The quasisymmetric Hölder equivalence problem

P. Pansu

September 2nd, 2021

The unit ball B in $\mathbb{C}^2$ is the prototype of

- a symplectic manifold with convex boundary;
- a complex manifold with strictly pseudoconvex boundary.

Furthermore, it carries a Riemannian metric which is invariant under all holomorphic automorphisms. It becomes *complex hyperbolic plane* $H^2_{\mathbb{C}}$.

The unit ball B in $\mathbb{C}^2$ is the prototype of

- a symplectic manifold with convex boundary;
- a complex manifold with strictly pseudoconvex boundary.

Furthermore, it carries a Riemannian metric which is invariant under all holomorphic automorphisms. It becomes *complex hyperbolic plane* $H^2_{\mathbb{C}}$. The boundary ∂B carries subRiemannian metrics, permuted by automorphisms, which are conformal to one another.

The unit ball B in $\mathbb{C}^2$ is the prototype of

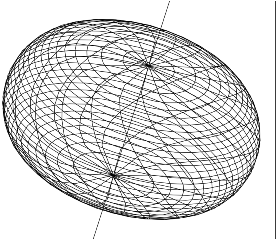
- a symplectic manifold with convex boundary;
- a complex manifold with strictly pseudoconvex boundary.

Furthermore, it carries a Riemannian metric which is invariant under all holomorphic automorphisms. It becomes *complex hyperbolic plane* $H^2_{\mathbb{C}}$. The boundary ∂B carries subRiemannian metrics, permuted by automorphisms, which are conformal to one another.

In B, there are two types of totally geodesic surfaces:

- complex lines, with curvature -1,
- lagrangian planes, with curvature $-\frac{1}{4}$.

Lagrangian planes intersect the boundary along Legendrian curves. Therefore the contact structure on the boundary is determined by the Riemannian geometry inside the ball.



W. Goldman

The unit ball B in $\mathbb{C}^2$ is the prototype of

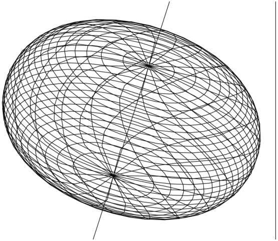
- a symplectic manifold with convex boundary;
- a complex manifold with strictly pseudoconvex boundary.

Furthermore, it carries a Riemannian metric which is invariant under all holomorphic automorphisms. It becomes *complex hyperbolic plane* $H^2_{\mathbb{C}}$. The boundary ∂B carries subRiemannian metrics, permuted by automorphisms, which are conformal to one another.

In B, there are two types of totally geodesic surfaces:

- complex lines, with curvature -1,
- lagrangian planes, with curvature $-\frac{1}{4}$.

Lagrangian planes intersect the boundary along Legendrian curves. Therefore the contact structure on the boundary is determined by the Riemannian geometry inside the ball.

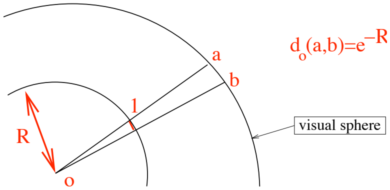


W. Goldman

What if one perturbs the Riemannian metric boundedly (i.e. in a biLipschitz manner), while keeping curvature negative? Is there a smooth boundary? With a contact structure?

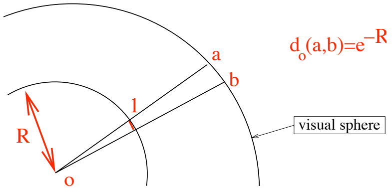
Facts.

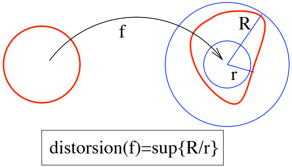
- Negatively curved manifolds M have a visual sphere ∂M, equipped with visual metrics (one for each point of M). Two such metrics are conformally related.
- If M is δ-pinched, polar coordinates define a $C^{\sqrt{-\delta}}$-Hölder homeomorphism from the round sphere $S \to \partial M$, with 1-Lipschitz inverse.
- Bi-Lipschitz maps between negatively curved Riemannian manifolds induce *quasisymmetric* maps between ideal boundaries.



Facts.

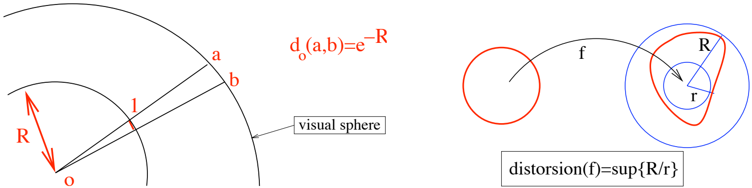
- Negatively curved manifolds M have a visual sphere ∂M, equipped with visual metrics (one for each point of M). Two such metrics are conformally related.
- If M is δ-pinched, polar coordinates define a $C^{\sqrt{-\delta}}$-Hölder homeomorphism from the round sphere $S \to \partial M$, with 1-Lipschitz inverse.
- Bi-Lipschitz maps between negatively curved Riemannian manifolds induce *quasisymmetric* maps between ideal boundaries.





Facts.

- Negatively curved manifolds M have a visual sphere ∂M, equipped with visual metrics (one for each point of M). Two such metrics are conformally related.
- If M is δ-pinched, polar coordinates define a $C^{\sqrt{-\delta}}$-Hölder homeomorphism from the round sphere $S \to \partial M$, with 1-Lipschitz inverse.
- Bi-Lipschitz maps between negatively curved Riemannian manifolds induce *quasisymmetric* maps between ideal boundaries.



So, the visual sphere ∂M is a metric space, it is not smooth, it merely has a quasisymmetric (weaker than biLipschitz) structure. And a Hölder structure too. One cannot make sense of a contact structure on it.

Definition

*Let M be a Riemannian manifold. Let $-1 \leq \delta < 0$. Say M is δ-*pinched *if sectional curvature ranges between -1 and δ. Define the* optimal pinching *$\delta(M)$ of M as the least $\delta \geq -1$ such that M is bi-Lipschitz equivalent to a δ-pinched complete simply connected Riemannian manifold.*

Definition

*Let M be a Riemannian manifold. Let $-1 \leq \delta < 0$. Say M is δ-*pinched *if sectional curvature ranges between -1 and δ. Define the* optimal pinching $\delta(M)$ *of M as the least $\delta \geq -1$ such that M is bi-Lipschitz equivalent to a δ-pinched complete simply connected Riemannian manifold.*

Example: Real hyperbolic spaces have $\delta(H^n_{\mathbb{R}}) = -1$. Complex hyperbolic plane $H^2_{\mathbb{C}}$ is $-\frac{1}{4}$-pinched, so $\delta(H^2_{\mathbb{C}}) \leq -\frac{1}{4}$. Is it true that $\delta(H^2_{\mathbb{C}}) = -\frac{1}{4}$?

Definition

*Let M be a Riemannian manifold. Let $-1 \leq \delta < 0$. Say M is δ-*pinched *if sectional curvature ranges between -1 and δ. Define the* optimal pinching $\delta(M)$ *of M as the least $\delta \geq -1$ such that M is bi-Lipschitz equivalent to a δ-pinched complete simply connected Riemannian manifold.*

Example: Real hyperbolic spaces have $\delta(H^n_{\mathbb{R}}) = -1$. Complex hyperbolic plane $H^2_{\mathbb{C}}$ is $-\frac{1}{4}$-pinched, so $\delta(H^2_{\mathbb{C}}) \leq -\frac{1}{4}$. Is it true that $\delta(H^2_{\mathbb{C}}) = -\frac{1}{4}$? Open.

Definition

*Let M be a Riemannian manifold. Let $-1 \leq \delta < 0$. Say M is δ-*pinched *if sectional curvature ranges between -1 and δ. Define the* optimal pinching $\delta(M)$ *of M as the least $\delta \geq -1$ such that M is bi-Lipschitz equivalent to a δ-pinched complete simply connected Riemannian manifold.*

Example: Real hyperbolic spaces have $\delta(H^n_{\mathbb{R}}) = -1$. Complex hyperbolic plane $H^2_{\mathbb{C}}$ is $-\frac{1}{4}$-pinched, so $\delta(H^2_{\mathbb{C}}) \leq -\frac{1}{4}$. Is it true that $\delta(H^2_{\mathbb{C}}) = -\frac{1}{4}$? Open.

Approach using boundary homeomorphisms:

Definition

$\alpha_{qs}(X) = \sup\{\alpha \in (0,1) \,|\, \exists$ *locally a C^α homeomorphism with Lipschitz inverse from Euclidean space to a metric space quasisymmetric to X*$\}$

Example: The visual boundary of complex hyperbolic plane is a sub-Riemannian 3-sphere, quasisymmetric to Heisenberg group *Heis*. Note that $\alpha_{qs}(Heis) \geq \frac{1}{2}$.

Definition

*Let M be a Riemannian manifold. Let $-1 \leq \delta < 0$. Say M is δ-*pinched *if sectional curvature ranges between -1 and δ. Define the* optimal pinching $\delta(M)$ *of M as the least $\delta \geq -1$ such that M is bi-Lipschitz equivalent to a δ-pinched complete simply connected Riemannian manifold.*

Example: Real hyperbolic spaces have $\delta(H^n_{\mathbb{R}}) = -1$. Complex hyperbolic plane $H^2_{\mathbb{C}}$ is $-\frac{1}{4}$-pinched, so $\delta(H^2_{\mathbb{C}}) \leq -\frac{1}{4}$. Is it true that $\delta(H^2_{\mathbb{C}}) = -\frac{1}{4}$? Open.

Approach using boundary homeomorphisms:

Definition

$\alpha_{qs}(X) = \sup\{\alpha \in (0,1) \,|\, \exists$ *locally a C^α homeomorphism with Lipschitz inverse from Euclidean space to a metric space quasisymmetric to X*$\}$

Example: The visual boundary of complex hyperbolic plane is a sub-Riemannian 3-sphere, quasisymmetric to Heisenberg group *Heis*. Note that $\alpha_{qs}(Heis) \geq \frac{1}{2}$.

Claim. $\sqrt{-\delta(M)} \leq \alpha_{qs}(\partial M)$. Proof on next slide.

Recall definitions

$\delta(M) =$ inf pinching of Riemannian manifolds N biLipschitz to M

$\alpha_{qs}(\partial M) =$ sup Hölder exponent α of homeos of round sphere $S \to X$ with Lipschitz inverse, with X quasisymmetric to ∂M.

Proof.

$$N \text{ biLipschitz to } N \implies \partial M \text{ quasisymmetric to } \partial N.$$

$$N\ \delta-\text{pinched} \implies \exists C^{\alpha} \text{ homeomorphism with Lipschitz inverse from round sphere to } \partial N \text{ with } \alpha = \sqrt{-\delta}.$$

Conclusion. $\sqrt{-\delta(M)} \leq \alpha_{qs}(\partial M)$.

Theorem (Pansu 2020)

For all sub-Riemannian n-manifolds X with Hausdorff dimension Q, $\alpha_{qs}(X) \leq \frac{n-1}{Q-1}$*.*
In particular, $\alpha_{qs}(Heis) \leq \frac{2}{3}$*.*

Theorem (Pansu 2020)

For all sub-Riemannian n-manifolds X with Hausdorff dimension Q, $\alpha_{qs}(X) \leq \frac{n-1}{Q-1}$*.*
In particular, $\alpha_{qs}(Heis) \leq \frac{2}{3}$*.*

(The consequence $\delta(H^2_{\mathbb{C}}) \geq -\frac{4}{9}$ was already known).

Remark: The obvious map $\mathbb{R}^3 \to Heis$ = subRiemannian Heisenberg group is $C^{1/2}$-Hölder continuous.

Question (Hölder equivalence problem, Gromov 1993)

Let X be a sub-Riemannian manifold. For which $\alpha \in (0,1)$ does there exist locally a homeomorphism from Euclidean space to X which is C^α-Hölder continuous ?

Remark: The obvious map $\mathbb{R}^3 \to Heis$ = subRiemannian Heisenberg group is $C^{1/2}$-Hölder continuous.

Question (Hölder equivalence problem, Gromov 1993)

Let X be a sub-Riemannian manifold. For which $\alpha \in (0,1)$ does there exist locally a homeomorphism from Euclidean space to X which is C^α-Hölder continuous ?

Definition

Let $\alpha(X) = \sup\{\alpha \in (0,1) \,|\, \exists$ *locally a* C^α *homeomorphism* $\mathbb{R}^n \to X\}$.

Remark: The obvious map $\mathbb{R}^3 \to Heis$ = subRiemannian Heisenberg group is $C^{1/2}$-Hölder continuous.

Question (Hölder equivalence problem, Gromov 1993)

Let X be a sub-Riemannian manifold. For which $\alpha \in (0,1)$ does there exist locally a homeomorphism from Euclidean space to X which is C^α-Hölder continuous ?

Definition

Let $\alpha(X) = \sup\{\alpha \in (0,1) \,|\, \exists \text{ locally a } C^\alpha \text{ homeomorphism } \mathbb{R}^n \to X\}$.

Theorem (Gromov 1993)

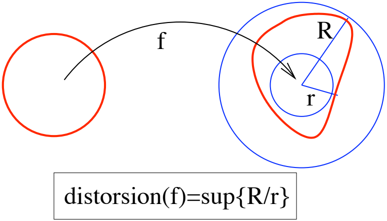
Let metric space X have dimension n, Hausdorff dimension Q. Then $\alpha(X) \leq \frac{n}{Q}$.
Let sub-Riem. X have dimension n, Hausdorff dimension Q. Then $\alpha(X) \leq \frac{n-1}{Q-1}$.
Let X be a $2m+1$-dimensional contact manifold. Then $\alpha(X) \leq \frac{m+1}{m+2}$ $(\leq \frac{2m}{2m+1})$.

Given a contact structure ξ, pick a smooth sub-Riemannian metric d. Then locally, d is unique up to biLipschitz equivalence. Local becomes global on compact manifolds. Every biLipschitz invariant notion becomes a concept in contact topology.

Given a contact structure ξ, pick a smooth sub-Riemannian metric d. Then locally, d is unique up to biLipschitz equivalence. Local becomes global on compact manifolds. Every biLipschitz invariant notion becomes a concept in contact topology.

The following classes of homeomorphisms can be thought of as classes of weakly differentiable contactomorphisms:

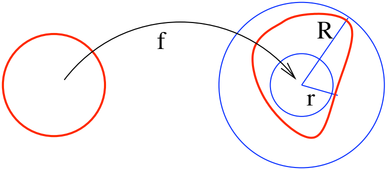
1. BiLipschitz maps.
2. Quasisymmetric maps.



Given a contact structure ξ, pick a smooth sub-Riemannian metric d. Then locally, d is unique up to biLipschitz equivalence. Local becomes global on compact manifolds. Every biLipschitz invariant notion becomes a concept in contact topology.

The following classes of homeomorphisms can be thought of as classes of weakly differentiable contactomorphisms:

1. BiLipschitz maps.
2. Quasisymmetric maps.



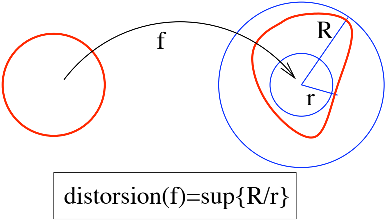
distorsion(f)=sup{R/r}

3. Sobolev L_1^p maps for p close to 4 (how close?).
4. Hölder C^α maps for $\alpha < 1$ close to 1 (how close?).

Given a contact structure ξ, pick a smooth sub-Riemannian metric d. Then locally, d is unique up to biLipschitz equivalence. Local becomes global on compact manifolds. Every biLipschitz invariant notion becomes a concept in contact topology.

The following classes of homeomorphisms can be thought of as classes of weakly differentiable contactomorphisms:

1. BiLipschitz maps.
2. Quasisymmetric maps.



3. Sobolev L^p_1 maps for p close to 4 (how close?).
4. Hölder C^α maps for $\alpha < 1$ close to 1 (how close?).

Question

What is the classification of contact structures in each of these categories?

Remark: The obvious map $\mathbb{R}^3 \to Heis$ = subRiemannian Heisenberg group is $C^{1/2}$-Hölder continuous, and its inverse is Lipschitz, so $\alpha(Heis) \geq \frac{1}{2}$.
Therefore, if $\alpha \leq \frac{1}{2}$, C^α maps are meaningless for contact geometry.

Theorem (Gromov 1993)

$\alpha(Heis) \leq \frac{2}{3}$.

Theorem (Wenger-Young 2018)

There exist proper degree 1 C^α maps $\mathbb{R}^3 \to Heis$ for all $\alpha < \frac{2}{3}$.

So perhaps C^α maps are meaningless for contact geometry as soon as $\alpha \leq \frac{2}{3}$. Who knows?

Remark: The obvious map $\mathbb{R}^3 \to Heis$ = subRiemannian Heisenberg group is $C^{1/2}$-Hölder continuous, and its inverse is Lipschitz, so $\alpha(Heis) \geq \frac{1}{2}$.
Therefore, if $\alpha \leq \frac{1}{2}$, C^α maps are meaningless for contact geometry.

Theorem (Gromov 1993)

$\alpha(Heis) \leq \frac{2}{3}$.

Theorem (Wenger-Young 2018)

There exist proper degree 1 C^α maps $\mathbb{R}^3 \to Heis$ for all $\alpha < \frac{2}{3}$.

So perhaps C^α maps are meaningless for contact geometry as soon as $\alpha \leq \frac{2}{3}$. Who knows?

Question

If two compact contact 3-dimensional contact manifolds are C^α biHölder-equivalent, for $\alpha \geq \frac{2}{3}$, are they contactomorphic?

Remark. The cohomology of a subRiemannian manifold is filtered by weights

- either by weights of differential forms,
- or by weights of Alexander-Spanier cochains.

Both are easily biLipschitz invariant.
The first one is finer.
The second filtration is C^α biHölder-invariant, for α close enough to 1.

Question

Show that the first one is invariant under Sobolev L^p_1 homeomorphisms, for p close enough to Q.

This convers quasisymmetric homeomorphisms.
Unfortunately, the filtration is trivial for contact manifolds (Rumin).

Thus the contact case seems the hardest.

Gromov's slogan:

Here
Euclidean
$\exists$ k-dimensional subset with Hausdorff dimension $\leq k$

There
SubRiemannian
$\forall$ k-dimensional subset, Hausdorff dimension $\geq k'$

then $\alpha(SubRiemannian) \leq \frac{k}{k'}$.

Gromov's slogan:

Here	There
Euclidean	SubRiemannian
$\exists$ k-dimensional subset with Hausdorff dimension $\leq k$	$\forall$ k-dimensional subset, Hausdorff dimension $\geq k'$

then $\alpha(SubRiemannian) \leq \frac{k}{k'}$.

In my argument,

Here	There
Euclidean	SubRiemannian
explicit family of subsets with Hausdorff dimension $\leq k$	for almost every image subset, Hausdorff dimension $\geq k'$

then $\alpha(SubRiemannian) \leq \frac{k}{k'}$.

Gromov's slogan:

Here
Euclidean
$\exists$ k-dimensional subset with Hausdorff dimension $\leq k$

There
SubRiemannian
$\forall$ k-dimensional subset, Hausdorff dimension $\geq k'$

then $\alpha(SubRiemannian) \leq \frac{k}{k'}$.

In my argument,

Here
Euclidean
explicit family of subsets with Hausdorff dimension $\leq k$

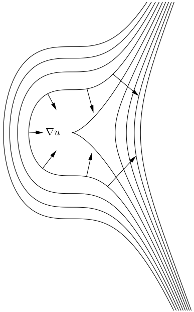
There
SubRiemannian
for almost every image subset, Hausdorff dimension $\geq k'$

then $\alpha(SubRiemannian) \leq \frac{k}{k'}$.

Furthermore, only quasisymmetric invariants are used, whence
$$\alpha_{qs}(SubRiemannian) \leq \frac{k}{k'}.$$

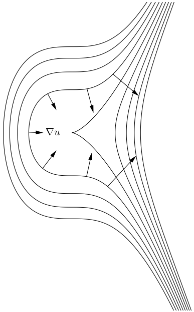
The *coarea formula* amounts to integrating along the level sets of a Lipschitz function $u : \mathbb{R}^n \to \mathbb{R}$. For every positive function ϕ on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \phi = \int_{\mathbb{R}} \left(\int_{u^{-1}(t)} \frac{\phi}{|\nabla u|} \right) dt$$



The *coarea formula* amounts to integrating along the level sets of a Lipschitz function $u:\mathbb{R}^n\to\mathbb{R}$. For every positive function ϕ on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n}\phi=\int_{\mathbb{R}}\Big(\int_{u^{-1}(t)}\frac{\phi}{|\nabla u|}\Big)\,dt$$



A special case: $\phi=|\nabla u|^n$.

$$\int_X|\nabla u|^n=\int_{\mathbb{R}}\Big(\int_{u^{-1}(t)}|\nabla u|^{n-1}\Big)\,dt.$$

Both sides are conformally invariant.

Let $f : \mathbb{R}^n \to X$ be a C^α Hölder homeomorphism with Lipschitz $f^{-1} : X \to \mathbb{R}^n$. Let $v : \mathbb{R}^n \to \mathbb{R}$ be a coordinate function. Then $u = v \circ f^{-1}$ is Lipschitz. Imagine that a *coarea inequality* holds:

$$\int_X Lip_u^Q \leq \int_{\mathbb{R}} \left(\int_{u^{-1}(t)} Lip_u^{Q-1} \right) dt \leq \mathrm{const.} \int_{\mathbb{R}} \mathcal{H}^{Q-1}(u^{-1}(t))\, dt. \qquad (1)$$

Here, Lip_u denotes the local Lipschitz constant. Since, for non constant u, $\int_X Lip_u^Q > 0$, this shows that there exists $t \in \mathbb{R}$ such that $\mathcal{H}^{Q-1}(u^{-1}(t)) > 0$, and therefore $u^{-1}(t)$ has Hausdorff dimension at least $Q - 1$.

Let $f:\mathbb{R}^n \to X$ be a C^α Hölder homeomorphism with Lipschitz $f^{-1}:X\to\mathbb{R}^n$. Let $v:\mathbb{R}^n\to\mathbb{R}$ be a coordinate function. Then $u=v\circ f^{-1}$ is Lipschitz. Imagine that a *coarea inequality* holds:

$$\int_X Lip_u^Q \leq \int_{\mathbb{R}} \left(\int_{u^{-1}(t)} Lip_u^{Q-1} \right) dt \leq \mathrm{const.} \int_{\mathbb{R}} \mathcal{H}^{Q-1}(u^{-1}(t))\, dt. \qquad (1)$$

Here, Lip_u denotes the local Lipschitz constant. Since, for non constant u, $\int_X Lip_u^Q > 0$, this shows that there exists $t\in\mathbb{R}$ such that $\mathcal{H}^{Q-1}(u^{-1}(t))>0$, and therefore $u^{-1}(t)$ has Hausdorff dimension at least $Q-1$. Whereas $f^{-1}(u^{-1}(t))=v^{-1}(t)$ has Hausdorff dimension $n-1$. This implies that $\alpha\leq\frac{n-1}{Q-1}$.

Unfortunately, Magnani 2002's *coarea inequality* goes in the opposite direction!

Let $f:\mathbb{R}^n \to X$ be a C^α Hölder homeomorphism with Lipschitz $f^{-1}:X\to\mathbb{R}^n$. Let $v:\mathbb{R}^n\to\mathbb{R}$ be a coordinate function. Then $u=v\circ f^{-1}$ is Lipschitz. Imagine that a *coarea inequality* holds:

$$\int_X Lip_u^Q \leq \int_{\mathbb{R}} \left(\int_{u^{-1}(t)} Lip_u^{Q-1} \right) dt \leq \text{const.} \int_{\mathbb{R}} \mathcal{H}^{Q-1}(u^{-1}(t))\, dt. \tag{1}$$

Here, Lip_u denotes the local Lipschitz constant. Since, for non constant u, $\int_X Lip_u^Q > 0$, this shows that there exists $t\in\mathbb{R}$ such that $\mathcal{H}^{Q-1}(u^{-1}(t))>0$, and therefore $u^{-1}(t)$ has Hausdorff dimension at least $Q-1$. Whereas $f^{-1}(u^{-1}(t))=v^{-1}(t)$ has Hausdorff dimension $n-1$. This implies that $\alpha\leq\frac{n-1}{Q-1}$.

Unfortunately, Magnani 2002's *coarea inequality* goes in the opposite direction!

Strategy: replace conformally invariant integrals $\int Lip_u^Q$ with *packing energy measures* which are quasisymmetric invariants and satisfy coarea inequality in the right direction. If possible, extend to vector valued maps u.

Here is a global question about noncompact contact manifolds.

A map $f : X \to Y$ between metric spaces is a *quasiisometry* if it is onto (up to bounded distance) and satisfies the biLipschitz inequality

$$d(f(x), f(x')) \leq L d(x, x')$$

for distances $d(x, x') \geq 1$.

Here is a global question about noncompact contact manifolds.

A map $f : X \to Y$ between metric spaces is a *quasiisometry* if it is onto (up to bounded distance) and satisfies the biLipschitz inequality

$$d(f(x), f(x')) \leq L d(x, x')$$

for distances $d(x, x') \geq 1$.

As P. Massot explained, on a contact manifold, every subRiemannian metric determines a Riemannian metric. Say a subRiemannian metric has *bounded geometry* if the corresponding Riemannian metric has bounded geometry (unit balls are uniformly biLipschitz to Euclidean unit balls).

Here is a global question about noncompact contact manifolds.

A map $f : X \to Y$ between metric spaces is a *quasiisometry* if it is onto (up to bounded distance) and satisfies the biLipschitz inequality

$$d(f(x), f(x')) \leq L d(x, x')$$

for distances $d(x, x') \geq 1$.

As P. Massot explained, on a contact manifold, every subRiemannian metric determines a Riemannian metric. Say a subRiemannian metric has *bounded geometry* if the corresponding Riemannian metric has bounded geometry (unit balls are uniformly biLipschitz to Euclidean unit balls).
If so, both metrics are quasiisometric.

Here is a global question about noncompact contact manifolds.

A map $f : X \to Y$ between metric spaces is a *quasiisometry* if it is onto (up to bounded distance) and satisfies the biLipschitz inequality

$$d(f(x), f(x')) \leq Ld(x, x')$$

for distances $d(x, x') \geq 1$.

As P. Massot explained, on a contact manifold, every subRiemannian metric determines a Riemannian metric. Say a subRiemannian metric has *bounded geometry* if the corresponding Riemannian metric has bounded geometry (unit balls are uniformly biLipschitz to Euclidean unit balls).
If so, both metrics are quasiisometric.

Question

Given a bounded geometry Riemannian 3-manifold M and a contact structure ξ on M, does there exist a subRiemannian metric (subordinate to ξ) of bounded geometry which is quasiisometric to M ?

Approach in case M is diffeomorphic to $\mathbb{R}^3$

Given a bounded geometry Riemannian metric, construct a contact structure with bounded covariant derivatives (a controlled Lutz theorem).
Then the subRiemannian metric obtained by restriction has bounded geometry and is quasiisometric to given Riemannian metric.

This gives a positive answer for the contact structure which is overtwisted in all neighborhoods of infinity.

How can one get the tight contact structure?

Proposition

Let X be a metric space. Let Γ be a family of subsets of X, equipped with a measure $d\gamma$. For each $\gamma \in \Gamma$, a probability measure m_γ is given on γ. Let $p \geq 1$. Assume that

$$\int_{\{\gamma\in\Gamma\,;\,\gamma\cap B\neq\emptyset\}} m_\gamma(\gamma\cap\ell B)^{1-p}\,d\gamma \leq \tau.$$

Then, for every function ϕ on the set of balls of X,

$$\Phi^p(X) \geq \frac{1}{\tau}\int_\Gamma \tilde{\Phi}^1(\gamma)^p\,d\gamma.$$

Proof Let $1_i(\gamma) = 1$ iff $\gamma \cap B_i \neq \emptyset$. The balls such that $1_i(\gamma) = 1$ cover γ, thus

$$\tilde{\Phi}^{1;\epsilon}(\gamma) \leq \sum_i \phi(B_i)1_i(\gamma) = \sum_i \phi(B_i)1_i(\gamma)m_\gamma(\gamma\cap\ell B_i)^{\frac{1-p}{p}}\,m_\gamma(\gamma\cap\ell B_i)^{\frac{p-1}{p}}.$$

Hölder's inequality gives

$$\tilde{\Phi}^{1;\epsilon}(\gamma)^p \quad\leq\quad \left(\sum_i \phi(B_i)^p 1_i(\gamma)m_\gamma(\gamma\cap\ell B_i)^{1-p}\right)\left(\sum_i m_\gamma(\gamma\cap\ell B_i)\right)^{p-1}.$$

Integrate over Γ.