Near-homology

Pierre Pansu, Université Paris-Saclay

May 12th, 2022

X simplicial complex, $\mathbf{k}$ a field. A k-cochain is a (skew-symmetric) $\mathbf{k}$-valued function f on the set of k-simplices. Its *coboundary* df is $df(\sigma) = f(\partial\sigma)$, e.g.,

- $k = 0$, $e = vv'$ edge, $df(e) = f(v) - f(v')$.
- $k = 1$, $\sigma = vv'v''$ triangle,
  $df(\sigma) = f(vv') + f(v'v'') + f(v''v) = f(vv') - f(vv'') + f(v'v'')$.

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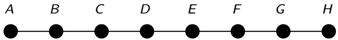

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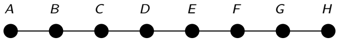

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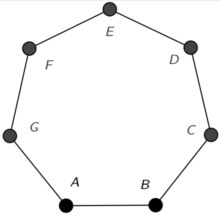

Example

X = n-cycle. H^0 =constant functions $\simeq \mathbf{k}$.
$H^1 \simeq \mathbf{k}$ by $f \mapsto \sum_{\mathrm{edges}\ e} f(e)$.

Pick an absolute value on **k**. Get norms on cochains: picking simplices uniformly at random, set $\|f\|_p = \mathbb{E}(|f|^p)^{1/p}$.

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Let $n = 2m+1$ be odd. In the n-stick with vertices $v_0, \ldots, v_n$, the 1-chain g equal to $\bar{1}$ on the middle edge $v_m v_{m+1}$ and 0 elsewhere is df where $f(v_i) = \bar{0}$ for $i \leq m$ and $f(v_i) = \bar{1}$ for $i \geq m+1$. Then

$$\|g\|_p = \frac{1}{n^{1/p}}, \quad \|f\|_p \sim \frac{1}{2^{1/p}}.$$

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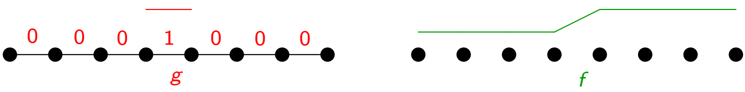

When $\mathbf{k} = \mathbb{F}_2$, the only other choice for f is $\bar{1} + f$ which has the same norm. When $\mathbf{k} = \mathbb{R}$, the optimal choice is $f(v_i) = \pm\frac{1}{2}$ which has norm $\frac{1}{2}$.

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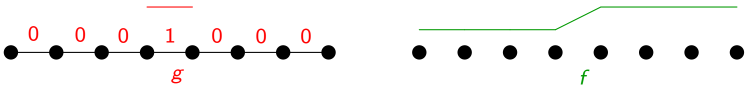

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When n is large, it is costly to solve $df = g$. We say that g is nontrivial in *near-cohomology*.

Definition

The **near-cohomology threshold in degree** *k is the norm of the inverse $\bar{d}^{-1}$ of $\bar{d} : C^k(X, \mathbf{k})/\mathrm{Ker}(d) \to dC^k(X, \mathbf{k})$. (It depends on p, i.e. on a choice of norm).*

Fact

Like cohomology, near-cohomology is a topological (even homotopy) invariant of simplicial complexes. Respectively, a quasiisometry invariant (for infinite complexes of locally bounded geometry).

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Remember: cohomology is not the only interesting information one can extract from cochain complexes. Apparently, near-cohomology is significant too.

Plan of lecture

1. *Relate the various ℓ^p near-cohomology thresholds for graphs to expansion.*
2. *Survey the theory of higher dimensional expanders.*
3. *Briefly mention one other instance where near-cohomology is relevant.*
4. *Start a more general discussion of normed chain complexes and the role played by near-homology.*

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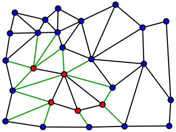
This is a geometric question. In a graph X with n vertices and e edge, A a subset of vertices, the **edge-boundary** ∂A of A is the set of edges that join A to its complement.



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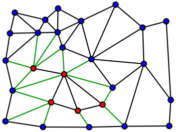

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The **Cheeger constant** $h(X)$ *of* X *is the largest* h *such that for every set* A *of vertices such that* $\mathbb{P}(A) < \frac{1}{2}$,

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Lemma

$$h(X) = (\|d^{-1}\|_{1\to 1})^{-1} \text{ over } \mathbb{F}_2.$$

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Here is an example of graph without near-cohomology.

Example

For the complete graph K_n on n vertices, $h(K_n) = \frac{n}{n-1}$ (n even) or $\frac{n+1}{n-1}$ (n odd).

▸ Proof

The ℓ^2 near-cohomology threshold $(\|d^{-1}\|_{2\to 2})^{-1}$ over $\mathbb{R}$ has an interpretation too.

Proposition

Let Δ be the self-adjoint operator corresponding to the quadratic form $f \mapsto \|df\|_2^2 = \langle f, \Delta f\rangle$. Let $\lambda_1 \leq \lambda_2 \leq \cdots$ denote its eigenvalues. If the graph X is connected, then $\lambda_1 = 0$ and

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This suggests a slight generalization: replacing graphs (resp. simplicial complexes) with weighted graphs (resp. weighted simplicial complexes).

Definition

Say a simplicial complex X has **pure dimension** *N if every vertex is contained in at least one N-simplex.*
A probability distribution on the simplices of a simplicial complex X of pure dimension N is **Garland** *if for every $k = 0, \ldots, N-1$, it is the same to pick a k-simplex at random, or to first pick a $k+1$-simplex σ at random and then a k-simplex of σ uniformly at random.*

For a graph, the uniform distribution on edges gives rise to the distribution on vertices which is proportional to degree.

ℓ^p norms on cochains are defined accordingly. Whence $h_{\mathbb{P}} = (\|d^{-1}\|_{1\to 1})^{-1}$ over $\mathbb{F}_2$ and $\lambda_{2,\mathbb{P}} = (\|d^{-1}\|_{2\to 2})^{-2}$ over $\mathbb{R}$.

Theorem (Cheeger, Buser, Dodziuk, Alon-Milman)

For a finite connected graph equipped with a Garland distribution $\mathbb{P}$,

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The right-hand inequality follows from two principles:

- Isoperimetry $\implies$ ℓ^1-Poincaré-Sobolev inequality.
  For this, write any real function f with vanishing median as an integral of $\{0,1\}$-valued functions, the characteristic functions of its superlevel sets.
- ℓ^1-Poincaré-Sobolev inequality $\implies$ ℓ^2-Poincaré-Sobolev inequality.
  For this, apply the ℓ^1 inequality to $g = f|f|$, and then Cauchy-Schwartz.

Details

An *expander* is an infinite family of finite graphs with uniformly bounded degree and Cheeger constant uniformly bounded below. Equivalently, with *spectral gaps* λ_2 uniformly bounded below.

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The uniform spectral gap says that spectrally, an expander behave as a complete graph. Therefore, in a problem where one should check all pairs of points (e.g. find the minimal distance between two points of a cloud), one can pick an expander and check only its edges. Hence expanders are used to *derandomize* algorithms. Fortunately, there are (several) deterministic constructions of expanders.

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The spectral gap measures the exponential speed at which the simple random walk on the graph is mixing (i.e. pushes forward any distribution close to the uniform distribution). For instance, the art of cards shuffling is closely related to expansion in the symmetric group.

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The spectral gap also controls coarse embeddings to Hilbert spaces. For instance, planar graphs cannot be expanders. Examples of finitely generated groups that cannot be coarsely embedded in Hilbert spaces (Gromov monsters) have been constructed by arranging that an expander embeds in their Cayley graphs.

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Proposition

Let $p \geq 1$. Let Γ be a discrete finitely generated group. Fix a finite generating system. There is an associated graph, the Cayley graph G. Then the following are equivalent:

1. $h(G) > 0$.
2. *d has a bounded inverse* $d\ell^p C^0(G) \to \ell^p C^1(G)/\mathrm{Ker}(d)$.
3. *The ℓ^p cohomology* $\ell^p H^1(G) := (\ell^p C^1(G) \cap \mathrm{Ker}(d))/d\ell^p C^0(G)$ *is Hausdorff.*
4. *Γ is non-amenable.*

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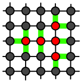
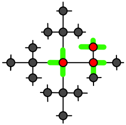
The free abelian group $\mathbb{Z}^d$ is amenable. It behaves like Euclidean space, which has vanishing Cheeger constant. Its ℓ^p cohomology is nonzero and non-Hausdorff for all $p > 1$.





The free group on at least 2 generators is non-amenable. It behaves like hyperbolic plane, which has positive Cheeger constant. Its ℓ^p cohomology is nonzero and Hausdorff for all $p > 1$.

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Conjecturally, this approximation factor cannot be improved. Some evidence is provided by a decade of work by A. Naor with various coauthors, showing that the integrality gap of the Goemans-Linial relaxation for SPARSEST CUT is at least $\sqrt{\log n}$ (2006-2018).

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$$\frac{1}{1+\epsilon} Q_G \leq Q_H \leq (1+\epsilon) Q_G.$$

Then every graph G is ϵ-close to a graph with less than $n(2+\epsilon)^2/\epsilon^2$ edges (J. Batson, D. Spielman and N. Srivastava 2014).

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Such a sparsification can be computed in nearly linear time. It follows that spectra are ϵ-close as well.

For a graph with few edges, eigenvalues can be fast computed using powers: solving $\Delta f_1 = f_0$ on functions with vanishing $\mathbb{E}(f)$ and iterating gives a sequence f_j such that $\|f_j\| \sim \lambda_2^{-j}\|f_0\|$, from which one can extract λ_2.

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The sparsification procedure relies on the concept of expansion. An expander is a graph which behaves spectrally like a complete graph, but with far less edges.
Random d-regular graphs are expanders, so one expects that decimating edges locally randomly does not disturb Q_G too much.

Definition

Let X be a finite simplicial complex of pure dimension N. Fix a Garland distribution on simplices of X, and the corresponding ℓ^p norms on cochains. For $k = 0, \ldots, n-1$, the k-th cocycle expansion constant of X is

$$h_k = (\|\bar{d}^{-1}\|_{1\to 1})^{-1},$$

where $\bar{d} : C^k(X, \mathbb{F}_2)/\mathrm{Ker}(d) \to dC^k(X, \mathbb{F}_2)$ is induced by the coboundary operator $d : C^k(X, \mathbb{F}_2) \to C^{k+1}(X, \mathbb{F}_2)$.

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Example For the complete N-dimensional complex on n vertices, let us use uniform distributions. Then for all $k = 0, \ldots, N-1$, $h_k \geq \dfrac{n}{n-k-1}$.

In this example, the degree, i.e. the number of simplices containing a given vertex, increases with n.

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where $\bar{d} : C^k(X, \mathbb{F}_2)/\mathrm{Ker}(d) \to dC^k(X, \mathbb{F}_2)$ is induced by the coboundary operator $d : C^k(X, \mathbb{F}_2) \to C^{k+1}(X, \mathbb{F}_2)$.

Example For the complete N-dimensional complex on n vertices, let us use uniform distributions. Then for all $k = 0, \ldots, N-1$, $h_k \geq \dfrac{n}{n-k-1}$.

In this example, the degree, i.e. the number of simplices containing a given vertex, increases with n.

Theorem (Kaufman-Kazhdan-Lubotzky 2016)

There exists an infinite family of 2-dimensional simplicial complexes with uniformly bounded degree such that h_0 and h_1 are bounded away from 0.

The construction is explicit. A simpler result is the fact that random 2-dimensional simplicial complexes have h_0 and h_1 are bounded away from 0 and edge-degree (number of simplices containing an edge) bounded.

A *tester* is a randomized algorithm that recognizes strings satisfying a property among a larger set of strings, by examining only a few bits, with a significant probability of being right.
The probability of rejecting a string should increase with the distance of the string to the subset of strings having the required property. It should vanish when this distance is 0. We use the normalized Hamming distance between strings, $d(f,g) = \mathbb{P}(f_i \neq g_i)$.

Definition

Let A, B be finite sets, $W \subset A^B$ a set of strings, and $P \subset W$ a subset. A (q,ϵ)-tester for P is a randomized algorithm that queries only q bits of a string f and rejects it with probability at least $\epsilon\, d(f,P)$.

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Example. *Linearity testing*. Let V be a finite dimensional $\mathbb{F}_2$-vectorspace. Strings are $\mathbb{F}_2$-valued functions on V, with P the subset of linear functions. The tester pick uniformly at random two elements $x, y \in V$, it rejects f if $f(x+y) \neq f(x) + f(y)$. This is a $(3,1)$-tester for linearity.

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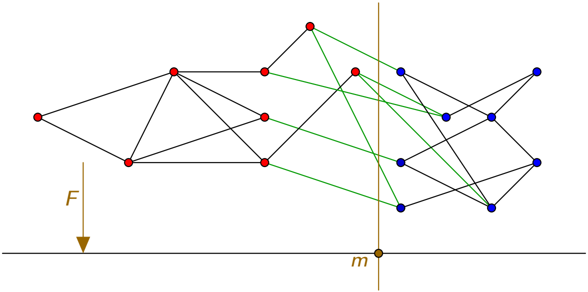
Theorem (Kaufman-Lubotzky 2014)

Let X be a finite simplicial complex. Let strings be $\mathbb{F}_2$-valued k-cochains on X. The k-cocycle tester picks a $k+1$-simplex σ at random and rejects a cochain f if and only if $df(\sigma)\neq 0$. This is a $(k+2,\epsilon)$-tester if and only if $\epsilon\geq h_k(X)$.

Application to Seidel equivalence

Remark. Let X be a connected graph and $F : X \to \mathbb{R}$ a continuous map. Let $m \in \mathbb{R}$ denote the median of F. Then m belongs to the images of at least a fraction $h(X)$ of the edges of X.



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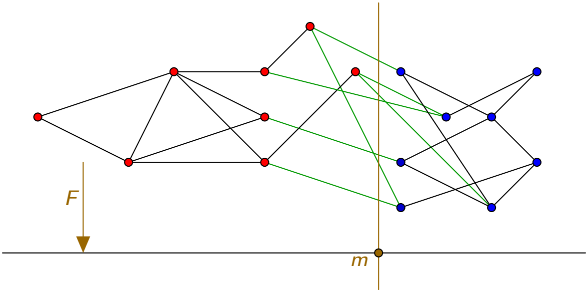

M. Gromov has given a multidimensional extension of this remark: if a sufficiently rich simplicial N-complex is continuously mapped to $\mathbb{R}^N$, then some point needs be hit by a lot of N-simplices.

Sufficiently rich is expressed by *cocycle expansion constants*, plus *sparsity*, and also *cosystoles* in case the cohomology does not vanish.

Let X be a simplicial complex. Fix a Garland probability distribution on simplices σ of X. The *local sparsity* of X controls the fraction of simplices of all dimensions that can meet at a vertex,

$$\mathrm{spar}(X) = \max_{v \in X^0} \mathbb{P}(v \in \sigma).$$

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When $H^k(X, \mathbb{F}_2) \neq 0$, the *k-cosystole* of X is the length of its shortest nonzero vector,

$$\theta_k(X) := \min\{\|\kappa\|_1 \, ; \, \kappa \in C^k(X, \mathbb{F}_2) \setminus dC^{k-1}(X, \mathbb{F}_2)\}.$$

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Theorem (Gromov 2010)

For every $h > 0, \theta > 0$ and N, for all sufficiently small $\epsilon > 0$, there exists $c(\epsilon, h, \theta, N) > 0$ such that for every simplicial complex X of pure dimension N, and every continuous map $F : X \to \mathbb{R}^N$, there exists a point in $\mathbb{R}^N$ which belongs to the images by F of at least a fraction

$$c(\mathrm{spar}(X), \min_{k=0,\dots,N-1} h_k(X), \min_{k=0,\dots,N-1} \theta_k(X), N)$$

of the N-simplices of X.

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For every n, $k = 1, \ldots, n-1$ and $\delta \in [-1, 0)$, there exists a $\mathbf{p}(n, \delta, k)$ such that if a simply connected complete Riemannian n-manifold is δ-pinched and $1 < p < \mathbf{p}(n, \delta, k)$, then $h_k > 0$ in ℓ^p cohomology.

This is sharp: for every n, $k = 1, \ldots, n-1$ and $\delta \in [-1, 0)$, there exists a Riemannian homogeneous space $M_{n,\delta,k}$ which is δ-pinched and has $h_k = 0$ in ℓ^p cohomology for p in an interval starting at $\mathbf{p}(n, \delta, k)$.

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There is a comparison theorem for the vanishing of ℓ^p cohomology, but it is not sharp.

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A *normed complex* is the data of a normed vectorspace and a bounded linear map $d : B \to B$ such that $d \circ d = 0$. An *isomorphism* of normed complexes $B_1 \stackrel{d_1}{\to} B_1$ and $B_2 \stackrel{d_2}{\to} B_2$ is a bijection $f : B_1 \to B_2$ such that $d_2 f_1 = f_1 d_1$ and f, f^{-1} are bounded. An *isometry* of normed complexes is an isometric isomorphism.

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Definition

Let $B_1 \stackrel{d_1}{\to} B_1$ and $B_2 \stackrel{d_2}{\to} B_2$ be two normed complexes. Consider all bounded homotopies, i.e.

- *bounded morphisms $F_1 : B_1 \to B_2$ and $F_2 : B_2 \to B_1$ such that*

$$d_2 F_1 = F_1 d_1, \quad d_1 F_2 = F_2 d_2,$$

- *bounded linear maps $Q_1 : B_1 \to B_1$ and $Q_2 : B_2 \to B_2$ such that*

$$1 - F_2 F_1 = d_1 Q_1 + Q_1 d_1, \quad 1 - F_1 F_2 = d_2 Q_2 + Q_2 d_2.$$

Denote by $q = \max\{|Q_1|, |Q_2|\}$, $f = \max\{1, |F_1||F_2|\}$. The **homotopy distance** $\mathrm{HomDist}(B_1, B_2)$ *is the infimum over all homotopies of* $\max\{\frac{q}{f} + \log f, \frac{f}{q} + \log q\}$.

Definition

$$\mathrm{HomDist}(B_1, B_2) = \inf \max\{\frac{q}{f} + \log f, \frac{f}{q} + \log q\}.$$

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When B_1 and B_2 are finite dimensional,

$$\mathrm{HomDist}(B_1, B_2) < \infty \iff \dim(H(B_1)) = \dim(H(B_2)),$$

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Definition

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Remark. NH is a homotopy covariant: if two chain complexes B_1 and B_2 are homotopic, then $\mathrm{NH}(B_1) \leq \exp(\mathrm{HomDist}(B_1, B_2))(1 + \mathrm{NH}(B_2))$. Details

Lemma

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This applies to the ℓ^2 norm on cochains on graphs. In order to remove the homology obstruction, we attach to a graph X the following complexes

$$B_{X,\mathbf{k},p} = H^0(X,\mathbf{k}) \oplus C^0(X,\mathbf{k}) \oplus dC^0(X,\mathbf{k}) \text{ equipped with } \ell^p \text{ norms.}$$

where $d : H^0(X,\mathbf{k}) \to C^0(X,\mathbf{k})$ embeds locally constant functions as 0-cochains. Then for all finite connected graphs,

$$HomDist(B_{X,\mathbb{R},2},0) = NH(B_{X,\mathbb{R},2}) = \lambda_2(X)^{-1/2}.$$

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Over $\mathbb{F}_2$, things are not as nice. Details

Example

Let K_3 be the complete graph on 3 vertices. Then

$$HomDist(B_{K_3,\mathbb{F}_2,1},0) = 1, \quad NH(B_{K_3,\mathbb{F}_2,1}) = \frac{1}{2}.$$

Definition

Let $r > 0$. Let $B \xrightarrow{d} B$ be a normed complex such that $|d| \leq r$, let B' and B'' be subcomplexes of B. Consider all bounded morphisms $f : B \to B$ such that $f(B') \subset B''$ and bounded maps $Q : B \to B$ such that, on B', $1 - f = dQ + Qd$. Take the infimum of $\log(1 + 2r|Q|)$. This defines the asymmetric embedded Hausdorff distance $\mathrm{HausDist}_{B,r}(B' \to B'')$. Make it symmetric by setting

$$\mathrm{HausDist}_{B,r}(B', B'') = \mathrm{HausDist}_{B,r}(B' \to B'') + \mathrm{HausDist}_{B,r}(B'' \to B').$$

Let $B_1 \xrightarrow{d_1} B_1$ and $B_2 \xrightarrow{d_2} B_2$ be two normed complexes. Consider all normed complexes $B \xrightarrow{d} B$ such that $|d| \leq r$, containing subcomplexes B' and B'' isometric to B_2 and B_2 respectively, and take the infimum of all embedded Hausdorff distances $\mathrm{HausDist}_{B,r}(B', B'')$. This defines the abstract **Hausdorff distance** $\mathrm{HausDist}_r(B_1, B_2)$.

Again, the role of the log is to achieve the triangle inequality. ▸ Details

Here is an other suggestion in order to bypass the homology obstruction. A *null complex* is a normed complex whose boundary operator is zero. If $B \overset{H,K}{\rightleftharpoons} N$ is a bounded homotopy to a null complex, then

$$B = \mathrm{Im}(K) \oplus \mathrm{Ker}(H),$$

where $\mathrm{Im}(K)$ is a null subcomplex of B which has the same homology as B and N, and $\mathrm{Ker}(h)$ is a subcomplex of B whose homology vanishes. Therefore one can think of $\mathrm{Ker}(H)$ as B with its homology removed.

Definition

Let $B_1 \overset{H_1,K_1}{\rightleftharpoons} N_1$ and $B_2 \overset{H_2,K_2}{\rightleftharpoons} N_2$ be normed complexes coming with homotopies to null-complexes. Define the near-homology distance between pairs as

$$\mathrm{NHDist}((B_1,N_1),(B_2,N_2)) := \mathrm{HomDist}(\mathrm{Ker}(H_1),\mathrm{Ker}(H_2)).$$

Then, in finite dimensions, NHDist is finite, and $\mathrm{NHDist}((B_1,N_1),(B_2,N_2)) = 0 \iff B_1$ and B_2 are isometric, up to direct sums with null complexes isomorphic to N_1 and N_2.

Lemma

$$h(X)=(\|d^{-1}\|_{1\to 1})^{-1} \text{ over } \mathbb{F}_2.$$

The support $x\mapsto A=\{v\,;\,x(v)\neq\bar{0}\}$ is a $1-1$ correspondence between 0-cochains (resp. 1-cochains) over $\mathbb{F}_2$ and subsets of vertices (resp. of edges). If A is a subset of vertices and x the corresponding 0-cochain, then the subset of edges corresponding to dx is ∂A.
Furthermore, $\|x\|_1=\#A/n$ and $\|dx\|_1=\#\partial A/e$.

Since G is connected, the kernel of d consists of the two 0-cochains x which are constant functions on the set of vertices.

If $y\in C^1$ belongs to dC^0, there exist exactly two 0-cochains $x_1,x_2\in C^0$ such that $dx_1=dx_2=y$. One of them (let us denote it by x) has least ℓ^1 norm. The support A of x satisfies $\#A\leq\frac{n}{2}$ and ∂A corresponds to y. Therefore the estimates

$$\|x\|_1\leq h^{-1}\,\|y\|_1 \quad\text{and}\quad \#\partial A/e\geq h\,\#A/n$$

are equivalent.

Back

Example

For the complete graph K_n on n vertices, $h(K_n) = \frac{n}{n-1}$ (n even) or $\frac{n+1}{n-1}$ (n odd).

Indeed, for every subset A, every vertex of A is connected to $n - \#A$ vertices of A^c, so

$$\mathbb{P}(\partial A) = \frac{\#\partial A}{e} = \frac{\#A(n-\#A)}{n(n-1)/2} = 2\frac{n}{n-1}\mathbb{P}(A)(1-\mathbb{P}(A)).$$

Since $\mathbb{P}(A) \leq \frac{1}{2}$ (resp. $\frac{n-1}{2n}$),

$$\frac{\mathbb{P}(\partial A)}{\mathbb{P}(A)} \geq \frac{n}{n-1} \quad (\text{resp}, \frac{n+1}{n-1}),$$

with equality if n is even and $\#A = \frac{n}{2}$ or n is odd and $\#A = \frac{n-1}{2}$.

Back

Proposition

X connected, Δ defined by $\|df\|_2^2 = \langle f, \Delta f\rangle$ with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$. Then

$$\lambda_1 = 0 \quad \textit{and} \quad \lambda_2 = (\|d^{-1}\|_{2\to 2})^{-2}.$$

Given $f \in C^0(G,\mathbb{R})$, let $f = \sum f_\lambda$ be the decomposition of f according to eigenspaces of Δ. Then

$$\|f\|_2^2 = \sum \|f_\lambda\|^2, \quad \|df\|_2^2 = \langle df, df\rangle = \langle f, \Delta f\rangle = \sum \lambda \|f_\lambda\|^2.$$

Since G is connected, $df = 0$ implies that f is a constant function on vertices, so the kernel of d, which coincides with the 0-eigenspace of Δ, is 1-dimensional.

If $f \in C^0(G,\mathbb{R})$ and $f_0 = 0$, then

$$\|df\|_2^2 \geq \lambda_2 \|f\|_2^2.$$

This shows that every $g \in dC^0$ has a primitive f such that $\|f\|_2 \leq \lambda_2^{-1/2}\|g\|_2$. Equality holds if $g = df$ where f belongs to the λ_2-eigenspace of Δ, so $\|d^{-1}\|_{2\to 2} = \lambda_2^{-1/2}$.

Back

Theorem

$h \geq \frac{1}{2}\lambda_2$.

If x is an $\mathbb{F}_2$-cochain, $f = |x|$ is a real cochain, and $\|x\|_1 = \|f\|_2^2$, $\|dx\|_1 = \|df\|_2^2$.

The real number m which minimizes $\mathbb{E}((f-m)^2)$, is $m = \mathbb{E}(f)$. Thus $d^{-1}df = f - \mathbb{E}(f)$, and

$$\|f - \mathbb{E}(f)\|_2^2 \leq (|d^{-1}|_{2\to 2})^2 \|f\|_2^2.$$

On the other hand,

$$\|f - \mathbb{E}(f)\|_2^2 = \mathbb{P}(x \neq 0)(1 - \mathbb{E}(f))^2 + \mathbb{P}(x = 0)\mathbb{E}(f)^2 = \|x\|_1(1 - \|x\|_1) \geq \frac{1}{2}\|d^{-1}dx\|_1.$$

so

$$\|d^{-1}dx\|_1 \leq 2(|d^{-1}|_{2\to 2})^2 \|x\|_1,$$

and

$$\|d^{-1}\|_{1\to 1} \leq 2(|d^{-1}|_{2\to 2})^2.$$

Back

Theorem

$h \leq 2\sqrt{\lambda_2}$.

First step.

$$\|d_{\mathbb{R}}^{-1}\|_{1\to 1} \leq \|d_{\mathbb{F}_2}^{-1}\|_{1\to 1},$$

i.e. the isoperimetric inequality (expressed in terms of $\mathbb{F}_2$ cochains) implies the Poincaré-Sobolev inequality (expressed in terms of $\mathbb{R}$ cochains).
Let $f \in C^0(G,\mathbb{R})$ be a function whose median vanishes. For $t \neq 0$, define $x_t \in C^0(G,\mathbb{F}_2)$ by

$$x_t(v) = \begin{cases} \bar{1} & \text{if } \frac{f(v)}{t} > 1, \\ \bar{0} & \text{otherwise,} \end{cases}$$

and set $f_t = |x_t|$. Then $|f| = \int_{\mathbb{R}} f_t\,dt$. Since the ℓ^1 norm is a norm,

$$\|f\|_1 = \||f|\|_1 \leq \int_{\mathbb{R}} \|f_t\|_1\,dt = \int_{\mathbb{R}} \|x_t\|_1\,dt \leq h^{-1}\int_{\mathbb{R}} \|dx_t\|_1\,dt = h^{-1}\int_{\mathbb{R}} \|df_t\|_1\,dt$$
$$\leq h^{-1}\|df\|_1.$$

Second step. The ℓ^1 Poincaré-Sobolev inequality implies its ℓ^2 version.

Let $g = f|f|$. Then the median of g is equal to 0 as well. One can apply the ℓ^1 Poincaré-Sobolev inequality to g and get

$$\|g\|_1 \leq h^{-1}\|dg\|_1.$$

One checks that $\|dg\|_1 \leq 2\|f\|_2\|df\|_2$, hence

$$\|f\|_2^2 = \|g\|_1 \leq h^{-1}2\|f\|_2\|df\|_2,$$

so

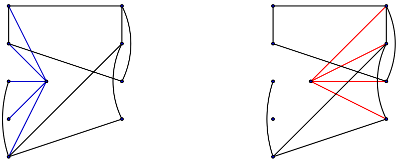
$$\|f\|_2 \leq 2h^{-1}\|df\|_2$$

and

$$\lambda_2^{-1/2} = \|d^{-1}\|_{2\to 2} \leq 2h^{-1}.$$

Back

A *Seidel switch* on a graph consists in picking a vertex, removing all edges that contain it and insert edges to all vertices that were not previously its neighbours.



Two graphs on the same vertex set are *Seidel equivalent* if they can be obtained one from the other after a finite number of Seidel switches.

Example

Seidel equivalence is $(3,1)$*-testable.*

Indeed, a graph with vertex set $\{1,\ldots,n\}$ can be viewed as a 1-cochain on the complete graph K_n. A Seidel switch at vertex v amounts to adding $d\chi_v$, where $\chi_v(v)=\bar{1}$ and $\chi_v=\bar{0}$ elsewhere. Therefore two graphs with vertex set $\{1,\ldots,n\}$ are Seidel equivalent if and only if the corresponding 1-cochains are cohomologous, and we have a $(3,1)$-tester for that, since $H^1(K_n,\mathbb{F}_2)=0$ and $h_1(K_n)\geq 1$.

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Lemma

$HomDist(B_1, B_3) \leq HomDist(B_1, B_2) + HomDist(B_2, B_3)$.

Let $(B_1, d_1, Q_1) \overset{F_1, F_2}{\rightleftharpoons} (B_2, d_2, Q_2)$ and $(B_2, d_2, Q_2') \overset{G_2, G_3}{\rightleftharpoons} (B_3, d_3, Q_3)$ be homotopies. Let $q = \max\{|Q_1|, |Q_2|\}$, $f = \max\{1, |F_1||F_2|\}$, $q' = \max\{|Q_2'|, |Q_3|\}$, $f' = \max\{1, |G_2||G_3|\}$. Then $(B_1, d_1, Q_1') \overset{G_2F_1, F_2G_3}{\rightleftharpoons} (B_3, d_3, Q_3')$ is a homotopy, where

$$Q_1' = Q_1 + F_2 Q_2' F_1, \quad Q_3' = Q_3 + G_2 Q_2 G_3.$$

Thus $q'' = \max\{|Q_1'|, |Q_3'|\}$ and $f'' = \max\{1, |G_2 F_1||F_2 G_3|\}$ satisfy

$$q'' \leq f' q + f q', \quad f'' \leq f f',$$

so $\dfrac{q''}{f''} + \log(f'') \leq \dfrac{q}{f} + \log(f) + \dfrac{q'}{f'} + \log(f')$.

The expression $\max\{\dfrac{q}{f} + \log f, \dfrac{f}{q} + \log q\}$ is necessary in order for this function to be nondecreasing in q and in f.

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Lemma

$HomDist(B,0) \geq \|\bar{d}^{-1}\|$.

Let

$$(B,d,Q) \overset{0,0}{\rightleftharpoons} (0,0,0)$$

be a homotopy to the trivial complex. This simply means that $1 = dQ + Qd$. If $y \in \mathrm{Im}(d)$, $y = dx$, then

$$y = dx = d(dQx + Qdx) = dQdx = dQy,$$

thus $|Qy| \geq |\bar{d}^{-1}y|$, and $|Q| \geq |\bar{d}^{-1}|$, so

$$HomDist(B,0) \geq \max\{|Q|, \frac{1}{|Q|} + \log(|Q|)\} \geq |\bar{d}^{-1}| = NH(B).$$

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Lemma

$\mathrm{NH}(B_1) \leq \exp(\mathrm{HomDist}(B_1, B_2))(1 + \mathrm{NH}(B_2))$.

Let $(B_1, d_1, Q_1) \overset{F_1,F_2}{\rightleftharpoons} (B_2, d_2, Q_2)$ be a homotopy of complexes. Let $c_2 = \mathrm{NH}(B_2)$.

If $y_1 \in d_1 B_1$, $y_2 = f_1(y_1) \in d_2 B_2$, so there exists $x_2 \in B_2$ such that $dx_2 = y_2$ and $|x_2| \leq c_2 |y_2|$. Let $x_1 = Q_1 y_1 + F_2(x_2)$. Then $dx_1 = y_1$ and

$$|x_2| \leq |Q_1||y_1| + |F_2|c_2|F_1||y_1| \leq (q + fc_2)|y_2|.$$

Therefore $\mathrm{NH}(B_1) \leq q + fc_2$.

Remember that $D = \mathrm{HomDist}(B_1, B_2) = \max\{\frac{q}{f} + \log f, \frac{f}{q} + \log q\}$.
If $q \leq f$, $D = \frac{q}{f} + \log f$ and $q + fc_2 \leq f + fc_2 \leq e^D(1 + c_2)$.
If $q \geq f$, $D = \frac{f}{q} + \log q$ and $q + fc_2 \leq q + qc_2 \leq e^D(1 + c_2)$ again.

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Proposition

If B is a Hilbert space, $HomDist(B,0)=NH(B) \iff H(B)=0$.

Indeed, homotopic complexes has the same homology, so
$HomDist(B,0)<\infty \implies H(B)=0$.

Conversely, assume that $H(B)=0$. The orthogonal projection $\pi : B\to \mathrm{Im}(d)$ has norm ≤ 1. The section σ of the projection $p : B\to B/\mathrm{Ker}(d)$ whose image is the orthogonal complement of $\mathrm{Ker}(d)$ has norm ≤ 1. Therefore $Q=\sigma\bar{d}^{-1}\pi$ satisfies $|Q|\leq|\bar{d}^{-1}|$. Since $H(B)=0$, $\mathrm{Im}(\sigma)=\mathrm{Ker}(\pi)$, and B is the orthogonal direct sum $B=\mathrm{Im}(\sigma)\oplus\mathrm{Im}(d)$. In this decomposition,

$$Qd=\begin{pmatrix}1&0\\0&0\end{pmatrix}\quad\text{and}\quad dQ=\begin{pmatrix}0&0\\0&1\end{pmatrix},$$

so $dQ+Qd=1$, $HomDist(B,0)\leq NH(B)$.

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Example

$HomDist(B_{K_3,\mathbb{F}_2,1},0)=1$, $NH(B_{K_3,\mathbb{F}_2,1})=\frac{1}{2}$.

The assertion $NH(B_{K_3,\mathbb{F}_2,1})=\frac{1}{2}$ is a special case of $h(K_n)=\frac{n+1}{n-1}$ for n odd.

Let $Q:B\to B$ satisfy $1=dQ+Qd$. Let e_1,e_2,e_3 be the natural basis of $C^0(K_3,\mathbb{F}_2)$. Then $|de_1|=|de_2|=|de_3|=\frac{2}{3}$, and $d(e_1+e_2+e_3)=\bar{0}$.
If Qde_1 has norm $<\frac{2}{3}$, then $Qde_1=e_1$. If furthermore Qde_2 has norm $<\frac{2}{3}$, then $Qde_2=e_2$. Then $Qd(e_3)=Qde_1+Qde_2=e_1+e_2$ has norm $\frac{2}{3}$. Thus one of the Qde_i must have norm $\frac{2}{3}$, thus $|Q|\geq 1$.

Conversely, let us denote by e_0 the nonzero element of $H^0(K_3,\mathbb{F}_2)$, so that $de_0=e_1+e_2+e_3$ and $e_0,e_1,e_2,e_3,de_1,de_2$ is a basis of $B_{K_3,\mathbb{F}_2,1}$. If $x=x_0e_0+x_1e_1+x_2e_2+x_3e_3+x_4de_1+x_5de_2$,

$$|x|=\frac{1}{3}(|x_0|+|x_1|+|x_2|+|x_3|)+\frac{2}{3}\max\{|x_4|,|x_5|,|x_4+x_5|\}.$$

Let us set $Q(x)=x_3e_3+x_4e_1+x_5e_2$. Then $dQ+Qd=1$, and

$$|Q(x)|=\frac{1}{3}(|x_3|+|x_4|+|x_5|)\leq\frac{1}{3}|x_3|+\frac{2}{3}\max\{|x_4|,|x_5|\}\leq|x|.$$

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Lemma

Let (B, d) be a normed complex. Let $r \geq |d|$. The embedded Hausdorff distance $\mathrm{HausDist}_{B,r}$ between subcomplexes of B satisfies the triangle inequality.

Let B_1, B_2, B_3 be subcomplexes of B. Let operators $F_1, F_2, Q_1, Q_2 : B \to B$ satisfy

$$F_1(B_1) \subset B_2, \quad F_2(B_2) \subset B_3,$$
$$1 - F_1 = d_1 Q_1 + Q_1 d_1, \quad 1 - F_2 = d_2 Q_2 + Q_2 d_2.$$

Then $F_3 = F_2 F_1$ maps B_1 to B_3 and satisfies $1 - F_3 = dQ_3 + Q_3 d$ for

$$Q_3 = Q_1 + Q_2 + Q_2 d Q_1 + Q_2 Q_1 d.$$

Since

$$|Q_3| \leq |Q_1| + |Q_2| + 2|d||Q_1||Q_2| \leq |Q_1| + |Q_2| + 2r|Q_1||Q_2|,$$

$$1 + 2r|Q_3| \leq 1 + 2r|Q_1| + 2r|Q_2| + 4r^2|Q_1||Q_2|$$
$$= (1 + 2r|Q_1|)(1 + 2r|Q_2|),$$

Corollary

Let $r > 0$. The Hausdorff distance $\mathrm{HausDist}_r$ between complete normed complexes $B \xrightarrow{d} B$ such that $|d| \leq r$ satisfies the triangle inequality.

Given isometries of B_1, B_2 to subcomplexes of B and isometries of B_2, B_3 to subcomplexes of $\bar{B}$, one constructs a complex $\bar{\bar{B}}$ that contains isometric copies of B and $\bar{B}$ which intersect along a common subcomplex B'' isometric to B_2. One starts with $B \oplus \bar{B}$ with the norm $|(x, \bar{x})| = |x| + |\bar{x}|$. By completeness, the subspace

$$D = \{(-x'', i(x''))\,;\, x'' \in B''\}$$

of $B \oplus \bar{B}$ is closed. Let $\bar{\bar{B}} = (B \oplus \bar{B})/D$, equipped with the quotient norm and the quotient operator $\bar{\bar{d}}$.

The embedded Hausdorff distances in $\bar{\bar{B}}$ are less that those in B and $\bar{B}$, so one can apply the previous Lemma.

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