Sobolev-Poincaré inequalities on Carnot groups

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Sobolev-Poincaré inequalities allow to control a function by its first derivatives.

Theorem (Classical Poincaré Inequality, global form)

If $1 \leq q \neq \nu$, there exists a constant C_q with the following effect. For all functions $u : \mathbb{R}^\nu \to \mathbb{R}$ with $Du \in L^q$,

$$\text{if } q < \nu, \text{ let } \frac{1}{p} = \frac{1}{q} - \frac{1}{\nu}; \text{ then } \quad \inf_{m \in \mathbb{R}} \|u - m\|_p \leq C_q \|Du\|_q,$$

$$\text{if } q > \nu, \text{ let } \alpha = 1 - \frac{\nu}{q}; \text{ then } \quad \sup_{x,y \in \mathbb{R}^n} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C_q \|Du\|_q.$$

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There are local versions too.

Theorem (Classical Poincaré Inequality, local form)

Let B denote the unit ball in $\mathbb{R}^\nu$ and $2B$ the twice larger concentric ball. If $1 \leq q \neq \nu$, there exists a constant C_q with the following effect. For all functions $u : 2B \to \mathbb{R}$ with $Du \in L^q(2B)$,

$$\text{if } q < \nu, \text{ let } \frac{1}{p} = \frac{1}{q} - \frac{1}{\nu}; \text{ then } \quad \inf_{m \in \mathbb{R}} \|u - m\|_{L^p(B)} \leq C_q \|Du\|_{L^q(2B)},$$

$$\text{if } q > \nu, \text{ let } \alpha = 1 - \frac{\nu}{q}; \text{ then } \quad \sup_{x,y \in B} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C_q \|Du\|_{L^q(2B)}.$$

Here, we are concerned with mappings $f : B \to G'$, where B is the unit ball in a Carnot group G of homogeneous dimension ν. We care only about the local form, second case, $q > \nu$. What should the condition $Du \in L^q(B)$ be replaced with?

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Say that a measurable mapping $f : B \to G'$ belongs to $W^{1,q}$ if there exists a measurable function $g \in L^p_{\mathrm{loc}}$ such that for all $z \in G'$, the composition $u = d(\cdot, z) \circ f$ belongs to $W^{1,q}_{\mathrm{loc}}$, $|D_h u| \leq g$ almost everywhere.

Here is today's theorem.

Theorem (Hölder bound for $W^{1,q}$ mappings)

If $f \in W^{1,q}(2B, G')$ for $q > \nu$ and $\alpha = 1 - \frac{\nu}{q}$, then f has a continuous representative and

$$\sup_{x,y \in B} \frac{d(f(x), f(y))}{|x-y|^\alpha} \leq C_q \|g\|_{L^q(2B)}.$$

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Indeed, pick a countable dense set of points $z \in G'$ and apply the scalar case to $u_z := d(\cdot, z) \circ f$. Countable in order to have simultaneously $|D_h u_z| \leq g$ for all z, almost everywhere. This gives Hölder continuity of u_z, hence of f.

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From now on, $G' = \mathbb{R}$.

The trick is to express u in the form $u = (Du) \star k$ for some matrix-valued function k. Indeed,

Proposition (Kernels of type 1 map L^q to C^α)

Let k be a kernel of type 1, i.e. a smooth function on $G \setminus \{e\}$ such that

1. $|k(g)| \leq C\,|g|^{1-\nu}$,
2. $|k(gh) - k(g)| \leq C\,|h||g|^{-\nu}$.

If $v \in L^q$, $v \star k \in C^\alpha$ and $\|v \star k\|_{C^\alpha} \leq C_q \|v\|_{L^q}$.

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Proof. We need estimate the integral

$$I = (v \star k)(g_0 h) - (v \star k)(g_0) = \int v(g_0 g^{-1})(k(gh) - k(g))\,dg.$$

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$$\begin{aligned} |I_1| &\leq \int_{B(2|h|)} |v(g_0 g^{-1})|(|k(gh)| + |k(g)|)\,dg \\ &\leq \|v\|_q \||k(\cdot h)| + |k|\|_{L^{q'}(B(2|h|))} \\ &\leq 2\|v\|_q \|k\|_{L^{q'}(B(C|h|))}. \end{aligned}$$

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By (2) and Hölder ($\frac{1}{q}+\frac{1}{q'}=1$),

$$|I_2| \leq \int_{B^c(2|h|)} |v(g_0 g^{-1})||k(gh)-k(g)|\,dg$$

$$\leq \|v\|_q \left(\int_{B^c(2|h|)} (|h||g|^{-\nu})^{q'}\right)^{1/q'}$$

$$\leq \|v\|_q |h|(2|h|)^{-\nu+(\nu/q')} \leq C\,\|v\|_q |h|^{\alpha}.$$

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Adding up,

$$|(v \star k)v(g_0 h) - (v \star k)(g_0)| \leq |I_1| + |I_2| \leq C\,\|v\|_q |h|^{\alpha},$$

which proves the Hölder estimate.

An exact formula like $u = (Du) \star k$ cannot exist for all u on $\mathbb{R}^\nu$ (add a constant to u) or on the unit ball B (the boundary interferes). But a slightly deformed version exists, up to an additive constant.

Proposition

Let $u : 2B \to \mathbb{R}$. Then, for $x \in B$,

$$u(x) - \fint_{2B} u = \int \langle Du(x), k_x(y) \rangle \, dy$$

where

$$k_x(y) = -\frac{y}{|y|} \ell_x(y) \quad \text{and} \quad \ell_x(y) \sim |y|^{1-\nu}.$$

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Indeed, write

$$u(x) - u(0) = \int_0^1 \frac{d}{dt} u(tx) \, dt = \int_0^1 \langle Du(tx), x \rangle \, dt.$$

$$\begin{aligned}\fint_{2B} u - u(0) &= c \int_{2B} \int_0^1 \langle Du(tx), x \rangle \, dt \, dx \\ &= c \int_{|y| \leq 2t \leq 2} t^{-\nu-1} \langle Du(y), y \rangle \, dt \, dy \\ &= \frac{c}{\nu} \int_{2B} \langle Du(y), y \rangle ((|y|/2)^{-\nu} - 1) \, dy.\end{aligned}$$

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For $v : G \to \mathbb{R}$ and $0 < t < 1$, let $I_t(v)(g) = t^{-\nu} v \circ \delta_{1/t}$. Let ϕ be a smooth positive compactly supported function with integral 1. Then $I_t \phi$ converges to the Dirac mass at e as $t \to 0$.

1. By expanding $u \star \phi - u = u \star \int_0^1 \frac{d}{dt} I_t \phi \, dt$, one gets an expression
$$u \star \phi - u = (D_h u) \star k,$$
where k is a kernel of type 1. So $u \star \phi - u \in C^{\alpha}$.

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4. Apply the Euclidean Poincaré inequality. Since $d_{\mathrm{Eucl}} \leq d_{\mathrm{CC}}$, this shows that $u \star \phi \in C^\alpha$, hence $u \in C^\alpha$.

We explain step 1 first in the Euclidean case.

$$\frac{\partial}{\partial x_i}(t^{-\nu}\frac{x_i}{t}\phi(\frac{x}{t})) = t^{-\nu-1}\phi(\frac{x}{t}) + t^{-\nu-1}\frac{x_i}{t}\frac{\partial\phi}{\partial x_i}(\frac{x}{t}),$$

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$$\begin{aligned}\frac{d}{dt}I_t\phi(x) = \frac{d}{dt}t^{-\nu}\phi(\frac{x}{t}) &= -\nu t^{-\nu-1}\phi(\frac{x}{t}) - \sum_{i=1}^{\nu}t^{-\nu-1}\frac{x_i}{t}\frac{\partial\phi}{\partial x_i}(\frac{x}{t})\\ &= -\sum_{i=1}^{\nu}\frac{\partial}{\partial x_i}(t^{-\nu}\frac{x_i}{t}\phi(\frac{x}{t}))\\ &= -\sum_{i=1}^{\nu}\frac{\partial}{\partial x_i}I_t(x_i\phi(x)).\end{aligned}$$

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$$u\star\phi - u = u\star\int_0^1 \frac{d}{dt}I_t\phi(x)\,dt = u\star\sum_{i=1}^{\nu}\frac{\partial}{\partial x_i}\psi_i = \sum_{i=1}^{\nu}\frac{\partial u}{\partial x_i}\star\psi_i,$$

where $\psi_i(x) = -\int_0^1 I_t(x_i\phi(x))\,dt$ satisfies $|\psi_i(x)| \leq C\,|x|^{-\nu+1}$.

In a general Carnot group, each adapted coordinate x_i has a weight w_i, and

$$\begin{aligned}\frac{d}{dt} I_t\phi(x) &= \frac{d}{dt} t^{-\nu}\phi(\ldots,\frac{x_i}{t^{w_i}},\ldots)\\ &= -\nu t^{-\nu-1}\phi(\delta_{1/t}x) - \sum_{i=1}^{\nu} t^{-\nu-1} w_i \frac{x_i}{t^{w_i}} \frac{\partial\phi}{\partial x_i}(\delta_{1/t}x)\\ &= -\sum_{i=1}^{\nu} \frac{\partial}{\partial x_i}(t^{-\nu} w_i \frac{x_i}{t^{w_i}} \phi(\delta_{1/t}x))\\ &= -\sum_{i=1}^{\nu} \frac{\partial}{\partial x_i} I_t(w_i x_i \phi(x)).\end{aligned}$$

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Whence an expression like

$$\sum_{i=1}^{\dim(G)}\frac{\partial u}{\partial x_i}\star\psi_i.$$

The other change derives from noncommutativity. If Y is a left-invariant vectorfield, let Y^R denote its image by $g \mapsto g^{-1}$. It is a right-invariant vectorfield. Then

$$Y(u \star v) = u \star Yv, \quad (Yu) \star v = -u \star Y^R v.$$

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We need replace each $\frac{\partial}{\partial x_i}$ with a sum of products $Y_j^R D_{ij}$ where Y_j^R are horizontal right-invariant vectorfields and D_{ij} are differential operators. This is a matter of linear algebra. One gets

$$u \star \phi - u = \sum_{i,j} u \star Y_j^R D_{ij}\psi_i = \sum_j Y_j u \star (-\sum_i D_{ij}\psi_i),$$

and checks that the right multipliers are kernels of type 1.

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To perform step 2, i.e. express vertical derivatives, one needs a similar algebraic trick, with $\frac{\partial}{\partial x_i}$ replaced with a left-invariant vectorfield Y_i: $Y_i = \sum_j Y_j^R D'_{ij}$ implies that

$$Y_i(u \star \phi) = u \star Y_i \phi = \sum_j u \star (\sum_j Y_j^R D'_{ij} \phi) = \sum_j Y_j u \star D'_{ij} \phi.$$

I explain the algebraic trick on the example of Engel's group, whose Lie algebra has basis X, Y, Z, T with $Z = [X, Y]$ and $T = [X, Z]$. X, Y are homogeneous of degree -1, Z of degree -2 and T of degree -3.

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Since T is central,

$$T = -T^R = -[X^R, Z^R] = -X^R Z^R + [X^R, Y^R] X^R = X^R(-Z^R + Y^R X^R) - Y^R(X^R X^R).$$

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Then $Z = -Z^R + aT$ where the vectorfield aT is homogeneous of degree -2, so is the function Ta. Since Ta is smooth, $Ta = 0$, so $aT = T \circ a$. Hence

$$Z = -X^R Y^R + Y^R X^R + T^R \circ a.$$

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$$Z = -X^R Y^R + Y^R X^R + T^R \circ a.$$

Finally, $X = -X^R + bZ + cT$ where the vectorfields bZ and cT are homogeneous of degree -1, so are the functions Zb and Tc, so both vanish. Thus

$$X = -X^R + Z \circ b + T \circ c$$

again has the required form thanks to above expressions for Z and T.