$\ell^{q,p}$ cohomology of certain Carnot groups

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Definition

X simplicial complex. Cochains are functions on simplices. When is an ℓ^p cocycle the coboundary of an ℓ^q cochain ? Set

$$\ell^{q,p}H^k(X) = \{\ell^p\ k\text{-cocycles}\}/d\{\ell^q\ k-1\text{-cochains}\}.$$

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Example. $X =$ tesselated plane. Then $\ell^{q,p}H^1(X) = 0$ if $\frac{1}{p} - \frac{1}{q} \geq \frac{1}{2}$. Indeed, Sobolev inequality allows to handle the case of finitely supported cocycles. It states that, for a smooth compactly supported function u on the plane, if $p < 2$,

$$\|u\|_q \leq C\,\|du\|_p.$$

Questions. Handle infinitely supported cocycles ? Pass from discrete to continuous and backward ?

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Example. $X=\mathbb{R}^n$. Then $L^{q,p}H^k(X)=0$ if $1<p\leq q<\infty$ and $\dfrac{1}{p}-\dfrac{1}{q}=\dfrac{1}{n}$.

Proof. Let $\Delta=d^*d+dd^*$. Then Δ has a pseudo-differential inverse which commutes with d. $T=d^*\Delta^{-1}$ has a homogeneous kernel of degree $1-n$, hence is bounded $L^p\to L^q$ provided $\frac{1}{p}-\frac{1}{q}=\frac{1}{n}$ (Calderon-Zygmund 1952). Finally $1=dT+Td$.

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Example. $X =$ ball in $\mathbb{R}^n$. Then $L^{q,p}H^k(X) = 0$ if $1 < p \leq q < \infty$ and $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}$.

Proof (Iwaniec-Lutoborsky 1993). Cartan's homotopy formula provides a homotopy T, $1 = dT + Td$, which has a homogeneous kernel of degree $1-n$. It does not require forms to be globally defined. Hölder $\Rightarrow$ q can be lessened. Works for convex sets.

Theorem (Leray, circa 1946)

Vanishing of $L^{q,p}H^{\cdot}$ of all simplices suffices to prove that $L^{q,p}H^{\cdot} = \ell^{q,p}H^{\cdot}$ for bounded geometry triangulated manifolds.

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Proposition (Rumin)

Proposition persists in the form $\ell^{q,p}H^{\cdot} = L^{q,p}_{\infty}H^{\cdot}$, for all $1 \leq p, q \leq \infty$, where $L^{p}_{\infty} = \{$forms all of whose derivatives are in $L^{p}\}$.

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Proposition (Pansu)

Proposition merely requires even weaker analytic information : it suffices that a closed form on ball of radius 2 have a primitive on unit ball with controlled norms.

"Loss on domain is allowed". Useful to prove quasiisometry invariance.

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Kohn's Laplacian. For a function u, let $d_R u = du_{|\mathfrak{g}_1}$ and let $\Delta u = d_R^* d_R$. This homogeneous of degree 2 under Carnot dilations δ_t.

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In general, there is no differential homogeneous Laplacian on forms, since left-invariant forms split under δ_t into several weight spaces.

Example. On the Heisenberg group, two weights on k-forms, k and $k+1$, since $\Lambda^k \mathfrak{g}^* = \Lambda^k \mathfrak{g}_1^* \oplus \Lambda^{k-1} \mathfrak{g}_1^* \otimes \mathfrak{g}_2^*$.

A pseudodifferential homogeneous Laplacian. Let $|\nabla| = \Delta^{1/2}$. Let $|\nabla|^N$ be the operator acting componentwise which is $|\nabla|^w$ on forms of weight w. Then $d^\nabla := |\nabla|^{-N} d |\nabla|^N$ is pseudodifferential of order 0, so is its Laplacian $\Delta^\nabla = (d^\nabla)^* d^\nabla + d^\nabla (d^\nabla)^*$. Both are homogeneous.

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Δ^∇ admits a pseudodifferential inverse (Helffer-Nourrigat 1979, Christ-Geller-Glowacki-Polin 1992), hence d^∇ admits a homotopy $K^\nabla := (d^\nabla)^* (\Delta^\nabla)^{-1}$, $1 = d^\nabla K^\nabla + K^\nabla d^\nabla$, hence a homogeneous homotopy $K := |\nabla|^N K^\nabla |\nabla|^{-N}$ for d.

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K^∇ is bounded on L^p (Folland 1975), thus K is bounded on

$$L^{N,p} := \{\alpha \, ; \, |\nabla|^{-N}\alpha \in L^p\},$$

and on $L^{N-m,p}$ for every constant m.

On functions,

$$L^p_\infty = \bigcap_{m=0}^{\infty} L^{N-m,p}.$$

On the space of forms whose weight lies between a and b, Ω_a^b,

$$\Omega_a^b \cap \bigcap_{m=a}^{\infty} L^{N-m,p} \subset \Omega_a^b \cap L^p_\infty \subset \bigcap_{m=b}^{\infty} L^{N-m,p}.$$

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$|\nabla|^{-\mu}$ is bounded from L^p to L^q if $\frac{1}{p} - \frac{1}{q} = \frac{\mu}{Q}$, $Q = \sum i \dim(\mathfrak{g}_i)$ (Folland 1975). Whence the graded Poincaré inequality

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Assume that $K(\Omega_a^b) \subset \Omega_{a'}^{b'}$. If $\mu = b - a'$, then

$$K(L^p_\infty \cap \Omega_a^b) \subset \Omega_{a'}^{b'} \cap \bigcap_{m=b}^{\infty} L^{N-m,p} \subset \Omega_{a'}^{b'} \cap \bigcap_{m=b}^{\infty} L^{N-m+\mu,q} = \Omega_{a'}^{b'} \cap \bigcap_{m=a'}^{\infty} L^{N-m,q} \subset L^q_\infty,$$

Proposition

$\ell^{q,p}H^k(G) = 0$ if $1 < p, q < \infty$, $\frac{1}{p} - \frac{1}{q} \geq \frac{b-a'}{Q}$, where b is the maximal weight in degree k, a' is the minimal weight in degree $k-1$.

Example. For Heisenberg group, $b = k+1$, $a' = k-1$, $b - a' = 2$. Unsharp.

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Strategy. Replace d with a subcomplex that uses less weights : Rumin's complex (1994, 1999).

Forms on G split into several weights under dilations. Let d_0 be the weight 0 (algebraic) part of d. Pick complements of $\ker(d_0)$ and $\mathrm{im}(d_0)$ and define d_0^{-1}. Then powers of $1 - d_0^{-1}d - dd_0^{-1}$ stabilize to a projector Π_E onto a subcomplex E. $\Pi_0 = 1 - d_0^{-1}d_0 - d_0 d_0^{-1}$ projects to a subspace E_R of forms where less weights occur. Set

$$d_R = \Pi_0 \circ d \circ \Pi_E.$$

Then Rumin's complex (E_R, d_R) is homotopic to de Rham's complex.

Example. For the 3-dimensional Heisenberg group,

- E_R^1 consists of horizontal 1-forms, $d_R : E_R^0 \to E_R^1$ is the horizontal gradient.
- E_R^2 consists of vertical 2-forms. Π_E extends a horizontal 1-form α in such a way that $d\Pi_E\alpha$ is vertical (unique choice). Hence $d_R\alpha = d\Pi_E\alpha$ involves second derivatives.

In general, for the $2m+1$-dimensional Heisenberg group, E_R has one weight in each degree, $w = k$ if $k \leq m$, $w = k+1$ if $k \geq m+1$, and $d_R : E_R^m \to E_R^{m+1}$ has order 2.

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Theorem (Pansu-Rumin)

Let G be a Carnot group of homogeneous dimension Q. Let $[a, b]$ be the scope of weights in Rumin k-forms, let $[a', b']$ be the scope of weights in Rumin $k-1$-forms.

1. $\ell^{q,p}H^k(G) = 0$ *provided* $1 < p, q < \infty$ *and*
$$\frac{1}{p} - \frac{1}{q} \geq \frac{b-a'}{Q}.$$
2. $\ell^{q,p}H^k(G) \neq 0$ *if* $1 \leq p, q \leq \infty$, $\frac{1}{p} - \frac{1}{q} < \frac{\max\{1, b'-a\}}{Q}$.

Example. For Heisenberg groups, $b - a' = b' - a = 1$, except in middle dimension where $b - a' = b' - a = 2$.

Example. For all Carnot groups, $b - a' = b' - a = 1$ in degrees 1 and $n = \dim(G)$.

Theorem (Baldi-Franchi-Pansu)

Closed L^p forms defined on the Heisenberg 2-ball have d_R-primitives on the unit ball which are L^q, provided $1 < p, q < \infty$ and $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{2m+2}$ (resp. $\frac{2}{2m+2}$ in degree $m+1$).

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Proof. No subRiemannian Cartan homotopy formula.

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Let K_R be the kernel of T_R. Write $K_R = K^1 + K^2$ where K_1 has small support and K_2 is smooth. Then $T_R = T^1 + T^2$,

$$1 = d_R T^1 + T^1 d_R + S$$

where S is smoothing. T^1 and therefore S map forms defined on ball of radius 2 to forms defined on unit ball. T^1 maps L^p to L^q like T_R. S wins the derivatives that Π_E looses.

Corollary (Baldi-Franchi-Pansu)

Rumin's complex can be used to compute $\ell^{q,p}$-cohomology of contact $2m+1$-manifolds with bounded geometry, provided $1 < p, q < \infty$ and $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{2m+2}$ (resp. $\frac{2}{2m+2}$ in degree $m+1$) : $\ell^{q,p}H^{\cdot} = L^{q,p}H^{\cdot}(d_c)$. Also, the complex (E_R, d_R) admits a smoothing homotopy.

Question. Cases when $p = 1$ or $q = \infty$? Work in progress with Baldi and Franchi.