Computing homology robustly: The geometry of normed chain complexes

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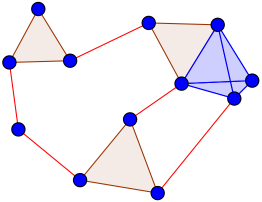
A simplicial complex is made of simplices of various dimensions. *Simplicial chains* are linear combinations of simplices. The *boundary* of a simplex is a chain, whence a linear map ∂ and its adjoint d, which satisfy $\partial \circ \partial = 0$ and $d \circ d = 0$.



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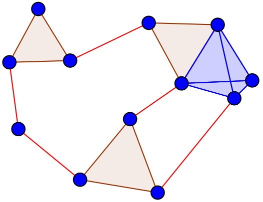

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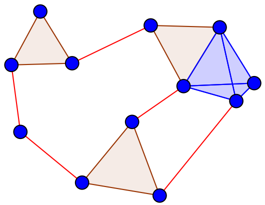

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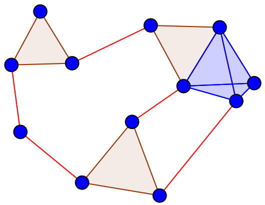

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Simplicial chains and cochains can be equipped with ℓ^p norms.

In general, a *normed chain complex* is a normed vector space B equipped with a linear map $d : B \to B$ such that $d\circ d = 0$.

When $F : B_1 \to B_2$ is a linear bijection, the robustness of the resolution of the equation

$$Fx = y$$

is governed by the *conditioning number*

$$\kappa(F) = |F||F^{-1}|.$$

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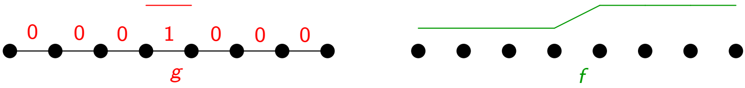
For normed chain complexes, we first turn d into a bijection $\bar{d} : B/\mathrm{Ker}(d) \to \mathrm{Im}(d)$, and set

$$\kappa(B) := |\bar{d}||\bar{d}^{-1}|.$$

Example. The n-stick satisfies $H^1 = 0$. The 1-cochain g equal to $\bar{1}$ on the central edge and 0 elsewhere can be written df where

$$\|g\|_p = 1, \quad \|f\|_p \sim n^{1/p}.$$



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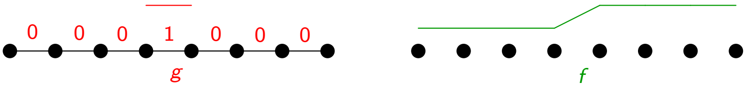
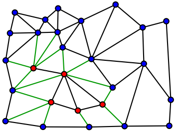


When n is large, solving $df = g$ is unstable. The computation of cohomology is ill-conditioned.

Definition

The **conditionning number** *of a graph* X *is* $\kappa(X, p, \mathbf{k}) = |\bar{d}||\bar{d}^{-1}|$ *where* $\bar{d} : C^0(X, \mathbf{k})/\mathrm{Ker}(d) \to dC^0(X, \mathbf{k})$. *(It depends on* p *and on the field* $\mathbf{k}$*).*

Isoperimetry = the art of cutting space apart.



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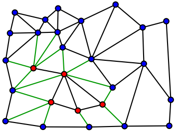

Definition

Cheeger's constant $h(X)$ *of a graph* X *is the largest* h *such that for every set* A *of vertices such that* $|A| \leq \frac{1}{2}|X|$,

$$|\partial A| \geq h\,|A|.$$

Here, ∂A *is the set of edges connecting* A *to its complement.*

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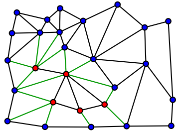

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Proposition

$$h(X) = \frac{2}{\kappa(X,1,\mathbb{F}_2)} = 2(\|\bar{d}\|_{1\to 1}\|\bar{d}^{-1}\|_{1\to 1})^{-1} \text{ over } \mathbb{F}_2.$$

Proposition

Let Δ be the self-adjoint operator corresponding to the quadratic form $f \mapsto \|df\|_2^2 = \langle f, \Delta f \rangle$. Let $\lambda_1 \leq \lambda_2 \leq \cdots$ denote its eigenvalues. If the graph X is connected, then $\lambda_1 = 0$ and

$$\lambda_2 = (\|\bar{d}^{-1}\|_{2\to 2})^{-2}.$$

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In particular,

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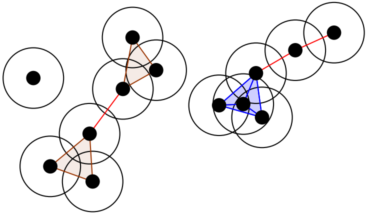
Morality. Normed chain complexes contain interesting information, beyond their mere homology.

Given a metric space X, a finite subset $Y \subset X$ and $r > 0$, the *Čech simplicial complex* Y_r has a simplex $(y_0, \ldots, y_k)$ each time $\bigcap_i B(y_i, r) \neq \emptyset$. Let $C_\cdot^r$ denote the simplicial chains of Y_r.



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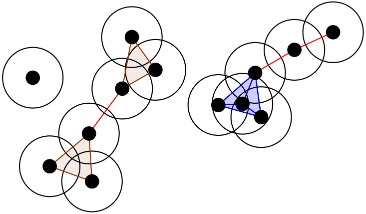

Theorem (Bobrowski-Weinberger 2017)

Fix $r < \frac{1}{2}$ and $1 \leq k \leq d$. Let Y be an n-sample picked at random on the standard d-torus. Then, with high probability, the k-homology of Y_r coincides with the homology of the torus as soon as

$$\omega_d r^d n \gg \log n + k \log \log n,$$

and this fails if $\omega_d r^d n \ll \log n + (k-2) \log \log n$. If $k = 0$, the threshold is $2^{-d} \log n$.

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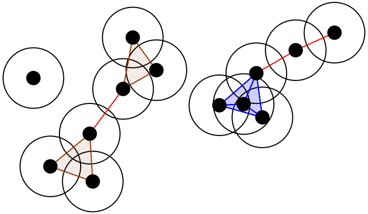

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Question. Can one say that the chain complexes $C_\cdot^r$ converge to some chain complex attached to the torus?

In order to define a distance between normed chain complexes, the first idea is to measure conditioning numbers of isomorphisms.

Definition

Let $B_1 \stackrel{d_1}{\to} B_1$ *and* $B_2 \stackrel{d_2}{\to} B_2$ *be normed chain complexes. The* **Banach-Mazur distance** $\mathrm{BMDist}(B_1, B_2)$ *is the infimum of* $\log(|F||F^{-1}|)$ *over all isomorphisms* $F : B_1 \to B_2$ *duch that* $Fd_1 = d_2F$.

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This is too restrictive: this implies $\dim(B_1) = \dim(B_2)$.

The second idea is too measure the size of homotopies.

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Let $B_1 \stackrel{d_1}{\to} B_1$ *and* $B_2 \stackrel{d_2}{\to} B_2$ *be normed chain complexes. Consider all bounded homotopies, i.e.*

- *bounded morphisms* $F_1 : B_1 \to B_2$ *and* $F_2 : B_2 \to B_1$ *such that*

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- *bounded operators* $Q_1 : B_1 \to B_1$ *and* $Q_2 : B_2 \to B_2$ *such that*

$$1 - F_2 F_1 = d_1 Q_1 + Q_1 d_1, \quad 1 - F_1 F_2 = d_2 Q_2 + Q_2 d_2.$$

Let $q = \max\{|Q_1|, |Q_2|\}$, $f = \max\{1, |F_1||F_2|\}$. *The* **homotopy distance** $\mathrm{HomDist}(B_1, B_2)$ *is the infimum over all homotopies of* $\min\{\frac{q}{f} + \log f, \frac{f}{q} + \log q\}$.

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The weird expression guarantees a triangle inequality.

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Let Null denote the set of null normed chain complexes (i.e. with $d = 0$). Denote by

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The **singular values** *of B are the numbers*

$$\sigma_j = \inf\{s \geq 0 \,;\, \exists L \subset \bar{B} \text{ subvectorspace such that } \dim(L) \geq j \text{ and } \forall \bar{x} \in L,\ |\bar{d}\bar{x}| \leq s|\bar{x}|\}.$$

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Fact. Each σ_j is continuous in homotopy distance.

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Fact. A prehilbertian chain complex is precompact $\iff$ its singular values form a finite sequence that tends to $+\infty$.

Proposition

Let B_i be precompact prehilbertian chain complexes. Then B_i converges to B $\iff$ for every j, $\sigma_j(B_i)$ tends to $\sigma_j(B)$.

Analogy between normed chain complexes and metric spaces.

Metric space	Normed chain complex
Gromov-Hausdorff distance	Homotopy distance
Point	?
Bounded	?
Precompact	?
Compactness criterion (Gromov)	?

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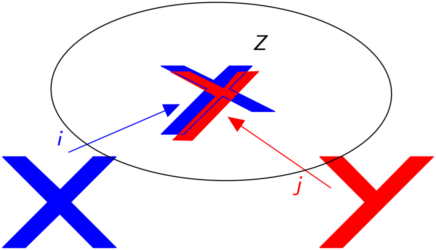
Let X, Y be metric spaces.

$$GHDist(X,Y) = \inf\{HDist_Z(i(X), j(Y))\,;\, Z \text{ metric space}, \\ i: X \to Z,\, j: Y \to Z \text{ isometric embeddings}\}.$$



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B is homotopic to a null complex $\iff \mathrm{ND}(B) < \infty$.

One can think of $\mathrm{ND}(B) = \mathrm{HomDist}(B, \mathit{Null})$ as an analogue of diameter.

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B is precompact $\implies$ B has a finite sequence of singular values that tends to $+\infty$ ($\iff$ if B is prehilbertian).

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Theorem (Gromov's compactness criterion)

A collection $\mathcal{T}$ of precompact metric spaces is precompact in Gromov-Hausdorff distance if and only if there is a function ν which serves as a covering number for all spaces in $\mathcal{T}$, i.e.

$$\forall \epsilon > 0, \quad \forall X \in \mathcal{T}, \quad N(X, \epsilon) \leq \nu(\epsilon).$$

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Definition

Let (B, d) be a normed chain complex that belongs to the closure of finite dimensional normed complexes. Its **profile** *is the smallest function $\pi = (\pi_d, \pi_c) : (0, +\infty) \to (0, +\infty)^2$ with the following property. For every $\epsilon > 0$, there exists a finite-dimensional normed complex (B', d') such that*

$$HomDist(B, B') < \epsilon, \quad \dim(B') \leq \pi_d(\epsilon), \quad \kappa(B', d') \leq \pi_c(\epsilon).$$

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$$HomDist(B, B') < \epsilon, \quad \dim(B') \leq \pi_d(\epsilon), \quad \kappa(B', d') \leq \pi_c(\epsilon).$$

Theorem

A collection of nonnull normed chain complexes is precompact if and only if a same profile serves for all and the distances to null complexes are bounded below.

Lemma

Let B be a prehilbertian chain complex. Then the profile of B is determined by the asymptotics of eigenvalues,

$$\pi_d(\epsilon) \leq \mathrm{Card}\{\lambda \in \mathrm{spectrum}(d^*d)\,;\, \lambda < \frac{1}{\epsilon^2}\}, \quad \pi_c(\epsilon) \leq \frac{1}{\epsilon\sqrt{\lambda_2}}.$$

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Conjecture

Consider finer and finer triangulations of a fixed compact manifold. The corresponding complexes of simplicial cochains in their weighted ℓ^p norms form a precompact family.

Here, the weight of a simplex is a function of its volume.

Let Y be a finite metric space. The *complete simplicial complex* Δ_Y on Y takes as simplices all tuples of points of Y. Pick a function of the diameter as a weight. Use weighted ℓ^p norms on cochains. This gives a normed chain complex $C^{\cdot}(Y)$.

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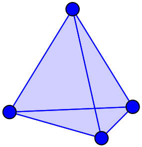

The complete simplicial complex on 4 points.

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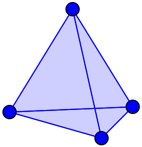

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Let (X,μ) be a metric measure space. Same construction with the same weight w and $L^p(\mu^{\otimes\cdot})$ norms yields a normed chain complex $C^\cdot(X)$.
Example. 1-cochains are functions c on $X \times X$. The squared weighted L^2 norm is

$$\int_{X\times X} w(|x-x'|)|c(x,x')|^2\, d\mu(x)\, d\mu(x').$$

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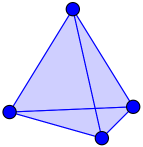

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Question. Given a metric measure space (X, μ) and a finite sample $Y \subset X$. Does $C^\cdot(Y)$ converge to $C^\cdot(X)$?