CONTINUOUS PRIMITIVES FOR HIGHER DEGREE DIFFERENTIAL FORMS IN EUCLIDEAN SPACES, HEISENBERG GROUPS AND APPLICATIONS

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ABSTRACT. It is shown that higher degree exact differential forms on compact Riemannian n-manifolds possess continuous primitives whose uniform norm is controlled by their L^n norm. A contact sub-Riemannian analogue is proven, with differential forms replaced with Rumin differential forms.

Il est démontré que les formes différentielles exactes de degré supérieur sur des variétés riemanniennes compactes de dimension n possèdent des primitives continues dont la norme uniforme est contrôlée par leur norme L^n. Un analogue dans des variétés sous-riemanniennes de contact est prouvé, les formes différentielles étant remplacées par des formes différentielles de Rumin.

CONTENTS

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1. Introduction

1.1. **The problem.** Poincaré's inequality in the flat n-torus $\mathbb{R}^n/\mathbb{Z}^n$ states that a function u whose gradient belongs to L^p, $p < n$, is itself in L^q, up to an additive constant c_u, provided that $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$. Moreover

$$\|u - c_u\|_q \leq C_{p,q} \|\nabla u\|_p.$$

Such an estimate fails to hold when $p = n$ and $q = \infty$ (see e.g. [**?**], p.484 and [**?**]). However, when one passes to higher degree differential forms, the limiting case holds. This discovery, at least for top degree differential forms, is due to J. Bourgain and H. Brezis, [**?**]. The statement takes the following form: *let ω be an exact n-form on the n-torus, which belongs to L^n, then there exists a bounded differential $(n-1)$-form ϕ on the torus such that $d\phi = \omega$ and*

$$\|\phi\|_\infty \leq C \|\omega\|_n.$$

Furthermore, Bourgain and Brezis show that the primitive can be taken to be *continuous*, with a similar estimate.

The global version on $\mathbb{R}^n$ itself is even slightly stronger: the primitive can be taken to be continuous and to *tend to zero at infinity*, with a similar estimate, as shown by L. Moonens & T. Picon [**?**], and T. De Pauw & M. Torres [**?**].

In this paper, we shall extend Moonens and Picon's result to differential forms of all degrees > 1 on $\mathbb{R}^n$, and prove a version for the Heisenberg group $\mathbb{H}^n$, with Rumin's complex d_c replacing de Rham's differential d. A special instance, in general Carnot groups for top degree Rumin forms, is considered by A. Baldi & F. Montefalcone [**?**].

In the next statement, $\mathcal{C}_0$ denotes spaces of continuous differential forms that tend to zero at infinity.

Theorem 1. *The following global Poincaré inequalities hold.*

i) *(Euclidean spaces:) if $2 \leq h \leq n$, then a d-exact h-form $\omega \in L^n(\mathbb{R}^n, \bigwedge^h)$ admits a primitive $\phi \in \mathcal{C}_0(\mathbb{R}^n, \bigwedge^{h-1})$ such that*

$$\|\phi\|_{\mathcal{C}_0(\mathbb{R}^n, \bigwedge^{h-1})} \leq C\|\omega\|_{L^n(\mathbb{R}^n, \bigwedge^h)};$$

ii) *(Heisenberg groups:) if $2 \leq h \leq 2n+1$, $h \neq n+1$, then a d_c-exact Rumin h-form $\omega \in L^{2n+2}(\mathbb{H}^n, E_0^h)$ admits a primitive $\phi \in \mathcal{C}_0(\mathbb{H}^n, E_0^{h-1})$ such that*

$$\|\phi\|_{\mathcal{C}_0(\mathbb{H}^n, E_0^{h-1})} \leq C\|\omega\|_{L^{2n+2}(\mathbb{H}^n, E_0^h)};$$

iii) *(Heisenberg groups:) a d_c-exact Rumin $(n+1)$-form $\omega \in L^{n+1}(\mathbb{H}^n, E_0^{n+1})$ admits a primitive $\phi \in \mathcal{C}_0(\mathbb{H}^n, E_0^n)$ such that*

$$\|\phi\|_{\mathcal{C}_0(\mathbb{H}^n, E_0^n)} \leq C\|\omega\|_{L^{n+1}(\mathbb{H}^n, E_0^{n+1})}.$$

Note the game with critical exponents: they are determined by the degree of volume growth, which is equal to n for Euclidean space $\mathbb{R}^n$, but $Q = 2n+2$ for Heisenberg group. Moreover, since Rumin's differential d_c from n-forms to $(n+1)$-forms has order 2, the critical exponent turns out to be $Q/2 = n+1$ instead of Q in this special degree.

We shall also prove a local version, from which Poincaré inequalities can be deduced on arbitrary compact Riemannian or compact contact subRiemannian manifolds. Detailed statements are given in Section 4. In particular, we recover Bourgain-Brezis' original result on the torus.

Question. Can one improve Theorem 1 by establishing a uniform modulus of continuity for primitives?

1.2. **The method.** Let us focus on Heisenberg groups (the Euclidean case follows the same lines, with less technicalities). A Poincaré inequality providing a bounded measurable primitive can be found in [**?**]. However one cannot take the results there as a black-box.

Following Bourgain-Brezis, the proof of Theorem 1 goes by duality. The general principle, that a closed unbounded operator with dense domain between Banach spaces has a closed range if and only if its adjoint does, is applied to Rumin's differential $d_c : \mathcal{C}_0 \to L^p$, $p = Q$ or $Q/2$. The adjoint operator maps $L^{p'}$ to the space $\mathcal{M}$ of *Rumin currents* of finite mass. The L^1-Poincaré inequalities proven in [**?**] provide $L^{p'}$ primitives ϕ to L^1-Rumin forms ω, $d_c\phi = \omega$, and

$$\|\phi\|_{p'} \leq C\|\omega\|_1.$$

By approximation of Rumin currents of finite mass with currents defined by L^1-Rumin forms, such an estimate can be upgraded to provide $L^{p'}$ primitives ϕ to L^1-Rumin currents T of the form $T = \partial_c S$, with

$$\|\phi\|_{p'} \leq C\|T\|_{\mathcal{M}}.$$

This is the required closed range estimate for the adjoint operator of d_c.

The procedure needs be modified in degree $n + 1$. Indeed, there is a technical point when identifying the domain of the adjoint: when handling the second order operator d_c, cut-off arguments require extra control on first derivatives. Hence the space $L^{p'}$ is replaced with the Beppo Levi space (see e.g. [**?**]) $BL^{1,Q'}(\mathbb{H}^n)$ of Rumin forms whose coefficients have their first derivatives in $L^{Q'}$. One therefore needs an L^1 Poincaré inequality of the form

$$\|\phi\|_{BL^{1,Q'}(\mathbb{H}^n)} \leq C\|\omega\|_1.$$

Unfortunately, one cannot use [**?**] as a black box. Instead, one follows the strategy of [**?**], based on the Gagliardo-Nirenberg inequalities of [**?**].

The local version builds upon Theorem 1 and the local $L^\infty - L^p$-Poincaré inequality of [**?**].

1.3. **Organization of the paper.** As already stressed above, all proofs are given for $\mathbb{H}^n$ since the case of $\mathbb{R}^n$ just requires less technicalities. After fixing notations in Section 2, a vademecum on Rumin's complex is provided in Section 3, focussing on Leibniz' formula and commutation properties of Rumin's Laplacian. Detailed statements of Theorem 1, its local version and its avatar for compact manifolds are given in section 4. The duality principle is formulated in Section 5. The spaces of currents to which this principle is applied are defined in Section 6, whose main result is a Poincaré inequality: solving d_c in $\mathcal{C}_0$ for a current datum. The proof of Theorem 1 appears in Section 7, except for degree $(n + 1)$-forms. In this special degree, a new Gagliardo-Nirenberg inequality is proven as Theorem 8.12; a Poincaré inequality, first for L^1 forms and then for currents of finite mass follows in Section 8. A density result needed to identify the domain of an adjoint operator is proven in Section 9. Section 10 contains the proof of Theorem 1 in the middle degree $n + 1$. Section 11 contains the proof of the

local version of Theorem 1, and Section 12 draws consequences for bouded geometry manifolds.

2. Heisenberg groups

2.1. **Definitions and preliminary results.** We denote by $\mathbb{H}^n$ the n-dimensional Heisenberg group, identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted by $p = (x, y, t)$, with both $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. If p and $p' \in \mathbb{H}^n$, the group operation is defined by

$$p \cdot p' = (x + x', y + y', t + t' + \frac{1}{2}\sum_{j=1}^{n}(x_j y'_j - y_j x'_j)).$$

The unit element of $\mathbb{H}^n$ is the origin, that will be denoted by e. For any $q \in \mathbb{H}^n$, the *(left) translation* $\tau_q : \mathbb{H}^n \to \mathbb{H}^n$ is defined as

$$p \mapsto \tau_q p := q \cdot p.$$

The Lebesgue measure in $\mathbb{R}^{2n+1}$ is a Haar measure in $\mathbb{H}^n$ (i.e., a bi-invariant measure on the group). It is denoted by $\mathcal{L}^{2n+1}$, and when we need to stress the integration variable p, will be denoted also by dp.

For a general review on Heisenberg groups and their properties, we refer to [**?**], [**?**], [**?**], and to [**?**]. We limit ourselves to fix some notations, following [**?**].

The Heisenberg group $\mathbb{H}^n$ can be endowed with the homogeneous norm (Cygan-Korányi norm)

$$\varrho(p) = (|p'|^4 + 16\, p_{2n+1}^2)^{1/4}, \tag{1}$$

and we define the gauge distance (a true distance, see [**?**], p. 638), that is left invariant i.e. $d(\tau_q p, \tau_q p') = d(p, p')$ for all $p, p' \in \mathbb{H}^n$) as

$$d(p, q) := \varrho(p^{-1} \cdot q). \tag{2}$$

Finally, the balls for the metric d are le so-called Korányi balls

$$B(p, r) := \{q \in \mathbb{H}^n;\ d(p, q) < r\}. \tag{3}$$

Notice that Korányi balls are smooth convex sets.

A straightforward computation shows that there exists $c_0 > 1$ such that

$$c_0^{-2}|p| \leq \rho(p) \leq |p|^{1/2}, \tag{4}$$

provided p is close to e. In particular, for $r > 0$ small, if we denote by $B_{\mathrm{Euc}}(e, r)$ the Euclidean ball centred at e of radius r,

$$B_{\mathrm{Euc}}(e, r^2) \subset B(e, r) \subset B_{\mathrm{Euc}}(e, c_0^2 r). \tag{5}$$

We denote by $\mathfrak{h}$ the Lie algebra of the left invariant vector fields of $\mathbb{H}^n$. The standard basis of $\mathfrak{h}$ is given, for $i = 1, \dots, n$, by

$$X_i := \partial_{x_i} - \frac{1}{2} y_i \partial_t, \quad Y_i := \partial_{y_i} + \frac{1}{2} x_i \partial_t, \quad T := \partial_t.$$

The only non-trivial commutation relations are $[X_i, Y_i] = T$, for $i = 1, \dots, n$. The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}$ spanned by $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$: $\mathfrak{h}_1 := \text{span}\ \{X_1, \dots, X_n, Y_1, \dots, Y_n\}$.
Coherently, from now on, we refer to $X_1, \dots, X_n, Y_1, \dots, Y_n$ (identified with first order differential operators) as the *horizontal derivatives*. Denoting by $\mathfrak{h}_2$ the linear span of T, the 2-step stratification of $\mathfrak{h}$ is expressed by

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

The stratification of the Lie algebra $\mathfrak{h}$ induces a family of non-isotropic dilations $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$, $\lambda > 0$ as follows: if $p = (x, y, t) \in \mathbb{H}^n$, then

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t). \tag{6}$$

Notice that the gauge norm (1) is positively δ_λ-homogenous, so that the Lebesgue measure of the ball $B(x, r)$ is r^{2n+2} up to a geometric constant (the Lebesgue measure of $B(e, 1)$).

The constant

$$Q := 2n + 2,$$

is said the *homogeneous dimension* of $\mathbb{H}^n$ with respect to δ_λ, $\lambda > 0$, It is well known that the topological dimension of $\mathbb{H}^n$ is $2n + 1$, since as a smooth manifold it coincides with $\mathbb{R}^{2n+1}$, whereas the Hausdorff dimension of $(\mathbb{H}^n, d)$ is Q.

The vector space $\mathfrak{h}$ can be endowed with an inner product, indicated by $\langle \cdot, \cdot \rangle$, making $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ and T orthonormal.

Throughout this paper, we write also

$$W_i := X_i, \quad W_{i+n} := Y_i \quad \text{and} \quad W_{2n+1} := T, \quad \text{for } i = 1, \dots, n. \tag{7}$$

As in [**?**], we also adopt the following multi-index notation for higher-order derivatives. If $I = (i_1, \dots, i_{2n+1})$ is a multi–index, we set

$$W^I = W_1^{i_1} \cdots W_{2n}^{i_{2n}}\ T^{i_{2n+1}}. \tag{8}$$

By the Poincaré–Birkhoff–Witt theorem, the differential operators W^I form a basis for the algebra of left invariant differential operators in $\mathbb{H}^n$. Furthermore, we set

$$|I| := i_1 + \cdots + i_{2n} + i_{2n+1}$$

the order of the differential operator W^I, and

$$d(I) := i_1 + \cdots + i_{2n} + 2i_{2n+1}$$

its degree of homogeneity with respect to group dilations.

Let $U \subset \mathbb{H}^n$ be an open set. We shall use the following classical notations: $\mathcal{E}(U)$ is the space of all smooth function on U, and $\mathcal{D}(U)$ is the space of all compactly supported smooth functions on U, endowed with the standard topologies (see e.g. [**?**]). The spaces $\mathcal{E}'(U)$ and $\mathcal{D}'(U)$ are their dual spaces of distributions. We recall that $\mathcal{E}'(U)$ is the class of compactly supported distributions.

Let $1 \leq p \leq \infty$ and $m \in \mathbb{N}$, $W^{m,p}_{\mathrm{Euc}}(U)$ denotes the usual Sobolev space.

We recall also the notion of (integer order) Folland-Stein Sobolev space (for a general presentation, see e.g. [**?**] and [**?**]).

Definition 2.1. *If $U \subset \mathbb{H}^n$ is an open set, $1 \leq p \leq \infty$ and $m \in \mathbb{N}$, then the space $W^{m,p}(U)$ is the space of all $u \in L^p(U)$ such that, with the notation of* (8)*,*

$$W^I u \in L^p(U) \quad \textit{for all multi-indices } I \textit{ with } d(I) \leq m,$$

endowed with the natural norm

$$\| u\|_{W^{k,p}(U)} := \sum_{d(I)\leq m} \|W^I u\|_{L^p(U)}.$$

Folland-Stein Sobolev spaces enjoy the following properties akin to those of the usual Euclidean Sobolev spaces (see [**?**], and, e.g. [**?**]).

Theorem 2.2. *If $U \subset \mathbb{H}^n$ is an open set, $1 \leq p \leq \infty$, and $k \in \mathbb{N}$, then*

i) *$W^{k,p}(U)$ is a Banach space.*

In addition, if $p < \infty$,

ii) *$W^{k,p}(U) \cap C^\infty(U)$ is dense in $W^{k,p}(U)$;*
iii) *if $U = \mathbb{H}^n$, then $\mathcal{D}(\mathbb{H}^n)$ is dense in $W^{k,p}(U)$;*
iv) *if $1 < p < \infty$, then $W^{k,p}(U)$ is reflexive.*
v) *$W^{k,p}_{\mathrm{Euc,loc}}(U) \subset W^{k,p}(U)$, i.e. for any compact set $K \subset U$ and for any $u \in W^{k,p}_{\mathrm{Euc,loc}}(U)$,*

$$\|u\|_{W^{k,p}(K)} \leq C_K \|u\|_{W^{k,p}_{\mathrm{Euc,loc}}(K)}.$$

vi) *$W^{2k,p}(U) \subset W^{k,p}_{\mathrm{Euc,loc}}(U)$, i.e. for any compact set $K \subset U$ and for any $u \in W^{2k,p}(U)$,*

$$\|u\|_{W^{k,p}_{\mathrm{Euc}}(U)(K)} \leq C_K \|u\|_{W^{2k,p}(K)}.$$

Theorem 2.3. *[see* [**?**]*, Theorem 5.15] If $p > Q$, then*

$$W^{1,p}(\mathbb{H}^n) \subset L^\infty(\mathbb{H}^n)$$

algebraically and topologically.

Definition 2.4. *If $U \subset \mathbb{H}^n$ is open and if $1 \le p < \infty$, we denote by $\overset{\circ}{W}{}^{k,p}(U)$ the completion of $\mathcal{D}(U)$ in $W^{k,p}(U)$.*

Remark 2.5. *If $U \subset \mathbb{H}^n$ is bounded, by (iterated) Poincaré inequality (see e.g.* [**?**]*), it follows that the norms*

$$\|u\|_{W^{k,p}(U)} \quad \textit{and} \quad \sum_{d(I)=k} \|W^I u\|_{L^p(U)}$$

are equivalent on $\overset{\circ}{W}{}^{k,p}(U)$ when $1 \le p < \infty$.

2.2. **Convolution in Heisenberg groups.** If $f : \mathbb{H}^n \to \mathbb{R}$, we set ${}^{\mathrm{v}}f(p) = f(p^{-1})$, and, if $T \in \mathcal{D}'(\mathbb{H}^n)$, then $\langle {}^{\mathrm{v}}T|\phi\rangle := \langle T|{}^{\mathrm{v}}\phi\rangle$ for all $\phi \in \mathcal{D}(\mathbb{H}^n)$. Obviously, the map $T \to {}^{\mathrm{v}}T$ is continuous from $\mathcal{D}'(\mathbb{H}^n)$ to $\mathcal{D}'(\mathbb{H}^n)$.

Following e.g. [**?**], p. 15, we can define a group convolution in $\mathbb{H}^n$: if, for instance, $f \in \mathcal{D}(\mathbb{H}^n)$ and $g \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$, we set

$$f * g(p) := \int f(q) g(q^{-1} \cdot p)\, dq \quad \text{for } q \in \mathbb{H}^n. \tag{9}$$

We recall that, if, say, g is a smooth function and P is a left invariant differential operator, then

$$P(f * g) = f * Pg.$$

We also recall that the convolution is well defined when $f, g \in \mathcal{D}'(\mathbb{H}^n)$, provided at least one of them has compact support.

In this case the following identities hold:

$$\langle f * g|\Phi\rangle = \langle g|{}^{\mathrm{v}}f * \Phi\rangle \quad \text{and} \quad \langle f * g|\Phi\rangle = \langle f|\Phi * {}^{\mathrm{v}}g\rangle \tag{10}$$

for any test function Φ.

Definition 2.6. *A kernel of type μ is a homogeneous distribution of degree $\mu - Q$ (with respect to group dilations δ_r), that is smooth outside of the origin.*

The convolution operator with a kernel of type μ is called an operator of type μ.

Proposition 2.7. *Let $K \in \mathcal{D}'(\mathbb{H}^n)$ be a kernel of type μ.*

i) *${}^{\mathrm{v}}K$ is again a kernel of type μ;*
ii) *WK and KW are associated with kernels of type $\mu - 1$ for any horizontal derivative W;*
iii) *If $\mu > 0$, then $K \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$.*

Theorem 2.8. *Suppose $0 < \alpha < Q$, and let K be a kernel of type α. Then*

i) *if $1 < p < Q/\alpha$, and $1/q := 1/p - \alpha/Q$, then there exists $C = C(p, \alpha) > 0$ such that*

$$\|u * K\|_{L^q(\mathbb{H}^n)} \le C \|u\|_{L^p(\mathbb{H}^n)}$$

for all $u \in L^p(\mathbb{H}^n)$.

ii) *If* $p \geq Q/\alpha$ *and* $B, B' \subset \mathbb{H}^n$ *are fixed balls with* $B \subset B'$*, then for any* $q \geq p$ *there exists* $C = C(B, B', p, q, \alpha) > 0$

$$\|u * K\|_{L^q(B')} \leq C\|u\|_{L^p(B)}$$

for all $u \in L^p(\mathbb{H}^n)$ *with* $\operatorname{supp} u \subset B$.

iii) *If* K *is a kernel of type 0 and* $1 < p < \infty$*, then there exists* $C = C(p) > 0$ *such that*

$$\|u * K\|_{L^p(\mathbb{H}^n)} \leq C\|u\|_{L^p(\mathbb{H}^n)}.$$

Theorem 2.9 ([**?**], Theorem 1)**.** *Let* $\Phi \in \mathcal{D}(\mathbb{H}^n, \mathfrak{h}_1)$ *be a smooth compactly supported horizontal vector field. Suppose* $G \in L^1_{\mathrm{loc}}(\mathbb{H}^n, \mathfrak{h}_1)$ *is* $\mathbb{H}$*-divergence free, i.e. if*

$$G = \sum_i G_i W_i, \quad \textit{then} \quad \sum_i W_i G_i = 0 \quad \textit{in } \mathcal{D}'(\mathbb{H}^n).$$

Then

$$|\langle G, \Phi\rangle_{L^2(\mathbb{H}^n, \mathfrak{h}_1)}| \leq C\|G\|_{L^1(\mathbb{H}^n, \mathfrak{h}_1)}\|\nabla_{\mathbb{H}}\Phi\|_{L^Q(\mathbb{H}^n, \mathfrak{h}_1)}.$$

3. Rumin's complex of differential forms

When dealing with differential forms in $\mathbb{H}^n$, the de Rham complex lacks scale invariance under anisotropic dilations (see (6)). M. Rumin, in [**?**] has defined a substitute of de Rham's complex for arbitrary contact manifolds, that recovers scale invariance under δ_λ. In the present section, we shall merely list a few properties of Rumin's complex that we need in this paper. We send a reader, interested to understand better Rumin's complex, to the Appendix of [**?**] for a quick review, or to [**?**] and [**?**], [**?**] for more details of the construction.

The dual space of $\mathfrak{h}$ is denoted by $\bigwedge^1 \mathfrak{h}$. The basis of $\bigwedge^1 \mathfrak{h}$, dual to the basis $\{X_1, \ldots, Y_n, T\}$, is the family of covectors $\{dx_1, \ldots, dx_n, dy_1, \ldots, dy_n, \theta\}$ where

$$\theta := dt - \frac{1}{2}\sum_{j=1}^{n}(x_j dy_j - y_j dx_j)$$

is the *contact form* in $\mathbb{H}^n$. We denote by $\langle \cdot, \cdot \rangle$ the inner product in $\bigwedge^1 \mathfrak{h}$ that makes $(dx_1, \ldots, dy_n, \theta)$ an orthonormal basis and by dV the associated volume form

$$dV := dx_1 \wedge \ldots \wedge dx_n \wedge dy_1 \wedge \ldots \wedge dy_n \wedge \theta.$$

Throughout this paper, $\bigwedge^h \mathfrak{h}$ denotes the h-th exterior power of the Lie algebra $\mathfrak{h}$. Keeping in mind that the Lie algebra $\mathfrak{h}$ can be identified with the tangent space to $\mathbb{H}^n$ at $x = e$ (see, e.g. [**?**], Proposition 1.72), starting from

$\bigwedge^h \mathfrak{h}$ we can define by left translation a fiber bundle over $\mathbb{H}^n$ that we can still denote by $\bigwedge^h \mathfrak{h} \simeq \bigwedge^h T^*\mathbb{H}^n$. Moreover, a scalar product in $\mathfrak{h}$ induces a scalar product and a norm on $\bigwedge^h \mathfrak{h}$.

We can think of h-forms as sections of $\bigwedge^h \mathfrak{h}$ and we denote by Ω^h the vector space of all smooth h-forms.

- For $h = 0, \dots, 2n+1$, the space of Rumin h-forms, E_0^h, is the space of smooth sections of a left-invariant subbundle of $\bigwedge^h \mathfrak{h}$ (that we still denote by E_0^h). Hence it inherits the inner product and the norm of $\bigwedge^h \mathfrak{h}$.
- If we denote by $\star$ the Hodge duality operator associated with the inner product in $E_0^\bullet$ and the volume form dV, then $\star E_0^h = E_0^{2n+1-h}$.

  In particular we have

  Remark 3.1. *If $\alpha \in E_0^h$, then $\star\star\alpha = (-1)^{(2n+1-h)h}\alpha = \alpha$. Thus*

  $$\alpha \wedge \phi = \phi \wedge (\star\star\alpha) = \langle \star\alpha, \phi\rangle \, dV.$$

  Moreover, if $\beta \in E_0^h$

  $$\begin{aligned}\langle \star\alpha, \star\beta\rangle \, dV &= \alpha \wedge \star\beta = (-1)^{h(2n+1-h)} \star\beta \wedge \alpha \\ &= \langle \star\star\beta, \alpha\rangle \, dV = \langle \beta, \alpha\rangle \, dV = \langle \alpha, \beta\rangle \, dV.\end{aligned}$$

- A differential operator $d_c : E_0^h \to E_0^{h+1}$ is defined. It is left-invariant, homogeneous with respect to group dilations. It is a first order homogeneous operator in the horizontal derivatives in degree $\neq n$, whereas *it is a second order homogeneous horizontal operator in degree n.*
- Altogether, operators d_c form a complex: $d_c \circ d_c = 0$.
- This complex is homotopic to de Rham's complex $(\Omega^\bullet, d)$. More precisely there exist a sub-complex (E, d) of the de Rham complex and a suitable "projection" $\Pi_E : \Omega^\bullet \to E^\bullet$ such that Π_E is a differential operator of order ≤ 1 in the horizontal derivatives.
- Π_E is a chain map, i.e.

  $$d\Pi_E = \Pi_E d.$$

- Let Π_{E_0} be the orthogonal projection on $E_0^\bullet$. Then

  $$\Pi_{E_0}\Pi_E\Pi_{E_0} = \Pi_{E_0} \qquad \text{and} \qquad \Pi_E\Pi_{E_0}\Pi_E = \Pi_E.$$

  (we stress that Π_{E_0} is an algebraic operator).

- The exterior differential d_c can be written as

$$d_c = \Pi_{E_0} d \Pi_E \Pi_{E_0}.$$

- The L^2-formal adjoint d_c^* of d_c on E_0^h satisfies

$$d_c^* = (-1)^h \star d_c \star . \tag{11}$$

Let us list a bunch of notations for vector-valued function spaces (for the scalar case, we refer to Section 2.1).

Definition 3.2. *If $U \subset \mathbb{H}^n$ is an open set, $0 \leq h \leq 2n+1$, $1 \leq p \leq \infty$ and $m \geq 0$, we denote by $L^p(U, \bigwedge^h \mathfrak{h})$, $\mathcal{E}(U, \bigwedge^h \mathfrak{h})$, $\mathcal{D}(U, \bigwedge^h \mathfrak{h})$, $W^{m,p}(U, \bigwedge^h \mathfrak{h})$ (by $\overset{\circ}{W}{}^{m,p}(U, \bigwedge^h \mathfrak{h})$) the space of all sections of $\bigwedge^h \mathfrak{h}$ such that their components with respect to a given left-invariant frame belong to the corresponding scalar spaces. The spaces $L^p(U, E_0^h)$, $\mathcal{E}(U, E_0^h)$, $\mathcal{D}(U, \mathbb{E}_0^h)$, $W^{m,p}(U, E_0^h)$ and $\overset{\circ}{W}{}^{m,p}(U, E_0^h)$ are defined in the same way.*

*In other words, for the definition of Sobolev spaces of nonnegative order, we identify a differential form with the vector-valued function of its coordinates with respect to a left-invariant frame, keeping in mind that these coordinates belong to $L^1_{\mathrm{loc}}(\mathbb{H}^n)$. This makes possible to avoid the use of the notion of currents associated with Rumin's complex (for this notion, see, e.g., [**?**], [**?**], [**?**]). With this identification, d_c and its L^2-formal adjoint d_c^* are identified with matrix-valued left-invariant differential operators.*

Clearly, all these definitions are independent of the choice of frame.

When d_c is second order (when acting on forms of degree n), $(E_0^\bullet, d_c)$ stops behaving like a differential module. This is the source of many complications. In particular, the classical Leibniz formula for the de Rham complex $d(\alpha \wedge \beta) = d\alpha \wedge \beta \pm \alpha \wedge d\beta$ is true in Rumin's complex only in special degrees, as shown in [**?**], Proposition A.1 and [**?**], Proposition 4.1. However, in general, the Leibniz formula fails to hold (see [**?**]-Proposition A.7). This causes several technical difficulties when we want to localize our estimates by means of cut-off functions.

The following Leibniz' formula for Rumin's differential forms holds.

Lemma 3.3 (see also [**?**], Lemma 4.1)**.** *If ζ is a smooth real function, then the following formulae hold in the sense of distributions:*

i) *if $h \neq n$, then on E_0^h we have*

$$[d_c, \zeta] = P_0^h(W\zeta),$$

where $P_0^h(W\zeta) : E_0^h \to E_0^{h+1}$ is a linear homogeneous differential operator of order zero with coefficients depending only on the horizontal derivatives of ζ. If $h \neq n+1$, an analogous statement holds if we replace d_c in degree h with d_c^* in degree $h+1$;

ii) *if $h = n$, then on E_0^n we have*

$$[d_c, \zeta] = P_1^n(W\zeta) + P_0^n(W^2\zeta),$$

where $P_1^n(W\zeta) : E_0^n \to E_0^{n+1}$ is a linear homogeneous differential operator of order 1 (and therefore horizontal) with coefficients depending only on the horizontal derivatives of ζ, and where $P_0^h(W^2\zeta) : E_0^n \to E_0^{n+1}$ is a linear homogeneous differential operator in the horizontal derivatives of order 0 with coefficients depending only on second order horizontal derivatives of ζ. If $h = n+1$, an analogous statement holds if we replace d_c in degree n with d_c^* in degree $n+1$.

Lemma 3.4. *If $0 \leq h \leq 2n$ and $\phi \in \mathcal{D}(\mathbb{H}^n, E_0^h)$ and $\alpha \in W^{1,1}_{\mathrm{loc}}(\mathbb{H}^n, E_0^{2n-h})$, then we have:*

$$\int_{\mathbb{H}^n} d_c\alpha \wedge \phi = (-1)^h \int_{\mathbb{H}^n} \alpha \wedge d_c\phi\,. \tag{12}$$

Proof. The statement follows by the formula of "integration by parts" of [**?**], Remarks 2.16, together with an approximation argument.

□

The wedge of two closed forms of complementary degrees has the following property.

Lemma 3.5. *Let $1 \leq h \leq 2n+1$ and let $\alpha \in L^p(\mathbb{H}^n, E_0^h)$ and $\omega \in L^{p'}(\mathbb{H}^n, E_0^{2n+1-h})$ be closed forms with $1 < p < Q$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then*

$$\int \alpha \wedge \omega = 0. \tag{13}$$

In addition, if $p = Q$, then (13) *still holds provided $2 \leq h \leq 2n+1$.*

Proof. Since the differential d_c is left-invariant, by mollification (convolution with smooth kernels), one can assume that α and ω are smooth.

Suppose first $1 < p < Q$. Without loss of generality, we can assume $h \neq n+1$ (indeed, if $h = n+1$, then $2n+1-h = n \neq n+1$ so we can consider ω in place of α in the following argument). By Theorem 1.1 of [**?**], there exists $\phi \in L^{pQ/(Q-p)}(\mathbb{H}^n, E_0^{h-1})$ such that $d_c\phi = \alpha$ and

$$\|\phi\|_{L^{pQ/(Q-p)}(\mathbb{H}^n, E_0^{h-1})} \leq C\|\alpha\|_{L^p(\mathbb{H}^n, E_0^h)}. \tag{14}$$

If $N > 0$, let now χ_N be a smooth cut-off function supported in $B(e, 2N)$, $\chi_N \equiv 1$ on $B(e, N)$, $|W\chi_N| \leq 2/N$. Obviously

$$\int_{\mathbb{H}^n} \chi_N \alpha \wedge \omega \to \int_{\mathbb{H}^n} \alpha \wedge \omega.$$

On the other hand, by Lemma 3.3,

$$\begin{aligned} &\int_{\mathbb{H}^n} \chi_N \alpha \wedge \omega \\ &\quad = \int_{\mathbb{H}^n} \chi_N d_c\phi \wedge \omega \\ &\quad = \int_{\mathbb{H}^n} d_c(\chi_N \phi) \wedge \omega + \int_{\mathbb{H}^n} P_0^{h-1}(W\chi_N)(\phi) \wedge \omega \\ &\quad = \int_{\mathbb{H}^n} P_0^{h-1}(W\chi_N)(\phi) \wedge \omega \qquad \text{by (12).} \end{aligned} \tag{15}$$

We are then left to prove that

$$\int_{\mathbb{H}^n} P_0^{h-1}(W\chi_N)(\phi) \wedge \omega \to 0 \qquad \text{as } N \to \infty. \tag{16}$$

Obviously,

$$P_0^{h-1}(W\chi_N)(\phi) \wedge \omega \to 0 \qquad \text{pointwise as } N \to \infty,$$

since $W\chi_N \equiv 0$ on $B(e, N)$. On the other hand, if $1 < p < Q$, by Hölder inequality and (14),

$$\begin{aligned} &\int_{\mathbb{H}^n} |P_0^{h-1}(W\chi_N)||\phi||\omega| \, dV \\ &\qquad \leq \|P_0^{h-1}(W\chi_N)\|_{L^Q(\mathbb{H}^n, E_0^1)} \cdot \|\phi\|_{L^{pQ/(Q-p)}(\mathbb{H}^n, E_0^{h-1})} \cdot \|\omega\|_{L^{p'}(\mathbb{H}^n, E_0^{2n+1-h})}, \end{aligned}$$

since $\frac{1}{Q} + \frac{Q-p}{pQ} + \frac{1}{p'} = 1$. But

$$\|P_0^{h-1}(W\chi_N)\|_{L^Q(\mathbb{H}^n, E_0^1)} \leq C \frac{1}{N} |B(e, 2N)|^{1/Q} \leq C.$$

This proves (16), and hence (13), when $1 < p < Q$.

The proof in the case $p = Q$ needs slightly different arguments. Indeed we distinguish the case $h = n + 1$, i.e., $\alpha \in L^Q(\mathbb{H}^n, E_0^{n+1})$, from the case $\alpha \in L^Q(\mathbb{H}^n, E_0^h)$ with $h \neq n + 1$. In the latter case, by [**?**], Theorem 1.8 (see also Theorem 4.3 below), there exists $\phi \in L^\infty(\mathbb{H}^n, E_0^{h-1})$ such that $d_c\phi = \alpha$ and

$$\|\phi\|_{L^\infty(\mathbb{H}^n, E_0^{h-1})} \leq C \|\alpha\|_{L^Q(\mathbb{H}^n, E_0^h)}. \tag{17}$$

Hence the proof can be carried out in the same way, using the estimate (17), since $\frac{1}{Q} + \frac{1}{\infty} + \frac{Q-1}{Q} = 1$. Suppose now $\alpha \in L^Q(\mathbb{H}^n, E_0^{n+1})$, hence

$\omega \in L^{Q/(Q-1)}(\mathbb{H}^n, E_0^n)$. Since $1 < \frac{Q}{Q-1} < Q$, we can apply Theorem 1.1 of [**?**] to ω, and we conclude arguing as we have done above for the case $1 < p < Q$ (replacing α by ω).

□

Remark 3.6. *The above lemma still holds for $p = 1$: see* [**?**].

3.1. **Rumin's Laplacian.**

Definition 3.7. *In $\mathbb{H}^n$, Rumin defined in* [**?**] *the operators $\Delta_{\mathbb{H},h}$ acting on E_0^h, by setting*

$$\Delta_{\mathbb{H},h} = \begin{cases} d_c d_c^* + d_c^* d_c & \text{if} \quad h \neq n, n+1; \\ (d_c d_c^*)^2 + d_c^* d_c & \text{if} \quad h = n; \\ d_c d_c^* + (d_c^* d_c)^2 & \text{if} \quad h = n+1. \end{cases}$$

Notice that $-\Delta_{\mathbb{H},0} = \sum_{j=1}^{2n} (W_j^2)$ is the usual sub-Laplacian of $\mathbb{H}^n$.

We stress that the order of $\Delta_{\mathbb{H},h}$ (with respect to group dilations) is 2 if $h \neq n, n+1$ and 4 if $h = n, n+1$. Notice also that, once a basis of E_0^h is fixed, the operator $\Delta_{\mathbb{H},h}$ can be identified with a matrix-valued map, still denoted by $\Delta_{\mathbb{H},h}$

$$\Delta_{\mathbb{H},h} = (\Delta_{\mathbb{H},h}^{ij})_{i,j=1,\dots,N_h} : \mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h}) \to \mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h}), \tag{18}$$

where $\mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h})$ is the space of vector-valued distributions on $\mathbb{H}^n$, and N_h is the dimension of E_0^h (see [**?**]).

Theorem 3.8 (see [**?**], Theorem 4.6)**.** *If $0 \leq h \leq 2n+1$, denote by a the order of $\Delta_{\mathbb{H},h}$ with respect to group dilations ($a = 2$ if $h \neq n, n+1$ and $a = 4$ if $h = n, n+1$). Then there exist*

$$K_{ij} \in \mathcal{D}'(\mathbb{H}^n) \cap \mathcal{C}^\infty(\mathbb{H}^n \setminus \{0\}) \qquad \text{for } i, j = 1, \dots, N_h, \tag{19}$$

with the following properties:

i) *if $a < Q$ then the K_{ij}'s are kernels of type a, for $i, j = 1, \dots, N_h$. If $a = Q$, then the K_{ij}'s satisfy the logarithmic estimate $|K_{ij}(p)| \leq C(1 + |\ln \rho(p)|)$ and hence belong to $L^1_{\text{loc}}(\mathbb{H}^n)$. Moreover, their horizontal derivatives $W_\ell K_{ij}$, $\ell = 1, \dots, 2n$, are kernels of type $Q-1$;*

ii) *when $\alpha = (\alpha_1, \dots, \alpha_{N_h}) \in \mathcal{D}(\mathbb{H}^n, \mathbb{R}^{N_h})$ (here again $N_h = \dim E_0^h$), if we set*

$$\Delta_{\mathbb{H},h}^{-1} \alpha := (\sum_j \alpha_j * K_{1j}, \dots, \sum_j \alpha_j * K_{N_h j}), \tag{20}$$

then

$$\Delta_{\mathbb{H},h} \Delta_{\mathbb{H},h}^{-1} \alpha = \alpha.$$

Moreover, if $a < Q$, also

$$\Delta_{\mathbb{H},h}^{-1}\Delta_{\mathbb{H},h}\alpha = \alpha.$$

iii) *if $a = Q$, then for any $\alpha \in \mathcal{D}(\mathbb{H}^n, \mathbb{R}^{N_h})$ there exists $\beta_\alpha := (\beta_1, \dots, \beta_{N_h}) \in \mathbb{R}^{N_h}$, such that*

$$\Delta_{\mathbb{H},h}^{-1}\Delta_{\mathbb{H},h}\alpha - \alpha = \beta_\alpha.$$

The following lemma states that d_c and $\Delta_{\mathbb{H},h}$ commute.

Lemma 3.9 ([**?**], Lemma 3.14)**.** *We notice that the Laplace operator commutes with the exterior differential d_c. More precisely, if $\alpha \in \mathcal{C}^\infty(\mathbb{H}^n, E_0^h)$ and $n \geq 1$,*

i) $d_c\Delta_{\mathbb{H},h}\alpha = \Delta_{\mathbb{H},h+1}d_c\alpha, \qquad h = 0, 1, \dots, 2n, \qquad h \neq n-1, n.$
ii) $d_c d_c^* d_c\Delta_{\mathbb{H},n-1}\alpha = \Delta_{\mathbb{H},n}d_c\alpha, \qquad (h = n-1).$
iii) $d_c\Delta_{\mathbb{H},n}\alpha = d_c d_c^*\Delta_{\mathbb{H},n+1}d_c\alpha \qquad (h = n).$
iv) $d_c d_c^*\Delta_{\mathbb{H},n}\alpha = \Delta_{\mathbb{H},n}d_c d_c^*\alpha \qquad (h = n).$

Coherently with formula (18), the matrix-valued operator $\Delta_{\mathbb{H},h}^{-1}$ can be identified with an operator (still denoted by $\Delta_{\mathbb{H},h}^{-1}$) acting on smooth compactly supported differential forms in $\mathcal{D}(\mathbb{H}^n, E_0^h)$. Moreover, when the notation will not be misleading, we shall denote by $\Delta_{\mathbb{H},h}^{-1}$ its kernel.

The commutation of d_c and d_c^* with $\Delta_{\mathbb{H},h}^{-1}$ follows from the previous lemma:

Lemma 3.10 ([**?**], Lemma 3.15)**.** *If $\alpha \in \mathcal{D}(\mathbb{H}^n, E_0^h)$ and $n \geq 1$,*

i) $d_c\Delta_{\mathbb{H},h}^{-1}\alpha = \Delta_{\mathbb{H},h+1}^{-1}d_c\alpha, \qquad h = 0, 1, \dots, 2n, \qquad h \neq n-1, n+1.$
ii) $d_c\Delta_{\mathbb{H},n-1}^{-1}\alpha = d_c d_c^*\Delta_{\mathbb{H},n}^{-1}d_c\alpha \qquad (h = n-1).$
iii) $d_c d_c^* d_c\Delta_{\mathbb{H},n+1}^{-1}\alpha = \Delta_{\mathbb{H},n+2}^{-1}d_c\alpha, \qquad (h = n+1).$
iv) $d_c^*\Delta_{\mathbb{H},h}^{-1}\alpha = \Delta_{\mathbb{H},h-1}^{-1}d_c^*\alpha \qquad h = 1, \dots, 2n+1, \qquad h \neq n, n+2.$
v) $d_c^*\Delta_{\mathbb{H},n+2}^{-1}\alpha = d_c^* d_c\Delta_{\mathbb{H},n+1}^{-1}d_c^*\alpha \qquad (h = n+2).$
vi) $d_c^* d_c d_c^*\Delta_{\mathbb{H},n}^{-1}\alpha = \Delta_{\mathbb{H},n-1}^{-1}d_c^*\alpha, \qquad (h = n).$

4. Poincaré inequalities in Heisenberg groups

In [**?**] we gave the definition of what we meant by Poincaré inequalities for differential forms in $\mathbb{H}^n$. We also distinguished from global Poincaré inequalities versus its interior (or local) version. More precisely we give the following definitions.

Definition 4.1. *If $1 \leq h \leq 2n+1$ and $1 \leq p \leq q \leq \infty$, we say that the global $\mathbb{H}$-Poincaré$_{p,q}$ inequality holds in E_0^h if there exists a constant*

C such that, for every d_c-exact differential h-form ω in $L^p(\mathbb{H}^n; E_0^h)$ there exists a differential $(h-1)$-form ϕ in $L^q(\mathbb{H}^n, E_0^{h-1})$ such that $d_c\phi = \omega$ and

$$\|\phi\|_{L^q(\mathbb{H}^n, E_0^{h-1})} \leq C \|\omega\|_{L^p(\mathbb{H}^n, E_0^h)} \quad \text{global } \mathbb{H}\text{-Poincaré}_{p,q}(h). \tag{21}$$

Definition 4.2. *Given $1 \leq h \leq 2n+1$ and $1 \leq p \leq q \leq \infty$, we say that the interior $\mathbb{H}$-Poincaré$_{p,q}$ inequality holds in E_0^h if there exists $\lambda > 1$ such that, if we set $B := B(e,1)$ and $B_\lambda := B(e,\lambda)$, there exists a constant $C > 0$ such that, for every d_c-exact differential h-form ω in $L^p(B_\lambda; E_0^h)$ there exists a differential $(h-1)$-form ϕ in $L^q(B, E_0^{h-1})$ such that $d_c\phi = \omega$ and*

$$\|\phi\|_{L^q(B, E_0^{h-1})} \leq C \|\omega\|_{L^p(B_\lambda, E_0^h)} \quad \text{interior } \mathbb{H}\text{-Poincaré}_{p,q}(h). \tag{22}$$

In [**?**] we proved the validity of Poincaré inequalities when $p > 1$, and in [**?**] we dealt with the case $p = 1$. Finally, in [**?**] we proved the following end-point Poincaré inequalities:

Theorem 4.3. *If $2 \leq h \leq 2n$, we have:*

i) *if $h \neq n+1$, then the global $\mathbb{H}$-Poincaré$_{Q,\infty}(h)$ holds;*
ii) *if $h = n+1$, then the global $\mathbb{H}$-Poincaré$_{n+1,\infty}(n)$ holds.*

Analogous statements hold for interior Poincaré inequalities on $\mathbb{H}^n$.

The aim of the present paper is to improve Theorem 4.3 by proving the following two statements, the first one of which is a global estimate and the second one that is its local counterpart.

Theorem 4.4. *The following global Poincaré inequalities hold:*

i) *if $2 \leq h \leq 2n+1$, $h \neq n+1$, then a d_c-exact form $\omega \in L^Q(\mathbb{H}^n, E_0^h)$ admits a primitive in $\phi \in \mathcal{C}_0(\mathbb{H}^n, E_0^{h-1})$ such that*

$$\|\phi\|_{\mathcal{C}_0(\mathbb{H}^n, E_0^{h-1})} \leq C\|\omega\|_{L^Q(\mathbb{H}^n, E_0^h)};$$

ii) *a d_c-exact form $\omega \in L^{Q/2}(\mathbb{H}^n, E_0^{n+1})$ admits a primitive in $\phi \in \mathcal{C}_0(\mathbb{H}^n, E_0^n)$ such that*

$$\|\phi\|_{\mathcal{C}_0(\mathbb{H}^n, E_0^n)} \leq C\|\omega\|_{L^{Q/2}(\mathbb{H}^n, E_0^{n+1})}.$$

Remark 4.5. *Statement ii) of Theorem 4.4 will follow from a more general result presented in Theorem 8.10 below.*

We are able to prove also an interior Poincaré inequality which is a local version of previous theorem (i.e. that there exists a continuous primitive). In the following statements, exact means the d or d_c of a form with distributional coefficients.

Theorem 4.6. *There exists $\lambda > 1$ such that, if we set $B := B(e,1)$ and $B_\lambda := B(e,\lambda)$, there exists $C > 0$:*

i) *if* $2 \leq h \leq 2n+1$, $h \neq n+1$, *then a* d_c*-exact form* $\omega \in L^Q(B_\lambda, E_0^h)$ *admits a primitive in* $\phi \in \mathcal{C}(\overline{B}, E_0^{h-1})$ *such that*

$$\|\phi\|_{\mathcal{C}(\overline{B},E_0^{h-1})} \leq C\|\omega\|_{L^Q(B_\lambda,E_0^h)}; \tag{23}$$

ii) *a* d_c*-exact form* $\omega \in L^{Q/2}(B_\lambda, E_0^{n+1})$ *admits a primitive in* $\phi \in \mathcal{C}(\overline{B}, E_0^n)$ *such that*

$$\|\phi\|_{\mathcal{C}(\overline{B},E_0^n)} \leq C\|\omega\|_{L^{Q/2}(B_\lambda,E_0^{n+1})}. \tag{24}$$

A result for general compact Riemannian and contact subRiemannian manifolds follows.

Theorem 4.7. *Let* M *be a compact* m*-dimensional manifold.*

i) *First assume that* M *is equipped with a smooth Riemannian metric. There exists* $C > 0$ *such that every exact* h*-form* ω *on* M *with* L^m *coefficients,* $h \geq 2$*, admits a continuous primitive* ϕ *such that*

$$\|\phi\|_{\mathcal{C}(M,\bigwedge^{h-1})} \leq C\|\omega\|_{L^m(M,\bigwedge^h)}. \tag{25}$$

Next assume that $m = 2n+1$ *is odd, that* M *carries a contact structure and a smooth subRiemannian metric. Let* $Q = 2n+2$.

ii) *if* $2 \leq h \leq 2n+1$, $h \neq n+1$, *then every* d_c*-exact form* $\omega \in L^Q(M, E_0^h)$ *admits a primitive in* $\phi \in \mathcal{C}(M, E_0^{h-1})$ *such that*

$$\|\phi\|_{\mathcal{C}(M,E_0^{h-1})} \leq C\|\omega\|_{L^Q(M,E_0^h)}; \tag{26}$$

iii) *every* d_c*-exact form* $\omega \in L^{Q/2}(M, E_0^{n+1})$ *admits a primitive in* $\phi \in \mathcal{C}(M, E_0^n)$ *such that*

$$\|\phi\|_{\mathcal{C}(M,E_0^n)} \leq C\|\omega\|_{L^{Q/2}(M,E_0^{n+1})}. \tag{27}$$

5. Bourgain-Brezis' duality argument for the global Poincaré inequality

The basic idea for proving the Poincaré inequalities stated in Theorem 4.4 is a duality argument inspired by Bourgain-Brezis.

Let us collect here a few facts of functional analysis as stated in Brezis' book [**?**].

Theorem 5.1 ([**?**], Section 2.7)**.** *Let* $A : \mathcal{D}(A) \subset E \to F$ *be a closed unbounded operator between Banach spaces. Assume that* $\mathcal{D}(A)$ *is dense in* E*. Then the adjoint* $A^* : \mathcal{D}(A^*) \subset F^* \to E^*$ *is uniquely defined, and closed.*

1. The following are equivalent:

(1) $A(\mathcal{D}(A)) \subset F$ *is closed.*
(2) $A^*(\mathcal{D}(A^*)) \subset E^*$ *is closed.*
(3) $A(\mathcal{D}(A)) = \mathrm{Ker}(A^*)^\perp$.

(4) $A^*(\mathcal{D}(A^*)) = \mathrm{Ker}(A)^\perp$.

2. The following are equivalent:

(1) $\exists C, \forall f^* \in \mathcal{D}(A^*)$, $\|f^*\|_{F^*} \leq C \|A^* f^*\|_{E^*}$.
(2) $A(\mathcal{D}(A)) = F$.
(3) $A^*(\mathcal{D}(A^*))$ *is closed and* $\mathrm{Ker}(A^*) = 0$.

3. The following are equivalent:

(1) $\exists C, \forall e \in \mathcal{D}(A)$, $\|e\|_E \leq C \|Ae\|_F$.
(2) $A^*(\mathcal{D}(A^*)) = E^*$.
(3) $A(\mathcal{D}(A))$ *is closed and* $\mathrm{Ker}(A) = 0$.

Combining all items of Theorem 5.1, one gets

Lemma 5.2. *Let $A : \mathcal{D}(A) \subset E \to F$ be a closed unbounded operator between Banach spaces. Assume that $\mathcal{D}(A)$ is dense in E. Then the following are equivalent.*

(1) $A(\mathcal{D}(A)) \subset F$ *is closed.*
(2) $\exists C, \forall e^* \in A^*(\mathcal{D}(A^*))$, $\exists f^* \in \mathcal{D}(A^*)$, $A^* f^* = e^*$ *and* $\|f^*\|_{F^*} \leq C \|e^*\|_{E^*}$.
(3) $A^*(\mathcal{D}(A^*)) \subset E^*$ *is closed.*
(4) $\exists C, \forall f \in A(\mathcal{D}(A))$, $\exists e \in \mathcal{D}(A)$, $Ae = f$ *and* $\|e\|_E \leq C \|f\|_F$.

Proof. Let $\tilde{F}$ denote the closure of $A(\mathcal{D}(A))$ in F. Let $\mathcal{D}(B) = \mathcal{D}(A)$ and $B : \mathcal{D}(B) \to \tilde{F}$ coincide with A. Then B is a closed operator and its domain is dense. Since $A(\mathcal{D}(A))^\perp = \mathrm{Ker}(A)$, $\mathcal{D}(B^*)$ identifies with the quotient $\mathcal{D}(A^*)/\mathrm{Ker}(A^*)$, and A^* factors via this quotient, yielding $B^* : \mathcal{D}(B^*) \to E^*$. Since $\mathrm{Ker}(B^*) = 0$, Lemma 5.1 2 gives

$$B(\mathcal{D}(B)) = \tilde{F} \iff \exists C, \forall [f^*] \in \mathcal{D}(B^*),\ \|[f^*]\|_{F^*/\mathrm{Ker}(A^*)} \leq C \|B^*[f^*]\|_{E^*}.$$

Since $B(\mathcal{D}(B)) = A(\mathcal{D}(A))$, this translates into (1) $\iff$ (2).

Let $\tilde{E} = E/\mathrm{Ker}(A)$. Since A is closed, $\mathrm{Ker}(A)$ is a closed subspace of E, so $\tilde{E}$ is a Banach space. Let $\mathcal{D}(\tilde{B}) = \mathcal{D}(A)/\mathrm{Ker}(A)$. The operator A factors via this quotient, yielding $\tilde{B} : \mathcal{D}(\tilde{B}) \to F$ which is a closed operator, with dense domain. Since $\mathrm{Ker}(\tilde{B}) = 0$, Lemma 5.1 3 gives

$$\tilde{B}(\mathcal{D}(\tilde{B})) = \tilde{E} \iff \exists C, \forall [e] \in \mathcal{D}(\tilde{B}),\ \|[e]\|_{E/\mathrm{Ker}(A)} \leq C \|\tilde{B}[e]\|_F.$$

Since $\tilde{B}(\mathcal{D}(\tilde{B})) = A(\mathcal{D}(A))$, this translates into (3) $\iff$ (4).

By Lemma 5.1 1, (1) $\iff$ (3), so all four properties are equivalent.

□

Next on the paper, we apply these general facts to the exterior differential on spaces of differential forms. Namely, let us denote by $\mathcal{C}_0$ the Banach

space of continuous functions vanishing at infinity with the L^∞-norm; in this paper, we shall deal with the space

$$E := \mathcal{C}_0(\mathbb{H}^n, E_0^{h-1}),$$

and the operator A will be the Rumin's exterior differential d_c (in a suitable weak sense).

In any case, the first step we have to do, if we want to use a duality argument, is to identify the dual E^* of E; this is done in the next section where we prove that E^* can be identified with the set of currents with finite mass.

6. Currents and measures

Definition 6.1. *Let $0 \le h \le 2n+1$. If $\Omega \subset \mathbb{G}$ is an open set, we say that T is a h-current on Ω if T is a continuous linear functional on $\mathcal{D}(\Omega, E_0^h)$ endowed with the usual topology. We write $T \in \mathcal{D}'(\Omega, E_0^h)$ and we denote by $\phi \to \langle T|\phi\rangle$ its action on $\mathcal{D}(\Omega, E_0^h)$.*

The definition of $\mathcal{E}'(\Omega, E_0^h)$ (compactly supported currents) is given analogously.

If $T \in \mathcal{D}'(\Omega, E_0^h)$, $h \ge 1$, we define its boundary $\partial_c T \in \mathcal{D}'(\Omega, E_0^{h-1})$ by the identity

$$\langle \partial_c T|\phi\rangle = \langle T|d_c\phi\rangle \qquad \text{for all } \mathcal{D}(\Omega, E_0^{h-1}).$$

Proposition 6.2. *If $\Omega \subset \mathbb{G}$ is an open set, and $T \in \mathcal{D}'(\Omega)$ is a (usual) distribution, then T can be identified canonically with a $(2n+1)$-current $\tilde{T} \in \mathcal{D}'(\Omega, E_0^{2n+1})$ through the formula*

$$\langle \tilde{T}|\alpha\rangle := \langle T|*\alpha\rangle \tag{28}$$

for any $\alpha \in \mathcal{D}(\Omega, E_0^{2n+1})$. Conversely, by (28), any $(2n+1)$-current $\tilde{T}$ can be identified with an usual distribution $T \in D'(\Omega)$.

Proof. See [**?**], Section 17.5, and [**?**], Proposition 4. □

Following [**?**], 4.1.7, we give the following definition.

Definition 6.3. *If $T \in \mathcal{D}'(\Omega, E_0^{2n+1})$, and $\varphi \in \mathcal{E}(\Omega, E_0^h)$, with $0 \le h \le 2n+1$, we define $T \llcorner \varphi \in D'(\Omega, E_0^{2n+1-h})$ by the identity*

$$\langle T \llcorner \varphi|\alpha\rangle := \langle T|\alpha \wedge \varphi\rangle$$

for any $\alpha \in \mathcal{D}(\Omega, E_0^{2n+1-h})$.

The following result is taken from [**?**], Propositions 5 and 6, and Definition 10, but we refer also to [**?**], Sections 17.3, 17.4 and 17.5.

Definition 6.4. *Let $1 \le h \le 2n+1$. A form $\alpha \in L^1_{\mathrm{loc}}(\mathbb{H}^n, E_0^{2n+1-h})$ can be identified with a current $T_\alpha \in \mathcal{D}'(\mathbb{H}^n, E_0^h)$ through the action*

$$\langle T_\alpha | \phi \rangle := \int_{\mathbb{H}^n} \alpha \wedge \phi = \int_{\mathbb{H}^n} \phi \wedge \alpha \qquad \text{for } \phi \in \mathcal{D}(\mathbb{H}^{\cdot} E_0^h). \tag{29}$$

Indeed $\alpha \wedge \phi = (-1)^{h(2n+1-h)} \phi \wedge \alpha$, and $2n+1-h$ is even if h is odd.

Remark 6.5. *If $\alpha \in L^1_{\mathrm{loc}}(\mathbb{H}^n, E_0^{2n+1-h})$, we notice that $\partial_c T_\alpha = 0$ if and only if $d_c \alpha = 0$ in the sense of distributions. Indeed, if $\phi \in \mathcal{D}(\mathbb{H}^n, E_0^{h-1})$ (and hence $\star\phi \in \mathcal{D}(\mathbb{H}^n, E_0^{2n+2-h})$), keeping in mind Remark 6.5, we have*

$$\begin{aligned}
\langle \partial_c T_\alpha | \phi \rangle &= \langle T_\alpha | d_c \phi \rangle = \int_{\mathbb{H}^n} \alpha \wedge d_c \phi = \int_{\mathbb{H}^n} \langle \star\alpha, d_c \phi \rangle \, dV \\
&= \int_{\mathbb{H}^n} \langle \star\alpha, \star\star d_c \phi \rangle \, dV = \int_{\mathbb{H}^n} \langle \alpha, \star d_c \phi \rangle \, dV = \int_{\mathbb{H}^n} \langle \alpha, \star d_c \star\star\phi \rangle \, dV \\
&= (-1)^h \int_{\mathbb{H}^n} \langle \alpha, d_c^* \star \phi \rangle \, dV
\end{aligned}$$

Proposition 6.6. *If $\alpha \in L^1_{\mathrm{loc}}(\mathbb{H}^n, E_0^{2n+1-h}) \cap d_c^{-1}(L^1_{\mathrm{loc}}(\mathbb{H}^n, E_0^{2n+2-h}))$, then*

$$\partial_c T_\alpha = (-1)^{h-1} T_{d_c \alpha}.$$

Proof. If $\phi \in \mathcal{D}(\mathbb{H}^{\cdot} E_0^{h-1})$, then

$$\begin{aligned}
\langle \partial_c T_\alpha | \phi \rangle &= \langle T_\alpha | d_c \phi \rangle = \int_{\mathbb{H}^n} \alpha \wedge d_c \phi \\
&= (-1)^{h-1} \int_{\mathbb{H}^n} d_c \alpha \wedge \phi = (-1)^{h-1} \langle T_{d_c \alpha} | \phi \rangle.
\end{aligned}$$

□

The notion of convolution can be extended by duality to currents.

Definition 6.7. *Let $0 \le h \le 2n+1$ and $\varphi \in \mathcal{D}(\mathbb{H}^n)$. Let $T \in \mathcal{E}'(\mathbb{H}^n, E_0^h)$ be given, and denote by ${}^{\mathrm{v}}\varphi$ the function defined by ${}^{\mathrm{v}}\varphi(p) := \varphi(p^{-1})$. Then we set*

$$\langle \varphi * T | \phi \rangle := \langle T | {}^{\mathrm{v}}\varphi * \phi \rangle$$

for any $\phi \in \mathcal{D}(\mathbb{H}^n, E_0^h)$.

Lemma 6.8. *If $\alpha \in L^1_{\mathrm{loc}}(\mathbb{H}^n, E_0^h)$, $1 \le h \le 2n+1$, and $J = {}^{\mathrm{v}}J \in \mathcal{D}(\mathbb{H}^n)$, then*

$$J * \partial_c T_\alpha = \partial_c (J * T_\alpha) = \partial_c T_{J*\alpha}. \tag{30}$$

Proof. By definition, T_α is a $(2n+1-h)$-current. If $\phi \in \mathcal{D}(\mathbb{H}^n, E_0^{2n-h})$ is a test form, then, by Definition 6.7,

$$\begin{aligned}
\langle J * \partial_c T_\alpha | \phi \rangle &= \langle \partial_c T_\alpha | J * \phi \rangle = \langle T_\alpha | d_c (J * \phi) \rangle = \langle T_\alpha | J * d_c \phi \rangle \\
&= \langle J * T_\alpha | d_c \phi \rangle = \langle \partial_c (J * T_\alpha) | \phi \rangle.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle \partial_c T_{J*\alpha}|\phi\rangle &= \langle T_{J*\alpha}|d_c\phi\rangle = \int \langle \star(J*\alpha), d_c\phi\rangle dV \\ &= \int \langle J*(\star\alpha), d_c\phi\rangle dV = \int \langle \star\alpha, J*d_c(\phi)\rangle dV \\ &= \langle T_\alpha|J*d_c\phi\rangle = \langle J*T_\alpha|d_c\phi\rangle = \langle \partial_c(J*T_\alpha)|\phi\rangle.\end{aligned}$$

□

Definition 6.9. *If $T \in \mathcal{D}'(\mathbb{H}^n, E_0^h)$, we define its mass $\mathcal{M}(T)$ by*

$$\mathcal{M}(T) := \sup\{\langle T|\phi\rangle,\ \phi \in \mathcal{D}(\mathbb{H}^n, E_0^h),\ |\phi| \leq 1\}.$$

Lemma 6.10. *If $\alpha \in L^1(\mathbb{H}^n, E_0^{2n+1-h})$, then*

$$\mathcal{M}(T_\alpha) = \|\alpha\|_{L^1(\mathbb{H}^n, E_0^{2n+1-h})}. \tag{31}$$

Moreover, if $\alpha \in L^1_{\mathrm{loc}}(\mathbb{H}^n, E_0^{2n+1-h})$ and $\mathcal{M}(T_\alpha) < \infty$, then

$$\alpha \in L^1(\mathbb{H}^n, E_0^{2n+1-h}) \qquad \textit{and } (31) \textit{ holds.}$$

Proof. The first assertion is basically contained in Exercise 4.26 of [**?**]. Suppose now $\alpha \in L^1_{\mathrm{loc}}(\mathbb{H}^n, E_0^{2n+1-h})$ and $\mathcal{M}(T_\alpha) < \infty$. Consider now a sequence of compactly supported smooth functions $(\rho_k)_{k\in\mathbb{N}}$, $0 \leq \rho_k \leq 1$, $\rho_k \to 1$ in $\mathbb{H}^n$ as $k \to \infty$ and set $\alpha_k := \rho_k\alpha \in L^1(\mathbb{H}^n, E_0^{2n+1-h})$. If $\phi \in \mathcal{D}(\mathbb{H}^n, E_0^h)$, $\|\phi\|_{L^\infty} \leq 1$, then

$$\langle T_{\alpha_k}|\phi\rangle = \int_{\mathbb{H}^n} \rho_k\alpha \wedge \phi = \int_{\mathbb{H}^n} \alpha \wedge \rho_k\phi \leq \mathcal{M}(T_\alpha),$$

so that $\mathcal{M}(T_{\alpha_k}) \leq \mathcal{M}(T_\alpha)$, and the assertion follows by Fatou's lemma.

□

As for Federer-Fleming currents, the mass of currents is lower semicontinuous with respect to weak* convergence.

Lemma 6.11. *If $1 \leq h \leq 2n+1$, $T, T_k \in \mathcal{D}'(\mathbb{H}^n, E_0^h)$ for $k \in \mathbb{N}$ and $T_k \to T$ in $\mathcal{D}'(\mathbb{H}^n, E_0^h)$, then*

$$\mathcal{M}(T) \leq \liminf_k M(T_k).$$

If $1 \leq h \leq 2n+1$, and Ξ_0^h is a left invariant basis of E_0^h, then the linear maps on E_0^h

$$\alpha \to (\xi_j^h)^*(\alpha) := \star(\alpha \wedge \star\xi_j^h)$$

belong to $(E_0^h)^*$ (the dual of E_0^h as a finite-dimensional vector space) and

$$(\xi_j^h)^*(\xi_i^h) = \star(\xi_i^h \wedge \star\xi_j^h) = \delta_{i,j} \star dV = \delta_{i,j},$$

i.e. $(\Xi_0^h)^* = \{(\xi_1^h)^*, \dots, (\xi_{N_h}^h)^*\}$ is a left invariant dual basis of $(E_0^h)^*$.

Let us also remind the notion of distribution section of a finite-dimensional vector bundle $\mathcal{F}$: a distribution section is a continuous linear map on the space of compactly supported sections of the dual vector bundle $\mathcal{F}^*$ (see, e.g., [**?**], p. 77). Then we can give the following remark.

Remark 6.12. *Let T be a current on E_0^h,*

$$T = \sum_j \tilde{T}_j \llcorner (\star \xi_j^h),$$

where $T_1, \dots, T_{N_h} \in \mathcal{D}'(\Omega)$. Then T can be seen as a section of $(E_0^h)^*$. Indeed, if $\alpha = \sum_i \alpha_i \xi_i^h \in \mathcal{D}(\Omega, E_0^h)$

$$\begin{aligned}\langle T|\alpha\rangle &= \sum_j \langle \tilde{T}_j \llcorner (\star\xi_j^h)|\alpha\rangle = \sum_j \langle \tilde{T}_j|\alpha \wedge (\star\xi_j^h)\rangle \\ &= \sum_j \langle T_j|\alpha_j\rangle = \sum_{i,j} \langle T_j|(\xi_j^h)^*(\alpha_i\xi_i^h)\rangle = \sum_j \langle T_j|(\xi_j^h)^*(\alpha)\rangle,\end{aligned}$$

where the dualities in the first line are meant as dualities between currents and test forms, where the dualities in the second line are meant as dualities between distributions and test functions. Thus we can write formally

$$T = \sum_j T_j (\xi_j^h)^* \tag{32}$$

and we can identify T with a vector-valed distribution $(T_1, \dots, T_{N_h})$.

We notice also that, if $\alpha = \sum_j \alpha_j \xi_j \in \mathcal{E}(\Omega, E_0^h)$, then

$$T_\alpha = \sum_j \alpha_j (\xi_j^h)^*.$$

We are now in position to identify the dual of $\mathcal{C}_0(\mathbb{H}^n, E_0^h)$.

Remark 6.13. *Let $T \in \mathcal{D}'(\mathbb{H}^n, E_0^h)$ be a current of finite mass $\mathcal{M}(T)$. By density of $\mathcal{D}(\mathbb{H}^n, E_0^h)$ in $\mathcal{C}_0(\mathbb{H}^n, E_0^h)$, T can be continued as a linear bounded functional on $\mathcal{C}_0(\mathbb{H}^n, E_0^h)$ with norm $\mathcal{M}(T)$. For sake of simplicity we still denote by T its extension and by $\langle T|\phi\rangle_{\mathcal{C}_0^*,\mathcal{C}_0}$ the action of the extension on $\phi \in \mathcal{C}_0(\mathbb{H}^n, E_0^h)$. By Riesz' representation theorem (see e.g.* [**?**]*, Theorem 1.2.4) there exists a vector-valued Borel measure $(\mu_1, \dots, \mu_{N_h}) \in \mathcal{D}'(\mathbb{H}^n)^{N_h}$ such that, for any $\phi \in \mathcal{C}_0(\mathbb{H}^n, E_0^h)$ (identified as above with $(\phi_1, \dots, \phi_{N_h}) \in \mathcal{C}_0(\mathbb{H}^n)^{N_h}$),*

$$\langle T|\phi\rangle_{\mathcal{C}_0^*,\mathcal{C}_0} = \sum_j \int_{\mathbb{H}^n} \phi_j \, d\mu_j. \tag{33}$$

In addition

$$\mathcal{M}(T) = \|T\|_{\mathcal{C}_0^*} = |\mu|(\mathbb{H}^n). \tag{34}$$

Conversely, if $\mu = (\mu_1, \dots, \mu_{N_h})$ is a finite vector-valued Borel measure, the map $T : \mathcal{D}(\mathbb{H}^n, E_0^h) \to \mathbb{R}$ defined by

$$T : \sum_j \phi_j \xi_j \to \sum_j \int_{\mathbb{H}^n} \phi_j \, d\mu_j \tag{35}$$

is a h-current of finite mass $\mathcal{M}(T) = |\mu|(\mathbb{H}^n)$.

In particular, if $T \in C_0^*$ (and therefore admits an expression as in (35)*)*

$$\langle T|\phi\rangle_{\mathcal{C}_0^*,\mathcal{C}_0} = \langle T|\phi\rangle_{\mathcal{D}',\mathcal{D}} \quad \text{for all } \phi \in \mathcal{D}(\mathbb{H}^n, E_0^h). \tag{36}$$

6.1. **Poincaré and Sobolev inequalities for currents via an approximation result.** Theorem $1.1-(2)$ in [**?**] contains the following result.

Theorem 6.14 (Global Poincaré and Sobolev inequalities in degree $h \neq n+1$)**.** *Let $h = 1, \dots, 2n$, $h \neq n+1$. For every d_c-exact h-form $\omega \in L^1(\mathbb{H}^n, E_0^h)$, there exists an $(h-1)$-form $\phi \in L^{Q/(Q-1)}(\mathbb{H}^n, E_0^{h-1})$, such that*

$$d_c\phi = \omega \qquad \text{and} \qquad \|\phi\|_{L^{Q/(Q-1)}}(\mathbb{H}^n, E_0^{h-1}) \leq C \, \|\omega\|_{L^1(\mathbb{H}^n, E_0^h)}.$$

Furthermore, if ω is compactly supported, so is ϕ.

Remark 6.15. *As in* [**?**]*, Section 1.2, we stress that Poincaré inequality fails to hold in top degree (see also* [**?**]*).*

We show that the previous result can be reformulated in terms of currents. This is done after we prove the following approximation result (see also [**?**] for Euclidean currents).

Proposition 6.16. *If $1 \leq h \leq 2n+1$ and $T \in \mathcal{D}'(\mathbb{H}^n, E_0^h)$ is a current of finite mass $\mathcal{M}(T)$ (identified with a vector-valued measure as in Proposition 6.13), then there exists a family $(\omega^\epsilon)_{\epsilon>0}$ of forms in $\mathcal{E}(\mathbb{H}^n, E_0^h) \cap L^1(\mathbb{H}^n, E_0^h)$ such that, if we set $T_\epsilon := T_{\star\omega^\epsilon} \in \mathcal{D}'(\mathbb{H}^n, E_0^h)$, then*

i) *$\mathcal{M}(T_\epsilon) < \infty$ for all $\epsilon > 0$;*
ii) *$(T_\epsilon)_{s>0}$ converges weakly* to T as $\epsilon \to 0$;*
iii) *$\|\omega^\epsilon\|_{L^1(\mathbb{H}^n, E_0^h)} = \mathcal{M}(T_\epsilon) \to \mathcal{M}(T)$ as $\epsilon \to 0$;*
iv) *if $T = \partial_c S$, $S \in \mathcal{D}'(\mathbb{H}^n, E_0^{h+1})$, then $d_c(\star\omega^\epsilon) = 0$ for all $\epsilon > 0$.*

If T is compactly supported, then the ω^ϵ's are supported in a neighborhood of $\operatorname{supp} T$.

Proof. We denote by μ the finite vector-valued Borel measure associated with T as in Proposition 6.13. Arguing as in [**?**], Section 2.1, if $\epsilon > 0$ let $J_\epsilon = {}^{\mathrm{v}}J_\epsilon$ is an usual (group) Friedrichs' mollifier, we define the fuctions

$$f^\epsilon(\eta) := J_\epsilon * \mu(\eta) := \int_{\mathbb{H}^n} J_\epsilon(\eta \cdot p^{-1}) \, d\mu(p) = \int_{\mathbb{H}^n} J_\epsilon(p \cdot \eta^{-1}) \, d\mu(p).$$

We point out that the family of functions $\{f^\epsilon\,,\ \epsilon>0\}$ is bounded in $(L^1(\mathbb{H}^n))^{N_h}$. Indeed, if $j=1,\dots,N_h$

$$\begin{aligned}\int_{\mathbb{H}^n}|f^\epsilon(\eta)|\,d\eta &= \int_{\mathbb{H}^n}|\int_{\mathbb{H}^n}J_\epsilon(\eta\cdot p^{-1})\,d\mu(p)|\,d\eta\\ &\le \int_{\mathbb{H}^n}(\int_{\mathbb{H}^n}J_\epsilon(\eta\cdot p^{-1})\,d|\mu|(p))\,d\eta\\ &= \int_{\mathbb{H}^n}(\int_{\mathbb{H}^n}J_\epsilon(\eta\cdot p^{-1})\,d\eta)\,d|\mu|(p)=|\mu|(\mathbb{H}^n).\end{aligned}\tag{37}$$

We notice that, for all $\epsilon>0$, we can associate with f^ϵ the measure $\mu^\epsilon(p):=f^\epsilon\,d\mathcal{L}^{2n+1}$. Moreover, $f^\epsilon\in(\mathcal{E}(\mathbb{H}^n))^{N_h}$. Indeed, if $j=1,\dots,N_h$ and W is a horizontal vector field, then the following identity holds in the sense of distributions

$$W(J_\epsilon*\mu_j)=J_\epsilon*W\mu_j={}^{\mathrm{v}}W^{\mathrm{v}}J_\epsilon*\mu_j,$$

so that, arguing as in (37), the f^ϵ's are smooth.

We can also write

$$f^\epsilon=(J_\epsilon*\mu_1,\dots,J_\epsilon*\mu_{N_h})=:(f^\epsilon_1,\dots,f^\epsilon_{N_h}).$$

Thus we can define a family of smooth forms in $L^1(\mathbb{H}^n,E^h_0)$

$$\omega^\epsilon=\sum_j f^\epsilon_j\xi_j$$

and a family of h-currents $T_\epsilon:=T_{\star\omega^\epsilon}$. By (31),

$$\mathcal{M}(T_\epsilon)=\|\star\omega^\epsilon\|_{L^1(\mathbb{H}^n,E^h_0)}=\|\omega^\epsilon\|_{L^1(\mathbb{H}^n,E^h_0)}=|\mu^\epsilon|<\infty.\tag{38}$$

Thus i) is proved.

We notice now that for all test form $\phi=\sum_j\phi_j\xi_j\in\mathcal{D}(\mathbb{H}^n,E^h_0)$, if we identify ϕ with the vector-valued function $(\phi_1,\dots,\phi_{N_h})$, we have

$$\langle T|J_\epsilon*\phi\rangle=\int_{\mathbb{H}^n}\langle f^\epsilon,\phi\rangle\,d\mathcal{L}^{2n+1}=:\int_{\mathbb{H}^n}\langle\phi,d\mu^\epsilon\rangle,\tag{39}$$

that we can also written as

$$\langle T|J_\epsilon*\phi\rangle=\int_{\mathbb{H}^n}\langle\omega^\epsilon,\phi\rangle\,d\mathcal{L}^{2n+1}.\tag{40}$$

Indeed, first of all, we notice that if $j=1,\dots,N_h$,

$$\int_{\mathbb{H}^n}(\int_{\mathbb{H}^n}J_\epsilon(q)\,|\phi_j(q^{-1}p)|\,d\mu_j(p))\,dq<\infty.$$

Thus, by (33),

$$\begin{aligned}
\langle T|J_\epsilon * \phi\rangle &= \sum_j \int_{\mathbb{H}^n} (J_\epsilon * \phi_j)(p)\, d\mu_j(p) \\
&= \sum_j \int_{\mathbb{H}^n} \Big(\int_{\mathbb{H}^n} J_\epsilon(q)\phi_j(q^{-1}p)\, dq\Big)\, d\mu_j(p) \\
&= \sum_j \int_{\mathbb{H}^n} \Big(\int_{\mathbb{H}^n} J_\epsilon(p\eta^{-1})\phi_j(\eta)\, d\eta\Big)\, d\mu_j(p) \\
&= \sum_j \int_{\mathbb{H}^n} \Big(\int_{\mathbb{H}^n} J_\epsilon(p\eta^{-1})\, d\mu_j(p)\Big)\phi_j(\eta)\, d\eta \\
&= \sum_j \int_{\mathbb{H}^n} \Big(\int_{\mathbb{H}^n} J_\epsilon(\eta p^{-1})\, d\mu_j(p)\Big)\phi_j(\eta)\, d\eta \qquad \text{(since } J = {}^{\mathrm{v}}J) \\
&= \sum_j \int_{\mathbb{H}^n} (J_\epsilon * \mu_j)(\eta)\phi_j(\eta)\, d\eta = \int_{\mathbb{H}^n} \langle f^\epsilon, \phi\rangle(\eta)\, d\eta = \int_{\mathbb{H}^n} \langle \phi, d\mu^\epsilon\rangle.
\end{aligned}$$

This proves (39).

Let us prove ii). Take again $\phi \in \mathcal{D}(\mathbb{H}^n, E_0^h)$: By (40), we have:

$$\langle T_\epsilon|\phi\rangle = \langle T_{\star\omega^\epsilon}|\phi\rangle = \int_{\mathbb{H}^n} \star\omega^\epsilon \wedge \phi = \int_{\mathbb{H}^n} \langle \omega^\epsilon, \phi\rangle\, d\mathcal{L}^{2n+1} = \langle T|J_\epsilon * \phi\rangle \to \langle T|\phi\rangle$$

as $\epsilon \to 0$, since $J_\epsilon * \phi \to \phi$ in $\mathcal{D}(\mathbb{H}^n, E_0^h)$.

In particular, $\int_{\mathbb{H}^n} \langle f^\epsilon, \phi\rangle\, d\mathcal{L}^{2n+1} \to \langle T|\phi\rangle$ for all $\phi \in \mathcal{D}(\mathbb{H}^n, E_0^h)$. Thanks to the density of $\mathcal{D}(\mathbb{H}^n, E_0^h)$ in $C_{\mathrm{comp}}(\mathbb{H}^n, E_0^h)$, by (37), $\mu^\epsilon \to \mu$ weak* in the sense of measures.

Now, we can prove iii). By [**?**], Theorem 1.59

$$\mathcal{M}(T) = |\mu|(\mathbb{H}^n) \le \liminf_{\epsilon\to 0} |\mu^\epsilon(\mathbb{H}^n)| = \liminf_{\epsilon\to 0} \mathcal{M}(T_\epsilon).$$

On the other hand, by (37)

$$\limsup_{\epsilon\to 0} \mathcal{M}(T_\epsilon) \le \mathcal{M}(T),$$

and iii) follows.

Finally, let us prove that $\star\omega^\epsilon$ is d_c-closed. Take again $\phi \in \mathcal{D}(\mathbb{H}^n, E_0^{h+1})$. Keeping in mind that $\partial_c T = 0$, by (39) we have:

$$\begin{aligned}
\int_{\mathbb{H}^n} \langle d_c \star \omega^\epsilon, \phi\rangle\, d\mathcal{L}^{2n+1} &= \int_{\mathbb{H}^n} \langle \star\omega^\epsilon, d_c^*\phi\rangle\, d\mathcal{L}^{2n+1} = \pm \int_{\mathbb{H}^n} \langle \star\omega^\epsilon, \star d_c \star \phi\rangle\, d\mathcal{L}^{2n+1} \\
&= \pm \int_{\mathbb{H}^n} \langle \omega^\epsilon, d_c \star \phi\rangle\, d\mathcal{L}^{2n+1} = \pm\langle T|J_\epsilon * (d_c \star \phi)\rangle \\
&= \pm\langle T|d_c(J_\epsilon * (\star\phi))\rangle = \pm\langle \partial_c T|J_\epsilon * (\star\phi)\rangle = 0,
\end{aligned}$$

and hence $d_c(\star\omega^\epsilon) = 0$. Thus iv) is proved.

□

Thanks to previous result, theorem 6.14 can reformulated in terms of currents as follows:

Theorem 6.17 (Global Poincaré and Sobolev inequalities for currents)**.** *Let $h = 1, \dots, 2n$. If $h \neq n$ and $T \in \mathcal{D}'(\mathbb{H}^n, E_0^h)$ is a current of finite mass of the form $T = \partial_c S$ with $S \in \mathcal{D}'(\mathbb{H}^n, E_0^{h+1})$, then there exists a form $\phi \in L^{Q/(Q-1)}(\mathbb{H}^n, E_0^{2n-h})$, such that*

$$\partial_c T_\phi = T \qquad \textit{and} \qquad \|\phi\|_{L^{Q/(Q-1)}(\mathbb{H}^n, E_0^{2n-h})} \leq C\, \mathcal{M}(T).$$

Furthermore, if T is compactly supported, so is ϕ.

Proof. Let us prove a). By Proposition 6.16 with $\epsilon = \epsilon_k \to 0$ as $k \to \infty$, with the notations therein, for any $k \in \mathbb{N}$ there exists a sequence $(\star\omega_k)_{k\in\mathbb{N}} \in \mathcal{E}(\mathbb{H}^n, E_0^{2n+1-h}) \cap L^1(\mathbb{H}^n, E_0^{2n+1-h})$ satisfying i), ii), iii), iv). In particular, $d_c \star \omega_k = 0$. Therefore, since $2n + 1 - h \neq n + 1$ (by hypothesis $h \neq n$), Theorem 6.14 implies that there exist $\phi_k \in L^{Q/(Q-1)}(\mathbb{H}^n, E_0^{2n-h})$, such that

$$d_c\phi_k = \star\omega_k \qquad \text{and} \qquad \|\phi_k\|_{L^{Q/(Q-1)}(\mathbb{H}^n, E_0^{2n-h})} \leq C\, \|\star\omega_k\|_{L^1(\mathbb{H}^n, E_0^{2n+1-h})}.$$

By Proposition 6.16, iii), we know also that

$$\|\star\omega_k\|_{L^1(\mathbb{H}^n, E_0^{2n+1-h})} = \|\omega_k\|_{L^1(\mathbb{H}^n, E_0^h)} \to \mathcal{M}(T)$$

as $k \to \infty$. Therefore the ϕ_k's are equibounded in $L^{Q/(Q-1)}(\mathbb{H}^n, E_0^{h-1})$ and hence we can assume that $\phi_k \to \tilde{\phi}$ weakly in $L^{Q/(Q-1)}(\mathbb{H}^n, E_0^{2n-h})$.

Thus

$$\|\tilde{\phi}\|_{L^{Q/(Q-1)}(\mathbb{H}^n, E_0^{2n-h})} \leq C\, \mathcal{M}(T).$$

By definition, with $\phi := (-1)^h \tilde{\phi}$ we can associate the current $T_\phi \in \mathcal{D}'(\mathbb{H}^n, E_0^{h+1})$. Eventually, we are left with the proof of $\partial_c T_\phi = T$. Let $\psi \in \mathcal{D}(\mathbb{H}^n, E_0^h)$ be a test form. We have:

$$\begin{aligned}
\langle \partial_c T_\phi | \psi \rangle = \langle T_\phi | d_c\psi \rangle &= \int_{\mathbb{H}^n} \phi \wedge d_c\psi = (-1)^h \lim_{k\to\infty} \int_{\mathbb{H}^n} \phi_k \wedge d_c\psi \\
&= (-1)^{2h} \lim_{k\to\infty} \int_{\mathbb{H}^n} d_c\phi_k \wedge \psi = \lim_{k\to\infty} \int_{\mathbb{H}^n} \star\omega_k \wedge \psi \\
&= \lim_{k\to\infty} \langle T_{\star\omega_k} | \psi \rangle = \langle T | \psi \rangle,
\end{aligned}$$

by 6.16, ii). This completes the proof of the statement.

□

7. Proof of Theorem 4.4–i): continuous primitives of forms in E_0^h, $h \neq n+1$

The proof of Theorem 4.4 will follows by duality. Having in mind Section 5, we start with a few definitions.

Definition 7.1. *Denote again by $\mathcal{C}_0$ the Banach space of continuous functions vanishing at infinity with the L^∞-norm. We set*

$$E := \mathcal{C}_0(\mathbb{H}^n, E_0^{h-1}),$$

By Remark 6.13, the dual space E^* can be identified with the set of $(h-1)$-currents with finite mass. If $2 \leq h \leq 2n+1$, $h \neq n+1$, we set

$$\mathcal{D}(A) := \{\psi \in E,\ d_c\psi \in L^Q(\mathbb{H}^n, E_0^h)\} \subset E,$$

and $A : \mathcal{D}(A) \to F$, where

$$F = L^Q(\mathbb{H}^n, E_0^h) \qquad \text{and} \qquad A\psi := d_c\psi.$$

Remark 7.2. *Notice that $\mathcal{D}(A)$ is dense since contains $\mathcal{D}(\mathbb{H}^n, E_0^{h-1})$ and A is closed since is a differential operator.*

In addition F^* can be identified with $L^{Q/(Q-1)}(\mathbb{H}^n, E_0^{2n+1-h})$ through the identity: if $\beta \in L^{Q/(Q-1)}(\mathbb{H}^n, E_0^h)$, then we put

$$f(\alpha) = \int_{\mathbb{H}^n} \beta \wedge \alpha$$

for $\alpha \in L^Q(\mathbb{H}^n, E_0^h)$.

Lemma 7.3. *Suppose $h \neq n+1$. If $\psi \in \mathcal{D}(A)$, then there exists a sequence $(\psi_k)_{k\in\mathbb{N}}$ in $\mathcal{D}(\mathbb{H}^n, E_0^{h-1})$ such that*

$$\psi_k \to \psi \quad \text{in } E \qquad \text{and} \qquad d_c\psi_k \to d_c\psi \quad \text{in } F. \tag{41}$$

Proof. Let show preliminarily that there exist a sequence

$$(\psi_k)_{k\in\mathbb{N}} \quad \text{in} \quad \mathcal{C}_{\text{comp}}(\mathbb{H}^n, E_0^{h-1})$$

satisfying (41). Let us take a sequence of smooth cut-off functions $(\phi_k)_{k\in\mathbb{N}}$, $0 \leq \phi_k \leq 1$, supp $\phi_k \subset B(e, 2k)$, $\psi \equiv 1$ in $B(e,k)$, $|W\phi_k| \leq C/k$, where C is independent of k. Set $\psi_k := \phi_k\psi$. Obviously, $\psi_k \to \psi$ in E. By Leibniz' formula (see Lemma 3.3), if $h \neq n$,

$$\begin{aligned}
\|d_c\psi_k - d_c\psi\|_F^Q &\lessapprox \|(\phi_k - 1)d_c\psi\|_F^Q + \int_{\mathbb{H}^n} |W\phi_k|^Q\, |\psi|^Q\, dp \\
&\lessapprox \|(\phi_k - 1)d_c\psi\|_F^Q + (\frac{1}{k})^Q \int_{k\leq\|p\|\leq 2k} |\psi|^Q\, dp \\
&\leq \|(\phi_k - 1)d_c\psi\|_F^Q + (\frac{1}{k})^Q |B(e,2k)| \sup_{k\leq\|p\|} |\psi| \to 0
\end{aligned}$$

as $k \to \infty$, since $\psi \in E$ and $d_c\psi \in F$.

□

Proposition 7.4. *We have: $\mathcal{D}(A^*)$ is dense in F^* and*

$$\mathcal{D}(A^*) = \{\beta \in F^*,\ \mathcal{M}(\partial_c T_\beta) < \infty\}. \tag{42}$$

In addition, if $\beta \in \mathcal{D}(A^*)$,

$$A^*\beta = \partial_c T_\beta. \tag{43}$$

Proof. Since F is reflexive, then $\mathcal{D}(A^*)$ is dense in F^* (see [**?**], Remark 15 of Section 2.6). To prove (42) we have first to show that, if $\beta \in F^*$ and there exists c_β such that

$$|\int_{\mathbb{H}^n} \beta \wedge d_c\psi| \leq c_\beta \|\psi\|_E \qquad \text{for all } \psi \in \mathcal{D}(A), \tag{44}$$

then $\mathcal{M}(\partial_c T_\beta) < \infty$. The assertion is straightforward since, if $\psi \in \mathcal{D}(\mathbb{H}^n, E_0^{h-1}) \subset \mathcal{D}(A)$, $\|\psi\|_E \leq 1$ then

$$|\langle \partial_c T_\beta | \psi \rangle| = |\langle T_\beta | d_c\psi \rangle| = |\int_{\mathbb{H}^n} \beta \wedge d_c\psi| \leq c_\beta.$$

This shows that $\mathcal{D}(A^*) \subset \{\beta \in F^*,\ \mathcal{M}(\partial_c T_\beta) < \infty\}$.

Let us prove now the reverse inclusion. Suppose $\beta \in F^*$ is such that $\mathcal{M}(\partial_c T_\beta) < \infty$. If $\psi \in \mathcal{D}(A)$, by Lemma 7.3 there exists a sequence $(\psi_k)_{k\in\mathbb{N}}$ in $\mathcal{D}(\mathbb{H}^n, E_0^{h-1})$ such that

$$\psi_k \to \psi \quad \text{in } E \qquad \text{and} \qquad d_c\psi_k \to d_c\psi \quad \text{in } F.$$

Hence, (44) holds. Indeed,

$$\begin{aligned} |\int_{\mathbb{H}^n} \beta \wedge d_c\psi| &= \lim_k |\int_{\mathbb{H}^n} \beta \wedge d_c\psi_k| = \lim_k |\langle \partial_c T_\beta | \psi_k \rangle| \\ &\leq \mathcal{M}(\partial_c T_\beta) \limsup_k \|\psi_k\|_E = \mathcal{M}(\partial_c T_\beta) \|\psi\|_E. \end{aligned}$$

This completes the proof of (42).

Again following [**?**], Section 2.6, if $\beta \in \mathcal{D}(A^*)$, then $A^*\beta$ is uniquely determined by the identity

$$\langle A^*(\beta), \psi \rangle_{E^*,E} = \langle \beta, A\psi \rangle_{F^*,F} \qquad \text{for } \psi \in \mathcal{D}(A).$$

Taking $\psi \in \mathcal{D}(\mathbb{H}^n, E_0^{h-1})$, then (43) follows.

□

Proof of Theorem 4.4-i). With the notations of Definition 7.1, let us show preliminarily that

$$A^*(\mathcal{D}(A^*)) \qquad \text{is closed in } E^*. \tag{45}$$

To this end, let $(T_n)_{n\in\mathbb{N}}$ a sequence in $A^*(\mathcal{D}(A^*))$ that converges to a $h-1$-current $T \in E^*$ (i.e. in the mass norm). Hence, $\mathcal{M}(T_n) = \|T_n\|_{E^*} \leq C_1$ for all $n \in \mathbb{N}$. Moreover, in particular $T_n \to T$ weakly in $\mathcal{D}'$, i.e.

$$\langle T|\sigma\rangle = \lim_{n\to\infty} \langle T_n|\sigma\rangle$$

for all $\sigma \in \mathcal{D}(\mathbb{H}^n, E_0^{h-1})$. By (42) and (43) there exists a corresponding sequence $(S_n)_{n\in\mathbb{N}}$ in $\mathcal{D}'(\mathbb{H}^n, E_0^h)$ such that

$$T_n = \partial_c S_n, \quad \text{with } S_n = T_{\beta_n},\ \beta_n \in L^{Q/(Q-1)}(\mathbb{H}^n, E_0^{2n+1-h}) \tag{46}$$

for any $n \in \mathbb{N}$. Since the $(h-1)$-currents $\partial_c S_n$'s satisfy $\mathcal{M}(\partial_c S_n) = \mathcal{M}(T_n) < \infty$, by Theorem 6.17 (that we can apply since $1 \leq h-1 \leq 2n$ and $h-1 \neq n$), again for any $n \in \mathbb{N}$ there exists $\phi_n \in L^{Q/(Q-1)}(\mathbb{H}^n, E_0^{2n+1-h})$ such that $\partial_c T_{\phi_n} = \partial_c S_n = \partial_c T_{\beta_n}$, and

$$\|\phi_n\|_{L^{Q/(Q-1)}(\mathbb{H}^n, E_0^{2n+1-h})} \leq C_2 \mathcal{M}(\partial_c S_n) = C\mathcal{M}(T_n) \leq C_1 C_2.$$

Since $Q/(Q-1) > 1$ we cas assume that

$$\phi_n \to \phi \qquad \text{weakly in } L^{Q/(Q-1)}(\mathbb{H}^n, E_0^{2n+1-h}).$$

Thus $T_{\phi_n} \to T_\phi$ weakly* in $\mathcal{D}'$ and $T_n = \partial_c S_n = \partial_c T_{\phi_n} \to \partial_c T_\phi$ weakly* in $\mathcal{D}'$; therefore, since also $T_n \to T$ weakly*, it follows that $T = \partial_c T_\phi$. To prove that $T \in A^*(\mathcal{D}(A^*))$, by (42) we have only to show that $\mathcal{M}(\partial_c T_\phi) < \infty$. Because of the lower semicontinuity of the mass with respect to the weak* convergence, we have

$$\mathcal{M}(\partial_c T_\phi) \leq \liminf_{n\to\infty} \mathcal{M}(\partial_c S_n) = \liminf_{n\to\infty} \mathcal{M}(T_n) = \mathcal{M}(T).$$

Thus (45) is proved.

By Theorem 5.1, 1), (45) implies that

$$A(\mathcal{D}(A)) = (\ker A^*)^\perp. \tag{47}$$

Moreover, by Lemma 5.2, (45) implies that there exists $C > 0$ such that for all $f \in A(D(A))$ there exists $e \in D(A)$, satisfying

$$Ae = f \quad \text{and} \quad \|e\|_E \leq C\,\|f\|_F. \tag{48}$$

We are left to show that $\{\alpha \in L^Q(\mathbb{H}^n, E_0^h),\ d_c\alpha = 0\} \subset A(D(A))$. This will be done by showing that

$$\{\alpha \in L^Q(\mathbb{H}^n, E_0^h),\ d_c\alpha = 0\} \subset (\ker A^*)^\perp. \tag{49}$$

Indeed, let $\alpha \in L^Q(\mathbb{H}^n, E_0^h)$ be a closed form, and take $\beta \in L^{Q/(Q-1)}(\mathbb{H}^n, E_0^{2n+1-h}) \in \ker A^*$, i.e. $\beta \in L^{Q/(Q-1)}(\mathbb{H}^n, E_0^{2n+1-h})$ is a closed form, by Remark 6.5. Then, by Proposition 3.5,

$$\int_{\mathbb{H}^n} \alpha \wedge \beta = 0,$$

i.e. (49) holds.

Thus, combining (49) and (47),

$$\{\alpha \in L^Q(\mathbb{H}^n, E_0^h),\ d_c\alpha = 0\} \subset A(\mathcal{D}(A)),$$

and hence, by (48), we have proved the theorem.

□

8. The main global result in degree $n+1$: Poincaré inequalities with Beppo Levi-Sobolev spaces

The proof of our main result when $h = n+1$ is technically more delicate since in this case the equation $d_c\phi = \omega$ involves a differential operator of order two. In this case, we prove a more general result involving the Beppo Levi-Sobolev space $BL^{1,Q/Q-1}(\mathbb{H}^n)$ that implies, as a corollary, statement ii) of Theorem 4.4. Let us start by mentioning some general facts regarding Beppo Levi-Sobolev spaces $BL^{1,p}(\mathbb{H}^n)$.

8.1. Beppo Levi-Sobolev spaces $BL^{1,p}(\mathbb{H}^n)$.

Definition 8.1. *If $1 < p < Q$, we denote by $BL^{1,p}(\mathbb{H}^n)$ the homogeneous Sobolev space (called also Beppo Levi space) defined as the completion of $\mathcal{D}(\mathbb{H}^n)$ with respect to the norm*

$$\|u\|_{BL^{1,p}(\mathbb{H}^n)} := \sum_{j=1}^{2n} \|W_j u\|_{L^p(\mathbb{H}^n)}.$$

Remark 8.2. *Since $BL^{1,p}(\mathbb{H}^n)$ is reflexive, it can be identified with its bidual via the canonical isomorphism $\tau(u)(f) = f(u)$ for all $BL^{1,p}(\mathbb{H}^n)$ and $f \in (BL^{1,p}(\mathbb{H}^n))^*$.*

Proposition 8.3. *If $1 < p < Q$, then*

$$(BL^{1,p}(\mathbb{H}^n))^* = \{T \in \mathcal{D}'(\mathbb{H}^n)\,;\, T = \sum_j W_j f_j\,,\ f_j \in L^{p'}(\mathbb{H}^n)\}.$$

If $F = (f_1, \dots, f_{2n})$ is a horizontal vector field, then we set

$$\operatorname{div}_{\mathbb{H}} F := \sum_j W_j f_j.$$

By Sobolev inequality with sharp exponents in Carnot groups (see e.g., [**?**]) it is easy to prove the following theorem.

Theorem 8.4. *If $1 < p < Q$, we have: $u \in BL^{1,p}(\mathbb{H}^n)$ if and only if*

$$u \in L^{pQ/(Q-p)}(\mathbb{H}^n) \quad \textit{and} \quad W_j u \in L^p(\mathbb{H}^n),\ j = 1, \cdots, 2n,$$

and the two norms are equivalent, i.e.,

$$\|u\|_{BL^{1,p}(\mathbb{H}^n)} \approx \|u\|_{L^{pQ/(Q-p)}(\mathbb{H}^n)} + \sum_j \|W_j u\|_{L^p(\mathbb{H}^n)}.$$

In particular, the embedding $BL^{1,p}(\mathbb{H}^n) \subset L^{pQ/(Q-p)}(\mathbb{H}^n)$ is continuous.

For sake of completeness, let us provide a full proof of the above statement. Let us state preliminarily the following (trivial) result.

Lemma 8.5. *Let X be a Banach space, and $X_0 \subset X$ a dense subspace. Let Y_0 be a normed space and Y its completion. If $T_0 : X_0 \to Y_0$ is a linear isomorphism such that*

$$\|u\|_{X_0} \approx \|T_0 u\|_{Y_0} \qquad \text{for } u \in X_0, \tag{50}$$

then T_0 can be continued as a linear isomorphism T between X and Y.

Proof. Take $u \in X$; then there exists a sequence $(u_k)_{k\in N}$ in X_0 converging to u in X. Then, by (50), $(Tu_k)_{k\in N}$ is a Cauchy sequence in Y_0 converging to an element of Y. We set $Tu := \lim_k T_0 u_k$. Trivially, the continuation T is a well defined bounded linear operator between X and Y. Clearly, T is injective. On the other hand, take $v \in Y$. By definition, there exists a sequence $(v_k)_{k\in N}$ in Y_0 converging to v in Y. Again by (50), $(T_0^{-1} v_k)_{k\in N}$ has a limit $u \in X$. By continuity, $Tu = \lim_k T T_0^{-1} v_k = \lim_k T_0 T_0^{-1} v_k = v$. Therefore, T is also surjective and hence is a linear isomorphism.

□

Proof of Theorem 8.4. With the notation of previous Lemma 8.5, set

$$X := L^{pQ/(Q-p)}(\mathbb{H}^n) \cap \{u \in \mathcal{D}'(\mathbb{H}^n)\,, |Wu| \in L^p(\mathbb{H}^n)\},$$

where we set $|Wu| := \sum_{j=1}^{2n} |W_j u|$, and (with a slight abuse of notation) endowed with the norm

$$\|u\|_X = \|u\|_{L^{pQ/(Q-p)}(\mathbb{H}^n)} + \|Wu\|_{L^p(\mathbb{H}^n)}.$$

A standard argument shows that X is a Banach space. Let us prove that $X_0 := \mathcal{D}'(\mathbb{H}^n)$ is dense in X. We notice first that $\mathcal{E}(\mathbb{H}^n) \cap X$ is dense in X. Indeed, if $\epsilon > 0$, let $J_\epsilon = {}^{\mathrm{v}}J_\epsilon$ denote again a (group) Friedrichs' mollifier. If $u \in X$, then $J_\epsilon * u \in \mathcal{E}(\mathbb{H}^n)$, and

$$J_\epsilon * u \to u \qquad \text{in } L^{pQ/(Q-p)}(\mathbb{H}^n),$$

and

$$W(J_\epsilon * u) = J_\epsilon * Wu \to Wu \qquad \text{in } L^p(\mathbb{H}^n)^{2n}.$$

as $\epsilon \to 0$. Therefore, from now on we can assume $u \in \mathcal{E}(\mathbb{H}^n) \cap X$. Let us fix a sequence of cut-off functions $\{\chi_k\}_{i\in\mathbb{N}} \subset \mathcal{D}(\mathbb{H}^n)$ such that for any

$k \in \mathbb{N}$ $\mathrm{supp}(\chi_k) \subset B(e, 2k)$, $\chi_k \equiv 1$ in $B(e,k)$, and $k\,|W\chi_k| \leq C$ for all $k \in \mathbb{N}$. Set $u_k := \chi_k u$. Obviously,

$$u_k \to u \qquad \text{in } L^{pQ/(Q-p)}(\mathbb{H}^n) \text{ as } k \to \infty.$$

Moreover, since

$$|Wu_k| \leq C\frac{1}{k}|u| + \chi_k |Wu|,$$

then

$$|Wu_k| \to |Wu| \qquad \text{in } L^p(\mathbb{H}^n) \text{ as } k \to \infty.$$

Again with the notations of previous Lemma 8.5, we choose $Y_0 := \mathcal{D}(\mathbb{H}^n)$ endowed with the norm $\|u\|_{Y_0} := \||Wu|\|_{L^p(\mathbb{H}^n)}$. The inequality (50) is nothing but the sharp Poincaré inequality of [**?**], and the statement is proved. □

Definition 8.6. *If $1 \leq h \leq 2n+1$, then a form α belongs to $BL^{1,p}(\mathbb{H}^n, E_0^h)$ if and only if all its components with respect to a fixed left invariant basis*

$$\Xi_0^h = \{\xi_1^h, \dots \xi_{N_h}^h\}$$

of E_0^h belong to $BL^{1,p}(\mathbb{H}^n)$.

Proposition 8.7. *The dual space $(BL^{1,p}(\mathbb{H}^n, E_0^h))^*$ can be identified with a family of currents $T \in \mathcal{D}'(\mathbb{H}^n, E_0^h)$ such that, with the notation of* (32)

$$T = \sum_j T_j (\xi_j^h)^*,$$

with $T_j \in \mathcal{D}'(\mathbb{H}^n)$, $j = 1, \dots, N_h$ of the form

$$T_j = \mathrm{div}_{\mathbb{H}}\, F_j.$$

with $F_j \in (L^{p'})^{2n}$, $j = 1, \dots, N_h$.

More precisely, by the density of $\mathcal{D}(\mathbb{H}^n, E_0^h)$ in $BL^{1,p}(\mathbb{H}^n, E_0^h)$, an element of $(BL^{1,p}(\mathbb{H}^n, E_0^h))^*$ is fully identified by its restriction to $\mathcal{D}(\mathbb{H}^n, E_0^h)$. In particular, if $T \in (BL^{1,p}(\mathbb{H}^n, E_0^h))^* \subset \mathcal{D}'(\mathbb{H}^n, E_0^h)$ and, if for sake of simplicity we write $\mathcal{W} := BL^{1,p}(\mathbb{H}^n, E_0^h)$, we have

$$\langle T|\phi\rangle_{\mathcal{W}^*,\mathcal{W}} = \langle T|\phi\rangle_{\mathcal{D}',\mathcal{D}}. \tag{51}$$

Proof. Let $T \in (BL^{1,p}(\mathbb{H}^n, E_0^h))^*$ be given, and consider a sequence $(\phi_k)_{k\in\mathbb{N}}$ of test forms, $\phi_k \to 0$ in $\mathcal{D}(\mathbb{H}^n, E_0^h)$. Since there exists a compact set K such that $\mathrm{supp}\ \phi_k \subset K$ for all $k \in \mathbb{N}$ and $\phi_k \to 0$ uniformly with all its derivatives, then

$$\|\phi_k\|_{BL^{1,p}(\mathbb{H}^n, E_0^h)} \to 0 \qquad \text{as } k \to \infty,$$

so that $\langle T|\phi_k\rangle_{\mathcal{W}^*,\mathcal{W}} \to 0$ as $k \to \infty$. This proves that T is a h-current.

Take now $\phi \in \mathcal{D}(\mathbb{H}^n)$. If $1 \leq k \leq N_h$, define

$$\langle T_k|\phi\rangle_{\mathcal{D}',\mathcal{D}} = \langle T|\phi\xi_k^h\rangle_{\mathcal{W}^*,\mathcal{W}} \leq C\|\phi\xi_k^h\|_{BL^{1,p}(\mathbb{H}^n,E_0^h)} = C\|\phi\|_{BL^{1,p}(\mathbb{H}^n,E_0^h)},$$

so that $T_k \in (BL^{1,p}(\mathbb{H}^n, E_0^h))^*$. By Proposition 8.7 there exists a horizontal vector field in $F_k \in (L^{p'})^{2n}$ such that

$$T_k(\xi_k^h)^* = (\mathrm{div}_{\mathbb{H}}\, F_k)(\xi_k^h)^*$$

and

$$T = \sum_j (\mathrm{div}_{\mathbb{H}}\, F_j)(\xi_j^h)^*. \tag{52}$$

□

In the sequel, we will often use the identification given in (8.7), omitting the subscript.

The next two assertions follow straightforwardly from Theorem 8.4 .

Proposition 8.8. *If $\phi_k \to \phi$ weakly in $BL^{1,p}(\mathbb{H}^n, E_0^h)$, then*

$$T_{\phi_k} \to T_\phi \qquad \textit{weak* in } \mathcal{D}'(\mathbb{H}^n, E_0^{2n+1-h}).$$

Lemma 8.9. *If $\omega \in L^{Q/2}(\mathbb{H}^n, E_0^{n+1})$, then $T_\omega \in (BL^{1,Q/(Q-1)}(\mathbb{H}^n, E_0^n))^*$ and*

$$\|T_\omega\|_{(BL^{1,Q/(Q-1)}(\mathbb{H}^n,E_0^n))^*} \leq C\|\omega\|_{L^{Q/2}(\mathbb{H}^n,E_0^{n+1})}.$$

Proof. Let $\phi \in \mathcal{D}(\mathbb{H}^n, E_0^n)$, then

$$\begin{aligned}\langle T_\omega|\phi\rangle &= \int_{\mathbb{H}^n} \omega \wedge \phi \leq \|\omega\|_{L^{Q/2}(\mathbb{H}^n,E_0^{n+1})}\|\phi\|_{L^{Q/(Q-2)}(\mathbb{H}^n,E_0^n)} \\ &\leq C\|\omega\|_{L^{Q/2}(\mathbb{H}^n,E_0^{n+1})}\|\phi\|_{BL^{1,Q/(Q-1)}(\mathbb{H}^n,E_0^n)},\end{aligned}$$

by Theorem 8.4 □

8.2. **Poincaré inequalities with the Beppo Levi-Sobolev space: the case of primitive in E_0^n.** One of the main results of this paper is contained in the following theorem.

Theorem 8.10. *Let Ω be a n-current identified with an element of*

$$(BL^{1,Q/(Q-1)}(\mathbb{H}^n, E_0^n))^*$$

(i.e. of the form described in Proposition 8.7).

If $\partial_c\Omega = 0$, then there exists $\phi \in \mathcal{C}_0(\mathbb{H}^n, E_0^n)$ such that $\partial_c T_\phi = \Omega$ and

$$\|\phi\|_{\mathcal{C}_0(\mathbb{H}^n,E_0^n)} \leq C\|\Omega\|_{(BL^{1,Q/(Q-1)}(\mathbb{H}^n,E_0^n))^*}.$$

As a corollary we get:

Proof of Theorem 4.4-ii). By Lemma 8.9, the statement follows from Theorem 8.10. □

The first step in order to prove Theorem 8.10, is a result that contains an improvement of Theorem 1.1 (2) in [**?**] in the case $h = n + 1$.

Theorem 8.11 (Global Poincaré and Sobolev inequalities in degree $h = n+1$)**.** *For every d_c-exact form $\omega \in L^1(\mathbb{H}^n, E_0^{n+1})$, there exists an n-form*

$$\phi \in BL^{1,Q/(Q-1)}(\mathbb{H}^n, E_0^n)$$

such that

$$d_c\phi = \omega \qquad \text{and} \qquad \|\phi\|_{BL^{1,Q/(Q-1)}(\mathbb{H}^n,E_0^n)} \leq C\,\|\omega\|_{L^1(\mathbb{H}^n,E_0^{n+1})}.$$

Furthermore, if ω is compactly supported, so is ϕ.

The proof of Theorem 8.11 follows the same line as the proof of Theorem 1.1 (2) in [**?**], after proving the following Gagliardo-Nirenberg type inequality, which generalizes Theorem 1.6 in [**?**] (see also Theorem 5.1 in [**?**]).

Theorem 8.12. *Let u be a smooth compactly supported Rumin n-form on $\mathbb{H}^n$. Assume that $d_c^*u = 0$. Then*

$$\|u\|_{BL^{1,Q/(Q-1)}(\mathbb{H}^n,E_0^n)} \leq C\|d_cu\|_{L^1(\mathbb{H}^n,E_0^{n+1})}. \tag{53}$$

Proof. Remember, by Theorem 8.4,

$$\|u\|_{BL^{1,Q/(Q-1)}(\mathbb{H}^n,E_0^n)} \approx \|u\|_{L^{Q/(Q-2)}(\mathbb{H}^n,E_0^n)} + \|Wu\|_{L^{Q/(Q-1)}(\mathbb{H}^n,E_0^n)}$$

In addition, by Theorem 1.6 in [**?**], we already know that

$$\|u\|_{L^{Q/(Q-2)}(\mathbb{H}^n,E_0^n)} \leq C\|d_cu\|_{L^1(\mathbb{H}^n,E_0^{n+1})};$$

thus we have but to show that

$$\|Wu\|_{L^{Q/(Q-1)}(\mathbb{H}^n,E_0^n)} \leq C\|d_cu\|_{L^1(\mathbb{H}^n,E_0^{n+1})}. \tag{54}$$

Arguing as in [**?**], if $\psi \in \mathcal{D}(\mathbb{H}^n, E_0^n)$

$$\begin{aligned}
\langle Wu, \psi\rangle_{L^2(\mathbb{H}^n,E_0^n)} &= -\langle u, W\psi\rangle_{L^2(\mathbb{H}^n,E_0^n)} \\
&= -\langle u, \Delta_{\mathbb{H},n}\Delta_{\mathbb{H},n}^{-1}W\psi\rangle_{L^2(\mathbb{H}^n,E_0^n)} \\
&= -\langle u, (\delta_cd_c + (d_c\delta_c)^2)\Delta_{\mathbb{H},n}^{-1}W\psi\rangle_{L^2(\mathbb{H}^n,E_0^n)} \\
&= -\langle u, \delta_cd_c\Delta_{\mathbb{H},n}^{-1}W\psi\rangle_{L^2(\mathbb{H}^n,E_0^n)} \\
&\quad - \langle u, (d_c\delta_c)^2)\Delta_{\mathbb{H},n}^{-1}W\psi\rangle_{L^2(\mathbb{H}^n,E_0^n)} \\
&=: I_1 + I_2.
\end{aligned} \tag{55}$$

Consider first the term I_2. Since u is coclosed, $I_2 = 0$:

$$\langle u, (d_c\delta_c)^2)\Delta_{\mathbb{H},n}^{-1}W\psi\rangle_{L^2(\mathbb{H}^n,E_0^n)} = \langle d_c^*u, d_c^*d_cd_c^*)\Delta_{\mathbb{H},n}^{-1}W\psi\rangle_{L^2(\mathbb{H}^n,E_0^{n-1})} = 0.$$

Consider now the term I_1:

$$\langle u, d_c^*d_c\Delta_{\mathbb{H},n}^{-1}W\psi\rangle_{L^2(\mathbb{H}^n,E_0^n)} = \langle d_cu, d_c\Delta_{\mathbb{H},n}^{-1}W\psi\rangle_{L^2(\mathbb{H}^n,E_0^{n+1})}.$$

By Theorem 3.8 and Proposition 2.7, $d_c\mathcal{K}W$ is a kernel of type 1. On the other hand, as proved in [**?**], Theorem 5.1, the components with respect to a given basis of the closed forms $d_c u$ are linear combinations of the components of a horizontal vector field with vanishing horizontal divergence. Thus we can apply Theorem 2.9 and we get eventually

$$\begin{aligned}|\langle Wu,\psi\rangle_{L^2(\mathbb{H}^n,E_0^n)}| &\leq C\|d_c u\|_{L^1(\mathbb{H}^n,E_0^{n+1})}\cdot\|\nabla_{\mathbb{H}} d_c\Delta_{\mathbb{H},n}^{-1}W\psi\|_{L^Q(\mathbb{H}^n,E_0^{n+1})}\\ &\leq C\|d_c u\|_{L^1(\mathbb{H}^n,E_0^{n+1})}\cdot\|\psi\|_{L^Q(\mathbb{H}^n,E_0^n)},\end{aligned}$$

by Theorem 2.8-iii). Thus (54) follows.

□

Proof of Theorem 8.11. As in [**?**], we can take $\phi := d_c^*\Delta_{\mathbb{H},n+1}^{-1}\omega$. In particular, Theorems 5.2-iii) in [**?**] shows that

$$\|\phi\|_{L^{Q/(Q-2)}(\mathbb{H}^n,E_0^n)}\leq C\|\omega\|_{L^1(\mathbb{H}^n,E_0^{n+1})}.$$

Repeating verbatim the proof of Theorems 5.2-i) in [**?**], combined now with Theorem 8.12, we arrive to show that $W\phi$ satisfies

$$\|W\phi\|_{L^{Q/(Q-1)}(\mathbb{H}^n,E_0^n)}\leq C\|\omega\|_{L^1(\mathbb{H}^n,E_0^{n+1})}.$$

To this end, we replace u in (53) by a suitable compactly supported smooth approximation of ϕ (see formula (33) in [**?**]). This completes the proof of Theorem 8.11.

□

Remark 8.13. *It is important to stress that in the statement of Theorem 5.2 in* [**?**] *it is required that ω has vanishing average. However, Pansu & Tripaldi in* [**?**] *proved that such an assumption can be removed.*

Theorem 8.11 can reformulated in terms of currents as follows:

Theorem 8.14 (Global Poincaré and Sobolev inequalities for currents)**.** *Let $h = 1,\dots,2n$. If $T \in \mathcal{D}'(\mathbb{H}^n,E_0^n)$ is a current of finite mass of the form $T = \partial_c S$ with $S \in \mathcal{D}'(\mathbb{H}^n,E_0^{n+1})$, then there exists a form $\phi \in BL^{1,Q/(Q-1)}(\mathbb{H}^n,E_0^n)$, such that*

$$\partial_c T_\phi = T \qquad \textit{and} \qquad \|\phi\|_{BL^{1,Q/(Q-1)}(\mathbb{H}^n,E_0^n)}\leq C\,\mathcal{M}(T).$$

Furthermore, if T is compactly supported, so is ϕ.

The proof of this statement is omitted since it can be carried out analogously to the proof of Theorem 6.17 combining Proposition 6.16 and Theorem 8.11.

9. Some more approximation results

Lemma 9.1. *If $\beta \in BL^{1,Q/(Q-1)}(\mathbb{H}^n, E_0^n)$ and $\mathcal{M}(\partial_c T_\beta) < \infty$, then there exists a sequence $(\beta_k)_{k\in\mathbb{N}}$ in $\mathcal{D}(\mathbb{H}^n, E_0^h)$ such that*

i) *$\beta_k \to \beta$ as $k \to \infty$ in $BL^{1,Q/(Q-1)}(\mathbb{H}^n, E_0^n)$;*
ii) *$\mathcal{M}(\partial_c T_{\beta_k}) \to \mathcal{M}(\partial_c T_\beta)$ as $k \to \infty$.*

Proof. If $\epsilon > 0$, let $J_\epsilon = {}^{\mathrm{v}}J_\epsilon$ be a (group) Friedrichs' mollifier. Since $\beta \in L^{Q/(Q-2)}(\mathbb{H}^n, E_0^n)$ then

$$J_\epsilon * \beta \to \beta \qquad \text{in } L^{Q/(Q-2)}(\mathbb{H}^n, E_0^n).$$

In particular,

$$T_{J_\epsilon * \beta} \to T_\beta \qquad \text{in } \mathcal{D}'(\mathbb{H}^n, E_0^{n+1}). \tag{56}$$

In addition, since $W\beta \in L^{Q/(Q-1)}(\mathbb{H}^n, E_0^n)$

$$W(J_\epsilon * \beta) = J_\epsilon * W\beta \to W\beta \qquad \text{in } L^{Q/(Q-1)}(\mathbb{H}^n, E_0^n).$$

Thus

$$J_\epsilon * \beta \to \beta \qquad \text{in } BL^{1,Q/(Q-1)}(\mathbb{H}^n, E_0^n).$$

If ϕ is a test form, $\|\phi\|_{L^\infty} \leq 1$, we have now

$$\begin{aligned}
&\langle \partial_c(J_\epsilon * T_\beta)|\phi\rangle_{\mathcal{D}',\mathcal{D}} = \langle J_\epsilon * \partial_c T_\beta|\phi\rangle_{\mathcal{D}',\mathcal{D}} \qquad \text{(by (30))}\\
&= \langle \partial_c T_\beta | J_\epsilon * \phi\rangle_{\mathcal{D}',\mathcal{D}} \qquad \text{(by Definition 6.7)}\\
&\leq \mathcal{M}(\partial_c T_\beta)\|J_\epsilon * \phi\|_{L^\infty} \leq \mathcal{M}(\partial_c T_\beta)\|\phi\|_{L^\infty},
\end{aligned}$$

so that

$$\mathcal{M}(\partial_c(J_\epsilon * T_\beta)) \leq \mathcal{M}(\partial_c T_\beta).$$

On the other hand, by (56) and Lemma 6.11,

$$\mathcal{M}(\partial_c T_\beta) \leq \liminf_{\epsilon\to 0} \mathcal{M}(\partial_c(J_\epsilon * T_\beta)),$$

and then

$$\mathcal{M}(\partial_c(J_\epsilon * T_\beta)) \to \mathcal{M}(\partial_c T_\beta) \qquad \text{as } \epsilon \to 0. \tag{57}$$

Therefore, from now on we can assume $\beta \in \mathcal{E}(\mathbb{H}^n, E_0^n)$. Let us fix a sequence of cut-off functions $\{\chi_k\}_{i\in\mathbb{N}} \subset \mathcal{D}(\mathbb{H}^n)$ such that for any $k \in \mathbb{N}$ $\mathrm{supp}(\chi_k) \subset B(e, 2k)$, $\chi_k \equiv 1$ in $B(e,k)$, and $k\,|W\chi_k| + k^2\,|W^2\chi_k| \leq C$ for all $k \in \mathbb{N}$.

Set $\beta_k := \chi_k\beta \in \mathcal{D}(\mathbb{H}^n E_0^n)$. By Theorem 8.4 $\beta \in L^{Q/(Q-2)(\mathbb{H}^n, E_0^n)}$, and hence, obviously,

$$\beta_k \to \beta \qquad \text{in } L^{Q/(Q-2)(\mathbb{H}^n, E_0^n)} \text{ as } k \to \infty. \tag{58}$$

In particular,

$$T_{\beta_k} \to T_\beta \qquad \text{in } \mathcal{D}'(\mathbb{H}^n, E_0^{n+1}) \text{ as } k \to \infty. \tag{59}$$

Moreover, again by Theorem 8.4, if W is a horizontal derivative, then $|W\beta|\in L^{Q/(Q-1)}(\mathbb{H}^n, E_0^n)$. By dominated convergence, $\chi_k W\beta \to W\beta$ in $L^{Q/(Q-1)}(\mathbb{H}^n, E_0^n)$ as $k\to\infty$. On the other hand, again by Theorem 8.4, $\beta\in L^{Q/(Q-2)}(\mathbb{H}^n, E_0^n)$ so that

$$\begin{aligned}
&\int_{\mathbb{H}^n} (|W\chi_k|\,|\beta|)^{Q/(Q-1)}\,dV\\
&\le k^{-Q/(Q-1)}\int_{B(e,2k)\setminus B(e,k)} |\beta|^{Q/(Q-1)}\,dV\\
&\le k^{-Q/(Q-1)}\Big(\int_{B(e,2k)\setminus B(e,k)} |\beta|^{Q/(Q-2)}\,dV\Big)^{(Q-2)/(Q-1)}\cdot\,|B(e,2k)|^{1/(Q-1)}\\
&= C\Big(\int_{B(e,2k)\setminus B(e,k)} |\beta|^{Q/(Q-2)}\,dV\Big)^{(Q-2)/(Q-1)}\to 0 \qquad \text{as } k\to\infty.
\end{aligned}$$

Thus

$$W\beta_k = \chi_k W\beta + (W\chi_k)\beta \to W\beta \tag{60}$$

in $L^p(\mathbb{H}^n, E_0^n)$. Combining (58) and (60) statement i) is proved

Now, let us prove ii). If ψ is a test form, $\|\psi\|_{L^\infty}\le 1$,

$$\begin{aligned}
&\langle \partial_c T_{\beta_k}|\psi\rangle = \langle T_{\chi_k\beta}|d_c\psi\rangle = \int_{\mathbb{H}^n}\chi_k\beta\wedge d_c\psi = \int_{\mathbb{H}^n} d_c(\chi_k\beta)\wedge\psi\\
&= \int_{\mathbb{H}^n} d_c\beta\wedge\chi_k\psi + \int_{\mathbb{H}^n} P_0(W^2\chi_k)\beta\wedge\psi + \int_{\mathbb{H}^n} P_1(W\chi_k)\beta\wedge\psi\\
&\le \mathcal{M}(\beta)\|\chi_k\psi\|_{L^\infty} + \frac{1}{k^2}\|\psi\|_{L^\infty}\int_{B(e,2k)\setminus B(e,k)}|\beta|\,dV\\
&+ \frac{1}{k}\|\psi\|_{L^\infty}\int_{B(e,2k)\setminus B(e,k)}|W\beta|\,dV\\
&\le \mathcal{M}(\beta) + \frac{1}{k^2}\Big(\int_{B(e,2k)\setminus B(e,k)}|\beta|^{Q/(Q-2)}\,dV\Big)^{(Q-2)/Q}\cdot|B(e,2k)|^{2/Q}\\
&+ \frac{1}{k}\Big(\int_{B(e,2k)\setminus B(e,k)}|W\beta|^{Q/(Q-1)}\,dV\Big)^{(Q-1)/Q}\cdot|B(e,2k)|^{1/Q}\\
&\le \mathcal{M}(\beta) + C\Big(\int_{B(e,2k)\setminus B(e,k)}|\beta|^{Q/(Q-2)}\,dV\Big)^{(Q-2)/Q}\\
&+ C\Big(\int_{B(e,2k)\setminus B(e,k)}|W\beta|^{Q/(Q-1)}\,dV\Big)^{(Q-1)/Q}.
\end{aligned}$$

Taking the supremum with respect to ψ, it follows that

$$\mathcal{M}(\partial_c T_{\beta_k}) \leq \mathcal{M}(\beta) + C(\int_{B(e,2k)\setminus B(e,k)} |\beta|^{Q/(Q-2)}\, dV)^{(Q-2)/Q}$$
$$+ C(\int_{B(e,2k)\setminus B(e,k)} |W\beta|^{Q/(Q-1)}\, dV)^{(Q-1)/Q}.$$

By Theorem 8.4 $\beta \in L^{Q/(Q-2)}(\mathbb{H}^n, E_0^n)$, and hence

$$\int_{B(e,2k)\setminus B(e,k)} |\beta|^{Q/(Q-2)}\, dV \to 0 \qquad \text{as } k \to \infty.$$

In addition, $|W\beta| \in L^{Q/(Q-1)(\mathbb{H}^n, E_0^n)}$, and hence

$$\int_{B(e,2k)\setminus B(e,k)} |W\beta|^{Q/(Q-1)}\, dV \to 0 \qquad \text{as } k \to \infty$$

Thus

$$\limsup_k\ \mathcal{M}(\partial_c T_{\beta_k}) \leq \mathcal{M}(\beta).$$

On the other hand, combining (59) and Lemma 6.11,

$$\mathcal{M}(\beta) \leq \liminf_k\ \mathcal{M}(\partial_c T_{\beta_k})$$

and statement ii) follows.

□

10. Continuous primitives of forms in E_0^n: proof of Theorem 8.10

Let us consider the following abstract setting:

Definition 10.1. *We set*

$$E := \mathcal{C}_0(\mathbb{H}^n, E_0^n), \qquad F := (BL^{1,Q/(Q-1)}(\mathbb{H}^n, E_0^n))^*.$$
$$\mathcal{D}(A) := \{\psi \in E, \partial_c T_\psi \in F\}, \qquad A\psi := \partial_c T_\psi,$$

where, according to Proposition 8.7, we use the identification of $(BL^{1,p}(\mathbb{H}^n, E_0^n))^*$ *with a space of* $n-$*currents.*

Again by Remark 6.13, the dual space of E *can be identified with the set of* n*-currents with finite mass.*

Lemma 10.2. *The domain* $\mathcal{D}(A)$ *is dense in* E *and the map* $A : \mathcal{D}(A) \to F$ *is closed.*

Proof. The first assertion follows since $\mathcal{D}(\mathbb{H}^n, E_0^n) \subset \mathcal{D}(A)$. Moreover, if $\psi \in \mathcal{D}(\mathbb{H}^n, E_0^n)$, then $d_c\psi \in \mathcal{D}(\mathbb{H}^n, E_0^{n+1}) \subset L^{Q/2}(\mathbb{H}^n, E_0^{n+1})$. Hence, by Lemma 8.9, $T_{d_c\psi} \in F$. By Proposition 6.6, up to a sign, $T_{d_c\psi} = \partial_c T_\psi$, hence we can conclude that $\psi \in \mathcal{D}(A)$.

Suppose now $(\psi_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{D}(A)$ such that $\psi_k \to \psi$ in E and $\partial_c T_{\psi_k} \to T$ in F. By definition, $\mathcal{D}(\mathbb{H}^n)$ is dense in $BL^{1,Q/(Q-1)}(\mathbb{H}^n)$, hence for any $\varphi \in \mathcal{D}(\mathbb{H}^n, E_0^n)$ we have

$$\begin{aligned}\langle \partial_c T_\psi | \varphi\rangle &= \langle T_\psi | d_c\varphi\rangle = \int_{\mathbb{H}^n} \psi \wedge d_c\varphi \\ &= \lim_k \int_{\mathbb{H}^n} \psi_k \wedge d_c\varphi = \lim_k \langle T_{\psi_k} | d_c\varphi\rangle = \langle T|\varphi\rangle.\end{aligned}$$

Therefore $\partial_c T_\psi = T \in (BL^{1,Q/(Q-1)}(\mathbb{H}^n))^*$.

□

Lemma 10.3. *If $\psi \in \mathcal{D}(A)$, then there exists a sequence*

$$(\psi_k)_{k\in\mathbb{N}} \quad \textit{in} \quad \mathcal{E}(\mathbb{H}^n, E_0^n) \cap E$$

such that

$$\psi_k \to \psi \quad \textit{in } E \quad \textit{and} \quad A\psi_k \to A\psi \quad \textit{weakly in } F \tag{61}$$

as $k \to \infty$.

Proof. Consider a family of group Friedrichs' mollifiers $(J_\epsilon)_{\epsilon>0}$ with $J_\epsilon = {}^{\mathrm{v}}J_\epsilon$. Obviously $J_\epsilon * \psi \in \mathcal{E}(\mathbb{H}^n, E_0^n) \cap E$. In addition, if we take $\varphi \in \mathcal{D}(\mathbb{H}^n, E_0^n)$, $\|\varphi\|_{BL^{1,Q/(Q-1)}(\mathbb{H}^n)(\mathbb{H}^n, E_0^n)} \leq 1$, by (30),

$$\begin{aligned}\langle \partial_c T_{J_\epsilon * \psi} | \varphi\rangle &= \langle J_\epsilon * \partial_c T_\psi | \varphi\rangle = \langle \partial_c T_\psi | J_\epsilon * \varphi\rangle \\ &\leq \|\partial_c T_\psi\|_F \|J_\epsilon * \varphi\|_{BL^{1,Q/(Q-1)}(\mathbb{H}^n, E_0^n)} \leq C\|\partial_c T_\psi\|_F.\end{aligned}$$

Therefore $\{\partial_c T_{J_\epsilon * \psi}\}$ is bounded in F and then we may assume that there is a subsequence weakly convergent in F. Then, since A is closed, (61) follows.

□

Proposition 10.4. *The domain $\mathcal{D}(A^*)$ is dense in F^* and*

$$\mathcal{D}(A^*) = \{\tau(\beta) \,:\, \beta \in BL^{1,Q/(Q-1)}(\mathbb{H}^n, E_0^n)\,; \mathcal{M}(\partial_c T_\beta) < \infty\}. \tag{62}$$

In addition, if $\beta \in \mathcal{D}(A^*)$, then, by definition, for any $\psi \in \mathcal{D}(A)$

$$\langle A^*\beta | \psi\rangle_{E^*,E} = \langle \beta | A\psi\rangle_{F^*,F}.$$

In particular, if $\psi \in \mathcal{D}(\mathbb{H}^n.E_0^n)$, then

$$\langle A^*\tau(\beta)|\psi\rangle_{E^*,E} = \langle A^*(\tau(\beta))|\psi\rangle_{\mathcal{D}',\mathcal{D}} = \langle \partial_c T_\beta | \psi\rangle_{\mathcal{D}',\mathcal{D}}, \tag{63}$$

i.e.

$$A^*(\tau(\beta)) = \partial_c T_\beta$$

in the sense of currents.

Proof. Since F is reflexive, then $\mathcal{D}(A^*)$ is dense in F^* (see [**?**], Remark 15 of Section 2.6). In the sequel, to avoid cumbersome notations, we shall also write

$$\mathcal{W} := BL^{1,Q/(Q-1)}(\mathbb{H}^n, E_0^n),$$

so that $F = \mathcal{W}^*$. To prove (62) we first show that $\mathcal{D}(A^*) \subset \{\tau(\beta)\ :, \beta \in \mathcal{W}\,; \mathcal{M}(\partial_c T_\beta) < \infty\}$ that is, if $\tau(\beta) \in F^*$ and there exists c_β such that

$$|\langle \tau(\beta)|A\psi\rangle_{F^*,F}| \leq c_\beta \|\psi\|_E \qquad \text{for all } \psi \in \mathcal{D}(A), \tag{64}$$

then $\mathcal{M}(\partial_c T_\beta) < \infty$.

First of all, Remark 8.2 says that

$$\langle \tau(\beta)|A\psi\rangle_{F^*,F} = \langle A\psi|\beta\rangle_{\mathcal{W}^*,\mathcal{W}}, \tag{65}$$

so that (64) becomes

$$|\langle \partial_c T_\psi|\beta\rangle_{\mathcal{W}^*,\mathcal{W}}| = |\langle A\psi|\beta\rangle_{\mathcal{W}^*,\mathcal{W}}| \leq c_\beta \|\psi\|_E \qquad \text{for all } \psi \in \mathcal{D}(A). \tag{66}$$

Now, if $\beta \in \mathcal{W}$ satisfies (66), by definition there exists a sequence $(\beta_N)_{N\in\mathbb{N}}$ of test forms converging to β in $\mathcal{W}$, and then, by Theorem 8.4, converging to β also in $L^1_{\text{loc}}(\mathbb{H}^n, E_0^n)$. Thus,

$$T_{\beta_N} \to T_\beta \qquad \text{in } \mathcal{D}'(\mathbb{H}^n, E_0^{n+1})$$

and

$$\partial_c T_{\beta_N} \to \partial_c T_\beta \qquad \text{in } \mathcal{D}'(\mathbb{H}^n, E_0^n).$$

Take now $\varphi \in \mathcal{D}(\mathbb{H}^n, E_0^n)$, $\|\varphi\|_E \leq 1$. By Lemma 3.4 we have:

$$\begin{aligned}
|\langle \partial_c T_\beta|\varphi\rangle_{\mathcal{D}',\mathcal{D}}| &= |\lim_N \langle \partial_c T_{\beta_N}|\varphi\rangle_{\mathcal{D}',\mathcal{D}}| = |\lim_N \langle T_{\beta_N}|d_c\varphi\rangle_{\mathcal{D}',\mathcal{D}}| \\
&= |\lim_N \int_{\mathbb{H}^n} \beta_N \wedge d_c\varphi| = |\lim_N \int_{\mathbb{H}^n} d_c\beta_N \wedge \varphi| \\
&= |\lim_N \langle T_\varphi|d_c\beta_N\rangle_{\mathcal{D}',\mathcal{D}}| = |\lim_N \langle \partial_c T_\varphi|\beta_N\rangle_{\mathcal{D}',\mathcal{D}}| \\
&= |\lim_N \langle A\varphi|\beta_N\rangle_{\mathcal{D}',\mathcal{D}}| = |\lim_N \langle A\varphi|\beta_N\rangle_{\mathcal{W}^*,W}| \\
&\qquad \text{(by (51), since } \partial_c T_\varphi = \pm T_{d_c\varphi}, \text{ and } d_c\varphi \in L^1_{\text{loc}}) \\
&= |\langle A\varphi|\beta\rangle_{\mathcal{W}^*,\mathcal{W}}| \leq c_\beta \|\varphi\|_E \leq c_\beta,
\end{aligned}$$

by (66), since $\mathcal{D}(\mathbb{H}^n, E_0^n) \subset \mathcal{D}(A)$. Taking the supremum with respect to φ we conclude that $\mathcal{M}(\partial_c T_\beta) < \infty$.

Now we have to prove the reverse inclusion: take $\beta \in BL^{1,Q/(Q-1)}(\mathbb{H}^n, E_0^n)$ such that $\mathcal{M}(\partial_c T_\beta) < \infty$. We must prove that (64) holds. By Lemma 10.3, there exists a sequence $(\psi_k)_{k\in\mathbb{N}}$ in $\mathcal{E}(\mathbb{H}^n, E_0^n) \cap E$ such that

$$\psi_k \to \psi \quad \text{in } E \quad \text{and} \quad A\psi_k \to A\psi \quad \text{weakly in } F \tag{67}$$

as $k \to \infty$. Hence

$$|\langle \tau(\beta)|A\psi\rangle_{F^*,F}| = \lim_k |\langle \tau(\beta)|A\psi_k\rangle_{F^*,F}|.$$

Suppose for a while we have proved (64) for $\psi_k \in \mathcal{E}(\mathbb{H}^n, E_0^n) \cap \mathcal{D}(A)$; then

$$|\langle \tau(\beta)|A\psi_k\rangle_{F^*,F}| \leq c_\beta \lim_k \|\psi_k\|_E = c_\beta \|\psi\|_E,$$

i.e. (64) holds for any $\psi \in E$. Therefore, we may suppose $\psi \in \mathcal{E}(\mathbb{H}^n, E_0^n) \cap \mathcal{D}(A)$.

By definition of $\mathcal{D}(A)$, $A\psi \in \mathcal{W}^*$. Let $(\beta_k)_{k\in\mathbb{N}}$ the approximation of β of Lemma 9.1. By (51), we have

$$\begin{aligned} |\langle \tau(\beta)|A\psi\rangle_{F^*,F}| &= |\langle A\psi|\beta\rangle_{\mathcal{W}^*,\mathcal{W}}| = \lim_k |\langle A\psi|\beta_k\rangle_{\mathcal{W}^*,\mathcal{W}}| \\ &= \lim_k |\langle A\psi|\beta_k\rangle_{\mathcal{D}',\mathcal{D}}| = \lim_k |\langle \partial_c T_\psi|\beta_k\rangle_{\mathcal{D}',\mathcal{D}}| \\ &= \lim_k |\langle T_\psi|d_c\beta_k\rangle_{\mathcal{D}',\mathcal{D}}| = \lim_k |\int_{\mathbb{H}^n} \psi \wedge d_c\beta_k|. \end{aligned} \tag{68}$$

Let now $k \in \mathbb{N}$ be fixed for a while, and let $\chi \in \mathcal{D}(\mathbb{H}^n)$, $0 \leq \chi \leq 1$ be a cut-off function, $\chi \equiv 1$ on $\operatorname{supp} \beta_k$. Then, keeping in mind that $\chi\psi$ is a test form,

$$\begin{aligned} |\int_{\mathbb{H}^n} \psi \wedge d_c\beta_k| &= |\int_{\mathbb{H}^n} (\chi\psi) \wedge d_c\beta_k| \\ &= |\int_{\mathbb{H}^n} d_c(\chi\psi) \wedge \beta_k| \qquad \text{(by Lemma 3.4)} \\ &= |\langle T_{\beta_k}|d_c(\chi\psi)\rangle| \qquad \text{(by (29))} \\ &= |\langle \partial_c T_{\beta_k}|\chi\psi\rangle| \leq \mathcal{M}(\partial_c T_{\beta_k})\|\chi\psi\|_E \leq \mathcal{M}(\partial_c T_{\beta_k})\|\psi\|_E. \end{aligned} \tag{69}$$

Thus, combining (69) with (68) and keeping in mind Lemma 9.1, ii), we obtain eventually

$$|\langle \tau(\beta)|A\psi\rangle_{F^*,F}| \leq \mathcal{M}(\partial_c T_\beta)\|\psi\|_E,$$

i.e. (64) is proved.

We are now left to prove (63). First, we remind that, following [**?**], Section 2.6, if $\tau(\beta) \in \mathcal{D}(A^*)$, then $A^*(\tau(\beta))$ is uniquely determined by the identity

$$\langle A^*(\tau(\beta))|\psi\rangle_{E^*,E} = \langle \tau(\beta)|A\psi\rangle_{F^*,F} \qquad \text{for } \psi \in \mathcal{D}(A).$$

Finally, take $\psi \in \mathcal{D}(\mathbb{H}^n, E_0^h)$. By Lemma 9.1, i) there exists a sequence compactly supported smooth forms $(\beta_k)_{k\in\mathbb{N}}$ converging to β in $\mathcal{W}$. Now,

by (65),

$$\begin{aligned}
&\langle A^*(\tau(\beta))|\psi\rangle_{E^*,E} = \langle \tau(\beta)|A\psi\rangle_{F^*,F} = \langle A\psi|\beta\rangle_{\mathcal{W}^*,\mathcal{W}} \\
&= \lim_k \langle A\psi|\beta_k\rangle_{\mathcal{W}^*,\mathcal{W}} = \lim_k \langle A\psi|\beta_k\rangle_{\mathcal{D}',\mathcal{D}} \\
&= \lim_k \langle \partial_c T_\psi|\beta_k\rangle_{\mathcal{D}',\mathcal{D}} = \lim_k \int_{\mathbb{H}^n} \psi \wedge d_c\beta_k \\
&= \lim_k \int_{\mathbb{H}^n} d_c\psi \wedge \beta_k = \int_{\mathbb{H}^n} d_c\psi \wedge \beta = \langle \partial_c T_\beta|\psi\rangle_{\mathcal{D}',\mathcal{D}},
\end{aligned}$$

and we are done.

□

In order to prove Theorem 8.11 we need a last result, that is akin to Lemma 3.5 which was used in the proof of Theorem 4.4. To avoid cumbersome notations, let us write again $\mathcal{W} := BL^{1,Q/(Q-1)}(\mathbb{H}^n, E_0^n)$.

Lemma 10.5. *Suppose* $\beta \in BL^{1,Q/(Q-1)}(\mathbb{H}^n, E_0^n)$ *and* $\Omega \in (BL^{1,Q/(Q-1}(\mathbb{H}^n, E_0^n))^*$ *are such that*

$$\partial_c T_\beta = 0 \qquad \textit{and} \qquad \partial_c \Omega = 0.$$

Then

$$\langle \Omega|\beta\rangle_{\mathcal{W}^*,\mathcal{W}} = 0.$$

Proof. By Remark 6.5, $d_c\beta = 0$ in the sense of distributions. If $\epsilon > 0$, let $J_\epsilon = {}^{\mathrm{v}}J_\epsilon$ be a (group) Friedrichs' mollifier. Then

$$J_\epsilon * \beta \to \beta \quad \text{in} \quad BL^{1,Q/(Q-1)}(\mathbb{H}^n, E_0^n)$$

and

$$d_c(J_\epsilon * \beta) = 0.$$

Thus, without lack of generality, we can assume $\beta \in \mathcal{W} \cap \mathcal{E}(\mathbb{H}^n, E_0^n)$. In particular, $\beta \in L^{Q/(Q-2)}(\mathbb{H}^n, E_0^n)$. Keeping in mind Theorem 2.8, let us set

$$\phi := d^* d_c d_c^* \Delta_c^{-1} \beta \in L^{Q/(Q-3)}(\mathbb{H}^n, E_0^{n-1}),$$

since $d^* d_c d_c^* \Delta_c^{-1}$ is a kernel of type 1. As in [**?**], Section 1.6,

$$d_c \phi = \beta.$$

Let us fix a sequence of cut-off functions $\{\chi_N\}_{N\in\mathbb{N}} \subset \mathcal{D}(\mathbb{H}^n)$ such that for any $N \in \mathbb{N}$, $\mathrm{supp}(\chi_N) \subset B(e, 2N)$, $\chi_N \equiv 1$ in $B(e, N)$, and

$$N\,|W\chi_N| + N^2\,|W^2\chi_N| \leq C$$

for all $N \in \mathbb{N}$.

We notice that $\chi_N \beta \in \mathcal{D}(\mathbb{H}^n, E_0^n)$ and

$$\chi_N \beta \to \beta \qquad \text{in } \mathcal{W}.$$

The proof of this fact is already done in Lemma 9.1.

Thus we can write

$$\langle \Omega|\beta\rangle_{\mathcal{W}^*,\mathcal{W}} = \lim_N \langle \Omega|\chi_N\beta\rangle_{\mathcal{W}^*,\mathcal{W}} = \lim_N \langle \Omega|\chi_N d_c\phi\rangle_{\mathcal{W}^*,\mathcal{W}}.$$

Therefore, the statement of the Lemma will follow by showing that

$$\lim_N \langle \Omega|\chi_N d_c\phi\rangle_{\mathcal{W}^*,\mathcal{W}} = 0. \tag{70}$$

First of all, by Lemma 3.3,

$$d_c(\chi_N\phi) = \chi_N d_c\phi + P_0^{n-1}(W\chi_N)\phi,$$

where $P_0^{n-1}(W\chi_N) : E_0^{n-1} \to E_0^n$ is a linear homogeneous differential operator of order zero with coefficients depending only on the horizontal derivatives of χ_N. Thus, by (51),

$$\begin{aligned}
&\langle \Omega|\chi_N d_c\phi\rangle_{\mathcal{W}^*,\mathcal{W}} \\
&\quad = \langle \Omega|d_c(\chi_N\phi)\rangle_{\mathcal{W}^*,\mathcal{W}} - \langle \Omega|P_0^{n-1}(W\chi_N)\phi\rangle_{\mathcal{W}^*,\mathcal{W}} \\
&\quad = \langle \Omega|d_c(\chi_N\phi)\rangle_{\mathcal{D}',\mathcal{D}} - \langle \Omega|P_0^{n-1}(W\chi_N)\phi\rangle_{\mathcal{W}^*,\mathcal{W}} \\
&\quad = \langle \partial_c\Omega|\chi_N\phi\rangle_{\mathcal{D}',\mathcal{D}} - \langle \Omega|P_0^{n-1}(W\chi_N)\phi\rangle_{\mathcal{W}^*,\mathcal{W}} \\
&\quad = -\langle \Omega|P_0^{n-1}(W\chi_N)\phi\rangle_{\mathcal{W}^*,\mathcal{W}},
\end{aligned} \tag{71}$$

since $\partial_c\Omega = 0$.

We notice now that $P_0^{n-1}(W\chi_N)\phi$ can be seen as a linear combination of the element of the basis Ξ_0^n of E_0^n of the form $(W_k\chi_N)\phi_\ell$, where the ϕ_ℓ are coordinates of ϕ with respect to the basis Ξ_0^{n-1}.

Thus, in order to prove (70), we have but to show that

$$(W_k\chi_N)\phi_\ell \to 0 \quad \text{in } BL^{1,Q/(Q-1)}(\mathbb{H}^n). \tag{72}$$

The proof of (72) will be articulated in three steps.

Step 1. Consider first

$$\begin{aligned}
&\|(W_k\chi_N)\phi_\ell\|^{Q/(Q-2)}_{L^{Q/(Q-2)}(\mathbb{H}^n)} \le N^{-Q/(Q-2)} \int_{B(e,2N)\setminus B(e,N)} |\phi|^{Q/(Q-2)}\, dp \\
&\le CN^{-Q/(Q-2)} \Big(\int_{B(e,2N)\setminus B(e,N)} |\phi|^{Q/(Q-3)}\, dp\Big)^{(Q-3)/(Q-2)} |B(e,2N)|^{1/(Q-2)} \\
&\le C\Big(\int_{B(e,2N)\setminus B(e,N)} |\phi|^{Q/(Q-3)}\, dp\Big)^{(Q-3)/(Q-2)} \to 0
\end{aligned}$$

as $N \to \infty$, since $\phi \in L^{Q/(Q-3)}(\mathbb{H}^n, E_0^{n-1})$.

Step 2. If W_i is a horizontal derivative, consider now the $L^{Q/(Q-1)}$-norm of $W_i((W_k\chi_N)\phi_\ell)$. It holds,

$$W_i((W_k\chi_N)\phi_\ell) = (W_iW_k\chi_N)\phi_\ell + (W_k\chi_N)W_i(\phi_\ell).$$

Step 2a Now,

$$\|(W_iW_k\chi_N)\phi_\ell\|^{Q/(Q-1)}_{L^{Q/(Q-1)}(\mathbb{H}^n)} \leq N^{-2Q/(Q-1)} \int_{B(e,2N)\setminus B(e,N)} |\phi|^{Q/(Q-1)}\, dp$$
$$\leq CN^{-2Q/(Q-1)}\Big(\int_{B(e,2N)\setminus B(e,N)} |\phi|^{Q/(Q-3)}\, dp\Big)^{(Q-3)/(Q-1)} |B(e,2N)|^{2/(Q-1)}$$
$$\leq C\Big(\int_{B(e,2N)\setminus B(e,N)} |\phi|^{Q/(Q-3)}\, dp\Big)^{(Q-3)/(Q-1)} \to 0$$

as $N \to \infty$, again since $\phi \in L^{Q/(Q-3)}(\mathbb{H}^n, E_0^{n-1})$.
Step 2b. Consider eventually the $L^{Q/(Q-1)}$-norm of

$$(W_k\chi_N)W_i(\phi_\ell) = (W_k\chi_N)W_i(d_c^*d_cd_c^*\Delta_c^{-1}\beta)_\ell =: (W_k\chi_N)\tilde{\phi}_\ell.$$

We notice now that the map $\beta \to \tilde{\phi}_\ell$ is associated with a kernel of type 0 and then is continuous from $L^{Q/(Q-2)}(\mathbb{H}^n)$ to itself (again by Theorem 2.8, since $Q/(Q-2) > 1$). Therefore

$$\|(W_k\chi_N)\tilde{\phi}_\ell\|^{Q/(Q-1)}_{L^{Q/(Q-1)}(\mathbb{H}^n)} \leq N^{-Q/(Q-1)} \int_{B(e,2N)\setminus B(e,N)} |\tilde{\phi}_\ell|^{Q/(Q-1)}\, dp$$
$$\leq CN^{-Q/(Q-1)}\Big(\int_{B(e,2N)\setminus B(e,N)} |\tilde{\phi}_\ell|^{Q/(Q-2)}\, dp\Big)^{(Q-2)/(Q-1)} |B(e,2N)|^{1/(Q-1)}$$
$$\leq C\Big(\int_{B(e,2N)\setminus B(e,N)} |\tilde{\phi}|^{Q/(Q-2)}\, dp\Big)^{(Q-2)/(Q-1)} \to 0$$

since $\tilde{\phi}_\ell \in L^{Q/(Q-2)}(\mathbb{H}^n)$.

This completes the proof of (72) and hence of the Lemma. □

We are now in a position to conclude the proof of Theorem 8.10.

Proof of Theorem 8.10. With the notations of Definition 10.1, let us show preliminarily that

$$A^*(\mathcal{D}(A^*)) \qquad \text{is closed in } E^*. \tag{73}$$

To this end, let $(T_k)_{k\in\mathbb{N}}$ a sequence of n-currents in $A^*(\mathcal{D}(A^*))$ that converges to a current $T \in E^*$ (i.e. in the mass norm). Hence, $\mathcal{M}(T_k) = \|T_k\|_{E^*} \leq C_1$ for all $k \in \mathbb{N}$. In particular $T_k \to T$ weakly* in $\mathcal{D}'(\mathbb{H}^n E_0^n)$, i.e.

$$\langle T|\sigma\rangle = \lim_{k\to\infty} \langle T_k|\sigma\rangle$$

for all $\sigma \in \mathcal{D}(\mathbb{H}^n, E_0^n)$. By (62) and (63) there exists a corresponding sequence $\beta_k \in BL^{1,Q/(Q-1)}(\mathbb{H}^n, E_0^n)$ such that

$$T_k = \partial_c T_{\beta_k}$$

for any $k \in \mathbb{N}$. Since the n-currents $\partial_c T_{\beta_k}$'s satisfy $\mathcal{M}(\partial_c T_{\beta_k}) = \mathcal{M}(T_k) < \infty$, by Theorem 8.14, for any $k \in \mathbb{N}$ there exists $\phi_k \in BL^{1,Q/(Q-1)}(\mathbb{H}^n, E_0^n)$ such that $\partial_c T_{\phi_k} = \partial_c T_{\beta_k}$, and

$$\|\phi_k\|_{BL^{1,Q/(Q-1)}(\mathbb{H}^n, E_0^n)} \leq C\mathcal{M}(T_k) \leq CC_1.$$

Since $Q/(Q-1) > 1$ we cas assume that

$$\phi_k \to \phi \qquad \text{weakly in } BL^{1,Q/(Q-1)}(\mathbb{H}^n, E_0^n).$$

Thus, by Proposition 8.8, $T_{\phi_k} \to T_\phi$ weakly* in $\mathcal{D}'(\mathbb{H}^n, E_0^{n+1})$ and $T_k = \partial_c T_{\phi_k} \to \partial_c T_\phi$ weakly* in $\mathcal{D}'$; therefore, since also $T_k \to T$ weakly*, it follows that $T = \partial_c T_\phi$. To prove that $T \in A^*(\mathcal{D}(A^*))$, by (62) we need merely show that $\mathcal{M}(\partial_c T_\phi) < \infty$. Because of the lower semicontinuity of the mass with respect to the weak* convergence, we have

$$\mathcal{M}(\partial_c T_\phi) \leq \liminf_{k\to\infty} \mathcal{M}(\partial_c T_{\beta_k}) = \liminf_{k\to\infty} \mathcal{M}(T_k) = \mathcal{M}(T).$$

Thus (73) is proved.

By Theorem 5.1, 1), (73) implies that

$$A(\mathcal{D}(A)) = (\ker A^*)^\perp. \tag{74}$$

Moreover, by Lemma 5.2, (73) implies that there exists $C > 0$ such that for all $f \in A(D(A))$ there exists $e \in D(A)$, satisfying

$$Ae = f \quad \text{and} \quad \|e\|_E \leq C\,\|f\|_F. \tag{75}$$

To complete the proof, taking into account (74), we only need to show that

$$\{\Omega \in (BL^{1,Q/(Q-1}(\mathbb{H}^n, E_0^n))^*,\ \partial_c\Omega = 0\} \subset (\ker A^*)^\perp. \tag{76}$$

Indeed, take $\Omega \in (BL^{1,Q/(Q-1}(\mathbb{H}^n, E_0^n))^*$ such that $\partial_c\Omega = 0$. For any $\beta \in BL^{1,Q/(Q-1}(\mathbb{H}^n, E_0^n)$ belonging to $\ker A^*$ (i.e. $\partial_c T_\beta = 0$), by Lemma 10.5

$$\langle \Omega | \beta \rangle = 0,$$

that proves (76). Hence, we have proved the theorem.

□

11. Continuous primitives of locally defined forms: proof of Theorem 4.6

From the global Poincaré inequalities proved in the previous sections, we can now quite easily obtain the interior ones, stated in Theorem 4.6. In the next section we need first to collect some results already proven in [**?**] and [**?**].

11.1. **Intermediate tools.** The proof of Theorem 4.6, when $h = n+1$, relies also on an approximated homotopy formula proved in [**?**] for closed forms ω. Indeed, if we take $\lambda > \lambda' > 1$ and set $B' := B(e, \lambda')$ and $B_\lambda := B(e, \lambda)$, in Theorem 5.19 in [**?**] formula (56), it is proved that

$$\omega = d_c T\omega + S\omega \qquad \text{in } B' \tag{77}$$

where

$$T : L^Q(B_\lambda, E_0^h) \to W^{1,Q}(B', E_0^{h-1}) \quad \text{for } h \neq n+1$$

and

$$T : L^{Q/2}(B_\lambda, E_0^{n+1}) \to W^{2,Q/2}(B', E_0^n).$$

Moreover, S is a regularizing operator, i.e. maps $L^p(B_\lambda, E_0^{n+1})$ into $\mathcal{E}(B', E_0^{n+1})$. In particular, S is bounded from $W^{m,p}(B_\lambda, E_0^{n+1})$ to $W^{s,p}(B', E_0^{n+1})$ for any m, s, $0 \le m < s$ and $1 < p < \infty$ (see also Theorem 5.14 and Theorem 5.15 of [**?**]).

In addition, again in [**?**] (see formula (37) and Lemmata 5.7 and 5.8) an operator K that inverts Rumin's differential d_c on closed forms of the same degree has been defined. More precisely, we have:

Lemma 11.1. *If ω is a smooth d_c-exact differential form in the ball B', then*

$$\omega = d_c K\omega \quad \text{if } 1 \le h \le 2n+1. \tag{78}$$

Morever,

i) *if $1 < p < \infty$ and $h = 1, \dots, 2n+1$, then $K : W^{1,p}(B', E_0^h) \to L^p(B', E_0^{h-1})$ is bounded;*
ii) *if $1 < p < \infty$ and $n+1 < h \le 2n+1$, then $K : L^p(B', E_0^h) \to L^p(B', E_0^{h-1})$ is compact;*
iii) *if $1 < p < \infty$ and $h = n+1$, then $K : L^p(B', E_0^{n+1}) \to L^p(B', E_0^n)$ is bounded.*

Let us recall the precise definition of the operator K. In [**?**] the authors proved that, starting from Cartan's homotopy formula, if $D \subset \mathbb{R}^N$ is a convex set and $1 < p < \infty$, $1 \le h \le N$, then there exists a bounded linear map:

$$K_{\mathrm{Euc}} : L^p(D, \textstyle\bigwedge^h) \to W^{1,p}_{\mathrm{Euc}}(D, \bigwedge^{h-1})$$

that is a homotopy operator, i.e.

$$\omega = dK_{\mathrm{Euc}}\omega + K_{\mathrm{Euc}} d\omega \qquad \text{for all } \omega \in C^\infty(D, \textstyle\bigwedge^h). \tag{79}$$

(see Proposition 4.1 and Lemma 4.2 in [**?**]).

The operator K defined in [**?**] is

$$K = \Pi_{E_0} \circ \Pi_E \circ K_{\mathrm{Euc}} \circ \Pi_E\,. \tag{80}$$

The following theorem provides a continuity result in $W^{k,p}$ of Iwaniec & Lutoborski's kernel K_{Euc}, though with a loss on domain. It has been proved in [**?**], Theorem 5.1.

Theorem 11.2. *Let $B_{\mathrm{Euc}}(0,a)$ and $B_{\mathrm{Euc}}(0,a')$, $0<a<a'$, be concentric Euclidean balls in $\mathbb{R}^N$. Then for $k\in\mathbb{N}$ and $p\in[1,\infty]$, Iwaniec and Lutoborski's homotopy K_{Euc} is a bounded operator*

$$K_{\mathrm{Euc}}: W^{k,p}_{\mathrm{Euc}}(B_{\mathrm{Euc}}(0,a'),\bigwedge^h)\to W^{k,p}_{\mathrm{Euc}}(B_{\mathrm{Euc}}(0,a),\bigwedge^{h-1})$$

(the spaces $W^{m,p}_{\mathrm{Euc}}(U,\bigwedge^h\mathfrak{h})$ and $W^{m,p}_{\mathrm{Euc}}(U,E^h_0)$ are defined replacing Folland-Stein Sobolev spaces by usual Sobolev spaces).

11.2. **The interior result.**

Theorem 11.3. *There exists $\lambda>1$, such that, if we set $B:=B(e,1)$ and $B_\lambda:=B(e,\lambda)$, then there exists a geometric constant $C>0$ such that*

i) *if $2\le h\le 2n+1$, $h\neq n+1$, then a d_c-exact form $\omega\in L^Q(B_\lambda,E^h_0)$ admits a primitive in $\phi\in\mathcal{C}(B,E^{h-1}_0)$ such that*

$$\|\phi\|_{\mathcal{C}(\overline{B},E^{h-1}_0)}\le C\|\omega\|_{L^Q(B_\lambda,E^h_0)}; \tag{81}$$

ii) *a d_c-exact form $\omega\in L^{Q/2}(B_\lambda,E^{n+1}_0)$ admits a primitive in $\hat\phi\in\mathcal{C}(B,E^n_0)$ such that*

$$\|\phi\|_{\mathcal{C}(\overline{B},E^{h-1}_0)}\le C\|\omega\|_{L^{Q/2}(B_\lambda,E^{n+1}_0)}. \tag{82}$$

Proof. We suppose first $h\neq n+1$. As above, we take $\lambda>\lambda'>1$ such that the interior Poincaré inequality proved in [**?**] (Theorem 5.19-i) with $p=q=Q$) holds for the pair B' and B_λ. That is, if $\omega\in L^Q(B_\lambda,E^h_0)$ is a closed form, then there exists $\tilde\phi\in L^Q(B',E^{h-1}_0)$ such that $d_c\tilde\phi=\omega$ in B' and

$$\|\tilde\phi\|_{L^Q(B',E^{h-1}_0)}\|\le C\|\omega\|_{L^Q(B_\lambda,E^h_0)}. \tag{83}$$

Let now $\zeta\in\mathcal{D}(B')$, and set $\omega':=d_c(\zeta\tilde\phi)$ continued by zero outside of B', $\zeta\equiv1$ in B. Keeping in mind (83), by Lemma 3.3

$$\begin{aligned}\|\omega'\|_{L^Q(\mathbb{H}^n,E^h_0)}&=\|\omega'\|_{L^Q(B',E^h_0)}\le C_\zeta(\|\tilde\phi\|_{L^Q(B',E^{h-1}_0)}+\|d_c\tilde\phi\|_{L^Q(B',E^h_0)})\\&\le C_\zeta\|\omega\|_{L^Q(B_\lambda,E^h_0)}.\end{aligned} \tag{84}$$

Since ω' is closed, Theorem 4.4 yields the existence of a continuous primitive ϕ of ω', $\phi\in\mathcal{C}_0(\mathbb{H}^n,E^{h-1}_0)$ such that

$$\|\phi\|_{\mathcal{C}_0(\mathbb{H}^n,E^{h-1}_0)}\le C\|\omega'\|_{L^Q(\mathbb{H}^n,E^h_0)} \tag{85}$$

On the other hand, in B

$$d_c\phi = \omega' = d_c(\zeta\tilde{\phi}) = d_c\tilde{\phi} = \omega.$$

Moreover, by (85) and (84),

$$\|\phi\|_{\mathcal{C}(\overline{B},E_0^{h-1})} \leq \|\phi\|_{\mathcal{C}_0(\mathbb{H}^n,E_0^{h-1})} \leq C\|\omega'\|_{L^Q(\mathbb{H}^n,E_0^h)} \leq C_\zeta\|\omega\|_{L^Q(B_\lambda,E_0^h)}.$$

This proves Theorem 4.6 when $h \neq n+1$.

Now we deal with the case $h = n+1$.

We use the homotopy formula (77). In particular, $S\omega$ is closed in B', so that we can apply Lemma 11.1 to ω. Set now

$$\Phi := (KS+T)\omega \qquad \text{in } B'.$$

Trivially, by (77) and Lemma 11.1, $d_c\Phi = d_cKS\omega + d_cT\omega = S\omega + d_cT\omega = \omega$ in B'. With the notations of (5), take $\lambda > \lambda' > \lambda'' > 1$ so that $c_0\sqrt{\lambda''} < \lambda'$. Then

$$\begin{aligned} B'' := B(e,\lambda'') \Subset B_E := B_{\mathrm{Euc}}(0, c_0\sqrt{\lambda''}) \\ \Subset B'_{\mathrm{Euc}} := B_{\mathrm{Euc}}(0, 2c_0\sqrt{\lambda''}) \Subset B' \Subset B_\lambda. \end{aligned} \tag{86}$$

Let us prove that

$$\|\Phi\|_{W^{1,Q/2}(B'',E_0^n)} \leq C\|\omega\|_{L^{Q/2}(B_\lambda,E_0^{n+1})}. \tag{87}$$

Indeed, we have

$$\|\Phi\|_{W^{1,Q/2}(B'',E_0^n)} \leq \|KS\omega\|_{W^{1,Q/2}(B'',E_0^n)} + \|T\omega\|_{W^{2,Q/2}(B'',E_0^n)}$$

By the continuity of the operator T we have that

$$\|T\omega\|_{W^{2,Q/2}(B'',E_0^n)} \leq \|\omega\|_{L^{Q/2}(B'',E_0^{n+1})}. \tag{88}$$

Let us consider now the term $\|KS\omega\|_{W^{2,Q/2}(B'',E_0^n)}$ and remember that, in B'', by Theorem 2.2, v), $W^{1,Q/2}_{\mathrm{Euc}}(B'') \subset W^{1,Q/2}(B'')$. By [**?**], Theorem 4.12, keeping in mind that Π_E is an operator of order 0 on $(n+1)$-forms

and it is a differential operator of order 1 on n-forms, we have

$$
\begin{aligned}
\|KS\omega\|_{W^{1,Q/2}(B'',E_0^n)} &\leq C\|KS\omega\|_{W^{1,Q/2}_{\mathrm{Euc}}(B'',E_0^n)} \\
&= C\|(\Pi_{E_0}\circ\Pi_E\circ K_{\mathrm{Euc}}\circ\Pi_E)S\omega\|_{W^{1,Q/2}_{\mathrm{Euc}}(B'',E_0^n)} \\
&\leq C\|(\Pi_E\circ K_{\mathrm{Euc}}\circ\Pi_E)S\omega\|_{W^{1,Q/2}_{\mathrm{Euc}}(B'',\bigwedge^n)} \\
&\leq C\|(K_{\mathrm{Euc}}\circ\Pi_E)S\omega\|_{W^{2,Q/2}_{\mathrm{Euc}}(B'',\bigwedge^n)} \\
&\leq C\|(K_{\mathrm{Euc}}\circ\Pi_E)S\omega\|_{W^{2,Q/2}_{\mathrm{Euc}}(B_E,\bigwedge^n)} \\
&\leq C\|\Pi_E S\omega\|_{W^{2,Q/2}_{\mathrm{Euc}}(B'_E,\bigwedge^{n+1})} \qquad \text{(by Theorem 11.2)} \\
&\leq C\|\Pi_E S\omega\|_{W^{2,Q/2}_{\mathrm{Euc}}(B',\bigwedge^{n+1})} \\
&\leq C\|S\omega\|_{W^{2,Q/2}_{\mathrm{Euc}}(B',E_0^{n+1})} \\
&\leq C\|S\omega\|_{W^{4,Q/2}(B',E_0^{n+1})} \qquad \text{(by Theorem 2.2-vi))} \\
&\leq C\|\omega\|_{L^{Q/2}(B_\lambda,E_0^{n+1})} \qquad \text{(by (86))},
\end{aligned}
\tag{89}
$$

where the last inequality uses the fact that S is a smoothing operator. Therefore, we get (87).

Let now $\zeta \in \mathcal{D}(B'')$, $\zeta \equiv 1$ in B, and set $\omega'' := d_c(\zeta\Phi)$ continued by zero outside of B''.

Keeping in mind (87) and by Leibniz formula stated in Lemma 3.3,

$$
\begin{aligned}
\|\omega''\|_{L^{Q/2}(\mathbb{H}^n,E_0^{n+1})} &= \|\omega''\|_{L^{Q/2}(B'',E_0^{n+1})} \\
&\leq C_\zeta(\|\Phi\|_{W^{1,Q/2}(B'',E_0^n)} + \|d_c\Phi\|_{L^{Q/2}(B'',E_0^{n+1})}) \\
&\leq C_\zeta\|\omega\|_{L^Q(B_\lambda,E_0^h)}.
\end{aligned}
\tag{90}
$$

Since ω'' is closed, Theorem 4.4 yields the existence of a continuous primitive $\hat{\phi}$ of ω'', $\hat{\phi} \in \mathcal{C}_0(\mathbb{H}^n, E_0^n)$ such that

$$
\|\hat{\phi}\|_{\mathcal{C}_0(\mathbb{H}^n,E_0^n)} \leq C\|\omega''\|_{L^{Q/2}(\mathbb{H}^n,E_0^{n+1})} \tag{91}
$$

On the other hand, in B

$$
d_c\hat{\phi} = \omega'' = d_c(\zeta\Phi) = d_c\Phi = \omega.
$$

Moreover, by (91) and (90),

$$
\|\hat{\phi}\|_{\mathcal{C}(\overline{B},E_0^n)} \leq \|\hat{\phi}\|_{\mathcal{C}_0(\mathbb{H}^n,E_0^n)} \leq C\|\omega''\|_{L^{Q/2}(\mathbb{H}^n,E_0^{n+1})} \leq C\|\omega\|_{L^{Q/2}(B_\lambda,E_0^{n+1})}.
$$

and we are done.

□

12. Compact Riemannian and contact subRiemannian manifolds

12.1. **A smoothing homotopy.**

Proposition 12.1. *There exists $\lambda > 1$ such that, if we denote by $B := B(e,1)$ and $B_\lambda := B(e,\lambda)$ two concentric Heisenberg balls, there exist operators $S : \mathcal{C}^\infty(B_\lambda, E_0^\bullet) \to \mathcal{C}^\infty(B, E_0^\bullet)$ and $T : \mathcal{C}^\infty(B_\lambda, E_0^\bullet) \to \mathcal{C}^\infty(B, E_0^{\bullet-1})$ such that $S + d_c T + T d_c$ equals the restriction operator from B_λ to B. Furthermore, T extends to a bounded operator on $\mathcal{C}^0$, i.e. from $\mathcal{C}(B_\lambda, E_0^\bullet)$ to $\mathcal{C}(B, E_0^{\bullet-1})$ and on $\mathcal{C}^1$, i.e., from $\mathcal{C}^1(B_\lambda, E_0^\bullet)$ to $\mathcal{C}^1(B, E_0^{\bullet-1})$. S extends to a bounded operator from $\mathcal{C}(B_\lambda, E_0^\bullet)$ to $\mathcal{C}^\ell(B, E_0^{\bullet-1})$ for every $\ell \in \mathbb{N}$.*

Proof. Pick $q > Q$. Then Theorem 5.14 of [**?**] provides such a homotopy, where T is bounded from L^q to $W^{1,q}$. Since $\mathcal{C}(B, E_0^\bullet) \subset L^q(B, E_0^\bullet)$ and Sobolev's embedding implies that $W^{1,q}(B, E_0^{\bullet-1}) \subset C(B, E_0^{\bullet-1})$, T is bounded from $\mathcal{C}^0$ to $\mathcal{C}^0$. A slight extension of the proof of Theorem 5.14 of [**?**] shows that T is bounded from $W^{1,q}$ to $W^{2,q}$, hence from $\mathcal{C}^1$ to $\mathcal{C}^1$. □

12.2. **A C^0 Poincaré inequality.**

Proposition 12.2. *There exists $\lambda > 1$ such that, if we denote by $B := B(e,1)$ and $B_\lambda := B(e,\lambda)$ two concentric Heisenberg balls, there exists an operator $P : \mathcal{C}^\infty(B_\lambda, E_0^\bullet) \to \mathcal{C}^\infty(B_\lambda, E_0^{\cdot-1})$ such that $d_c P + P d_c$ equals the restriction operator from B_λ to B. Furthermore, P extends to a bounded operator on $\mathcal{C}^0$, i.e. from $\mathcal{C}(B_\lambda, E_0^\bullet)$ to $\mathcal{C}(B_\lambda, E_0^{\cdot-1})$.*

Proof. The proof of Theorem 5.19 of [**?**] provides such an operator. It is of the form $P = KS + T$ where S, T is the homotopy occurring in Proposition 12.1 and K is cooked up from the Euclidean homotopy of Iwaniec-Lutoborsky and Rumin's homotopy between de Rham and Rumin complexes. For every $q > Q$, P is bounded from $L^q(B_\lambda, E_0^\bullet)$ to $W^{1,q}(B_\lambda, E_0^{\bullet-1})$, hence on $\mathcal{C}^0$. □

12.3. **Leray's acyclic covering theorem.** Leray's acyclic covering theorem relates the de Rham cohomology of a manifold and the simplicial cohomology of a simplicial complex, the nerve of an acyclic covering. Acyclic means that de Rham cohomology of all intersections of pieces of the covering vanishes. The nerve is the complex with one vertex per piece, and with a simplex through a set of vertices each time the corresponding pieces have a nonempty common intersection.

An application of Leray's acyclic covering theorem to Poincaré inequalities on bounded geometry Riemannian manifolds is given in [**?**]. A persistent version of Leray's acyclic covering theorem is described in [**?**]. It provides the same conclusion under weaker requirements on the covering. The analytical ingredients are

- A smoothing homotopy, as in Proposition 12.1, to circumvent the obstacle arising from the Leibniz formula.

- Poincaré inequalities provided by linear operators, with losses on domains allowed, as in Proposition 12.2.

The context here is slightly different from [**?**] and [**?**]. In these papers, manifolds are noncompact and are assumed to have vanishing cohomology. Here, manifolds are compact and the vanishing assumption is replaced with the following fact: there exists a radius r_0 such that balls of radius $< r_0$ have vanishing cohomology.

There is a second difference. To prove Theorem 4.7 in degree h, one uses the vector of exponents $(\infty, \dots, \infty, Q)$, meaning that the L^Q norm is used on h-forms (replace Q with $Q/2$ if $h = n+1$), and the L^∞ norm on spaces of continuous j-forms is used when $j < h$. To apply the machinery of [**?**], one would need a linear operator P that solves d_c on a pair of concentric Heisenberg balls (B, B_λ), with estimates: P is bounded from $L^Q(B_\lambda, E_0^h)$ to $\mathcal{C}(B, E_0^{h-1})$. As observed by Bourgain and Brezis, such a linear operator cannot exist. Theorem 4.6 merely provides a nonlinear map $\omega \mapsto \phi$. It turns out that this does not seriously affect the argument.

Morally (this is the point of view adopted in [**?**]), Leray's method goes as follows. Pick a suitable finite covering $\{U_i\}$ by small enough balls, so that each of them comes with a nearly isometric contactomorphism to an open set in Heisenberg group. Given a d_c-closed L^Q h-form ω on M, pick a continuous primitive on each U_i, using Theorem 4.6. This yields a 0-cochain ω^1 of the covering, with values in spaces of continuous $(h-1)$ forms on pieces U_i. Let $\delta\omega^1$ denote the 1-cochain which on a pair (i, i') is equal to the restriction to $U_i \cap U_{i'}$ of $\omega_i^1 - \omega_{i'}^1$. It is d_c-closed. Pick a primitive on each $U_i \cap U_{i'}$, using the operator P of Proposition 12.2. This yields a 1-cochain ω^2 of the covering with values in spaces of continuous $(h-2)$-forms on intersections $U_i \cap U_{i'}$. And so on. After h steps, one gets a h-cochain with values in spaces of constant functions, i.e. a combinatorial h-cochain.

Conversely, given a j-cocycle ψ with values in spaces of continuous j-forms on $h+1$-uple intersections $U_i \cap U_{i'} \cap \cdots$, using a smooth partition of unity to extend forms to intersections of less pieces and adding them up, one produces a $(j-1)$-cochain $\epsilon\psi$ such that $\delta\epsilon\psi = \psi$. In order to be able to apply d_c to $\epsilon\phi$, one applies to it the smoothing operator S of Proposition 12.1. Given a combinatorial h-cocycle $\kappa \in \ell^Q K^h$, i.e. a skew-symmetric real valued function on h-simplices of the nerve which satisfies $\delta\kappa = 0$ and is Q-summable, one defines $\kappa^1 = d_c S \epsilon \kappa$, which is a $(h-1)$-cocycle with values in spaces of smooth 1-forms on h-uple intersections, and which is again Q-summable. And so on. After h steps, one gets a 0-cocycle with values in spaces of smooth h-forms, i.e. a smooth globally defined L^Q h-form.

The machinery shows that both constructions are inverses of each other in cohomology. Indeed, the nonlinear step occurs only once, at the very beginning, so its nonlinear character does not affect the needed identities. The constructions define a biLipschitz bijection in cohomology. Since one of them is linear, one gets a linear and bounded bijection between de Rham cohomology

$$(\ker(d_c) \cap L^Q(M, E_0^h))/d_c(C(M, E_0^{h-1}) \cap d_c^{-1}(L^Q))$$

and combinatorial cohomology

$$(\ker(\delta) \cap \ell^Q K^h)/\delta(\ell^\infty K^{h-1} \cap \delta^{-1}\ell^Q).$$

It also shows that exact cohomology

$$(\mathrm{im}(d_c) \cap L^Q(M, E_0^h))/d_c(C(M, E_0^{h-1}) \cap d_c^{-1}(L^Q))$$

is mapped to exact cohomology

$$(\mathrm{im}(\delta) \cap \ell^Q K^h)/\delta(\ell^\infty K^{h-1}) \cap \delta^{-1}\ell^Q),$$

which is 0. Indeed, the nerve is finite, all cochains belong to ℓ^Q or ℓ^∞, so exact cohomology vanishes. It follows that exact cohomology vanishes too on the de Rham/Rumin side: every exact L^Q h-form is the d_c of a continuous $(h-1)$-form, with estimates. This proves Theorem 4.7.

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REFERENCES

1. Robert A. Adams and John J. F. Fournier, *Sobolev spaces*, second ed., Pure and Applied Mathematics (Amsterdam), vol. 140, Elsevier/Academic Press, Amsterdam, 2003. MR 2424078
2. Luigi Ambrosio, Nicola Fusco, and Diego Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000. MR 1857292
3. Luigi Ambrosio and Paolo Tilli, *Topics on analysis in metric spaces*, Oxford Lecture Series in Mathematics and its Applications, vol. 25, Oxford University Press, Oxford, 2004. MR 2039660
4. A. Baldi and F. Montefalcone, *The distributional divergence of horizontal vector fields vanishing at infinity on Carnot groups*, Matematiche (Catania) **78** (2023), no. 1, 239–271. MR 4622357

5. Annalisa Baldi, *Sobolev-Poincaré inequalities for differential forms and currents in* $\mathbb{R}^n$, Bruno Pini Mathematical Analysis Seminar 2019, Bruno Pini Math. Anal. Semin., vol. 10, Univ. Bologna, Alma Mater Stud., Bologna, 2019, pp. 14–27. MR 4075926
6. Annalisa Baldi, Marilena Barnabei, and Bruno Franchi, *A recursive basis for primitive forms in symplectic spaces and applications to Heisenberg groups*, Acta Math. Sin. (Engl. Ser.) **32** (2016), no. 3, 265–285. MR 3456421
7. Annalisa Baldi, Bruno Franchi, and Pierre Pansu, *Gagliardo-Nirenberg inequalities for differential forms in Heisenberg groups*, Math. Ann. **365** (2016), no. 3-4, 1633–1667. MR 3521101
8. _____, *L^1-Poincaré inequalities for differential forms on Euclidean spaces and Heisenberg groups*, Adv. Math. **366** (2020), 107084. MR 4070308
9. _____, *Poincaré and Sobolev inequalities for differential forms in Heisenberg groups and contact manifolds*, J. Inst. Math. Jussieu **21** (2022), no. 3, 869–920. MR 4404128
10. _____, *Cohomology of annuli, duality and L^∞-differential forms on Heisenberg groups*, J. Funct. Anal. **285** (2023), no. 2, Paper No. 109944, 48. MR 4571869
11. Annalisa Baldi, Bruno Franchi, Nicoletta Tchou, and Maria Carla Tesi, *Compensated compactness for differential forms in Carnot groups and applications*, Adv. Math. **223** (2010), no. 5, 1555–1607.
12. Annalisa Baldi, Bruno Franchi, and Maria Carla Tesi, *Fundamental solution and sharp L^p estimates for Laplace operators in the contact complex of Heisenberg groups*, Ricerche Mat. **55** (2006), no. 1, 119–144.
13. _____, *Compensated compactness in the contact complex of Heisenberg groups*, Indiana Univ. Math. J. **57** (2008), 133–186.
14. Andrea Bonfiglioli, Ermanno Lanconelli, and Francesco Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer Monographs in Mathematics, Springer, Berlin, 2007. MR MR2363343
15. Jean Bourgain and Haïm Brezis, *On the equation* $\operatorname{div} Y = f$ *and application to control of phases*, J. Amer. Math. Soc. **16** (2003), no. 2, 393–426 (electronic). MR 1949165 (2004d:35032)
16. Haim Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011. MR 2759829
17. Sagun Chanillo and Jean Van Schaftingen, *Subelliptic Bourgain-Brezis estimates on groups*, Math. Res. Lett. **16** (2009), no. 3, 487–501. MR 2511628 (2010f:35042)
18. Thierry De Pauw and Monica Torres, *On the distributional divergence of vector fields vanishing at infinity*, Proc. Roy. Soc. Edinburgh Sect. A **141** (2011), no. 1, 65–76. MR 2773439
19. J. Deny and J. L. Lions, *Les espaces du type de Beppo Levi*, Ann. Inst. Fourier (Grenoble) **21** (1945), 305–370. MR 74787
20. Jean Dieudonné, *Éléments d'analyse. Tome III: Chapitres XVI et XVII*, Cahiers Scientifiques, Fasc. XXXIII, Gauthier-Villars Éditeur, Paris, 1970. MR MR0270377 (42 #5266)
21. Herbert Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969. MR MR0257325 (41 #1976)
22. Gerald B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat. **13** (1975), no. 2, 161–207. MR MR0494315 (58 #13215)

23. Gerald B. Folland and Elias M. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes, vol. 28, Princeton University Press, Princeton, N.J., 1982. MR MR657581 (84h:43027)
24. Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano, *Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields*, Houston J. Math. **22** (1996), no. 4, 859–890. MR 1437714
25. _____, *Regular submanifolds, graphs and area formula in Heisenberg groups*, Adv. Math. **211** (2007), no. 1, 152–203. MR MR2313532 (2008h:49030)
26. Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine, *Riemannian geometry*, third ed., Universitext, Springer-Verlag, Berlin, 2004. MR 2088027
27. Mikhael Gromov, *Carnot-Carathéodory spaces seen from within*, Sub-Riemannian geometry, Progr. Math., vol. 144, Birkhäuser, Basel, 1996, pp. 79–323. MR MR1421823 (2000f:53034)
28. Tadeusz Iwaniec and Adam Lutoborski, *Integral estimates for null Lagrangians*, Arch. Rational Mech. Anal. **125** (1993), no. 1, 25–79. MR MR1241286 (95c:58054)
29. David Jerison, *The Poincaré inequality for vector fields satisfying Hörmander's condition*, Duke Math. J. **53** (1986), no. 2, 503–523. MR MR850547 (87i:35027)
30. Guozhen Lu, *The sharp Poincaré inequality for free vector fields: an endpoint result*, Rev. Mat. Iberoamericana **10** (1994), no. 2, 453–466. MR 1286482
31. Laurent Moonens and Tiago H. Picon, *Solving the equation* $\operatorname{div} v = F$ *in* $\mathcal{C}_0(\mathbb{R}^n, \mathbb{R}^n)$, Proc. Edinb. Math. Soc. (2) **61** (2018), no. 4, 1055–1061. MR 3866518
32. Pierre Pansu, *Cup-products in* $l^{q,p}$*-cohomology: discretization and quasi-isometry invariance*, arXiv 1702.04984, 2017.
33. Pierre Pansu and Michel Rumin, *On the* $\ell^{q,p}$ *cohomology of Carnot groups*, Ann. H. Lebesgue **1** (2018), 267–295. MR 3963292
34. Pierre Pansu and Francesca Tripaldi, *Averages and the* $\ell^{q,1}$ *cohomology of Heisenberg groups*, Ann. Math. Blaise Pascal **26** (2019), no. 1, 81–100 (en).
35. Michel Rumin, *Formes différentielles sur les variétés de contact*, J. Differential Geom. **39** (1994), no. 2, 281–330. MR MR1267892 (95g:58221)
36. Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. MR MR1232192 (95c:42002)
37. François Trèves, *Topological vector spaces, distributions and kernels*, Academic Press, New York, 1967. MR MR0225131 (37 #726)
38. Neil S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–483. MR 0216286
39. Nicholas Th. Varopoulos, Laurent Saloff-Coste, and Thierry Coulhon, *Analysis and geometry on groups*, Cambridge Tracts in Mathematics, vol. 100, Cambridge University Press, Cambridge, 1992. MR MR1218884 (95f:43008)
40. Davide Vittone, *Lipschitz graphs and currents in Heisenberg groups*, Forum Math. Sigma **10** (2022), Paper No. e6, 104. MR 4377000
41. Richard L. Wheeden and Antoni Zygmund, *Measure and integral*, second ed., Pure and Applied Mathematics (Boca Raton), CRC Press, Boca Raton, FL, 2015, An introduction to real analysis. MR 3381284

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