# On Besicovitch's $\frac{1}{2}$-problem [after Preiss and Tišer]

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$$\underline{D}(E,x) = \liminf_{r\to 0} \frac{\mathrm{H}^1(E\cap B(x,r))}{2r}.$$

**Theorem (Besicovitch 1938)**

*If $M = \mathbb{R}^2$, any subset $E$ of finite $\mathcal{H}^1$-measure such that $\underline{D}(E,x) > \frac{3}{4}$ at $\mathcal{H}^1$-almost every point of $E$ is rectifiable.*

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**Theorem (Preiss, Tišer 1992)**

*For arbitrary metric spaces $M$, any subset $E \subset M$ of finite $\mathcal{H}^1$-measure such that $\underline{D}(E,x) > \frac{2+\sqrt{46}}{12}$ at $\mathcal{H}^1$-almost every point of $E$ is rectifiable.*

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## Definition

*Say that $M$ satisfies the Besicovitch pair condition with parameter $\sigma \in (0,1)$ if whenever $\mu$ is a measure on $M$ satisfying $\mu(S) \leq \mathrm{diam}(S)$ for every subset $S$, then there exists $\tau > 0$ such that $\forall \lambda > 0$, $\exists \delta > 0$ such that for every pair of Borel subsets $E_1$ and $E_2$ for which*

- $0 < dist(E_1, E_2) < \delta$, *and*
- $\mu(B(x,s)) > 2\sigma s$ *for every* $x \in E_1 \cup E_2$ *and every* $0 < s < \lambda$,

*there exists a subset $U \subset M$ intersecting both $E_1$ and $E_2$ and such that*

$$\mu(U \setminus (E_1 \cup E_2)) > \tau\, \mathrm{diam}(U).$$

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The proof of the theorem splits into two steps :

1. Every metric space satisfies BPC($\sigma$) for $\sigma = \frac{2+\sqrt{46}}{12}$.
2. BPC($\sigma$) $\Longrightarrow$ the theorem with bound $\sigma$.

## Proof of BPC

Let $\mu$ be a measure on $M$ satisfying $\mu(S) \leq \mathrm{diam}(S)$ for every subset $S$. Fix $\lambda > 0$. Let $E_1$ and $E_2$ be subsets for which

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Pick $x \in E_1$, $y \in E_2$ with $d(x,y) = d(E_1, E_2) := r$. Then (up two switching $x$ and $y$),

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For BPC($\sigma$), it suffices to set $U = B(x, 2r)$ and prove that $\mu(B(x,2r) \setminus (E_1 \cup E_2)) > 4\tau\, r$ for a $\tau > 0$ that depends only on $\sigma$.

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Thus one can pack inside $B(x,2r)$ two disjoint balls of not too small radii centered at points of $E_1$. Their measures are not too small, this contradicts (1) if $\sigma \geq \frac{2+\sqrt{46}}{12}$.

When the BPC will be used, the measure $\mu$ will be a constant multiple of $\mathcal{H}^1 \llcorner A$ for a suitable $A$. The fact that $\mu(S) \leq \mathrm{diam}(S)$ for all $S \subset M$ comes from the

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*Let $M$ be a metric space with $\mathcal{H}^1(M) < \infty$. At $\mathcal{H}^1$-almost all points $x \in M$,*

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For some $\tilde{\sigma} > \sigma$, this allows to pick compact sets $B \subset A \subset M$ of positive $\mathcal{H}^1$ measure such that $\mathcal{H}^1(A \setminus B) < \frac{\tau}{15}\mathcal{H}^1(B)$ (where $\tau$ is the constant in BPC($\sigma$)) and

$$S \cap A \neq \emptyset,\ \mathrm{diam}(S) \text{ small} \implies \mathcal{H}^1(S) \leq \frac{\tilde{\sigma}}{\sigma}\mathrm{diam}(S),$$

$$x \in B,\ s \text{ small} \implies \mathcal{H}^1(B(x,s) \cap A) \geq 2\tilde{\sigma}s,$$

and set $\mu(S) = \frac{\sigma}{\tilde{\sigma}}\mathcal{H}^1(S \cap A)$.

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**Strategy**. Add to $B$ the convex hulls of its intersections with a suitable collection of convex sets. Show that the result is connected (or at least contains a large connected subset). Replace convex hulls with line segments. The result is a continuum of finite $\mathcal{H}^1$ measure, hence rectifiable. Then remove (slightly larger) convex sets and show that the remainder, which is contained in $M$, has positive $\mathcal{H}^1$ measure.

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Among the family $\mathcal{V}$ of convex subsets $W$ of $X$ intersecting $B$ and for which $\mu(W \setminus B) > \tau\,\mathrm{diam}(W)$, pick a disjointed sequence $W_i$ such that

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Hence there exists a point $z$ in $B \setminus \bigcup_i W_i^7$ where the density of $A \setminus B$ vanishes. In particular, for some small $s$, $\mu(B(z,2s) \setminus B) < \frac{\sigma\tau}{14} s$.

Let $Q = B \cup \bigcup_i conv(B \cap W_i^2)$. This is a compact set. One uses BPC($\sigma$) to show that the connected component $C$ of $Q$ containing $z$ has diameter $> \frac{\sigma s}{2}$.

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Otherwise, there is a clopen $H$ of $Q$ containing $z$ and contained in $B(z, \frac{\sigma s}{2})$. Then BPC($\sigma$) applies to $E_1 = B \cap H$ and $E_2 = B \setminus H$, yielding a set $U$ intersecting both $E_1$ and $E_2$ and such that

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But inside $W_i^2$, $Q$ is convex, so its disconnectedness cannot happen there, contradiction.

In the sequel, one sticks to those $W_i$'s such that $W_i^3 \cap C \neq \emptyset$. Then

$$\sum_i \mathrm{diam}(W_i^3) \leq \frac{7}{\tau} \sum_i \mu(W_i \setminus B) \leq \frac{7}{\tau} \mu(B(z, 2s) \setminus B) < \frac{7}{\tau} \frac{\sigma \tau s}{14} \leq \mathrm{diam}(C).$$

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Replace each $W_i^3 \cap C$ with a line segment. This produces a continuum $D$ satisfying

- $\mathrm{diam}(D) = \mathrm{diam}(C)$,
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Remove from $M$ the union of a $\mathcal{H}^1$ measure maximizing sequence of rectifiable subsets. Apply previous result to remainder, conclude that it has measure 0, so $M$ is rectifiable.