the discrete to the continuous setting (requires analysis: interior estimates on balls),
- in the continuous setting, global estimates.

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X Riemannian manifold.

$$L^{q,p}H^k(X) = \{L^p \text{ } k\text{-closed forms } \}/d\{L^q \text{ } k-1\text{-forms } \omega \text{ such that } d\omega \in L^p\}.$$

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Example. $X = \mathbb{R}^n$. Then $L^{q,p}H^k(X) = 0$ if $1 < p \leq q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$.

Proof. Let $\Delta = d^*d + dd^*$. Then Δ has a pseudodifferential inverse which commutes with d. $T = d^*\Delta^{-1}$ has a kernel which is homogeneous of degree $1-n$, hence is bounded $L^p \to L^q$ as soon as $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$ (Calderon-Zygmund 1952). Finally, $1 = dT + Td$.

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Case where $p = 1$. On n-forms, $d^*\Delta^{-1}$ is not bounded from L^1 to $L^{n/(n-1)}$. Otherwise, by duality, Sobolev inequality ($Sobol_{\infty,n}$) would hold. But is does not: if $n \geq 2$, $\mathbb{R}^n$ is n-parabolic.

Theorem (Bourgain-Brezis-Mironescu 2004)

*Let ω be a **closed** compactly supported $n-1$-form on $\mathbb{R}^n$. Then for all 1-forms α on $\mathbb{R}^n$ such that $\nabla\alpha \in L^n$,*

$$|\int_{\mathbb{R}^n} \alpha \wedge \omega| \leq C\,\|\omega\|_1 \|\nabla\alpha\|_n.$$

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The *averages* of a L^1 k-form ω are the integrals $\int_{\mathbb{R}^n} \beta\wedge\omega$, where β has constant coefficients.

Corollary

*$d^*\Delta^{-1} : L^1 \to L^{n/(n-1)}$ is bounded on $n-1$-forms which are **closed and of vanishing averages**.*

Proof of BBM's inequality [after van Schaftingen]

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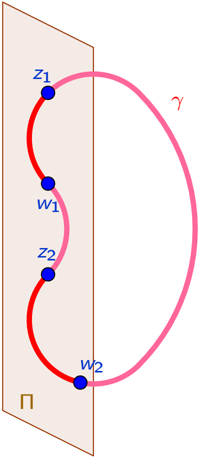
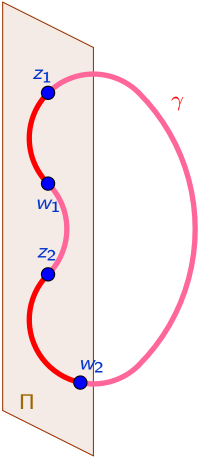
Let us approximate ω with forms which are Poincaré-dual to loops γ and let us prove that

$$|\int_{\gamma}\alpha| \leq C\,\ell(\gamma)\|\nabla\alpha\|_{n},$$

for $\alpha = u\,dx_1$.



Use the projection along parallel hyperplanes $\Pi = \{x_1 = t\}$.



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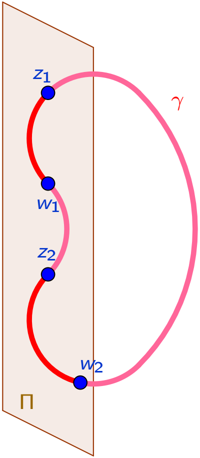
Since $u \in C^{(n-1)/n}(\Pi)$ and $\gamma \cap \Pi = \sum_{i=1}^{m} z_i - w_i$, by Morrey-Sobolev,

$$|u(z_i) - u(w_i)| \leq C \, \|\nabla \alpha\|_{L^n(\Pi)} |z_i - w_i|^{1/n}.$$



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By Hölder,

$$\sum_{i=1}^m |u(z_i) - u(w_i)| \leq C \, \|\nabla\alpha\|_{L^n(\Pi)} \ell(\gamma)^{1/n} m^{(n-1)/n}.$$

We integrate with respect to x_1, then apply Hölder, we get an estimate of $\int_\gamma \alpha$ by $\|\nabla\alpha\|_n$ and $\int_{\mathbb{R}} m(x_1)\, dx_1 \leq \ell(\gamma)$.

The *Heisenberg group* $\mathbb{H}^n$ has Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_1 = \mathbb{R}^{2n}$ et $[\cdot, \cdot] : \mathfrak{g}_1 \to \mathfrak{g}_2 = \mathbb{R}$ is symplectic. By left-translation, $\mathfrak{g}_1$ defines a distribution, hence a subRiemannian metric. Vectors or curves tangent to it are called *horizontal*. Curves of finite length must be horizontal. For a function u, $\nabla_{\mathbb{H}} u$ denotes the *horizontal gradient*.

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Following Chanillo-van Schaftingen 2009, let us prove that for every horizontal loop γ in Heisenberg group $\mathbb{H}^n$ and every horizontal 1-form α,

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for $Q = 2n + 2$. It suffices to treat $\alpha = u\,dx_1$.

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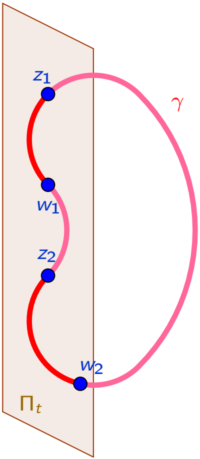
Trick: for every $\lambda > 0$, there exists a smooth function u_λ on $\mathbb{H}^n$ such that

$$\|u - u_\lambda\|_{L^\infty(\Pi_t)} \leq C\,\lambda^{1/Q} M(t),$$
$$\|\nabla_{\mathbb{H}} u_\lambda\|_{L^\infty(\mathbb{H}^1)} \leq C\,\lambda^{(1/Q)-1} M(t),$$

where M is the maximal function of $t \mapsto \|(\nabla_{\mathbb{H}} u)_{|\Pi_t}\|_Q$ (based on Jerison 1986).



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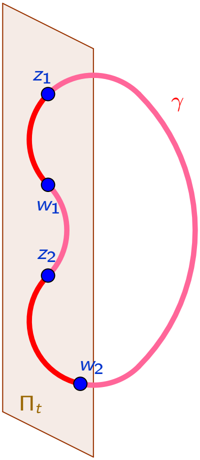

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$$|\sum_{i=1}^{m}((u-u_\lambda)(z_i)-(u-u_\lambda)(w_i))| \leq 2mC\,\lambda^{1/Q}M$$

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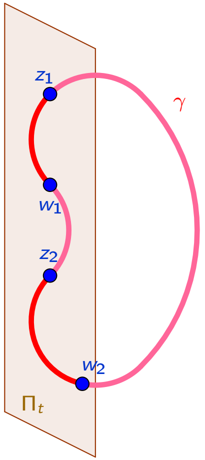

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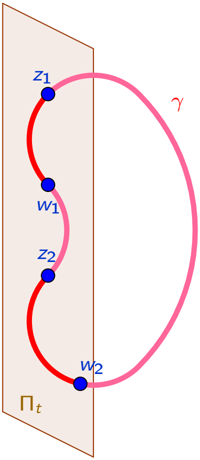

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Theorem (Baldi-Franchi-Pansu-Tripaldi)

Let $Q = 2n+2$. Then $\ell^{q,p}H^k(\mathbb{H}^n) = 0$ in the following cases:

- *If $k \neq n$, $1 \leq p \leq Q$, $\frac{1}{p} - \frac{1}{q} \geq \frac{1}{Q}$.*
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Corollary

Quasiconformal classification of 3-dimensional subRiemannian Lie groups: $\mathbb{H}^1$ and $\widetilde{Mot}(\mathbb{R}^2)$ are distinguished by $L^{4,2}H^2$ (but not by $L^{\infty,4}H^1$ or $L^{2,1}H^3$).

This completes the classification in the simply connected case.

Question

Let G be the (subRiemannian) quotient of $\mathbb{H}^1$ by a cyclic subgroup of its center. Is G quasiconformal to the group $Mot(\mathbb{R}^2)$ of planar Euclidean motions?