# $L^p$-cohomology

P. Pansu

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### Definition

*Let $M$ be a Riemannian manifold. Let $p > 1$. $L^p$-*cohomology *of $M$ is the cohomology of the complex of $L^p$-differential forms on $M$ whose exterior differentials are $L^p$ as well,*

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$$\begin{aligned} H^{k,p} &= \textit{closed } k\textit{-forms in } L^p / d((k-1)\textit{-forms in } L^p), \\ R^{k,p} &= \textit{closed } k\textit{-forms in } L^p / \textit{closure of } d((k-1)\textit{-forms in } L^p), \\ T^{k,p} &= \textit{closure of } d((k-1)\textit{-forms in } L^p) / d((k-1)\textit{-forms in } L^p). \end{aligned}$$

*$R^{k,p}$ is called the* reduced cohomology*. $T^{k,p}$ is called the* torsion*.*

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More generally, for $p > 1$, $T^{1,p} = 0$ and $H^{1,p}$ is equal to the Besov space $B^{1/p}_{p,p}(\mathbb{R}/2\pi\mathbb{Z})$ mod constants.

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$T^{1,p}$ *is non zero and thus infinite dimensional.*
Indeed, the 1-form $\frac{dt}{t}$ (cut off near the origin) is in $L^p$ for all $p > 1$ but it is not the differential of a function in $L^p$.

# What are our favourite spaces ?

- $L^p$-cohomology has been used (L. Saper, S. Zucker) to study manifolds with thin ends. The answer is related to the topology of a compactification.

- In this talk: manifolds with large ends, e.g. groups. $L^p$-cohomology is related to analytic features of a compactification.

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*The obvious map $\mathbb{Z} \to \mathbb{R}$ is uniform. Any homomorphism between groups (with left invariant metrics) is uniform. The parametrization of a cusp by a punctured disk is not uniform.*

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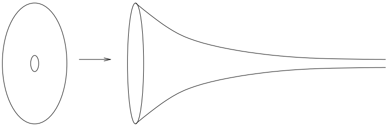

## Proposition

*Among contractible Riemannian manifolds admitting a cocompact isometric group action, $L^p$-cohomology is natural under uniform maps.*

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# $L^p$-cohomology of discrete groups

$L^p$-cohomology can be discretized. It makes sense for discrete groups, and cannot see any difference between a cocompact lattice in a semi-simple Lie group $G$, the Lie group $G$ itself or the Riemannian homogeneous space $G/K$, $K$ compact.

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In conclusion,

- $L^p$-cohomology is a tool to investigate discrete groups.
- It shares nearly all properties of usual cohomology.
- Nevertheless, it is not easy to calculate it.
- In the case of cocompact lattices in Lie groups, it can probably be computed by analytic means.

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$L^2$-invariants are a huge subject. In this course, we explain a few applications of $L^p$-cohomology, $p \neq 2$, to negatively curved Riemannian manifolds and groups.

1. Hopf Euler characteristic conjecture
2. Cannon conjecture on hyperbolic groups with boundary a 2-sphere
3. Classification of hyperbolic compactly generated groups
4. Coarse and large scale conformal embeddings
5. Curvature pinching

# Outline

1. Hopf Euler characteristic conjecture
2. Cannon conjecture on groups with boundary a 2-sphere
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## Remark

- *Compact 2-dimensional negatively curved manifolds have negative Euler characteristic.*
- $2m$*-dimensional compact hyperbolic manifolds have Euler characteristic proportional to* $(-1)^m$*.*
- *This generalizes to all compact negatively curved locally symmetric spaces (Gauss-Bonnet).*

# Hopf conjecture

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## Conjecture (H. Hopf)

*If* $M$ *is* $2m$*-dimensional compact negatively curved, then* $(-1)^m\chi(M) > 0$*.*

The conjecture is often extended to all aspherical manifolds. See M. Davies and B. Eckman's surveys in *Guido's book of conjectures*, and W. Lück's survey in *Bulletin of the Manifold Atlas*.

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## Theorem

*(M. Gromov, 1991). This is true provided* $M$ *also admits a* Kähler *metric.*

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## Corollary

*Let $M^{2m}$ be a complete Kähler manifold with Kähler form $\omega$. Then wedging with $\omega$ maps $L^2$-harmonic forms to $L^2$-harmonic forms, and this induces an* injection *in reduced $L^2$-cohomology $R^{k,2}(M) \to R^{k+2}(M)$ for all $k < m$.*

# Role of negative curvature

## Proposition

*(M. Gromov). Let $\tilde{M}$ be a complete simply connected negatively curved Riemannian manifold. Let $k \geq 2$.*

- *(Coning of cycles). Every $k-1$-cycle $z$ spans a $k$-chain $c$ with $vol(c) \leq const.\ vol(z)$.*

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thus $\omega \wedge \alpha = 0$ in $R^{k+2,2}(\tilde{M})$. If $\alpha$ is harmonic, conclude that $\alpha = 0$ in $R^{k,2}(\tilde{M})$.

# Vanishing of torsion

Let $\alpha$ be a $k$-form in $L^2$, $k < m$. Let us show that $\|\alpha\|_2^2 \leq C \langle \Delta\alpha, \alpha \rangle$.

$$\begin{aligned} \|\alpha\|^2 &= \|\omega \wedge \alpha\|^2 = \langle \omega \wedge \alpha, db \wedge \alpha \rangle = \langle \omega \wedge \alpha, d(b \wedge \alpha) \rangle + \langle \omega \wedge \alpha, b \wedge d\alpha \rangle \\ &\leq \|b\|_\infty \|\alpha\| (\|d^*(\omega \wedge \alpha)\| + \|d\alpha\|). \end{aligned}$$

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We show that $T^{k,2} = 0$ by induction on $k$. On 0-forms, $d^\alpha = 0$, hence $\|\alpha\| \leq C\,\|d\alpha\|$, and $dL^2\Omega^0$ is closed in $L^2\Omega^1$. If $\alpha \in L^2\Omega^2$, $\alpha = d\beta$ is an exact 2-form, there exists $\beta' \in L^2\Omega^1$ of minimal norm such that $d\beta' = \alpha$. Since $d^*\beta' = 0$, $\|\beta'\| \leq C\,\|d\beta'\| = \|\alpha\|$, and $dL^2\Omega^1$ is closed in $L^2\Omega^2$, and so on.

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We show that $T^{k,2} = 0$ by induction on $k$. On 0-forms, $d^\alpha = 0$, hence $\|\alpha\| \leq C\,\|d\alpha\|$, and $dL^2\Omega^0$ is closed in $L^2\Omega^1$. If $\alpha\in L^2\Omega^2$, $\alpha = d\beta$ is an exact 2-form, there exists $\beta'\in L^2\Omega^1$ of minimal norm such that $d\beta' = \alpha$. Since $d^*\beta' = 0$, $\|\beta'\| \leq C\,\|d\beta'\| = \|\alpha\|$, and $dL^2\Omega^1$ is closed in $L^2\Omega^2$, and so on.

Furthermore $\Delta$ is invertible on $(\ker\Delta)^\perp$. Indeed, Hodge decomposition $L^2\Omega^m = \ker\Delta\oplus\overline{dL^2\Omega^{m-1}}\oplus\overline{d^*L^2\Omega^{m+1}}$, which holds in general for complete Riemannian manifolds, reads $L^2\Omega^m = \ker\Delta\oplus\overline{dL^2\Omega^{m-1}}\oplus\overline{\star dL^2\Omega^{m-1}}$. Since $\Delta$ commutes with $d$ and $\star$, inequality $\|\alpha\|^2 \leq C^2\langle\Delta\alpha,\alpha\rangle$ persists on $(\ker\Delta)^\perp$ in degree $m$.

# $L^2$-Betti numbers

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### Example (W. Lück)

*If $M$ admits a tower of finite degree $d_j$ normal coverings $M_j$ such that $\bigcap_j \pi_1(M_j) = \{1\}$, then*

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## Proposition

*Let $\tilde{M}$ cover a compact manifold $M$. Then*

$$\chi(M) = \sum_k (-1)^k b^{k,2}(M).$$

**Proposition**

*(Relative index theorem, M. Gromov-B. Lawson). Let $\tilde{M}$ be a simply connected nonpositively curved Riemannian manifold. Then there exists $k$ such that $H^{k,2}(\tilde{M}) \neq 0$.*

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Proof of Gromov's theorem. Assume $M$ is compact and admits both a negatively curved metric and a Kähler metric. Then all $b^{k,2}(M)$ vanish except $b^{m,2}(M)$, which is nonzero, thus $(-1)^m\chi(M) = b^{m,2}(M) > 0$.

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In conclusion, we have used

- Lefschetz mechanism, $L^2$-Betti numbers.
- Vanishing of $L^\infty$-cohomology.
- Cup-product $H^{k,2} \otimes H^{2,\infty} \to H^{k+2,\infty}$.

# Further remarks

Gromov calls Kähler manifolds $(M, \omega)$ such that $\omega = db$, $b \in L^\infty$, *Kähler hyperbolic manifolds*. Hermitian symmetric spaces are Kähler hyperbolic. Teichmüller space is Kähler hyperbolic (S. Kruskal).

For such manifolds, Gromov's theorem determines the sign of the Euler characteristics of all the sheaves $\Omega^q$ of exterior powers of the complex cotangent bundle.

Nonvanishing of $L^2$-cohomology is equivalent to 0 belonging to the $L^2$ spectrum of $\Delta$. J. Lott's *zero in the spectrum conjecture* states that this should be the case for the universal coverings of all compact aspherical manifolds. There is a non-aspherical counterexample (M. Farber and S. Weinberger).

# Outline

1. Hopf Euler characteristic conjecture

   Interlude: more on $L^\infty/L^p$ cohomology of hyperbolic groups
2. Cannon conjecture on groups with boundary a 2-sphere
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$H^{k,\infty}(M) = 0 \Leftrightarrow$ linear filling in dimension $k$,
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**Conclusion**. $H^{2,\infty}(M) = 0 \Rightarrow$ linear filling of loops $\Rightarrow G$ hyperbolic.
Conversely, $G$ hyperbolic $\Rightarrow$ linear filling in all dimensions $k \geq 2 \Rightarrow H^{k,\infty}(M) = 0$.

Lemma

$H^{k,\infty} = 0 \Leftrightarrow \forall$ *bounded closed k-form* $\alpha$, $\exists \beta$ *with* $\|\beta\|_\infty \leq C\,\|\alpha\|_\infty$ *s. t.* $d\beta = \alpha$.

**Proof**. Use norm $\|\alpha\|_\infty + \|d\alpha\|_\infty$ to complete smooth differential forms into a Banach space $\mathcal{C}$. Then $d : \mathcal{C} \to \mathcal{C}$ is continuous, with kernel $\mathcal{Z}$. It descends to a continuous operator $\bar{d} : \mathcal{C}/\mathcal{Z} \to \mathcal{Z}$. If $L^\infty$-cohomology vanishes, this is a continuous bijection between Banach spaces, hence an isomorphism. Get a smooth $\beta$...

**Proof of Duality Lemma**.

A $k$-cycle $S$ of finite mass $M(S) := \sup\{\langle S, \beta\rangle\,;\, \|\beta\|_\infty \leq 1\}$ defines a functional on smooth bounded $k$-forms, which vanishes on exact forms. If $L^\infty$-cohomology vanishes, one can define a functional $T$ on smooth bounded exact $k+1$-forms by $\langle T, \alpha\rangle = \langle S, \beta\rangle$ for some $\beta$ such that $d\beta = \alpha$. Then $T$ is bounded above by $C\,M(S)\|\cdot\|_\infty$. By Hahn-Banach, $T$ extends to a linear functional on smooth $k+1$-forms, i.e. a current, of finite mass $M(T) \leq C\,M(S)$ and such that $\partial T = S$.

The converse follows from the symmetry between differential forms and currents, $\|\alpha\|_\infty = \sup\{\langle T, \alpha\rangle\,;\, M(T) \leq 1\}$.

### Example

*Let $M$ be n-dimensional, 1-connected, curvature $-a^2 \leq K \leq -1$. Then $H^{1,p}(M) \neq 0$ provided $p > (n-1)a$.*

In horospherical coordinates $(r, \theta)$, $u = u(\theta)$ has $|\nabla u| \leq C\, e^{-r}$, $vol \leq e^{(n-1)ar}$.

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This is sharp. Functions with $\|\nabla u\|_p < \infty$ on $H^n_{\mathbb{R}}$ have a limit $u_\infty$ along almost every ray, $u \in L^p \Leftrightarrow u_\infty = 0$. If $p \leq n-1$, $\|\nabla u_{\partial B(r)}\|_p$ tends to 0, hence $H^{1,p}(H^n_{\mathbb{R}}) = 0$ if $p \leq n-1$.

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1. *There exists $p(G)$ such that $H^{1,p}(G) \neq 0$ for $p > p(G)$.*
2. $1, p$*-cohomology classes on hyperbolic groups have well-defined boundary values, so $H^{1,p}(G)$ identifies with a function space on $\partial G$.*

Fix a visual metric on $\partial G$. Pick a maximal ball packing of $\partial G$ at each scale $2^{-k}$ and work on the incidence graph $X$ whose vertices $v$ correspond to balls $B(x_v, r_v)$. Then $X$ is quasi-isometric to $G$. Given a Lipschitz function $u$ on $\partial G$, set $u(v) = u(x_v)$. Then $|\nabla u|(v) \sim r_v$. If metric is $Q$-Ahlfors-regular, $\|\nabla u\|_p < \infty$ if $p > Q$.

# Outline

1. Hopf Euler characteristic conjecture
2. Cannon conjecture on groups with boundary a 2-sphere
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**Theorem (W. Thurston 1979, G. Perelman 2002)**

*A closed 3-manifold group which is infinite, does not split as a free product and does not contain* $\mathbb{Z}^2$ *is a cocompact lattice in* $SO(3,1)$.

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**Conjecture (J. Cannon, circa 1988)**

*Let $G$ be a hyperbolic group whose ideal boundary is a topological 2-sphere. Then $G$ is virtually a cocompact lattice in $SO(3,1)$.*

Before Perelman, it was thought of as an alternative to Thurston's methods.
Avatar of Riemann mapping theorem: uniqueness of a (coarse) conformal structure on the 2-sphere with a large conformal group.

## Remark

*The ideal boundary of a hyperbolic group carries a natural conformal structure (technically, a* quasi-Möbius structure*), but no canonical metric.*

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*Define the* conformal dimension *of a hyperbolic group as the least Hausdorff dimension of an Ahlfors-regular metric in the natural quasi-Möbius structure.*

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**Theorem (S. Keith-T. Laakso, M. Bonk-B. Kleiner 2005)**

*Let* $G$ *be a hyperbolic group whose ideal boundary is a 2-sphere. If conformal dimension is achieved, then* $G$ *is virtually a cocompact lattice in* $SO(3,1)$.

# $L^p$-dimension

## Remark

*For hyperbolic groups, $H^{1,p}$ is nonzero for $p$ large, but zero for $p$ small.*

## Definition

*Define the* $L^p$-dimension *of a group as the least $p > 1$ such that its $H^{1,p}$ is nonzero.*

For instance, $L^p$-dimension of hyperbolic $n$-space is $n - 1$.

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**Examples (M. Bourdon-H. Pajot)**

*There exist hyperbolic groups for which conformal dimension $> 2 \geq L^p$-dimension (and which do not split over virtually cyclic subgroups). For such groups, conformal dimension cannot be achieved.*

# Bourdon-Pajot's examples

Let $G = A \star_C B$ where $A$ and $B$ are hyperbolic, $C$ is quasi-convex and malnormal in both $A$ and $B$. Then $G$ is hyperbolic, $A$ and $B$ are quasi-convex in $G$, hence

$$\mathrm{ConfDim}(G) \geq \max\{\mathrm{ConfDim}(A), \mathrm{ConfDim}(B)\}.$$

Mayer-Vietoris implies that

$$b^{1,2}(G) \geq b^{1,2}(A) + b^{1,2}(B) - b^{1,2}(C) > 0,$$

if $b^{1,2}(A) > b^{1,2}(C)$. For instance, let $A$ and $B$ be lattices of isometries of Bourdon buildings, $A$ of large covolume and negative Euler characteristic, $B$ of large conformal dimension, and $C$ a free group on 2 generators. Then $H^{1,2}(G) \neq 0$, hence $L^p - \dim(G) \leq 2 < \mathrm{ConfDim}(G)$.

For a Bourdon building with $p$-sided right-angled faces and thickness $q$, Euler characteristic is proportional to $q^2 - p(q-1)$, conformal dimension equals $1 + \frac{\log q}{\arg\cosh(\frac{p-2}{2})}$.

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In conclusion, we have used

- Mayer-Vietoris and $L^2$-Betti numbers.
- Expression of $H^{1,p}$ as a function space on the ideal boundary.

# Further remarks

Bourdon-Kleiner show that conformal dimension is equal to the infimum of $p$ such that the continuous part of $L^p$ cohomology separates points of the ideal boundary.

Continuous part means continuous functions on $\partial G$ which continuously extend to functions on $G$ with finite $\ell^p$ norm of gradient.

## Theorem (Bourdon 2008)

*Consider the continuous part of $L^p$ cohomology of $G$ as a Banach algebra $\mathcal{A}_p(G)$ with norm $\|\cdot\|_\infty + \|\cdot\|_{H^{1,p}(G)}$.*
*Let $G_1$, $G_2$ be hyperbolic groups which do not split over finite groups. Let $p > \max\{\mathrm{ConfDim}(G_1), \mathrm{ConfDim}(G_1)\}$. Then any Banach algebra isomorphism $\mathcal{A}_p(G_1) \to \mathcal{A}_p(G_2)$ is induced by a quasi-isometry.*

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**Question**. Is $\partial G$ retrievable from the whole $H^{1,p}(G)$, and not its continuous part ?

In 2015, Bourdon-Kleiner implemented succesfully Kleiner's strategy for Coxeter groups. They fully describe the equivalence relations on $\partial G$ induced by $H^{1,p}(G)$ when $L^p - \dim(G) \leq p \leq ConfDim(G)$ in this special case.

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# Curvature pinching

### Remark

*Rank one symmetric spaces are hyperbolic spaces over the reals $H^n_{\mathbb{R}}$, the complex numbers $H^m_{\mathbb{C}}$, the quaternions $H^m_{\mathbb{H}}$, and the octonions $H^2_{\mathbb{O}}$.*
*Real hyperbolic space has sectional curvature $-1$. Other rank one symmetric spaces are $-\frac{1}{4}$-pinched, i.e. their sectional curvature ranges between $-1$ and $-\frac{1}{4}$.*

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*Real hyperbolic space has sectional curvature $-1$. Other rank one symmetric spaces are $-\frac{1}{4}$-*pinched*, i.e. their sectional curvature ranges between $-1$ and $-\frac{1}{4}$.*

## Definition

*Define the* optimal pinching *$\delta(G)$ of a discrete (or Lie) group $G$ as the least $\delta > -1$ such that $G$ is bi-uniformly equivalent to a $\delta$-pinched Riemannian manifold.*

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## Conjecture

*The optimal pinching of $SU(m,1)$, $Sp(m,1)$ ($m \geq 2$) and $F_4^{-20}$ is $-\frac{1}{4}$.*

**Theorem**

*If $M^n$ is simply connected and $\delta$-pinched for some $\delta \in [-1, 0)$, then*

$$p < 1 + \frac{n-k}{k-1}\sqrt{-\delta} \quad \Rightarrow \quad T^{k,p}(M) = 0.$$

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This is sharp. For instance, consider the semidirect product $G = \mathbb{R}^3 \rtimes_\alpha \mathbb{R}$ where $\alpha = diag(1, 1, 2)$.

- It admits a $-\frac{1}{4}$-pinched left-invariant Riemannian metric, therefore $\delta(G) \leq -\frac{1}{4}$.
- It has $T^{2,p}(G) \neq 0$ for $2 < p \leq 4$. This implies that $\delta(G) = -\frac{1}{4}$.

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**Theorem**

$T^{2,p}(H^2_{\mathbb{C}}) = 0$ *for* $2 < p < 4$.

Proof of torsion comparison theorem

Use the gradient vectorfield $\xi$ of a Busemann function and its flow $\phi_t$, whose derivative is controlled by sectional curvature. For $\alpha$ a closed $k$-form in $L^p$,

$$\phi_t^*\alpha = \alpha + d\left(\int_0^t \phi_s^*\iota_\xi\alpha\, ds\right)$$

has a limit as $t \to +\infty$ under the assumptions of the theorem. This boundary value map injects $H^{k,p}$ into a function space of closed forms on the ideal boundary, showing that $H^{k,p}$ is Hausdorff.

# Proofs of vanishing theorems

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## Proof of torsion vanishing for $H^2_{\mathbb{C}}$

For $p \notin \{4/3, 2, 4\}$, differential forms $\alpha$ on $H^2_{\mathbb{C}}$ split into components $\alpha_+$ and $\alpha_+$ which are contracted (resp. expanded) by $\phi_t$. Then

$$B_t : \alpha \mapsto \int_0^t \phi_s^*\iota_\xi\alpha_+\, ds - \int_{-t}^0 \phi_s^*\iota_\xi\alpha_-\, ds$$

converges as $t \to +\infty$ to a bounded operator $B$ on $L^p$. $P = 1 - dB - Bd$ retracts the $L^p$ de Rham complex onto a complex of differential forms on $Heis^3$ with missing components and weakly regular coefficients. If $2 < p < 4$, this complex is nonzero in degrees 1 and 2, but it is so small that its cohomology can be shown to be Hausdorff.

# Non-vanishing of torsion

Use Poincaré duality. Let $p' = p/p-1$ denote the conjugate exponent. In order to prove that a closed $k$-form $\alpha$ is nonzero in cohomology, it suffices to construct a sequence $\psi_j$ of $(n-k)$-forms such that $\| d\psi_j \|_{L^{p'}}$ tends to zero but $\int \alpha \wedge \psi_j$ does not tend to zero.

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In conclusion, we have used

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## Conjecture

- *For rank 1 symmetric spaces, $T^{k,p} = 0$ except for at most 1 value of $p$ in each degree.*
- *For higher rank symmetric spaces, $H^{k,p} = 0$ for $k < rank$, $T^{k,p} = 0$ for $k = rank$.*
- *For $k = rank$, $R^{k,p} \neq 0$ for $p$ large, and $R^{k,p}$ is a function space on the maximal boundary.*
- *For each $p > 1$, there exists $k$ such that $H^{k,p} \neq 0$.*

## $L^p$-cohomology of $G$