Poincaré inequalities for differential forms on Heisenberg groups

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Definition

X simplicial complex. Cochains are functions on simplices. When is an ℓ^p cocycle the coboundary of an ℓ^q cochain ? Set

$$\ell^{q,p}H^k(X) = \{\ell^p\ k\text{-}cocycles\}/d\{\ell^q\ k-1\text{-}cochains\}.$$

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Example. $X =$ tesselated plane. Then $\ell^{q,p}H^1(X) = 0$ if $\frac{1}{p} - \frac{1}{q} \geq \frac{1}{2}$. Indeed, Sobolev inequality allows to handle the case of finitely supported cocycles. It states that, for a smooth compactly supported function u on the plane, if $p < 2$,

$$\|u\|_q \leq C\,\|du\|_p.$$

Questions. Handle infinitely supported cocycles ? Pass from discrete to continuous and backward ?

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X Riemannian manifold. Set

$$L^{q,p}H^k(X) = \{L^p \text{ closed } k\text{-forms}\}/d\{L^q\ k-1\text{-forms } \omega \text{ such that } d\omega \in L^p\}.$$

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Example. $X = \mathbb{R}^n$. Then $L^{q,p}H^k(X) = 0$ if $1 < p \leq q < \infty$ and $\dfrac{1}{p} - \dfrac{1}{q} = \dfrac{1}{n}$.

Proof. Let $\Delta = d^*d + dd^*$. Then Δ has a pseudo-differential inverse which commutes with d. $T = d^*\Delta^{-1}$ has a homogeneous kernel of degree $n-1$, hence is bounded $L^p \to L^q$ provided $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$ (Calderon-Zygmund). Finally $1 = dT + Td$.

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Example. X = ball in $\mathbb{R}^n$. Then $L^{q,p}H^k(X) = 0$ if $1 < p \leq q < \infty$ and $\dfrac{1}{p} - \dfrac{1}{q} \leq \dfrac{1}{n}$.

Proof (Iwaniec-Lutoborsky). Poincaré's homotopy formula provides a homotopy T, $1 = dT + Td$, which has a homogeneous kernel of degree $n-1$. It does not require forms to be globally defined. Hölder $\Rightarrow$ q can be lessened. Works for convex sets.

Proposition (Leray)

Vanishing of $L^{q,p}H^{\cdot}$ of all simplices suffices to prove that $L^{q,p}H^{\cdot} = \ell^{q,p}H^{\cdot}$ for bounded geometry triangulated manifolds.

Example. (U_i) covering by stars of vertices. ω closed 1 form on X. $\omega_{|U_i} = du_i$, $u_i - u_j = \kappa_{ij}$ is constant on simplex $U_i \cap U_j$, it is a 1-cocycle of the triangulation. Conversely, pick partition of unity χ_i. Given cocycle κ, set $u_i = \sum_j \chi_j \kappa_{ij}$. Then $du_i - du_j = 0$ on $U_i \cap U_j$, hence defines a closed 1-form.

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Theorem (Pansu)

Proposition merely requires even weaker analytic information : it suffices that a closed form on ball of radius 2 have a primitive on unit ball with controlled norms.

"Loss on domain is allowed".

To handle Carnot groups G, need homogeneous Laplacian : Rumin ?

Forms on G split into several weights under dilations. Let d_0 be the weight 0 (algebraic) part of d. Pick complements of $\mathrm{ker}(d_0)$ and $\mathrm{im}(d_0)$ and define d_0^{-1}. Then powers of $1 - d_0^{-1}d - dd_0^{-1}$ stabilize to a projector Π_E onto a subcomplex E.
$\Pi_0 = 1 - d_0^{-1}d_0 - d_0 d_0^{-1}$ projects to a subspace E_R of forms where less weights occur. Set

$$d_R = \Pi_0 \circ d \circ \Pi_E .$$

Then Rumin's complex (E_R, d_R) is homotopic to the de Rham complex.

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Example. Heisenberg group $\mathbb{H}^1$.

- E_R^1 consists of horizontal 1-forms, $d_R : E_R^0 \to E_R^1$ is the horizontal gradient.
- E_R^2 consists of vertical 2-forms. Π_E extends a horizontal 1-form α in such a way that $d\Pi_E\alpha$ is vertical (unique choice). Hence $d_R\alpha = d\Pi_E\alpha$ involves second derivatives.

More generally, for Heisenberg group $\mathbb{H}^n$, E_R has exactly one weight in each degree, hence d_R is homogeneous under Heisenberg dilations. It has order 1, except in degree n where it has order 2.

One can make choices in a contact invariant manner : (E_R, d_R) is invariantly defined for contact manifolds.

In presence of a sub-Riemannian metric, adjoints are defined. Let $\Delta_R := d_R^* d_R + d_R d_R^*$ be replaced with $\Delta_R := (d_R^* d_R)^2 + d_R d_R^*$ or $d_R^* d_R + (d_R d_R^*)^2$ in degrees n and $n+1$. Then Δ_R is maximally hypoelliptic, hence $d_R^* \Delta_R^{-1}$ has a smooth (away from the origin), homogeneous kernel.

Proposition (Coifman-Weiss, Koranyi-Vagi 1971)

$T_R := d_R^* \Delta_R^{-1}$ is bounded L^p to L^q provided $1 < p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{2n+2}$ (resp. $\frac{2}{2n+2}$ in degree $n+1$).

Again, $1 = d_R T + T d_R$.

Need local version.

Theorem (Baldi-Franchi-Pansu)

Closed L^p forms defined on the Heisenberg 2-ball have d_R-primitives on the unit ball which are L^q, provided $1 < p, q < \infty$ and $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{2n+2}$ (resp. $\frac{2}{2n+2}$ in degree $n+1$).

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Proof. The sub-Riemannian analogue of Poincaré's homotopy formula is not compatible with Rumin's construction.

Instead, use Rumin's homotopy Π_E and then apply Iwaniec-Lutoborsky. Since Π_E is differential, this requires a preliminary smoothing homotopy.

Let K_R be the kernel of T_R. Write $K_R = K^1 + K^2$ where K_1 has small support and K_2 is smooth. Then $T_R = T^1 + T^2$,

$$1 = d_R T^1 + T^1 d_R + S$$

where S is smoothing. T^1 and therefore S map forms defined on ball of radius 2 to forms defined on unit ball. T^1 maps L^p to L^q like T_R. S wins the derivatives that Π_E looses.

Leray's method requires control on $\|d_R(\chi\omega)\|_p$ in terms of $\|\omega\|_p$ and $\|d_R\omega\|_p$. This fails when d_R is second order. The way around this is

Theorem (Baldi-Franchi-Pansu)

Let $1 < p, q < \infty$ and $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{2n+2}$ (resp. $\frac{2}{2n+2}$ in degree $n+1$). On contact manifolds with C^{k+1}-bounded geometry, global smoothing homotopies $1 = d_R T + T d_R + S$ are defined, where $T : W^{s,p} \to W^{s+1,q}$ and $S : W^{s,p} \to W^{s+k,q}$, $-k \leq s \leq 0$.

Corollary

Rumin's complex can be used to compute $\ell^{q,p}$-cohomology of contact manifolds with bounded geometry for this range of p, q.

Corollary

$\ell^{q,p} H^{\cdot}(\mathbb{H}^n) = 0$ provided $1 < p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} \geq \frac{1}{2n+2}$ (resp. $\frac{2}{2n+2}$ in degree $n+1$).

Questions.

1. Sobolev inequality (i.e. for compactly supported forms) ? Yes.
2. Sharpness ? Probably yes.
3. Cases when $p = 1$ and $q = \infty$? Work in progress with Baldi and Franchi.
4. Other Carnot groups ? Work in progress with Rumin, but sharp intervals are rarely attained.