A uniformization theorem for closed convex polyhedra in Euclidean 3-space.

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ABSTRACT. We introduce a notion of discrete-conformal equivalence of closed convex polyhedra in Euclidean 3-space. Using this notion, we prove a uniformization theorem for closed convex polyhedra in Euclidean 3-space.

CONTENTS

INTRODUCTION

In this paper, we introduce an equivalence relation on the class of closed convex polyhedra in the Euclidean 3-space $\mathbb{E}^3$. This equivalence relation has the property that, if P and Q are two convex polyhedra inscribed in the unit sphere, then P is equivalent to Q if and only if there exists a Möbius transformation on the sphere that maps the vertex set of P to the vertex set of Q. This property suggests this equivalence relation as a concept of discrete conformality.

Inspired by Riemann's mapping theorem and the more general uniformization theorem of Poincaré and Koebe, we prove a uniformization theorem for closed convex polyhedra in Euclidean 3-space in the following sense.

Theorem: (Uniformization of polyhedra) *Every closed convex polyhedron in $\mathbb{E}^3$ is discrete-conformally equivalent to a closed convex polyhedron inscribed in the unit sphere. This polyhedron is unique up to Möbius transformations on the sphere.*

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In a special case, we further characterize the equivalence relation by simple transformations on the vertices of the polyhedra. More specifically, if two polyhedra P and Q share a common Delaunay triangulation $\mathcal{T}$ (to be defined below), then P and Q are conformally equivalent if and only if there exists a real valued function $u_{\mathcal{T}}$ on the vertices of P such that, for every edge ij in the Delaunay triangulation between vertices i and j, its length in Q is related to its length in P by

$$l_Q(ij) = l_P(ij)\, e^{\frac{1}{2}(u_{\mathcal{T}}(i)+u_{\mathcal{T}}(j))}.$$

We conjecture more generally that P and Q are discrete-conformally equivalent if and only if there exists a finite sequence of closed convex polyhedra $P = P_1, P_2, \ldots, P_{n-1}, P_n = Q$ such that, for $k = 1, \ldots, n-1$ the polyhedra P_k and P_{k+1} share a common Delaunay triangulation $\mathcal{T}_k$ and there exists a real valued function $u_{\mathcal{T}_k}$ on the vertices of P_k with the following property. For every edge ij in the Delaunay triangulation between vertices i and j, its length in P_{k+1} is related to its length in P_k by

$$l_{P_{k+1}}(ij) = l_{P_k}(ij)\, e^{\frac{1}{2}(u_{\mathcal{T}_k}(i)+u_{\mathcal{T}_k}(j))}.$$

This work arose out of a general interest in understanding the relationship between different concepts of discrete conformality that have been developed in the last decades.

The concept of discrete conformality by a vertex scaling as above, first appeared in a paper by Luo in 2004 [1]. Luo introduces a discrete scalar curvature on polyhedral surfaces and describes a discrete analog of Yamabe flow in this setting. A *polyhedral surface* is a pair $(S, \mathcal{T})$ of a surface S and a triangulation $\mathcal{T}$ of S, together with a positive real valued function ρ on the set of edges of $\mathcal{T}$ such that the edge lengths of any triangle in $\mathcal{T}$ define an isometric Euclidean triangle. The function ρ is called a *polyhedral metric* on $(S, \mathcal{T})$.

Given a polyhedral metric ρ on $(S, \mathcal{T})$, let u be a real valued function defined on the vertex set of $(S, \mathcal{T})$, Luo defines a discrete-conformal change of ρ by the vertex scaling

$$u * \rho(vv') = \rho(vv')\, e^{\frac{1}{2}(u(v)+u(v'))}$$

on edges of $\mathcal{T}$. If $u * \rho$ defines a polyhedral metric, we say that ρ and $u * \rho$ are discrete-conformally equivalent.

The first hint that the concept of conformality could make sense also in a discrete setting appeared in the theory of circle packings in the 1980's. A *circle packing* is a connected collection of circles in the plane whose interiors are disjoint. A classical result in this area is Koebe's circle packing theorem [2].

Theorem: (Koebe) *For every connected simple planar graph G there is a circle packing in the plane whose intersection graph is G.*

The *intersection graph* of a circle packing is the graph having a vertex for each circle, and an edge for every pair of circles that are tangent. Let S be an oriented surface, i.e. a connected topological 2-manifold, with a metric. Given a collection $C = \{c_v\}$ of circles in S and a simplicial 2-complex K triangulating an oriented surface, the pair (C, K) is said to be a *circle packing for a simplicial 2-complex* K, denoted C_K, if

1. for each vertex v in K there exists exactly one circle c_v in C and vice versa,
2. if $\langle u, v \rangle$ is an edge of K, then the two circles c_u and c_v form a tangent pair and
3. if $\langle u, v, w \rangle$ forms a positively oriented face of K, then the three circles c_u, c_v and c_w form a positively oriented tangent triple in S.

We say that an abstract simplicial 2-complex K is a combinatorial closed disc if it triangulates a topological closed disc; that is, K is finite, simply connected, and has nonempty boundary.

For circle packings for a combinatorial closed disk K, Thurston observed the following rigidity property.

Theorem: (Thurston) *Let K be a combinatorial closed disc. Then there exists a univalent circle packing C_K, i.e. the interior of the circles are disjoint, in the unit disk such that every boundary circle is a horocycle. This circle packing is unique up to Möbius transformations of the disk.*

An ϵ *hexagonal circle packing* of the plane is a circle packing with all circles of radius ϵ associated to a tiling of the plane into equilateral triangles of edge lenth 2ϵ and all vertices of degree 6. Let C_ϵ be a portion of an ϵ hexagonal circle packing of the plane cut out by a domain in $\mathbb{C}$. The above theorem gives a "map" f_ϵ that associates with C_ϵ a "maximal" circle packing in the unit disk that is unique up to Möbius transformations.

Thurston's observation proved very fruitful. In a 1985 leture on the occasion of the proof of the Bieberbach conjecture, William Thurston conjectured that the above "map" converges to the Riemann map when ϵ goes to zero [3]. In 1987 Rodin and Sullivan proved this conjecture, interpreting the map f_ϵ as a map between the centers of circles [4].

Thurston's observation led to an increased interest in circle packings in the late 80's. Significant contributions and generalizations of the concept of circle packings were in particular made by his student, Oded Schramm.

Schramm is certainly best known for his invention of the Schramm-Loewner Evolution (SLE), and his subsequent collaboration with Lawler and Werner. Schramm's early work however was around Koebe's theorem and Thurston's discrete version of the Riemann mapping theorem. The transition from circle packing to SLE was through a sequence of papers concerning probability on graphs, many of them written jointly together with Benjamini. An interesting short survey on Schramm's work was written by Rohde [5].

One of Schramm's last works concerned analogs of Koebe's theorem in higher dimensions. Koebe's Theorem says that a graph can be circle packed in the plane $\mathbb{R}^2$ if and only if it is planar. Which graphs can be sphere packed in $\mathbb{R}^d$, $d > 2$? Using tools from non-linear potential theory, Schramm and Benjamini prove that the lattice grid $\mathbb{Z}^{d+1}$ or the 3-regular tree $\times \mathbb{Z}$ cannot be quasi-sphere packed in $\mathbb{R}^d$, for all d [6]. In first approximation, a *quasi-sphere packing* is a packing of domains, the ratio between the inner radius and the outer radius of each domain, is uniformly bounded over the elements of the packing.

Inspired from Benjamini and Schramm, Pansu proposed a concept of large scale conformal maps between metric spaces [7]. Roughly speacking, a map between metric spaces is large scale conformal if it maps every packing by sufficiently large balls to a collection of large quasi-balls which can be split into the union of boundedly many packings. This notion allows to transfer some techniques and results from conformal geometry to discrete spaces like finitely generated groups.

Another theory on conformal changes of metrics inspired from circle packings was developed by Lee for graphs [8]. Consider a locally finite, connected graph G. A *conformal metric* on G is a map $r : V(G) \to \mathbb{R}_{>0}$. This metric endows G with a graph distance as follows: Given an edge uv in $E(G)$, let

$$\mathrm{len}_r(uv) := \frac{1}{2}(r(u) + r(v)).$$

This induces a length

$$\mathrm{len}_r(\gamma) := \sum_{k>0} \mathrm{len}_r(v_k v_{k+1})$$

for every path $\gamma = \{v_1, v_2, \dots\}$ in G. Given any pair of vertices u and v in G, the distance $\mathrm{dist}_r(u, v)$ between u and v is defined as the infimum of the length of all paths between u and v in G. This endows G with a path metric dist_r.

One realizes that in the concept of discrete conformality coming from circle packings, the metric is changed by modifying the radii of circles. Hence, the metric is represented in form of an addition of two radii. In the other concept of discrete conformality coming from Yamabe flow, the metric is changed multiplicatively.

Closely related to circle packings is the concept of circle patterns. Let G be an immersed connected planar graph in the plane, and let $w : E(G) \to (0, \pi)$ be a weight on the edges $E(G)$ such that for all edges incident to a face f of G we have

$$\sum_{e \text{ incident to } f} w(e) = 2\pi.$$

An *immersed planar circle pattern* in the plane with adjacency graph G and intersection angles w is a collection of circles for each vertex, such that the following conditions hold.

1. For each edge uv in $E(G)$, the two circles associated to u, v in $V(G)$ intersect with exterior intersection angle $w(uv)$.
2. The circles corresponding to the vertices adjacent to the same face of G intersect in a single point.
3. Consider a counterclockwise cyclic order of the intersection points from (2) on the circle corresponding to an interior vertex v of G. This order agrees with the counterclockwise cyclic order of the cycle of faces of G adjacent to v.

Notice that for each circle packing for a simplicial 2-complex K there is an associated orthogonal circle pattern. Simply add a circle for each triangular face which passes through the three touching points.

Circle patterns on the sphere are closely related to ideal polyhedra.

Theorem: *Let Σ be a cellular decomposition of the sphere. Let $w : \Sigma^{(1)} \to (0, \pi)$ be a weight on the edges of Σ such that for all edges incident to a face f of Σ we have*

$$\sum_{e \textit{ incident to } f} w(e) = 2\pi,$$

and for every simple circuit $e_1, \ldots, e_k$ of edges in Σ that does not bound a single face of Σ we have

$$\sum_i w(e_i) > 2\pi.$$

Then there exists an immersed planar circle pattern C_Σ in the sphere with adjacency graph $\Sigma^{(1)}$ and intersection angles w. This circle pattern is unique up to Möbius transformations on the sphere.

We may interpret the sphere as the ideal boundary of the hyperbolic space $\mathbb{H}^3$ in the Poincaré model. If we carve out all hyperbolic half-planes defined by the circles in C_Σ on the ideal boundary of $\mathbb{H}^3$, we obtain an ideal convex polyhedron P_{C_Σ} in $\mathbb{H}^3$ with the dihedral angle at an edge e of P_{C_Σ} given by $w(e)$.

In that sense, the above theorem is nothing but a reformulation of Igor Rivin's celebrated theorem on the characterization of ideal polyhedra in hyperbolic 3-space [9]. This theorem can be stated as follows.

Suppose that a convex ideal polyhedron P in $\mathbb{H}^3$ is given. Let P^* denote the *dual polyhedron* of P, i.e. the abstract polyhedron that has a vertex for every face of P and two vertices in P^* are connected by an edge e^* if and only if the correspoding faces in P are share an edge e. Assign to each edge e^* of P^* a weight $w(e^*)$ equal to the exterior dihedral angle at the corresponding edge e in P. Then the following result holds:

Theorem: (Rivin) *The dual polyhedron P^* of a convex ideal polyhedron P in $\mathbb{H}^3$ satisfies the following conditions:*

1. $0 < w(e^*) < \pi$ *for all edges e^* of P^*.*
2. *If the edges $e_1^*, e_2^*, \ldots, e_k^*$ form the boundary of a face of P^*, then $w(e_1^*) + w(e_2^*) + \cdots + w(e_k^*) = 2\pi$.*
3. *If $e_1^*, e_2^*, \ldots, e_k^*$ form a simple circuit which does not bound a face of P^*, then $w(e_1^*), w(e_2^*), \ldots, w(e_k^*) > 2\pi$.*

Conversely, any abstract polyhedron P^* with weighted edges satisfying the conditions 1 - 3 is the dual polyhedron of a convex polyhedron P with the exterior dihedral angles equal to the weights.

A convex polyhedron with prescribed dihedral angles is also determined uniquely up to ambient isometry, this is shown by Rivin using a variational priciple in [10].

Rivin's characterization of ideal convex polyhedra in hyperbolic 3-space is a generalization of Andreev's theorem [11]. In fact, Thurston was led to study circle packings during his program on the geometrization conjecture. Andreev's theorem is an essential ingredient in his proof of the hyperbolization theorem [12] [13] [14].

The full geometrization conjecture of Thurston was proven by Perelman in 2003 using Ricci flow with surgery [15] [16] [17]. The analog of Ricci flow for scalar curvature is the Yamabe flow, whose discretization is the setting in which the other concept of discrete conformality by vertex scaling first appeared.

A hint that the concept of discrete conformality by vertex scaling and the concept of discrete conformality associated to circle packings are related, appears in a paper by Bobenko, Pinkall and Springborn [18]. In their paper they address the following question: Given a polyhedral surface $(S, \mathcal{T}, \rho)$ with N vertices and a set of complete angles $(\theta_1, \dots, \theta_N)$ (i.e. the sum of angles around vertices), satisfying some necessary conditions, does there exist a conformal factor u such that $u * \rho$ is a polyhedral metric and has complete angle θ_i at each vertex? Bobenko, Pinkall and Springborn give a partial answer using a variational principle. Their functional is closely related to a family of functionals developed within the theory of circle packings and circle patterns. To this family belongs for example the functional of Rivin introduced in his paper on "Euclidean structures on simplicial surfaces and hyperbolic volume" [10] and the functional of Colin de Verdière that gives an existence and uniquness proof of circle packings [19].

This observation raises the question if both concepts of discrete conformality are just two sides of the same story. The uniformization theory of convex polyhedra in Euclidean 3-space aims to shed some light on this.

Let $\mathcal{P}_{\text{ideal}}$ be the space of ideal convex polyhedra in $\mathbb{H}^3$, this space is equivalent to the space of convex Euclidean polyhedra inscribed in the unit sphere if $\mathbb{H}^3$ is the Klein model of the hyperbolic 3-space. Hence, let us interpret $\mathcal{P}_{\text{ideal}}$ as the space of convex Euclidean polyhedra inscribed in the unit sphere. Let $\mathcal{C}$ be the set of circle patterns covering the unit sphere. To every circle pattern C in $\mathcal{C}$ corresponds a unique convex Euclidean polyhedron P_C in $\mathcal{P}_{\text{ideal}}$ by cutting off all half-planes defined by the circles in C. Conversely, every convex polyhedron inscribed in the unit sphere corresponds to a unique circle pattern covering the unit sphere.

As a consequence of the results presented in this paper, we obtain the following theorem.

Theorem: *Let C_1 and C_2 be two circle patterns in $\mathcal{C}$ and let P_{C_1} and P_{C_2} be the corresponding convex polyhedra inscribed in the unit sphere. There exists a Möbius transformation f on the sphere mapping the circle pattern C_1 onto the circle pattern C_2 if and only if P_{C_1} and P_{C_2} share a common Delaunay triangulation $\mathcal{T}$, and there exists a function $u_{\mathcal{T}}$ defined on the vertex set of P_{C_1} with the following property. For every edge ij in the Delaunay triangulation between vertices i and j, its length in P_{C_2} is related to its length in P_{C_1} by*

$$l_{P_{C_2}}(ij) = l_{P_{C_1}}(ij)\, e^{\frac{1}{2}(u_{\mathcal{T}}(i)+u_{\mathcal{T}}(j))}.$$

Moreover, f and $u_{\mathcal{T}}$ are related by $u_{\mathcal{T}} = \log |df|_V$, where V is the vertex set of P_{C_1}.

This theorem not only relates the above mentioned concepts of discrete conformality to each other, but also suggests the introduction of a third variant of discrete conformality, namely *discrete Möbius geometry*, which we discuss in the last chapter of this work.

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ALEXANDROV'S THEORY ON CLOSED CONVEX POLYHEDRA

We will consider closed convex polyhedra in Euclidean 3-space $\mathbb{E}^3$ and hyperbolic 3-space $\mathbb{H}^3$. A *closed convex polyhedron* in $\mathbb{E}^3$ or $\mathbb{H}^3$ is the convex hull of a finite set of points in $\mathbb{E}^3$ or $\mathbb{H}^3$. This definition includes doubly-covered closed convex polygons. By a *closed polygon* we mean any domain in $\mathbb{E}^2$ or $\mathbb{H}^2$ that is bounded by finitely many geodesic line segments.

The boundary of a closed convex polyhedron is composed of finitely many closed convex polygons in the respective 2-dimensional space. In the following, we will not explicitely stipulate that the polyhedron under consideration is closed and convex.

The polygons bounding a polyhedron are the *faces* of the polyhedron. The sides and vertices of the faces of a polyhedron are the *edges* and *vertices* of the polyhedron.

In the same manner one could define the vertices of a polyhedron P as the minimal number of points, whose convex hull agrees with P.

A *convex polyhedron with vertices at infinity* in $\mathbb{H}^3$ is the convex hull of a finite set of points, some of them lying on the ideal boundary of $\mathbb{H}^3$. A convex polyhedron with all vertices on the ideal boundary is called an *ideal convex polyhedron.*

Polyhedral surface. A *polyhedral surface* is a surface S together with a flat cone metric ρ on S that has finitely many cone points. A *cone point* is a point v in S that admits a circle centered at v with circumference less than $2\pi r$, where r is its radius. A *marking* μ of a surface S is a labelling of a finite set of points in S. We denote a marked surface by S_μ.

Given two points x and y on the boundary of a Euclidean or hyperbolic polyhedron P, there exists a polygonal path from x to y on the boundary

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of P. The infimum of the lengths of polygonal paths from x to y defines a *distance* $\rho_P(x, y)$ between them. To distinguish whether we are dealing with Euclidean or hyperbolic geometry, we denote the hyperbolic distance by $\hat{\rho}_P(x, y)$. This construction associates with every Euclidean polyhedron P a marked Euclidean polyhedral surface (S_μ, ρ_P) homeomorphic to the sphere. The marking μ is inherited from the vertices of P. Analogously, with every hyperbolic polyhedron P this construction associates a marked hyperbolic polyhedral surface $(S_\mu, \hat{\rho}_P)$ homeomorphic to the sphere.

An *ideal polyhedral surface* is a complete hyperbolic surface of finite area, homeomorphic to the N times punctured sphere. We denote a surface homeomorphic to the N times punctured sphere by S_N. Analogously, every ideal polyhedron P gives rise to a marked ideal polyhedral surface $(S_\mu, \hat{\rho}_P)$.

The *complete angle* at a point x in a polyhedral surface (S, ρ) or $(S, \hat{\rho})$ is the number

$$\lim_{\epsilon \to 0} \frac{C_\epsilon(x)}{\epsilon},$$

where C_ϵ is the circumference of a circle of radius ϵ at x. The notion of a complete angle is an intrinsic property of the polyhedral surface.

Let θ be the complete angle at a point x, the difference $2\pi - \theta$ is the *curvature* at x. A polyhedral surface that has a positive curvature at every point is said to be a *polyhedral surface of positive curvature*.

A polyhedral surface arising as the boundary of a convex polyhedron has a positive curvature everywhere. Conversely, does every polyhedral surface of positive curvature arise from a convex polyhedron in $\mathbb{E}^3$ or $\mathbb{H}^3$?

An affirmative answer was given by Alexandrov in the 1940's. In fact, Alexandrov showed that every polyhedral surface of positive curvature defines a unique polyhedron in $\mathbb{E}^3$ or $\mathbb{H}^3$ up to congruence [20].

Around 2000 Igor Rivin showed that every ideal polyhedral surface defines a unique ideal polyhedron up to congruence [21].

We will give a complete exposition of Alexandrov's and Rivin's beautiful theory below.

Development. A *development* is a finite collection of closed polygons in $\mathbb{E}^2$ or $\mathbb{H}^2$ together with a set of rules for "gluing" them together along their edges. The rule for gluing satisfies the following conditions:

1. The correspondence of "gluing" two segments is an isometry.
2. It is possible to pass from each polygon to any other polygon by traversing polygons with glued sides.
3. Each side of every polygon is glued to exactly one side of another polygon.

The sides and vertices of the polygons within a development are the *edges* and *vertices* of the development, where identified sides and vertices are considered the same. We denote a development by R.

Every development defines an underlying marked polyhedral surface, which we denote by (S_μ, ρ_R). The marking μ labels the points on the surface S corresponding to the vertices of R. In other words, a development is a polyhedral surface plus a subdivision into geodesic polygons.

Several developments can define the same polyhedral surface. One may think of a development as a "coordinate representation" of a polyhedral surface. Different cuttings of a polyhedral surface into polygons correspond to different coordinate representations of the same polyhedral surface.

Two developments R and R' can be obtained from each other by *cutting and gluing* if the polygons in R can be cut into polygons and glued along edges such that we obtain the development R'. One observes:

Theorem: *Two developments R and R' are related by cutting and gluing if and only if (S_μ, ρ_R) and $(S_\mu, \rho_{R'})$ are isometric by an isometry homotopic to the identity.*

We will use the above ideas to turn the space of closed polyhedral surfaces into a manifold by "cutting" polyhedral surfaces into triangles. Those representations will turn out to be convenient coordinate charts for our space.

Every convex polyhedron is naturally associated with a development. The *face development* of a polyhedron P is the development R_P whose polygons are the faces of the polyhedron P.

A rigidity property of convex polyhedra. It is a fundamental result of rigidity theory that convex polyhedra in $\mathbb{E}^3$ or $\mathbb{H}^3$ with congruent corresponding faces must be congruent to each other. This result is attributed to Augustin Cauchy who published this result in 1813 [22]. His proof is widely considered as one of the most elegant arguments of geometry.

Cauchy's argument is based on the following combinatorial observation.

Lemma: (Cauchy) *Suppose that some edges of a closed convex polyhedron are labeled by either a plus or minus signs. Sign changes may occur between labeled edges around a vertex. It is impossible to have at least four sign changes at every vertex.*

A slightly stronger statement is this purely topological observation.

Lemma: *Suppose a "net of edges" is given on a surface homeomorphic to the sphere, i.e, suppose that finitely many edges (each of which is homeomorphic to a straight line segment) are given, and these edges are pairwise disjoint, except possibly at their endpoints, the "vertices of the net". Assume further that none of the regions separated by the net of the sphere is bounded by only two edges. Assign pluses and minuses to the edges of the net. Let V be the total number of vertices and N the total number of sign changes at all vertices. Then*

$$N \leq 4V - 8.$$

By a *net* we mean an arbitrary finite collection of simple (i.e., not self-intersecting) polygonal lines on a polyhedral surface with no common points except possibly endpoints. Each polygonal line is called an *edge* of the net and endpoints of edges are the *vertices* of the net. A net may devide the polyhedral surface into several regions, i.e. points that can be joined with one another by a polygonal line not intersecting the net. A net may also consist of several disjoint parts. A part that cannot be further decomposed is called a connected component of the net.

In the proof of the stronger Cauchy lemma we make use of the following generalized Euler Theorem. A detailed proof can be found in Alexandrovs book on page 61 [20].

Theorem: (The Generalized Euler Theorem) *Given a net on the boundary of a closed convex polyhedron, let v be the number of vertices, e the number of edges, c the number of connected components, and f the number of regions into which the net divides the polyhedron. Then*

$$v - e + f = c + 1.$$

In particular, if the net is connected, then $v - e + f = 2$.

Proof of the stronger Cauchy lemma: N is the total number of sign changes as one moves around the vertices of the net. We observe that counting the number of sign changes as one moves around each of the regions separated on the surface by the edges of the net gives the same number N.

Indeed, orient the surface and start going around the regions of the surface in the direction prescribed by this orientation. Suppose we are going along the contour of a region with vertices A, B, C, D, and E. Assume we have passed the edge AB in the order A to B. Now, we are standing at the vertex B and are about to pass the edge BC in the order B to C. The edge BC follows the edge AB as we go around the region in the same order as in the order given by moving around B. Thus, the sign change from AB to BC is counted in both cases. We proceed moving along the contour of the region until we arrive at the edge from where we started. If two edges AB and BC separate two distinct regions, then in one region they are passed from AB to BC and in the other from BC to AB, which is the same as moving around the vertex B. Nets may have edges that do not separate two distinct regions. The region may have a free endpoint in its interior or an edge that bounds the same region on both sides. Every such edge will be passed twice when going along the region: first in one direction, then, in the opposite direction. If D is a free endpoint of the edge DE, then when going along the contour, after passing the edge DE in the direction E to D, we again pass the edge DE in the opposite direction. This is the same as moving around the vertex D.

When counting the edges of a region, we count edges of this region twice if they do no separate two distinct regions from another. Notice that the number of sign changes when going along the contour of a region cannot be bigger than the number n of its edges. Furthermore, this number of sign changes is always even, since when completing a full cycle, we return to the initial sign.

Let F_n be the total number of regions with n edges. From the above follows that the total number N of sign changes has the upper bound

$$N \leq 2F_3 + 4F_4 + 4F_5 + \dots .$$

Let V, E, and F be the number of vertices, edges and regions in a net. By the generalized Euler formula

$$V - E + F \geq 2.$$

This is equivalent to

$$4V - 8 \geq 4E - 4F.$$

Since each edge either belong to two regions or is counted twice, we have

$$2E = \sum_n nF_n.$$

The total number F of regions is

$$F = \sum_n F_n.$$

Substituting both equations in the inequality above gives

$$4V - 8 \geq \sum_n 2(n-2)F_n = 2F_3 + 4F_4 + 6F_5 + \dots .$$

The right-hand-side is clearly bigger than the total number N of sign changes. Hence,

$$4V - 8 \geq N.$$

To use Cauchy's lemma to prove the rigidity result of convex polyhedra in $\mathbb{E}^3$ or $\mathbb{H}^3$, requires some preparation.

The *link* of a vertex of a polyhedron in $\mathbb{E}^3$ or $\mathbb{H}^3$ is the spherical polygon obtained by intersecting an infinitesimal sphere centered at the vertex with the polyhedron, and rescaling so that the sphere has radius 1. The edge lengths in the link are precisely the face angles at the vertex. The link of a convex polyhedron is a convex polygon.

For a vertex at the boundary at infinity, there exists a one-parameter family of horospheres centered at that vertex. A sufficiently small horosphere intersects with the polyhedron in a Euclidean polygon, this polygon we call the *link* of the vertex at the boundary at infinity. Notice that the link in this case is only defined up to similarity. We may normalize the edge length of the link by setting the length of the longest side equal to 1.

The following characterization of convex spherical polygons is shown in detail by Alexandrov in his book on convex polyhedra, pp. 155 – 157 [20].

Theorem: *If none of the angles of a spherical polygon exceeds π, then the polygon is convex.*

Lemma: *Let P and Q be two convex polygons with the same number of vertices $p_1, p_2, \dots, p_n$ and $q_1, q_2, \dots, q_n$. Suppose that the following conditions are satisfied:*

1. *all their corresponding sides except p_np_1 and q_nq_1 are of equal length, i.e.,*
$$|p_1p_2| = |q_1q_2|, \ldots, |p_{n-1}p_n| = |q_{n-1}q_n|;$$
2. *the angles between these sides in P do not exceed those in Q, i.e.,*
$$\angle p_2 \leq \angle q_2, \ldots, \angle p_{n-1} \leq \angle q_{n-1},$$
and the strict inequality holds at least once.

Then the "exceptional" side of the first polygon is less than the "exceptional" side of the second, i.e.,
$$|p_np_1| < |q_nq_1|.$$

Proof: The proof goes by induction on the number of vertices of the polygons.

For triangles the lemma reduces to the well known result that if two triangles $p_1p_2p_3$ and $q_1q_2q_3$ have two sides of equal lengths, let's say $|p_1p_2| = |q_1q_2|$ and $|p_2p_3| = |q_2q_3|$, and the angles between these two sides differ, then the third side is shorter in that triangle where the angle is smaller.

Suppose the theorem is true for $(n-1)$-gons. We prove that it is then also true for n-gons. Given two n-gons P and Q satisfying the conditions of the theorem, we distinguish two cases. Either (1) all angles at $p_2, \ldots, p_{n-1}$ are strictly smaller than the angles at $q_2, \ldots, q_{n-1}$ or (2) some of those angles are equal.

Consider the second case. Let p_k and q_k be corresponding vertices at which the angles are equal. Cut off the triangles $p_{k-1}p_kp_{k+1}$ and $q_{k-1}q_kq_{k+1}$ from P and Q respectively. The triangles $p_{k-1}p_kp_{k+1}$ and $q_{k-1}q_kq_{k+1}$ are congruent since two sides and one angle are equal. Hence, the remaining $(n-1)$-gons clearly satisfy the assumptions of the theorem and therefore $|p_1p_{n-1}| < |q_1q_{n-1}|$.

Now, assume that all angles at $p_2, p_3, \ldots, p_{n-1}$ of the polygon P are strictly smaller than the corresponding angles of the polygon Q. Let p_k be a point of the polygon P not lying on the prolongation of the side p_1p_n. Construct the triangle $T = p_kp_1p_n$. This way, the polygon P is cut into three parts, two polygons P_1 and P_2 and the triangle T.

Transform the triangle T continuously such that the angle at p_k increases while the lengths of the segments p_kp_1 and p_kp_n remain the same. By the elementary result for triangles, we have that $|p_1p_n| < |p_1'p_n'|$, where p_1' and p_n' are the points corresponding to p_1 and p_n after the transformation.

Transform T in such a way that the angle at p_k' agrees with the angle at q_k.

If the transformed polygon P' is convex, then P' satifies the assumptions of the second case above. Hence,
$$|p_1'p_n'| < |q_1q_n|.$$

In particular,

$$|p_1p_n| < |q_1q_n|,$$

which was to be shown.

However, the polygon P' may indeed fail to be convex. Interestingly, Cauchy missed to see this in his proof in 1813. The correction of this part of the proof is attributed to Ernst Steinitz (1871 – 1928) and Isaac Jacob Schönberg (1903 – 1990).

Notice that increasing the angle at p_k may indeed also increase the angle at p_1 and p_n. If an angle at p_1 or p_n exceeds π before p_k reaches the size of the angle at q_k, then P' is not convex. In this case we may stop the transformation process as soon as the angle at p_1 or p_n reaches π.

Assume that the angle at p_1 reaches π before the angle at p_n and before the angle at p_k reaches the size of the angle at q_k. Then the tranformed polygon P' is convex and the segments $p'_1p'_2$ and $p'_1p'_n$ form a single segment in P'. We can construct the triangle $p'_2p'_kp'_n$ and begin to increase the length of the segment $p'_2p'_n$ such that the angle at p'_k increases. The vertex p''_1 will be the point on the transformed segment $p''_2p''_n$ such that $|p''_1p''_2| = |p_1p_2|$.

As previously, during the transformation, the angle at p'_k increases while the angles at p'_2 and p'_n increase or decrease. There are three different possibilities of what may happen.

1. The angle at p'_k can be increased to the angle at q_k without violating the convexity and such that the angle at p'_2 stays smaller than the angle at q_2.
2. The angle at p'_2 increases and reaches the size of the angle at q_2 before p'_k reaches the size of the angle at q_k.
3. The angle at p'_n reaches π.

In the first two cases, there exists an angle in the convex polygon P'' that agrees with a corresponding angle in Q. Hence, we can conclude the proof as we did previously.

In the third case, we again stop the transformation as soon as the angle at p'_n reaches π. In this case, the segments $p''_2p''_1$, $p''_1p''_n$ and $p''_np''_{n-1}$ form a single segment and we can construct the triangle $p''_2p''_kp''_{n-1}$. When transforming this triangle such that the angle at p''_k increases, no violation of convexity can occur, since one of the angles at p''_2, p''_k and p''_{n-1} will reach the size of the corresponding angle in Q. Finally we may use the arguments above to conclude that

$$|p_1p_n| < |q_1q_n|.$$

Lemma: *If two combinatorially equivalent links have corresponding sides of equal length but not all the corresponding angles are equal, then there are at least four sign changes in the differences between the corresponding angles as we go around the links.*

Proof: Let $\delta_i = \angle p_i - \angle q_i$, if not all δ_i vanish, then there must exist a sign change. Indeed, suppose $\angle p_2 \leq \angle q_2, \ldots, \angle p_{n-1} \leq \angle q_{n-1}$ with at least one strict inequality, then by the theorem above $|p_1p_n| < |q_1q_n|$, which contradicts the assumption.

Suppose there exists a sign change, then the number of sign changes when travelling around the polygons is even. Hence, it suffices to show that there cannot be exactly two sign changes. Assume we have exactly two sign changes; number the vertices such that the angles at $p_1, \ldots, p_m$ are greater than the angles at $q_1, \ldots, q_m$, and the angles at $p_{m+1}, \ldots, p_n$ are less than the angles at $q_{m+1}, \ldots, q_n$.

Bisect the sides $p_1p_n, p_mp_{m+1}, q_1q_n$ and q_mq_{m+1} by adding points i, j, k and l respectively and polygonal lines ij and kl between them. This gives us two pairs of polygons both satisfying the assumptions of the theorem above. Applying the lemma to the first pair $p_1p_2 \ldots p_mij$ and $q_1q_2 \ldots q_mkl$ we conclude that $|ij| < |kl|$. Applying the lemma to the second pair, we conclude that $|kl| < |ij|$, leading to the desired contradiction.

The previous lemma can be reformulated in a more convenient form.

Lemma: *If two convex polyhedral angles, distinct from dihedral angles and possibly degenerate, have corresponding planar angles of equal measure while not all of their dihedral angles are equal, then there are at least four sign changes in the differences between the corresponding dihedral angles as we move around the vertices.*

Proof: If the two polyhedral angles are non-degenerate, then the associated links satisfy the assumptions of the previous lemma. Hence, there are at least four sign changes in the differences between corresponding dihedral angles as we move around the vertices.

Assume that at least one of the polyhedral angles is degenerate. Let V_P be a degenerate polyhedral angle and denote the other by V_Q. Let p_1 and p_2 be the two edges of V_P at which the dihedral angles are equal to zero. If in V_Q the dihedral angle at the edge q_1 corresponding to p_1 in V_P is zero, then so is the dihedral angle at q_2. Hence, the polyhedral angles V_P and V_Q are congruent.

Assume that the dihedral angle at q_1 is different from zero. Then also the dihedral angle at q_2 is different from zero. In particular, the dihedral angles at q_1 and q_2 are bigger than the dihedral angles at p_1 and p_2. The sum of the planar angles between q_1 and q_2 is the same from both sides. Hence, there exists an edge between q_1 and q_2 on both sides with dihedral angle smaller than π. However, the corresponding edges in V_P have dihedral angle equal to π since V_P is degenerate. Thus, there are at least four sign changes in the difference between the corresponding dihedral angles as we move around the vertices.

Let us return to Cauchy's rigidity theorem. As previously, we include among convex polyhedra doubly-covered convex polygons. For the application of this theorem, it will turn out to be convenient to admit the partition of faces of the polyhedron into finitely many smaller polygons, each of which is counted as a "face". We may distinguish two sorts of edges and vertices: "genuine" and "fictitious". A *fictitious edge* is an edge at which the dihedral angle equals π. A *fictitious vertex* is a vertex at which the polyhedral angle is in fact a dihedral angle, in particular a plane when the vertex lies inside a genuine face of the polyhedron.

We say that two convex polyhedra *have the same structure* if there exists a one-to-one correspondence between the faces, edges and vertices that preserves the incidence relation.

Theorem: (Cauchy) *If two closed convex polyhedra in* $\mathbb{R}^3$ *or* $\mathbb{H}^3$ *have the same structure and corresponding planar angles on corresponding faces are equal, then the dihedral angles at the corresponding edges are also equal.*

Proof: Let P and Q be two polyhedra satisfying the conditions of the theorem. Assign either a plus or minus sign to the edges of P at which the dihedral angle is larger or smaller than the corresponding dihedral angle of Q. If the corresponding dihedral angles agree, no sign is assigned to the edge.

Given two corresponding vertices p and q, then either both are genuine or both are fictitious, since the curvatures at p and q are equal.

If p and q are genuine vertices, then by the preceeding lemma either all dihedral angles are equal, or there are at least four sign changes as we move around p.

If p and q are fictitious vertices, we may split the discussion in two cases.

1. There are genuine edges incident to both p and q, but to a genuine edge incident to p corresponds a genuine edge incident to q.
2. For at least one of the vertices there are either no genuine incident edges, or all genuine incident edges correspond to each other.

Consider the first case. Recall that a polyhedral angle at a fictitious vertex is in fact a dihedral angle. Hence, if both p and q admit a genuine edge, they both admit precisely two genuine edges which are the prolongation of each other. Let e_1 and e_3 be the genuine edges incident to p and l_2 and l_4 be the genuine edges incident to q, numbered in such a way that the ordered list e_1, e_2, e_3, e_4 corresponds to traversing p in anti-clockwise direction. By assumption, the edges e_2, e_4 and l_1, l_3 corresponding to l_2, l_4 and e_1, e_3, respectively, are fictitious. A genuine edge has dihedral angle less than π and a fictitious edge has dihedral angle equal to π. Therefore, the signs of the difference between the dihedral angles at e_1, e_2, e_3 and e_4 are $-, +, -, +$, which yields precisely four sign changes.

The second case may be further split up into

1. neither p nor q have incident genuine edges;
2. only one of the vertices has genuine edges;
3. both p and q have genuine edges which correspond to each other.

If all edges incident to p and q are fictitious, all dihedral angles are equal to π. Hence, there does not occur any labeling around p and the vertices p and q and all edges incident to these vertices can be removed.

Assume that p has a genuine edge e_1. Since p is in fact dihedral, e_1 admits a prolongation e_2 and p has precisely two genuine edges. Let l_1 and l_2 be the corresponding edges incident to q. If q has any genuine edge, then l_1 and l_2 are precisely its genuine edges. All other edges incident to p and q

are fictitious and can be removed. The edges l_1 and l_2 are prolongations of each other, since the planar angles between them agree with the planar angles between the edges e_1 and e_2 which are π. The dihedral angles at l_1 and l_2 are equal. Therefore, the labels attached to e_1 and e_2 agree. We may remove the vertices p and q by assigning the edge $e_1 \cup e_2$ to the edge $l_1 \cup l_2$.

We conclude that all fictitious vertices can be removed. Hence, if there exist sign assignments to the edges of the polyhedron P as above, there would exist at least four sign changes at every vertex contained in some labeled edge. This is impossible by Cauchy's lemma. Hence there are no labeled edges at all, which means that the corresponding dihedral angles of P and Q are equal.

Cauchy's theorem can be used straightforwardly to obtain an analog rigidity statement for convex polyhedra in hyperbolic three-space with some or all vertices at the boundary at infinity. This result is due to Hodgson and Rivin [23]. Henceforth, we distinguish two types of vertices, vertices inside the hyperbolic three-space and vertices on the ideal boundary.

Theorem: (Hodgson, Rivin) *A convex polyhedron in $\mathbb{H}^3$ with some or all vertices at the boundary at infinity is determined up to congruence by the type of its vertices and the edge length of the link of its vertices.*

Proof: Let P and P' be two convex polyhedra in $\mathbb{H}^3$ with some or all vertices at the boundary at infinity, whose corresponding links have equal edge lengths. Apply Cauchy's lemma to each pair of links. We conclude that all angles in corresponding links are equal. Hence, P and P' have corresponding equal dihedral angles.

This implies that P and P' have congruent *ends* corresponding to their vertices at the ideal boundary. If we truncate the ends of P and P' along suitably chosen corresponding planes, we obtain two compact convex polyhedra Q and Q' with equal face angles. By Cauchy's theorem, Q and Q' are congruent. Hence, P and P' are also congruent since they are obtained from Q and Q' by attaching congruent ends.

Cauchy's Theorem may be formulated in a stronger form as follows [20].

Theorem: (Aleksandrov) *Every isometry φ from the boundary of a closed convex polyhedron P in $\mathbb{R}^3$ or $\mathbb{H}^3$ onto the boundary of another closed convex polyhedron Q can be realized as a motion or a motion and a reflection, i.e. there is a motion, or a motion followed by a reflection, which takes each point of the boundary of P to its image under the mapping φ.*

This stronger form of Cauchy's Theorem resulted from the work of Aleksandrov and was published in the 1940's.

An analogous statement can be given for convex polyhedra in $\mathbb{H}^3$ with some or all vertices at the boundary at infinity [23].

Theorem: (Hodgson, Rivin) *Every isometry φ from the boundary of a convex polyhedron P in $\mathbb{H}^3$ with some or all vertices at the boundary at infinity onto the boundary of another convex polyhedron Q can be realized as a motion or a motion and a reflection, i.e. there is a motion, or a* *motion followed by a reflection, which takes each point of the boundary of P to its image under the mapping φ.*

In the following, we return to the usual usage of the notion of a face of a polyhedron as its genuine face without further subdivision.

Let φ be an isometry from the face development R_P of a closed convex polyhedron P onto the face development R_Q of a closed convex polyhedron Q. The image of the 1-skeleton of R_P in R_Q induces a partition of the polygons in R_Q. Analogously, the pre-image of the 1-skeleton of R_Q in R_P induces a partition of the polygons in R_P.

Lemma: *The partition induced by φ cuts R_P and R_Q into collections of convex polygons. φ induces a correspondence between those polygons that preserves the incidence relation such that corresponding polygons are congruent.*

Proof: Let P_1 be a polygon in R_P and pick a polygon Q_1 in R_Q that is partly covered by $\varphi(P_1)$. We like to show that $Q_1 \cap \varphi(P_1)$ is a convex polygon.

Let q_1 and q_2 be two points in $Q_1 \cap \varphi(P_1)$ and let p_1 and p_2 be their pre-image in P_1 via φ, respectively. Since P_1 is convex, the segment p_1p_2 lies in P_1; hence, its image $\varphi(p_1p_2)$ lies in $\varphi(P_1)$. Since φ is an isometry and q_1 and q_2 lie on the same polygon Q_1, $\varphi(p_1p_2)$ agrees with the segment q_1q_2. Thus, q_1q_2 lies in $Q_1 \cap \varphi(P_1)$ and $Q_1 \cap \varphi(P_1)$ is convex.

The image of edges of R_P are shortest arcs in R_Q. Shortest arcs in R_Q are polygonal lines with at most one segment in each polygon in R_Q. Hence, $Q_1 \cap \varphi(P_1)$ is cut out from Q_1 by finitely many straight line segments and therefore forms a bounded polygon.

The isometry φ is cellular with respect to the partition of R_P and R_Q induced by φ. Indeed, every element of the partition of R_P will be of the form $P_1 \cap \varphi^{-1}(Q_1)$ for some polygons P_1 and Q_1 in R_P and R_Q respectively. This polygon is mapped to the element $\varphi(P_1) \cap Q_1$ of the partition of R_Q.

Proof of Alexandrov's stronger Cauchy Theorem: Given an isometry φ from the boundary of a closed convex polyhedron P onto the boundary of a closed convex polyhedron Q, φ is cellular with respect to its induced partition of R_P and R_Q in polygons as described above. The corresponding polygons are congruent and in particular have equal angles. We can apply Cauchy's Lemma as discussed above to conclude that all polyhedral angles along their edges must be equal.

Let P_1 be a polygon in the partition of R_P induced by φ and let Q_1 be the corresponding polygon in the partition of R_Q. Since P_1 and Q_1 are congruent, there exists a motion ψ that assigns to every point of the boundary of P corresponding to P_1 its image under φ in Q. If the image of P under this motion and Q lie on opposite sides of the plane determined by Q_1, reflect P along this plane.

Since all dihedral angles along edges are equal, the image under ψ of an adjacent polygon P_2 of P_1 in the partition of R_P lies in the plane determined by the corresponding polygon Q_2 in the partition of R_Q. Since

$\psi(P_1)$ and Q_1 agree on all edges, $\psi(P_2)$ and Q_2 clearly agree on the edge shared with Q_1. $\psi(P_2)$ also lies on the same side of this edge as Q_2, hence $\psi(P_2)$ and Q_2 agree.

Since the partition of R_P is connected, φ can be realized by a motion or a motion and a reflection.

Isometric embedding of polyhedral surfaces. We now return to the previously stated question: Does every polyhedral surface of positive curvature in $\mathbb{E}^3$ or $\mathbb{H}^3$ arise as the boundary of a convex polyhedron? Alexandrov gave the following answer [20]:

Theorem: (Alexandrov) *Let (S, ρ) be a polyhedral surface with N cone points of strictly positive curvature, homeomorphic to the sphere. Then (S, ρ) can be realized as the boundary of a closed convex polyhedron P_N with N vertices. This polyhedron is unique up to congruence.*

We suggested to think of developments of polyhedral surfaces as coordinate representations of the polyhedral surface. In this section we will make this more precise by cutting polyhedral surfaces into triangles. From this we will establish that the space of polyhedra with N vertices up to congruence and the space of polyhedral surfaces of positive curvature with N cone points are homeomorphic.

Let $T_{PL}(N)$ be the space of polyhedral metrics on the sphere with N marked points. The set of marked points is required to contain all points on the polyhedral surface of non-zero curvature. The polyhedral metrics ρ in $T_{PL}(N)$ are considered up to isometry homotopic to the identity. A *triangulation* of a marked sphere S_μ is a triangulation $\mathcal{T}$ of the sphere with vertices the N marked points on the sphere. A *geodesic triangulation* of a marked polyhedral surface $T_{PL}(N)$ is a triangulation of S_μ whose edges are minimizing geodesics.

Let $\mathcal{T}$ be a triangulation of the marked sphere S_μ with N marked points and $E(\mathcal{T})$ the set of edges associated to $\mathcal{T}$. Let $E(\mathcal{T})^*$ be the set of positive real valued functions on $E(\mathcal{T})$, satisfying

$$x(e_i) + x(e_j) > x(e_k)$$

for every triangle $ijk \in \mathcal{T}$ with edges e_i, e_j and e_k. For a function $x \in E(\mathcal{T})^*$ let ρ_x be a polyhedral metric on the sphere constructed by isometrically gluing Euclidean triangles $ijk \in \mathcal{T}$ of edge lengths $x(e_i), x(e_j), x(e_k)$ along corresponding edges. This provides an injective map

$$\iota : E(\mathcal{T})^* \to T_{PL}(N)$$

mapping x to ρ_x.

Let $U_\mathcal{T}$ be the image of ι in $T_{PL}(N)$ and $\vartheta_\mathcal{T}$ the inverse of ι on $U_\mathcal{T}$. The pair $(U_\mathcal{T}, \vartheta_\mathcal{T})$ is a *length coordinate chart* on $T_{PL}(N)$.

Theorem: *The set of length coordinate charts*

$$\{(U_\mathcal{T}, \vartheta_\mathcal{T}) \,|\, \mathcal{T} \textit{ triangulation of } S_\mu\}$$

forms a real-analytic atlas on $T_{PL}(N)$.

Proof: Every polyhedral metric on the sphere with N points of non-zero curvature admits a geometric triangulation, hence $\bigcup_{\mathcal{T}} U_{\mathcal{T}}$ covers $T_{PL}(N)$. If two triangulations $\mathcal{T}$ and $\mathcal{T}'$ are related by a diagonal flip along an edge e, then the transition function $\vartheta_{\mathcal{T}'}\vartheta_{\mathcal{T}}^{-1}$ is the identity in all components except for the component associated to e. Let ABC and ADC be the triangles in $\mathcal{T}$ adjacent to e. The length of the diagonally switched edge is

$$\sqrt{x_{AB}^2+x_{DC}^2-2x_{AB}x_{DC}\cos(\arccos(\frac{x_{AC}^2+x_{AB}^2-x_{BC}^2}{2x_{AC}x_{AB}})+\arccos(\frac{x_{AC}^2+x_{DC}^2-x_{AD}^2}{2x_{AC}x_{DC}}))}.$$

Hence, the transition function $\vartheta_{\mathcal{T}'}\vartheta_{\mathcal{T}}^{-1}$ is real analytic.

Given two arbitrary triangulations of S_μ, there exists a sequence of triangulations of S_μ, each related to the next by a single diagonal flip along an edge. Hence, all transition functions are real analytic.

The set of closed simply-connected marked polyhedral surfaces with N marked points of strictly positive curvature $T_{PL}^{\text{con}}(N)$ forms a subset of $T_{PL}(N)$. For some coordiante chart $\vartheta_{\mathcal{T}}$ of $T_{PL}(N)$ let $\sum_j \alpha_{ij}$ be the complete angle at i, where j enumerates the angles at the i-th vertex. The complete angle is a continuous function of the edge length. The set $T_{PL}^{\text{con}}(N)$ is determined by the inequalities

$$\sum_j \alpha_{ij} < 2\pi \qquad (i = 1, \ldots, N).$$

Hence, $T_{PL}^{\text{con}}(N)$ is an open submanifold of $T_{PL}(N)$.

Lemma: *If $N > 3$, then the manifold $T_{PL}^{\text{con}}(N)$ is a proper subset of $T_{PL}(N)$ and consequently has a topological boundary in $T_{PL}(N)$.*

Proof: We first prove that in any triangulation of an element in $T_{PL}(N)$ with $N > 3$, there are at least three triangles touching at a single vertex. In fact, we will show that, if at most two triangles touch at each vertex, then N is at most three.

Let $\mathcal{T}$ be a geodesic triangulation of (S_μ, ρ), such that at most two triangles touch at each vertex. Let ABC be a triangle in $\mathcal{T}$. Suppose that some triangle ABD is glued to the side AB of the triangle ABC. There are two triangles touching at the vertices A and B. If there are no other triangles touching at A and B, then the sides AC and BC must be glued to AD and BC respectively. If AC were glued to BC, then A and B would be identified, which is impossible by our convention of triangulation. Hence, (S_μ, ρ) is a doubly covered triangle.

Let $\mathcal{T}$ be an abstract triangulation of the sphere with more than three vertices. Construct a polyhedral surface (S_μ, ρ) by assigning length 1 to all edges of $\mathcal{T}$. By the argument above, there exists a vertex A of $\mathcal{T}$ where at least three triangles touch. Construct a sequence of polyhedral surfaces $\{(S_\mu, \rho_t)\}_{t\in[1,2]}$ by taking $\rho_t(e) = 1/t$ for every edge e adjacent to A and constant equal to 1 otherwise. Notice that the angle at A in every triangle increases to π as t approaches 2. Since there are at least three triangles touching at A, there exists a $t' \in [1, 2)$ such that the sum of the angles at

A is greater than or equal to 2π. Such a development is not contained in T_{PL}^{con}, which was to be proven.

Theorem: (Alexandrov) *Let (S, ρ) be a polyhedral surface with N cone points of strictly positive curvature, homeomorphic to the sphere. Then (S, ρ) can be realized as the boundary of a closed convex polyhedron P_N with N vertices. This polyhedron is unique up to congruence.*

Outline of the proof. Let $\mathcal{P}^N$ be the space of closed convex polyhedra with N vertices in $\mathbb{R}^3$, parametrized by the positions of their vertices. Three vertices are sent by an isometry to the origin, the positive x-axis and the half-plane $y > 0$ of the xy-plane, respectively. If the polyhedron does not degenerate into a doubly-covered polygon, then a fourth point not contained in the xy-plane is mapped into the half-space $z > 0$ by reflecting along the $z = 0$ plane if needed. This eliminates the action of the isometry group of $\mathbb{R}^3$. There are $3N$ variable coordinates, however three vertices are constant in three, two and one coordinates respectively. Therefore, we have $3N - 6$ variable coordinates and $\mathcal{P}^N$ is a $3N - 6$ dimensional manifold.

The boundary of every closed convex polyhedron with N vertices can be viewed as a polyhedral surface with N cone points of strictly positive curvature homeomorphic to the sphere. Formally this gives a map $\mathfrak{g} : \mathcal{P}^N \to T_{PL}^{\text{con}}(N)$. In the following we show that $\mathfrak{g}$ is a (1) continuous, (2) injective and (3) closed map and (4) that every connected component of $T_{PL}^{\text{con}}(N)$ admits a preimage in $\mathcal{P}^N$.

$\mathcal{P}^N$ and $T_{PL}^{\text{con}}(N)$ are manifolds of equal dimension, by (1) and (2) and the invariance of domain principle of Brouwer, $\mathfrak{g}$ is an open map. Since $\mathfrak{g}$ is also closed, we conclude together with (4) that $\mathfrak{g}$ is a homeomorphism from $\mathcal{P}^N$ onto $T_{PL}^{\text{con}}(N)$.

Remark: The fact that a polyhedron in $\mathbb{R}^3$ is determined by the geometry of its surface, is particular to polyhedra in three dimensional space. A polygon is not at all determined by the length of its edges. Also, in higher dimensions such a correspondence does not hold in general. The dependends of the theory on the dimension reveals itself in the usage Brouwer's invariance of domain principle. That the dimension of the space of closed convex polyhedra with N vertices has the same dimension as the space of polyhedral surfaces with N cone points of strictly positive curvature, is particular to $\mathbb{R}^3$.

A *triangulation* of a polyhedron P with N vertices is a geometric triangulation of its polyhedral surface (S_μ, ρ_P).

Theorem: *The map $\mathfrak{g}$ is continuous.*

Proof: Let $\mathcal{T}$ be a triangulation of a closed convex polyhedron P. There exists an $\epsilon > 0$ such that, if the vertices of some polyhedron Q are at distances less than ϵ from the vertices in $\mathcal{T}$ (with exactly one vertex of Q corresponding to a vertex of $\mathcal{T}$), then there is a unique triangulation of Q close to $\mathcal{T}$ that has the same structure. The continuity of $\mathfrak{g}$ follows immediately.

Theorem: *The map $\mathfrak{g}$ is injective.*

Proof: The injectivity follows from the above proven theorem that, if there exists an isometry φ between the boundary of two closed convex polyhedra P and Q, i.e. their associated marked polyhedral surfaces (S_μ, ρ_P) and (S_μ, ρ_Q) are isometric, then there exists a motion or a motion and a reflection from P to Q realizing φ. That is to say, P and Q are equal in $\mathcal{P}^N$ since we eliminated the action of the isometry group of $\mathbb{R}^3$.

Theorem: *The image of* $\mathfrak{g}$ *is closed.*

Proof: Let $\rho_1, \ldots, \rho_i, \ldots$ be a sequence of metrics in $T_{PL}^{\mathrm{con}}(N)$, converging to a metric ρ. For each i let P_i be a closed convex polyhedron in $\mathcal{P}^N$ such that $\mathfrak{g}(P_i) = \rho_i$. There exists a closed convex polyhedron P_∞ in $\mathcal{P}^N$ such that $\mathfrak{g}(P_\infty) = \rho$.

Indeed, since the sequence of polyhedral surfaces associated with $\{P_i\}_{i\geq 1}$ converges, the distances between vertices of polyhedra in $\{P_i\}_{i\geq 1}$ are uniformly bounded. Hence, the polyhedra are contained in some ball centered at the origin.

Pick a subsequence $\{\rho_{i_k}\}_{k\geq 1}$ such that all ρ_{i_k} admit a triangulation $\mathcal{T}_{i_k}$ of the same combinatorics. The sequence of triangulations $\{\mathcal{T}_{i_k}\}_{k\geq 1}$ converges to a unique triangulation $\mathcal{T}$ of ρ. Enumerate the vertices of the triangulations such that the sequence of vertices associated with i converges to the i-th vertex in $\mathcal{T}$. This enumeration gives N sequences in $\mathbb{R}^3$ associated with the vertices of the polyhedra. Pick a subsequence $\{P_{i_{k_l}}\}_{l\geq 1}$ such that those sequences converge. Let P_∞ be the boundary of the convex hull of the N limit points.

For notational simplicity we replace $\{P_i\}_{i\geq 1}$ with the sequence $\{P_{i_{k_l}}\}_{l\geq 1}$. The enumeration of vertices in $\{\mathcal{T}_i\}_{i\geq 1}$ induces an enumeration of edges in $\{\mathcal{T}_i\}_{i\geq 1}$. Given $\mathcal{T}_i$, the map $\mathfrak{g}$ induces a triangulation $\mathcal{T}_i'$ of P_i. We say that $\mathcal{T}_i$ is plotted on P_i. Let n_i denote the n-th edge in $\mathcal{T}_i'$. We will make use of the following statement, proved as a lemma below.

There exists an integer C*, such that for every* i *the number of intersecting points of* n_i *with edges of* P_i *is bounded by* C*, i.e.,*

$$\sup_i \#\{n_i \cap P_i^{(0)}\} \leq C.$$

If for $\{1_i\}_{i\geq 1}$ there exist infinitely many polyhedra in $\{P_i\}_{i\geq 1}$ such that 1_i is an edge of P_i, take this infinite subset $\{P_{i_k}\}_{k\geq 1}$. Since the edges of $\{P_{i_k}\}_{k\geq 1}$ converge, 1_{i_k} converges. Continue with $\{2_{i_k}\}_{k\geq 1}$. Otherwise, since the number of intersecting points of 1_i with P_i is uniformly bounded in i, there exists a subsequence $\{P_{i_k}\}_{k\geq 1}$ such that each edge 1_{i_k} has the same number of intersecting points with P_{i_k} and all intersecting points converge. The segments between adjacent intersecting points converge as well. In particular, 1_{i_k} converges. Continue with $\{2_{i_k}\}_{k\geq 1}$. Define $\mathcal{T}' := \lim_k \mathcal{T}_{i_k}'$, by continuity of $\mathfrak{g}$, $\mathcal{T} = \mathfrak{g}(\mathcal{T}')$. Hence, P_∞ is a closed convex polyhedron in P^N such that $\mathfrak{g}(P_\infty) = \rho$.

Lemma: *There exists an integer* C*, such that for every* i *the number of intersecting points of* n_i *with edges of* P_i *is bounded by* C*, i.e.,*

$$\sup_i \#\{n_i \cap P_i^{(0)}\} \leq C.$$

Proof: We will prove that there exists an integer C_0 uniform in i, such that if an edge of P_i does not coincide with any edge in $\mathcal{T}_i'$, then the number of intersections of this edge with all edges in $\mathcal{T}_i'$ is smaller than C_0. We will conclude from this the lemma above.

Let $\{l_{P_i}(e)\,|\,e \text{ edge of } P_i, i \geq 1\}$ be the set of edge lengths in $\{P_i\}_{i\geq 1}$ and let $\cup_{i\geq 1} \cup_{ABC\in\mathcal{T}_i'} \{h_A, h_B, h_C\}$ be the set of altitudes of triangles of $\{\mathcal{T}_i'\}_{i\geq 1}$ in $\{\rho_i\}_{i\geq 1}$. Let $L := \sup\{l_{P_i}(e)\,|\,e \text{ edge of } P_i, i \geq 1\}$ and $h := \inf \cup_{i\geq 1} \cup_{ABC\in\mathcal{T}_i'} \{h_A, h_B, h_C\}$. Notice that $h > 0$ since $\{\rho_i\}_{i\geq 1}$ converges. Let $m \in \mathbb{Z}^{>0}$ be the maximum of the degree of vertices in $\mathcal{T}_i'$, this maximum exists since all triangulations in $\{\mathcal{T}_i'\}_{i\geq 1}$ have the same combinatorics.

Assume there exists an edge e in a polyhedron $P_k \in \{P_i\}_{i\geq 1}$ such that the number of intersections with the edges of $\mathcal{T}_k'$ is

$$C_0 > \frac{2Lm}{h} + m.$$

The edge e of the polyhedron is partitioned by the C_0 intersecting points into $C_0 + 1$ segments. We will show that among these segments there are at least m successive segments, each of length less than $h/2$.

If that were not the case, then in every collection of m successive segments there would exist a segment of length greater than or equal to $h/2$. The number of collections of m successive segments among all $C_0 + 1$ segments is equal to $[\frac{C_0+1}{m}]$, i.e. the integer part of the fraction $\frac{C_0+1}{m}$. Hence, the length of the edge e would be greater than or equal to $[\frac{C_0+1}{m}]\,\frac{h}{2}$. This is impossible, since

$$\left[\frac{C_0+1}{m}\right]\frac{h}{2} \geq \left[\frac{C_0}{m}\right]\frac{h}{2} \geq \left[\frac{2L}{h}+1\right]\frac{h}{2} > L.$$

Set

$$C_0 := \frac{2Lm}{h} + m.$$

Let EF be the first segment of m successive segments of length less than $h/2$, cut out of e by a triangle ABC in $\mathcal{T}_k'$. Drop a perpendicular FG from F to AB and a perpendicular CH from C to AB. The triangles AGF and ACH are similar. Since $FG \leq EF < h/2 \leq \frac{1}{2}CH$, we have $FA < \frac{1}{2}AC$. Hence, point F is closer to A than to C. A similar argument holds for point E.

Leaving the triangle ABC the edge e of P_k enters a neighboring triangle ACD of $\mathcal{T}_k'$ where it again has a segment of length less than $h/2$. Hence, in this triangle also the segment passes closer to point A than to the other points. Since we have m such segments, while at most m triangles touch at A, it follows that the edge e of P_k goes around A and returns to the side AB (even intersects it again). This is impossible, since A is a vertex of P_k and therefore must be incident with an edge joining A with another vertex of P_k. According to the above, this edge must then intersect e, which is impossible, since edges meet only at vertices.

We have thus proved that the number of intersections of an arbitrary edge of any polyhedron P_k in $\{P_i\}_{i\geq 1}$ with the edges of the triangulation $\mathcal{T}_k'$

does not exceed C_0. There exists a integer C_1 such that the total number of edges in an arbitrary polyhedron P_k in $\{P_i\}_{i\geq 1}$ does not exceed C_1. The total number of intersecting points in a polyhedron P_k in $\{P_i\}_{i\geq 1}$ is therefore at most C_0C_1. Hence, for each edge in $\mathcal{T}'_k$, there are at most C_0C_1 intersections with edges of P_k. Set $C = C_0C_1$.

We say that a marked polyhedral surface (S_μ, ρ) is *realizable* if there exists a polyhedron P whose boundary surface (S_μ, ρ_P) is isometric to (S_μ, ρ).

Theorem: *Every connected component of $T_{PL}^{\mathrm{con}}(N)$ contains a realizable polyhedral surface.*

Proof: Notice that N is at least 3, since a polyhedral surface homeomorphic to the sphere has total curvature equal to 4π, while the curvature at each vertex is less than 2π. Given a polyhedral surface in $T_{PL}^{\mathrm{con}}(3)$, let A, B, C be the vertices of the polyhedral surface and connect them by shortest arcs. This splits the topological sphere into two triangles, each of which contains no interior point with cone angle other than 2π. Therefore, the triangles can be developed on the plane (a detailed exposition of this fact is given by Alexandrov, p. 79). Superposing them so that the corresponding sides coincide, we obtain a doubly-covered triangle that realizes the polyhedral surface in $T_{PL}^{\mathrm{con}}(3)$. Hence, every polyhedral surface in $T_{PL}^{\mathrm{con}}(3)$ is realizable as a doubly-covered triangle. We will show that if every polyhedral surface with less than N vertices is realizable, then also every polyhedral surface in $T_{PL}^{\mathrm{con}}(N)$ is realizable.

The manifold $T_{PL}^{\mathrm{con}}(N)$ is an open submanifold of the manifold of polyhedral surfaces $T_{PL}(N)$. The topological boundary of $T_{PL}^{\mathrm{con}}(N)$ in $T_{PL}(N)$ consists of polyhedral surfaces with cone angles less than or equal to 2π and some cone angle equal to 2π. Every polyhedral surface in $\partial T_{PL}^{\mathrm{con}}(N)$ is isometric to a polyhedral surface in $T_{PL}^{\mathrm{con}}(N')$ for $N' < N$, which is realizable by the induction hypothesis. Hence, $\partial T_{PL}^{\mathrm{con}}(N)$ is realizable.

Let $C \subset T_{PL}^{\mathrm{con}}(N)$ be a connected component. We will see in the lemma below that ∂C contains a point that admits a neighborhood devoid of any points of any other connected component of $T_{PL}^{\mathrm{con}}(N)$. Let (S_μ, ρ) be such a polyhedral surface in ∂C.

Since (S_μ, ρ) is realizable, there exists a convex polyhedron P such that $\mathfrak{g}(P) = \rho$. However, not all marked points on S_μ are vertices of P, since the cone angle at some of the marked points equals 2π. Let $A_1, \ldots, A_l$ be points on P that correspond to the remaining marked points on S_μ. These points either lie in the interior of a face of P or on an edge of P. Let $A_{l+1}, \ldots, A_N$ be the vertices of P. Move the points $A_1, \ldots, A_l$ away from the polyhedron P for a sufficiently small distance. Let Q be the convex hull of the moved points $A_1, \ldots, A_l$ and $A_{l+1}, \ldots, A_N$. Now Q is a convex polyhedron with N vertices close to the points $A_1, \ldots, A_l, A_{l+1}, \ldots, A_N$ on the boundary of P. In particular, the complete angles at the points $A_1, \ldots, A_l, A_{l+1}, \ldots, A_N$ of (S_μ, ρ_Q) are less than 2π and $\mathfrak{g}(Q)$ belongs to $T_{PL}^{\mathrm{con}}(N)$. Since $\mathfrak{g}(Q)$ is close to $\mathfrak{g}(P)$, and close to $\mathfrak{g}(P)$, there are no elements of $T_{PL}^{\mathrm{con}}(N)$ except for those in C, $\mathfrak{g}(Q)$ belongs to C. Thus, each connected component of $T_{PL}^{\mathrm{con}}(N)$ contains a realizable polyhedral surface.

Lemma: *For $N > 3$, the boundary of every connected component of*

$T_{PL}^{\text{con}}(N)$ contains a point that admits a neighborhood devoid of any points of any other connected component of $T_{PL}^{\text{con}}(N)$.

Proof: Recall that given a coordinate chart $\vartheta_{\mathcal{T}}$ on $T_{PL}(N)$, the submanifold $T_{PL}^{\text{con}}(N)$ is determined by the inequalities

$$\sum_j \alpha_{ij} < 2\pi \qquad (i = 1, 2, \ldots, N),$$

with α_{ij} denoting the j-th angle at the i-th vertex. Hence, the topological boundary of $T_{PL}^{\text{con}}(N)$ in $T_{PL}(N)$ is composed of pieces of N surfaces $F_1, \ldots, F_N$ determined by the equations

$$\sum_j \alpha_{ij} = 2\pi \qquad (i = 1, 2, \ldots, N).$$

Let $C \subset T_{PL}^{\text{con}}(N)$ be a connected component of $T_{PL}^{\text{con}}(N)$. Let $\rho \in \partial C$ be a point on the boundary of C that belongs to the least number of surfaces $F_1, \ldots, F_N$, say $F_1, \ldots, F_l$. The corresponding polyhedral surface (S_μ, ρ) has cone angles equal to 2π at the marked points $A_1, \ldots, A_l$ and cone angles less than 2π at the marked points $A_{l+1}, \ldots, A_N$.

Let $\vartheta_{\mathcal{T}}$ be a coordinate chart of $T_{PL}(N)$ containing ρ. There exists an $\epsilon > 0$ such such that all polyhedral surfaces in the ϵ-ball $B_\epsilon(\vartheta_{\mathcal{T}}(\rho))$ have cone angles less than 2π at $A_{l+1}, \ldots, A_N$. In other words, $\vartheta_{\mathcal{T}}^{-1}(B_\epsilon(\vartheta_{\mathcal{T}}(\rho)))$ contains no points of any surface F_i other than $F_1, \ldots, F_l$.

Consider the triangles in $\mathcal{T}$ containing the vertex A_1. Let e be an edge of one of them subtended by A_1. Notice that the cone angle at A_1 is an increasing function in the length $\vartheta_{\mathcal{T}}(\rho)_e$ of e. Label the edges in $\mathcal{T}$ such that e is the first edge. Let $\delta_1, \delta_2 > 0$ be small enough such that the disks

$$\{\vartheta_{\mathcal{T}}(\rho)_e - \delta_1\} \times B_{\delta_2}((\vartheta_{\mathcal{T}}(\rho)_2, \ldots, \vartheta_{\mathcal{T}}(\rho)_E))$$

and

$$\{\vartheta_{\mathcal{T}}(\rho)_e + \delta_1\} \times B_{\delta_2}((\vartheta_{\mathcal{T}}(\rho)_2, \ldots, \vartheta_{\mathcal{T}}(\rho)_E))$$

are contained in $B_\epsilon(\vartheta_{\mathcal{T}}(\rho))$, and all polyhedral surfaces in the first disk have cone angles smaller than 2π at A_1, and all polyhedral surfaces in the second disk have cone angles greater than 2π at A_1.

Let ρ_1 be a point in the first disk and ρ_2 its translation along the e axis in the second disk. Since the cone angle is increasing monotonically along the e axis when moving ρ_1 to ρ_2, there exists a unique point where the cone angle equals 2π. Hence, the face F_1 divides the cylinder

$$\vartheta_{\mathcal{T}}^{-1}(\,[\vartheta_{\mathcal{T}}(\rho)_e - \delta_1, \vartheta_{\mathcal{T}}(\rho)_e + \delta_1] \times B_{\delta_2}((\vartheta_{\mathcal{T}}(\rho)_2, \ldots, \vartheta_{\mathcal{T}}(\rho)_E))\,)$$

into two pieces V_1 and V_2, where all polyhedral surfaces in V_2 have cone angles greater than 2π at A_1 and all polyhedral surfaces in V_1 have cone angles less than 2π at A_1.

The polyhedral surface ρ lies on the boundary ∂C of the connected component C of $T_{PL}^{\text{con}}(N)$. Clearly, $C \cap V_1$ is non-empty. We will show that in fact $V_1 \subset C$.

Assume the contrary. Let ρ_1 be a polyhedral surface in V_1 not belonging to C, and let ρ_2 be a polyhedral surface in V_1 that belongs to C. ρ_1 and ρ_2 project to polyhedral surfaces ρ_1' and ρ_2' in the disk

$$\{\vartheta_{\mathcal{T}}(\rho)_e - \delta_1\} \times B_{\delta_2}((\vartheta_{\mathcal{T}}(\rho)_2, \ldots, \vartheta_{\mathcal{T}}(\rho)_E)).$$

We obtain a continuous polygonal line $L = \vartheta_{\mathcal{T}}(\rho_1)\vartheta_{\mathcal{T}}(\rho_1') \cup \vartheta_{\mathcal{T}}(\rho_1')\vartheta_{\mathcal{T}}(\rho_2') \cup \vartheta_{\mathcal{T}}(\rho_2')\vartheta_{\mathcal{T}}(\rho_2)$ in V_1 that joins ρ_1 to ρ_2. Since ρ_2 belongs to C and ρ_1 does not, this line must intersect the boundary ∂C.

Being a boundary point of $T_{PL}^{\mathrm{con}}(N)$, the intersecting point of the line L with ∂C must lie on some surface F_i. Since line L stays in V_1, it cannot lie on the face F_1. Line L also lies in the ball $B_\epsilon(\vartheta_{\mathcal{T}}(\rho))$ and no point in $B_\epsilon(\vartheta_{\mathcal{T}}(\rho))$ lies on a surface other than $F_1, \ldots, F_l$. Hence, the intersecting point of L with ∂C lies on at most $l-1$ surfaces. However, ρ was a point on ∂C that intersects the minimal number of surfaces F_i, which was l. We come to a contradiction, which shows that V_1 is entirely contained in the connected component C of $T_{PL}^{\mathrm{con}}(N)$.

Since V_2 does not contain any point of $T_{PL}^{\mathrm{con}}(N)$ at all, the cylinder

$$\vartheta_{\mathcal{T}}^{-1}(\,[\vartheta_{\mathcal{T}}(\rho)_e - \delta_1, \vartheta_{\mathcal{T}}(\rho)_e + \delta_1] \times B_{\delta_2}((\vartheta_{\mathcal{T}}(\rho)_2, \ldots, \vartheta_{\mathcal{T}}(\rho)_E))\,)$$

is the required neighborhood of ρ.

RIVIN'S THEORY ON IDEAL CONVEX POLYHEDRA

Isometric embedding of ideal polyhedral surfaces. Does every ideal polyhedral surface arise from the boundary of an ideal hyperbolic polyhedron? Rivin gave the following answer [21]:

Theorem: (Rivin) *Let $(S_N, \hat\rho)$ be an ideal polyhedral surface. Then $(S_N, \hat\rho)$ can be isometrically embedded in $\mathbb{H}^3$ as the boundary of a convex polyhedron P with all vertices on the sphere at infinity.*

The proof needs some specific techniques related to the fact that we are dealing with geodesics between ideal points. Nevertheless, the proof follows essentially the same philosophy as Alexandrov's.

Proof: (Outline) Let $\mathcal{P}_{\mathrm{ideal}}^N$ be the space of convex ideal polyhedra with N vertices in $\mathbb{H}^3$, this space is parametrized by the positions of their vertices on the sphere at infinity. Three of their vertices are fixed at 0, 1, and ∞. This eliminates the action of the isometry group of $\mathbb{H}^3$. There are $2N$ variable coordinates, however three vertices are fixed. Therefore, we have $2N-6$ variable coordinates and $\mathcal{P}_{\mathrm{ideal}}^N$ is a $2N-6$ dimensional manifold.

Let S_μ be a surface homeomorphic to the N-times punctured sphere, together with a marking μ, that is, a labelling of the punctures.

Let $T(N)$ be the set of hyperbolic metrics $\hat\rho$ on the surface S_μ, such that the hyperbolic surface $(S_\mu, \hat\rho)$ is complete and of finite volume. Two hyperbolic metric metrics $\hat\rho_1$ and $\hat\rho_2$ in $T(N)$ are identified if $(S_\mu, \hat\rho_1)$ and $(S_\mu, \hat\rho_2)$ are isometric by an isometry homotopic to the identity.

The set $T(N)$ is parametrized by *shears* along the edges of a geodesic triangulation of S_μ. This notion measures the shift between two abutting ideal triangles and will be introduced below. We will see that this parametrization turns $T(N)$ into a $2N-6$ dimensional connected manifold.

The boundary of every convex ideal polyhedron with N vertices can be viewed as a complete hyperbolic surface of finite area, homeomorphic to the N times punctured sphere. Formally this gives a map $\mathfrak{h} : \mathcal{P}^N_{\text{ideal}} \to T(N)$. In the following, we show that $\mathfrak{h}$ is a (1) continuous, (2) injective and (3) closed map.

$\mathcal{P}^N_{\text{ideal}}$ and $T(N)$ are manifolds of equal dimension, by (1) and (2) and the invariance of domain principle of Brouwer, $\mathfrak{h}$ is an open map. Since $\mathfrak{h}$ is also closed, we conclude, together with the fact that $T(N)$ is connected, that $\mathfrak{h}$ is a homeomorphism from $\mathcal{P}^N_{\text{ideal}}$ onto $T(N)$.

We will need some more notions of hyperbolic geometry to make the above precise.

Let ABC be an ideal triangle in $\mathbb{H}^2$. Let h_A be a horocycle centered at A, define $\mathcal{D}_{ABC}(h_A)$ to be the length of the arc of h_A cut out by the triangle ABC. The difference in size between arcs of two horocycles h_A and h'_A cut out by ABC gives information on the distance between the arcs. More precisely:

Lemma: Let h_A and h'_A be two horocycles at A. The hyperbolic distance between h_A and h'_A is equal to $|\log(\mathcal{D}_{ABC}(h_A))/\log(\mathcal{D}_{ABC}(h'_A))|$.

Proof: Let ABC be the triangle $A=\infty$, $B=0$ and $C=1$ in the upper half-space model. The horocycles h_A and h'_A are horizontal lines through i/y and i/y', respectively. Hence, the length of the arcs of h_A and h'_A cut out by ABC is $1/y$ and $1/y'$ respectively and the distance between h_A and h'_A is $|\log(y/y')|$.

Two ideal triangles ABC and ADC can slide with respect to each other along the common side AC. For any choice of horocycle h_A, the number $\mathfrak{s}_{AC} := \log(\mathcal{D}_{ABC}(h_A)/\mathcal{D}_{ADC}(h_A))$ measures the *shear* between the triangles ABC and ADC along AC. The shear $\mathfrak{s}_{AC}$ does not depend on which of the vertices A or C is taken as the center of the horocycles.

Intuitively, two triangles ABC and ADC are joined along AC without a shear, if for any horocycle at A the arcs cut out by ABC and ADC have the same "distance" to A.

The *cross-ratio* of four points z_1, z_2, z_3, z_4 in the complex plane is the number

$$[z_1, z_2, z_3, z_4] := \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_2)(z_3-z_4)}.$$

The notion of cross-ratio of four points and shear between two triangles are related.

Lemma: The shear between two triangles ABC and ABD is equal to the log of the absolute value of the cross-ratio $[C, B, D, A]$.

Proof: Let ABC be the triangle $A=\infty$, $B=1$ and $C=0$. In this case, the shear between ABC and ABD is $\log|D|$.

A *triangulation* of an ideal polyhedral surface is a triangulation whose vertices are at the cusps of the hyperbolic surface. A *geometric triangulation* of an ideal polyhedral surface is a triangulation of the ideal polyhedral surface whose edges are geodesics. Let $\mathcal{T}$ be a geometric triangulation of an ideal polyhedral surface $(S_\mu, \hat{\rho})$ with N cusps. To each edge of $\mathcal{T}$, associate the shear of the two abutting triangles of $\mathcal{T}$. This information determines the geometry of $(S_\mu, \hat{\rho})$ completely. Conversely, an assignment of real numbers to the edges of $\mathcal{T}$ specifies a complete hyperbolic structure on S_μ if and only if the shears around any cusp add up to zero. Hence the set $T(N)$ is naturally parametrized by $\mathbb{R}^{|E(\mathcal{T})|-N}$. According to the Euler formula, $|E(\mathcal{T})| - N = 2N - 6$, so the dimension of this space depends only on the number of cusps.

Lemma: *Any triangulation of a complete hyperbolic surface with cusps can be straightened to a geodesic triangulation.*

Proof: We need to show that, if A, B, C and D are cusps of a complete hyperbolic surface such that A and B are connected by a path γ_1 in $\mathcal{T}$ and C and D are connected by a path γ_2 in $\mathcal{T}$, then the corresponding geodesics also do not intersect.

The path γ_1 and γ_2 do not intersect in S_N if and only if their lifts to the universal cover $\mathbb{H}^2$ of S_N do not intersect. However, if two paths between the ideal boundary of $\mathbb{H}^2$ do not intersect, then the corresponding minimizing geodesics do not intersect either.

The *shear coordinate system* on $T(N)$ corresponding to the triangulation $\mathcal{T}$ of $(S_\mu, \hat{\rho})$ is the map $\varphi_{\mathcal{T}} : T(N) \to \mathbb{R}^{2N-6}$, associating a particular metric with its shear along the straightened edges of $\mathcal{T}$.

Notice that $\varphi_{\mathcal{T}} : T(N) \to \mathbb{R}^{2N-6}$ is a homeomorphism. Given a point x in $\mathbb{R}^{2N-6}$, we can compute the remaining N shears from the condition that shears must add up to zero around vertices. Hence, $T(N)$ is connected.

Theorem: *The set of shear coordinate charts*

$$\{(T(N), \varphi_{\mathcal{T}}) \,|\, \mathcal{T} \text{ triangulation of } S_\mu\}$$

forms a real analytic atlas on $T(N)$. This turns $T(N)$ into a $2N-6$ dimensional, connected, real analytic manifold.

Proof: Let $\mathcal{T}$ and $\mathcal{T}'$ be two triangulations of S_μ related by flipping an edge e. Let ABC and ADC be the triangles in $\mathcal{T}$ adjacent to e. Flipping the diagonal corresponds to permuting the arguments of the cross-ratio. A permutation of the cross-ratio is a fractional linear transformation of the cross-ratio itself. Hence the transition function $\varphi_{\mathcal{T}} \varphi_{\mathcal{T}'}^{-1}$ is real analytic.

Two arbitrary triangulations of S_μ can be obtained from each other by a sequence of edge flips.

Hence, the set of shear coordinate charts forms a real analytic atlas.

We already introduced the map $\mathfrak{h} : \mathcal{P}^N_{\text{ideal}} \to T(N)$ that formalizes the operation of viewing the boundary of a convex ideal polyhedron with N vertices as a complete hyperbolic surface up to isometry homotopic to the identity. In the following, we show that $\mathfrak{h}$ is a (1) continuous, (2) injective and (3) closed map.

For the next theorems we will view the ideal hyperbolic polyhedron P in the upper half-space model. P is the intersection of the half-spaces defined by its faces. We think of the ideal boundary as the Riemann sphere $\bar{\mathbb{C}}$. Through an isometry of $\mathbb{H}^3$ and re-labelling, we can transform P in such a way that the face f_1 rises above the real axis and the vertices v_1, v_2 and v_3 are at $0, 1$ and ∞, respectively. Furthermore, we assume that the polyhedron P lies above the half-plane $\mathrm{Im}(z) \geq 0$.

The polyhedron P casts a shadow on the ideal boundary of $\mathbb{H}^3$ under the orthogonal projection. The edges of P are mapped to straight line segments and the faces of P to convex polygons in $\mathbb{C}$.

This gives a tessalation in $\mathbb{C}$ of the set of vertices $\{v_1, v_2, \ldots, v_N\} \backslash \{v_3\}$ of P in $\mathbb{C}$. In fact, every triangulation of this tessalation, is a Delaunay triangulation. A *Delaunay triangulation* of a finite set of points V in $\mathbb{C}$ is a triangulation of the convex hull of V into triangles such that no point in V is inside the circumcircle of any other triangle. We call the tessalation in $\mathbb{C}$ induced from P, the *Delaunay tessalation induced by* P. Every triangulation of the Delaunay tessalation induced by P corresponds to a triangulation of the boundary of P, we call such a triangulation a *Delaunay triangulation of the convex ideal polyhedron* P.

Let $\mathcal{T}$ be a Delaunay triangulation of P. Let ABC and ADC be two abutting triangles in $\mathcal{T}$. If A, B, C and D are transformed by way of hyperbolic isometry in such a way that A is mapped to ∞, B to 1, C to 0 and D to $z \in \mathbb{C}$, then the cross-ratio

$$\frac{B-A}{B-C} : \frac{D-A}{D-C}$$

equals z. Hence, the following theorem follows from the above discussion of shears in the Poincaré model.

Theorem: *The shear between two abutting triangles ABC and ADC in $\mathcal{T}$ is given by the logarithm of the absolute value of the complex cross-ratio*

$$\frac{B-A}{B-C} : \frac{D-A}{D-C}.$$

Using the continuity of cross-ratios, we may now prove that the map g is continuous.

Theorem: *The map $\mathfrak{h}$ is continuous.*

Proof: If the Delaunay tessalation of an ideal hyperbolic polyhedron P is already a triangulation, then any small pertubation of the vertices of P does not change the combinatorics of the Delaunay tessalation. In this case, the continuity of $\mathfrak{h}$ follows directly from the continuity of cross ratios.

If the Delaunay tessalation of P is not a triangulation, then any small pertubation of the vertices of P does change the combinatorics of the Delaunay tessalation in $\mathbb{C}$. However, in both conditions there exists a uniform $\epsilon > 0$ such that, if a point D is closer than ϵ to the circumcircle of a triangle ABC, then A, B, C and D are co-circular.

In other words, for a sufficiently small ϵ, the Delaunay tessalation of the perturbed polyhedron P^ϵ is combinatorially equivalent to the Delaunay tessalation of P with some diagonals added to the non-triangular faces.

This shows that, for a sufficiently small pertubation of P, there always exists a coordinate system in which $\mathfrak{h}$ is clearly continuous. Since all transition maps are continuous, we conclude that $\mathfrak{h}$ is continuous.

Theorem: *The map $\mathfrak{h}$ is injective.*

Proof: If $\mathfrak{h}(P)$ equals $\mathfrak{h}(Q)$, then P and Q are isometric according to the Cauchy Rivin rigidity theorem. Hence P and Q agree in $\mathcal{P}^N_{\text{ideal}}$.

Theorem: *The map $\mathfrak{h}$ is closed.*

Proof: Let $\hat{\rho}_{P_1}, \ldots, \hat{\rho}_{P_i}, \ldots$ be a sequence of hyperbolic metrics on a surface S_μ homeomorphic to the N-times punctured sphere and converging to a metric $\hat{\rho}$ such that $\mathfrak{h}(P_i) = \hat{\rho}_{P_i}$.

Choose a subsequence $\{P_{i_k}\}$ such that all polyhedra have the same combinatorics. The vertices and faces of the polyhedra are labelled such that $v_1(P_{i_k}) = 0$, $v_2(P_{i_k}) = 1$, $v_3(P_{i_k}) = \infty$ and $f_1(P_{i_k})$ rises above the real axis. Since the Riemann sphere $\bar{\mathbb{C}}$ is compact, there exists a limiting tessalation $\mathcal{T}$. If $\mathcal{T}$ is non-degenerate, then by the continuity of $\mathfrak{h}$ it follows that there exists P_∞ such that $\mathfrak{h}(P_\infty) = \hat{\rho}$.

We will show that $\mathcal{T}$ is always non-degenerate. Suppose it is not, then there exist two abutting triangles ABC and ADC in $\mathcal{T}$ such that ADC is collapsed and ABC is not. Such two triangles exist, since the face f_1 is not collapsing. Map A to ∞, B to 1 and C to 0. Then the shear between the two triangles equals $\log |D|$. Since $\hat{\rho}$ is a non-degenerate metric, $|D|$ must stay away from 0 and ∞. But this means that ADC is non-degenerate after all.

A CONFORMAL EQUIVALENCE RELATION FOR CONVEX POLYHEDRA

Discrete conformality of convex polyhedra. A *Delaunay triangulation of a development* R is a Delaunay triangulation of every polygon in R. The following lemma is a classical property of Delaunay triangulations in the plane. A proof can be found in Aurenhammer's book on Voronoi diagrams [24].

Lemma: *If a finite set of points in the plane admits two Delaunay triangulations, then there exists a sequence of Delaunay triangulations between them, such that each is related to the next by a diagonal switch.*

Hence, if a Euclidean development R admits two distinct Delaunay triangulations, then they differ by a finite number of diagonal switches between two abbuting triangles within a polygon in R that share the same circumcircle.

Every Euclidean development R has a unique set of circumcircles attached to its vertices, by taking the circumcircles of a Delaunay triangulation of R. A Euclidean triangle with its circumcircle can be viewed as an ideal hyperbolic triangle in the Klein model. This construction does not depend on the chosen Delaunay triangulation and associates with every Euclidean development R with N vertices a marked ideal polyhedral surface $(S_\mu, \hat{\rho}_R)$ with a cusp for each vertex of the development. Indeed, by the following theorem the associated hyperbolic surface with cusps is complete, since the shear coordinates add up to zero around vertices.

Theorem: *Let R be a Euclidean development with N vertices and $\mathcal{T}$ a Delaunay triangulation of R. Let ijk and ilj be two triangles in $\mathcal{T}$ abutting along the edge ij. The hyperbolic structure $\hat{\rho}_R$ on S_μ is the unique complete hyperbolic structure on S_μ with shear*

$$\log \frac{\vartheta_{\mathcal{T}}(\rho_R)_{il}}{\vartheta_{\mathcal{T}}(\rho_R)_{ik}} : \frac{\vartheta_{\mathcal{T}}(\rho_R)_{jl}}{\vartheta_{\mathcal{T}}(\rho_R)_{jk}}$$

along the edge ij of $\mathcal{T}$.

Proof: The map associating every Euclidean development R with a hyperbolic surface with cusps $(S_\mu, \hat{\rho}_R)$, can be described in upper half-space model as follows. Consider $\mathbb{C}$ as the sphere at infinity of the hyperbolic 3-space $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_{>0}$. Let ijk and ijl be two abutting triangles in R. Embed $ijk \cup ijl$ into the sphere at infinity by an isometry f. The hyperbolic metric $\hat{\rho}_R$ on $ijk \cup ijl$ is the hyperbolic metric of the ideal hyperbolic triangles in $\mathbb{H}^3$, having the same vertices as ijk and ijl, glued by the same isometry f, considered as a hyperbolic motion of $\mathbb{H}^3$.

The shear of $(S_\mu, \hat{\rho}_R)$ along the edge ij is the logarithm of the absolute value of the complex cross-ratio of the four vertices z_i, z_j, z_k and z_l of the triangles ijk and ijl in $\mathbb{C}$. Clearly,

$$\log \left| \frac{z_i - z_l}{z_i - z_k} : \frac{z_j - z_l}{z_j - z_k} \right| = \log \frac{\vartheta_{\mathcal{T}}(\rho_R)_{il}}{\vartheta_{\mathcal{T}}(\rho_R)_{ik}} : \frac{\vartheta_{\mathcal{T}}(\rho_R)_{jl}}{\vartheta_{\mathcal{T}}(\rho_R)_{jk}}.$$

A *Delaunay triangulation of a convex polyhedron* P is a triangulation of its boundary coming from a Delaunay triangulation of its face development R_P.

Note: A Delaunay triangulation of an ideal convex polyhedron P in $\mathbb{H}^3$, is a Delaunay triangulation of a convex polyhedron if P is viewed as a Euclidean convex polyhedron inscribed in the sphere.

Given a convex Euclidean polyhedron P. We associate with P the marked ideal polyhedral surface $(S_\mu, \hat{\rho}_{R_P})$ coming from the face development of P. In the following, we denote $(S_\mu, \hat{\rho}_{R_P})$ by $(S_\mu, \hat{\rho}_P)$. Formally we obtain a function $\mathfrak{f} : \mathcal{P}^N \to T(N)$ mapping P to $(S_\mu, \hat{\rho}_P)$.

Definition: Two closed convex polyhedra P and Q with N vertices are *discrete-conformally equivalent* if and only if $(S_\mu, \hat{\rho}_P)$ and $(S_\mu, \hat{\rho}_Q)$ are isometric.

Theorem: *Let P and Q be two convex polyhedra inscribed in the unit sphere that are discrete-conformally equivalent. Then there exists a Möbius transformation on the sphere that maps the vertex set of P to the vertex set of Q.*

Proof: If P is inscribed in the unit sphere, then the association $P \to (S_\mu, \hat{\rho}_P)$ defined above is nothing but interpreting P as a convex ideal polyhedron in the Klein model and moving to the boundary. Hence, if P and Q are discrete-conformally equivalent, there exists a hyperbolic isometry from P onto Q. According to the rigidity theory of Cauchy, Alexandrov and Rivin, this isometry can be realized as a motion or a motion and a reflection in $\mathbb{H}^3$. Equally, there exists a Möbius transformation on the sphere mapping the vertex set of P to the vertex set of Q.

The above rigidity theorem allows us to classify Euclidean polyhedra up to discrete conformality.

Theorem: (Uniformization) *Every closed convex polyhedron in $\mathbb{E}^3$ is discrete-conformally equivalent to a closed convex polyhedron inscribed in the unit sphere. This polyhedron is unique up to Möbius transformations on the sphere.*

The uniquness part was proven above. The existence follows from Rivin's isometric embedding of ideal polyhedra in hyperbolic 3-space.

Theorem: *Given a convex polyhedron P in $\mathbb{E}^3$, there exists a convex polyhedron Q inscribed in the unit sphere that is discrete-conformally equivalent to P.*

Proof: Let $(S_\mu, \hat{\rho}_P)$ be the marked ideal polyhedral surface associated with P. According to Rivin's isometric embedding theorem, $(S_\mu, \hat{\rho}_P)$ can be isometrically embedded in $\mathbb{H}^3$ as the boundary of a convex hyperbolic polyhedron Q with all vertices on the sphere at infinity. The polyhedron Q may be interpreted as a convex Euclidean polyhedron inscribed in the sphere if viewed in the Klein model. This interpretation is just the inverse of the map $Q \to (S_\mu, \hat{\rho}_Q)$. Hence, $(S_\mu, \hat{\rho}_P)$ is isometric to $(S_\mu, \hat{\rho}_Q)$ and P and Q are discrete-conformally equivalent.

Characterization of discrete conformality. The notion of discrete conformality passes through hyperbolic geometry. In the following we characterize discrete conformality of Euclidean polyhedra that share a common Delaunay triangulation by elementary transformations on vertices.

Theorem: *Let P and Q be two polyhedra that share a common Delaunay triangulation $\mathcal{T}$, then P and Q are discrete-conformally equivalent if and only if there exists a real valued function $u_\mathcal{T}$ on the vertices of P so that if, e is an edge in $\mathcal{T}$ between the vertices i and j, then the length $l_P(e)$ and $l_Q(e)$ of e in P and Q are related by*

$$l_Q(e) = l_P(e)\, e^{\frac{1}{2}(u_\mathcal{T}(i)+u_\mathcal{T}(j))}.$$

We will first give an alternative description of the function $\mathfrak{f} : \mathcal{P}^N \to T(N)$ using Penner's theory on decorated Teichmüller spaces [25]. To shorten notation, we call $u_\mathcal{T}$ a *conformal factor* and write $u_\mathcal{T} * P = Q$ if P and Q are related by the conformal factor $u_\mathcal{T}$ as above.

A *decorated ideal triangle* is an ideal triangle ABC together with a choice of horocycles h_A, h_B and h_C. The *Penner distance* between two distinct horocycles h_A and h_B is

$$l^P_{AB} := l^P(h_A, h_B) := e^{\lambda_{AB}/2},$$

where $\lambda_{AB} := \lambda(h_A, h_B)$ is the signed distance between two distinct horocycles.

Two decorated ideal triangles (ABC, h_A, h_B, h_C) and (ADC, h_A, h_D, h_C) can be glued along the edge AC by an isometry preserving the horocycles h_A and h_C.

Recall that S_μ is a surface homeomorphic to the 2-sphere with N punctures, together with its marking. A *decorated hyperbolic metric* on S_μ is a complete finite area hyperbolic metric $\hat{\rho}$ on S_μ, together with horoballs h_i at every cusp i. Two decorated hyperbolic metrics on S_μ are equivalent if there exists an isometry homotopic to the identity between them that preserves the horoballs.

Let $T_D(N)$ be the set of equivalence classes of decorated hyperbolic metrics on S_μ. Let $T(N)$ be the set of complete, finite volume hyperbolic structures on S_μ as introduced above. The mapping

$$\begin{aligned} T_D(N) &\to T(N) \times \mathbb{R}^N_{>0} \\ (\hat{\rho}, \{h_i\}_{i=1}^N) &\mapsto (\hat{\rho}, (w_1, \dots, w_N)) \end{aligned}$$

is a bijection, where w_i is the sum of the lengths $\mathcal{D}_{ijk}(h_i)$ of horoarcs cut out by the ideal triangles at i.

Let $\mathcal{T}$ be a triangulation of S_μ and $E(\mathcal{T})$ the set of edges associated to $\mathcal{T}$. Let $E(\mathcal{T})^*$ be the set of positive real valued functions on $E(\mathcal{T})$. For $x \in E(\mathcal{T})^*$ let $(\hat{\rho}, w)$ be a decorated hyperbolic metric on S_μ constructed by isometrically gluing decorated hyperbolic triangles $ijk \in \mathcal{T}$ with Penner distances $l^P_{ij} = x(ij)$, $l^P_{jk} = x(jk)$ and $l^P_{ik} = x(ik)$ along corresponding edges. This provides an injective map

$$\iota : E(\mathcal{T})^* \to T_D(N)$$

mapping x to $(\hat{\rho}, w)$.

Let $U_\mathcal{T}$ be the image of ι in $T_D(N)$ and $\varphi_\mathcal{T}$ the inverse of ι on $U_\mathcal{T}$. The pair $(U_\mathcal{T}, \varphi_\mathcal{T})$ is a *Penner coordinate chart* on $T_D(N)$.

Theorem: *The set of Penner coordinate charts*

$$\{(U_\mathcal{T}, \varphi_\mathcal{T}) \,|\, \mathcal{T} \textit{ triangulation of } S_\mu\}$$

forms a real analytic atlas on $T_D(N)$. This turns $T_D(N)$ into an $|E|$ dimensional, real analytic manifold.

Proof: For any two triangulations of S_μ there exists a sequence of triangulations of S_μ between them, such that each is related to the next by a diagonal switch along an edge. Hence, it is enough to show that the transition function $\varphi_\mathcal{T}\varphi^{-1}_{\mathcal{T}'}$ is real analytic if $\mathcal{T}$ and $\mathcal{T}'$ are related by a diagonal switch.

In this case,

$$\varphi_\mathcal{T}\varphi^{-1}_{\mathcal{T}'}(x_0, x_1, \dots, x_n) = (\frac{x_1x_3 + x_2x_4}{x_0}, x_1, x_2, \dots, x_n),$$

which is real analytic.

Let $\mathfrak{p} : T_D(N) \to T(N)$ be the projection, mapping $(\hat{\rho}, \{h_i\}_{i=1}^N)$ to $\hat{\rho}$, and let $\mathfrak{g} : \mathcal{P}^N \to T^{\mathrm{con}}_{PL}(N)$ be Alexandrov's homeomorphism. We aim for an alternative description of $\mathfrak{f}$ by constructing a function $F : T^{\text{con}}_{PL}(N) \to T_D(N)$ such that $\mathfrak{p} \circ F \circ \mathfrak{g} = \mathfrak{f}$. As usual we denote by ρ_P the image of the polyhedron P under $\mathfrak{g}$ in $T^{\text{con}}_{PL}(N)$. Let $D_{PL}(\mathcal{T})$ be the set of polyhedral metrics ρ_P in $T^{\text{con}}_{PL}(N)$ such that $\mathcal{T}$ is isotopic to a Delaunay triangulation of the associated polyhedron P. The sets $D_{PL}(\mathcal{T})$ for different isotopy classes of triangulations of S_μ form a covering of $T^{\text{con}}_{PL}(N)$. Let $F_\mathcal{T} = \varphi^{-1}_\mathcal{T} \circ \vartheta_\mathcal{T}$, define a function F on $T^{\text{con}}_{PL}(N)$ by setting $F(\rho) = F_\mathcal{T}(\rho)$ if $\rho \in D_{PL}(\mathcal{T})$.

Lemma: The function $F : T^{\text{con}}_{PL}(N) \to T_D(N)$ is well-defined.

Proof: Suppose $\rho_P \in D_{PL}(\mathcal{T}) \cap D_{PL}(\mathcal{T}')$, i.e. both $\mathcal{T}$ and $\mathcal{T}'$ are Delaunay triangulations of P. Then there exists a sequence of Delaunay triangulations $\mathcal{T} = \mathcal{T}_1, \ldots, \mathcal{T}_n = \mathcal{T}'$ of P such that $\mathcal{T}_i$ is obtained from $\mathcal{T}_{i+1}$ by a diagonal switch. In particular $F_\mathcal{T}(\rho_P) = F_{\mathcal{T}'}(\rho_P)$ follows from $F_{\mathcal{T}_i}(\rho_P) = F_{\mathcal{T}_{i+1}}(\rho_P)$ for $i = 1, 2, \ldots, n-1$. Hence, assume that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by a diagonal switch at an edge e.

Let $\vartheta_\mathcal{T}(\rho_P) = (x_0, x_1, \ldots, x_n)$. Since both $\mathcal{T}$ and $\mathcal{T}'$ are Delaunay triangulations of P, the triangles abutting at e share a common circumcircle. In this case the transition function is of the form

$$\vartheta_{\mathcal{T}'}\vartheta^{-1}_\mathcal{T}(x_0, x_1, \ldots, x_n) = (\frac{x_1x_3 + x_2x_4}{x_0}, x_1, x_2, \ldots, x_n).$$

On the other hand, according to Penner [25] the λ-lengths satisfy the Ptolemy relation for decorated ideal triangles. Hence,

$$\varphi_{\mathcal{T}'}\varphi^{-1}_\mathcal{T}(x_0, x_1, \ldots, x_n) = (\frac{x_1x_3 + x_2x_4}{x_0}, x_1, x_2, \ldots, x_n).$$

This shows,

$$\vartheta_{\mathcal{T}'}\vartheta^{-1}_\mathcal{T}(x_0, x_1, \ldots, x_n) = \varphi_{\mathcal{T}'}\varphi^{-1}_\mathcal{T}(x_0, x_1, \ldots, x_n),$$

which is

$$F_\mathcal{T}(\rho_P) = \varphi^{-1}_\mathcal{T} \circ \vartheta_\mathcal{T}(\rho_P) = \varphi^{-1}_{\mathcal{T}'} \circ \vartheta_{\mathcal{T}'}(\rho_P) = F_{\mathcal{T}'}(\rho_P).$$

In fact, the function F maps ρ_P to $(\hat\rho_P, \{w_i\}_{i=1}^N)$ where $\hat\rho_P$ is the image of P under $\mathfrak{f}$. To see this, we will study the shear coordinates of the underlying ideal polyhedral surface of a decorated ideal polyhedral surface $(\hat\rho, \{w_i\}_{i=1}^N)$ if $(\hat\rho, \{w_i\}_{i=1}^N)$ is given in Penner coordinates.

Theorem: *Let $(\hat\rho, w) \in T_D(N)$ and let $\varphi_\mathcal{T}$ be a coordinate chart containing $(\hat\rho, w)$, then the shear coordinate between two abutting triangles ilj and ikj in $\mathcal{T}$ of $\hat\rho$ is given by*

$$\log \frac{\varphi_\mathcal{T}((\hat\rho, w))_{il}}{\varphi_\mathcal{T}((\hat\rho, w))_{ik}} : \frac{\varphi_\mathcal{T}((\hat\rho, w))_{jl}}{\varphi_\mathcal{T}((\hat\rho, w))_{jk}}.$$

Proof: Recall that $\varphi_\mathcal{T}((\hat\rho, w))_{il} = e^{\lambda_{il}/2}$, where λ_{il} is the signed distance between the horospheres h_i and h_l. Hence,

$$\log \frac{\varphi_\mathcal{T}((\hat\rho, w))_{il}}{\varphi_\mathcal{T}((\hat\rho, w))_{ik}} : \frac{\varphi_\mathcal{T}((\hat\rho, w))_{jl}}{\varphi_\mathcal{T}((\hat\rho, w))_{jk}} = \frac{1}{2}(\lambda_{il} - \lambda_{lj} + \lambda_{jk} - \lambda_{ki}).$$

Let us focus first only on the decorated triangle ijk. The axis of symmetry through the point i of the ideal triangle ijk splits the signed distance λ_{jk} between the horocycles h_i and h_l into the sum of two numbers p^k_{ij} and p^j_{ki}, being the signed distance between the base point of the axis of symmetry and the horocycle h_k and h_j, respectively. Doing the same for λ_{ij} and λ_{ki} gives $\lambda_{ij} = p^i_{jk} + p^j_{ki}$, $\lambda_{jk} = p^j_{ki} + p^k_{ij}$ and $\lambda_{ki} = p^k_{ij} + p^i_{jk}$. Solving for p^j_{ki} gives

$$p^j_{ki} = \frac{1}{2}(\lambda_{ij} + \lambda_{jk} - \lambda_{ki}).$$

Doing the same for the triangle ijl gives

$$p^j_{il} = \frac{1}{2}(\lambda_{ij} + \lambda_{jl} - \lambda_{il}).$$

Hence,

$$\log \frac{\varphi_{\mathcal{T}}((\hat\rho, w))_{il}}{\varphi_{\mathcal{T}}((\hat\rho, w))_{ik}} : \frac{\varphi_{\mathcal{T}}((\hat\rho, w))_{jl}}{\varphi_{\mathcal{T}}((\hat\rho, w))_{jk}} = p^j_{ki} - p^j_{il}.$$

But the right-hand-side is nothing but the shear between the two triangles ijk and ijl.

We are now prepared to prove the alternative description of the function $\mathfrak{f}$.

Theorem: *Let $\mathfrak{f} : \mathcal{P}^N \to T(N)$ be the map associating a convex polyhedron with the ideal polyhedral surface $(S_\mu, \hat\rho_P)$, let $\mathfrak{g} : \mathcal{P}^N \to T_{PL}(N)$ be Alexandrov's homeomorphism and let $p : T_D(N) \to T(N)$ be the projection on the underlying hyperbolic surface. Then,*

$$p \circ F \circ \mathfrak{g} = \mathfrak{f}.$$

Proof: Let $\mathcal{T}$ be a Delaunay triangulation of P. Let ijk and ilj be two triangles in $\mathcal{T}$ abutting along the edge ij. The ideal polyhedral surface $\mathfrak{f}(P) = \hat\rho_P$ has shear coordinates

$$\log \frac{\vartheta_{\mathcal{T}}(\rho_P)_{il}}{\vartheta_{\mathcal{T}}(\rho_P)_{ik}} : \frac{\vartheta_{\mathcal{T}}(\rho_P)_{jl}}{\vartheta_{\mathcal{T}}(\rho_P)_{jk}}$$

along the edge ij of $\mathcal{T}$. Let $(\hat\rho, w) = F \circ \mathfrak{g}(P)$, the shear coordinates of the decorated ideal polyhedral surface $(\hat\rho, w)$ along the edge ij is

$$\log \frac{\varphi_{\mathcal{T}}((\hat\rho, w))_{il}}{\varphi_{\mathcal{T}}((\hat\rho, w))_{ik}} : \frac{\varphi_{\mathcal{T}}((\hat\rho, w))_{jl}}{\varphi_{\mathcal{T}}((\hat\rho, w))_{jk}}.$$

But $\mathfrak{g}(P)$ lies in $D_{PL}(\mathcal{T})$, hence $(\hat\rho, w) = F_{\mathcal{T}}(\rho_P) = \varphi_{\mathcal{T}}^{-1}\vartheta_{\mathcal{T}}(\rho_P)$. Hence,

$$\log \frac{\varphi_{\mathcal{T}}((\hat\rho, w))_{il}}{\varphi_{\mathcal{T}}((\hat\rho, w))_{ik}} : \frac{\varphi_{\mathcal{T}}((\hat\rho, w))_{jl}}{\varphi_{\mathcal{T}}((\hat\rho, w))_{jk}} = \log \frac{\vartheta_{\mathcal{T}}(\rho_P)_{il}}{\vartheta_{\mathcal{T}}(\rho_P)_{ik}} : \frac{\vartheta_{\mathcal{T}}(\rho_P)_{jl}}{\vartheta_{\mathcal{T}}(\rho_P)_{jk}},$$

and $\hat\rho = \hat\rho_P$.

Let us return to the main theorem of this section.

Theorem: *Let P and Q be two polyhedra that share a common Delaunay triangulation $\mathcal{T}$, then P and Q are discrete-conformally equivalent if and* *only if there exists a real valued function $u_\mathcal{T}$ on the vertices of P so that, if e is an edge in $\mathcal{T}$ between the vertices i and j, then the length $l_P(e)$ and $l_Q(e)$ of e in P and Q are related by*

$$l_Q(e) = l_P(e)\, e^{\frac{1}{2}(u_\mathcal{T}(i)+u_\mathcal{T}(j))}.$$

Proof: We have seen that the hyperbolic structure $\hat{\rho}_Q$ on S_μ is the unique complete hyperbolic structure on S_μ with shear

$$\log \frac{\vartheta_\mathcal{T}(\rho_Q)_{il}}{\vartheta_\mathcal{T}(\rho_Q)_{ik}} : \frac{\vartheta_\mathcal{T}(\rho_Q)_{jl}}{\vartheta_\mathcal{T}(\rho_Q)_{jk}}$$

along the edge ij of $\mathcal{T}$. If there exists a conformal factor $u_\mathcal{T}$ such that $Q = u_\mathcal{T} * P$, this number equals

$$\log \frac{e^{\frac{1}{2}(u_\mathcal{T}(i)+u_\mathcal{T}(l))}\vartheta_\mathcal{T}(\rho_P)_{il}}{e^{\frac{1}{2}(u_\mathcal{T}(i)+u_\mathcal{T}(k))}\vartheta_\mathcal{T}(\rho_P)_{ik}} : \frac{e^{\frac{1}{2}(u_\mathcal{T}(j)+u_\mathcal{T}(l))}\vartheta_\mathcal{T}(\rho_P)_{jl}}{e^{\frac{1}{2}(u_\mathcal{T}(j)+u_\mathcal{T}(k))}\vartheta_\mathcal{T}(\rho_P)_{jk}},$$

which equals

$$\log \frac{\vartheta_\mathcal{T}(\rho_P)_{il}}{\vartheta_\mathcal{T}(\rho_P)_{ik}} : \frac{\vartheta_\mathcal{T}(\rho_P)_{jl}}{\vartheta_\mathcal{T}(\rho_P)_{jk}}.$$

Hence, $(S_\mu, \hat{\rho}_P)$ is isometric to $(S_\mu, \hat{\rho}_Q)$.

If P and Q are discrete-conformally equivalent, i.e. $\mathfrak{f}(P) = \mathfrak{f}(Q)$, then $\mathfrak{p} \circ F \circ \mathfrak{g}(P) = \mathfrak{p} \circ F \circ \mathfrak{g}(Q)$. In other words, we obtain an ideal polyhedral surface with two decorations $\{h_i^P\}_{i=1}^N$ and $\{h_i^Q\}_{i=1}^N$ corresponding to P and Q respectively.

Let $\lambda^i_{P\to Q}$ be the signed distance between the horoballs h_i^P and h_i^Q at the i-th cone point of the given ideal polyhedral surface, which is negative if and only if the horoball h_i^P is smaller than the horoball h_i^Q. Given an edge ij of $\mathcal{T}$, the signed distances between horoballs $\lambda^P_{ij} = \lambda(h_i^P, h_j^P)$ and $\lambda^Q_{ij} = \lambda(h_i^Q, h_j^Q)$ are related by

$$\lambda^Q_{ij} = \lambda^P_{ij} + \lambda^i_{P\to Q} + \lambda^j_{P\to Q}.$$

In particular,

$$e^{\lambda^Q_{ij}/2} = e^{\lambda^P_{ij}/2}\, e^{\frac{1}{2}(\lambda^i_{P\to Q}+\lambda^j_{P\to Q})}.$$

Since $F \circ \mathfrak{g}(P) = \varphi_\mathcal{T}^{-1}\vartheta_\mathcal{T}(\rho_P)$ as well as $F \circ \mathfrak{g}(Q) = \varphi_\mathcal{T}^{-1}\vartheta_\mathcal{T}(\rho_Q)$, we have $e^{\lambda^P_{ij}/2} = l_P(ij)$ and $e^{\lambda^Q_{ij}/2} = l_Q(ij)$. Hence, if we define

$$u_\mathcal{T}(i) := \lambda^i_{P\to Q},$$

for every vertex $i = 1, \ldots, N$ of the polyhedron P, then $u_\mathcal{T}$ is a conformal factor satisfying $u_\mathcal{T} * P = Q$.

Concepts of discrete conformality and Möbius geometry. Let $\mathcal{P}_{\text{ideal}}$ be the space of ideal convex polyhedra in $\mathbb{H}^3$, this space is equivalent to the space of convex Euclidean polyhedra inscribed in the unit sphere if $\mathbb{H}^3$ is the Klein model of the hyperbolic 3-space. Hence, let us interpret $\mathcal{P}_{\text{ideal}}$ as the space of convex Euclidean polyhedra inscribed in the unit sphere. Let $\mathcal{C}$ be the set of circle patterns covering the unit sphere. To every circle pattern C in $\mathcal{C}$ corresponds a unique convex Euclidean polyhedron P_C in $\mathcal{P}_{\text{ideal}}$ by cutting off all half-planes defined by the circles in C. Conversely, every convex polyhedron inscribed in the unit sphere corresponds to a unique circle pattern covering the unit sphere.

Theorem: *Let C_1 and C_2 be two circle patterns in $\mathcal{C}$ and let P_{C_1} and P_{C_2} be the corresponding convex polyhedra inscribed in the unit sphere. There exists a Möbius transformation f on the sphere mapping the circle pattern C_1 onto the circle pattern C_2 if and only if P_{C_1} and P_{C_2} share a common Delaunay triangulation $\mathcal{T}$ and there exists a function $u_{\mathcal{T}}$ defined on the vertex set of P_{C_1} such that for every edge ij in the Delaunay triangulation between vertices i and j, its length in P_{C_2} is related to its length in P_{C_1} by*

$$l_{P_{C_2}}(ij) = l_{P_{C_1}}(ij)\, e^{\frac{1}{2}(u_{\mathcal{T}}(i)+u_{\mathcal{T}}(j))}.$$

Moreover, f and $u_{\mathcal{T}}$ are related by $u_{\mathcal{T}} = \log |df|_V$, where V is the vertex set of P_{C_1}.

Proof: It only remains to prove the relation of $u_{\mathcal{T}}$ and the Möbius transformation f. Let x and y be two distinct vertices of P_{C_1}. Let $\{x_n\}$ and $\{y_n\}$ be sequences on the unit sphere converging to x and y, respectively, but not containing the points x and y. The Euclidean length cross-ratio is invariant under Möbius transformations on the sphere. Hence

$$\frac{|x-x_n|}{|x-y_n|} : \frac{|y-x_n|}{|y-y_n|} = \frac{|f(x)-f(x_n)|}{|f(x)-f(y_n)|} : \frac{|f(y)-f(x_n)|}{|f(y)-f(y_n)|}.$$

A rearrangment gives

$$\frac{|f(x)-f(y_n)|}{|x-y_n|}\,\frac{|f(y)-f(x_n)|}{|y-x_n|} = \frac{|f(x)-f(x_n)|}{|x-x_n|}\,\frac{|f(y)-f(y_n)|}{|y-y_n|}.$$

Taking the limit $n \to \infty$ results in

$$\frac{|f(x)-f(y)|^2}{|x-y|^2} = |df(x)||df(y)|.$$

This shows that P_{C_1} and P_{C_2} are discrete-conformally equivalent with $u_{\mathcal{T}} = \log |df|_V$.

As we mentioned in the introduction, this theorem suggests a third variant of discrete conformality, namely *discrete Möbius geometry.*

Roughly speaking, a Möbius structure on a set X is an equivalence class of metrics on X, where two metrics are equivalent if they define the same crossratio. Let $\mathcal{M}$ be a Möbius structure on a set X. The pair $(X, \mathcal{M})$ is called a Möbius space.

If X is a strongly hyperbolic metric space, then its ideal boundary carries a natural Möbius structure as observed by Nica and Spakula.

Let X be a finite set, let d_f be the pull-back metric of the Euclidean distance on X induced by an embedding f of X into the sphere.

Theorem: *The metric spaces* (X, d_{f_1}) *and* (X, d_{f_2}) *are Möbius equivalent if and only if the associated convex polyhedra* P_{f_1} *and* P_{f_2} *are discrete-conformally equivalent.*

Proof: If (X, d_{f_1}) and (X, d_{f_2}) are Möbius equivalent, then $(S_\mu, \hat{\rho}_{P_{f_1}})$ is isometric to $(S_\mu, \hat{\rho}_{P_{f_2}})$, i.e. P_{f_1} and P_{f_2} are discrete-conformally equivalent. If P_{f_1} and P_{f_2} are discrete-conformally equivalent, then there exists a Möbius transformation on the sphere mapping the vertex set of P_{f_1} onto the vertex set of P_{f_2}, hence (X, d_{f_1}) and (X, d_{f_2}) are Möbius equivalent.

Note: It follows from previous considerations that two metric spaces (X, d_{f_1}) and (X, d_{f_2}) with N points are already Möbius equivalent if only a small subset of $2N - 6$ cross-ratios is preserved, namely the cross-ratios along edges within a quadrilateral of a common Delaunay triangulation of P_{f_1} and P_{f_2}.

DIRECTIONS OF FURTHER RESEARCH

Characterization of discrete conformality. It would be more elegant to have a definition of discrete-conformal equivalence of convex polyhedra by elementary transformations on vertices. We conjecture that P and Q are discrete-conformally equivalent if and only if there exists a finite sequence of closed convex polyhedra $P = P_1, P_2, \ldots, P_{n-1}, P_n = Q$ such that, for $k = 1, \ldots, n-1$ the polyhedra P_k and P_{k+1} share a common Delaunay triangulation $\mathcal{T}_k$ and there exists a real valued function $u_{\mathcal{T}_k}$ on the vertices of P_k with the following property. For every edge ij in the Delaunay triangulation between vertices i and j, its length in P_{k+1} is related to its length in P_k by

$$l_{P_{k+1}}(ij) = l_{P_k}(ij)\, e^{\frac{1}{2}(u_{\mathcal{T}_k}(i) + u_{\mathcal{T}_k}(j))}.$$

Difficulties arise from the fact that a Delaunay triangulation of a Euclidean convex polyhedron P is not a Delaunay triangulation of the associated marked polyhedral surface (S_μ, ρ_P). Hence, the function $\mathfrak{f}$ can be discontinuous when passing from a "cell" $D_{PL}(\mathcal{T})$ to another.

Variational principles. The uniformization theory of convex polyhedra may shed some light on the relationships between the different variational principles developed in the context of discrete conformality. Glickenstein suggested a formal framework in [26]. The natural appearance of real analytic cell decompositions in the work of Gu, Luo, Sun and Wu [27], may suggest the theory of moment maps as a general setting. According to Atiyah, Guillemin and Sternberg, the image of the moment map of a hamiltonian torus action on a compact connected symplectic manifold is always a polytope [28] [29].

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