COHOMOLOGY OF ANNULI, DUALITY AND L^∞-DIFFERENTIAL FORMS ON HEISENBERG GROUPS

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ABSTRACT. In the last few years the authors proved Poincaré and Sobolev type inequalities in Heisenberg groups $\mathbb{H}^n$ for differential forms in the Rumin's complex. The need to substitute the usual de Rham complex of differential forms for Euclidean spaces with the Rumin's complex is due to the different stratification of the Lie algebra of Heisenberg groups. The crucial feature of Rumin's complex is that d_c is a differential operator of order 1 or 2 according to the degree of the form.

Roughly speaking, Poincaré and Sobolev type inequalities are quantitative formulations of the well known topological problem whether a closed form is exact. More precisely, for suitable p and q, we mean that every exact differential form ω in L^p admits a primitive ϕ in L^q such that $\|\phi\|_{L^q} \le C\,\|\omega\|_{L^p}$. The cases of the norm L^p, $p \geq 1$ and $q < \infty$ have been already studied in a series of papers by the authors. In the present paper we deal with the limiting case where $q = \infty$: it is remarkable that, unlike in the scalar case, when the degree of the forms ω is at least 2, we can take $q = \infty$ in the left-hand side of the inequality. The corresponding inequality in the Euclidean setting $\mathbb{R}^N$ ($p = N$ and $q = \infty$) was proven by Bourgain & Brezis.

CONTENTS

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1. Introduction

1.1. **Euclidean spaces and de Rham complex.** To begin with, let us consider preliminarily the Euclidean space $\mathbb{R}^N$, $N > 1$ and the differential forms of the de Rham complex $(\Omega^\bullet, d)$ on $\mathbb{R}^N$. It is well known that closed forms $\omega \in \Omega^\bullet$ are exact, i.e. $d\omega = 0$ implies that there exists $\phi \in \Omega^{\bullet -1}$ such that $d\phi = \omega$. This can expressed by saying that the cohomology groups

$$H^h_{\rm dR} := (\Omega^h \cap \ker d)/d\Omega^{h-1} \qquad \text{are trivial for } 1 \le h \le N. \tag{1}$$

Global Poincaré and Sobolev inequalities in $(\Omega^\bullet, d)$ are meant to give a quantitative meaning to (1). More precisely, if $1 \le p, q \le \infty$, we say that a *global* Poincaré inequality holds on $\mathbb{R}^N$, if there exists a positive constant $C = C(p,q)$ such that for every exact h-form ω on $\mathbb{R}^N$, belonging to L^p, there exists a $(h-1)$-form ϕ such that $d\phi = \omega$ and

$$\|\phi\|_{L^q} \le C\,\|\omega\|_{L^p}.$$

Shortly, we shall write that Poincaré$_{p,q}(h)$ inequality holds, or that the $L^{q,p}$-cohomology vanishes. For further comments and applications, we refer to [28]. Notice that a homogeneity argument shows that, if $1 \le p < N$, then we can take $p \le q \le pN/(N-p)$.

In addition, we say that a *global* Sobolev inequality holds on $\mathbb{R}^N$, if for every exact compactly supported h-form ω on $\mathbb{R}^N$, belonging to L^p, there exists a compactly supported $(h-1)$-form ϕ such that $d\phi = \omega$ and

$$\|\phi\|_{L^q} \le C\,\|\omega\|_{L^p}.$$

Again, we shall write that Sobolev$_{p,q}(h)$ holds.

We point out that if u is a scalar function on $\mathbb{R}^N$ (i.e. $u \in \Omega^0$), then Poincaré$_{p,q}(1)$ and Sobolev $_{p,q}(1)$ for du are nothing but the usual Poincaré and Sobolev inequalities.

Besides global inequalities, it is natural to consider *local* inequalities, where the Euclidean space $\mathbb{R}^N$ is replaced (for instance) with a Euclidean ball. If $1 < p < N$, a local Poincaré inequality in de Rham complex has been proved by Iwaniec & Lutoborsky [24], and a Sobolev inequality for bounded convex sets has been proved by Mitrea, Mitrea & Monniaux [26].

The notions of Poincaré and Sobolev inequalities can be weakened through the notions of *interior inequalities*. More precisely, we say that an interior Poincaré$_{p,q}(h)$ inequality holds on $\mathbb{R}^N$ if there exists a fixed $\lambda \ge 1$ large enough such that for every $r > 0$ small enough there exists a constant $C = C(M, p, q, r, \lambda)$ such that for every $x \in \mathbb{R}^N$ and every exact h-form ω on $B(x, \lambda r)$, belonging to L^p, there exists a $(h-1)$-form ϕ on $B(x,r)$ such that $d\phi = \omega$ on $B(x,r)$ and

$$\|\phi\|_{L^q(B(x,r))} \le C\,\|\omega\|_{L^p(B(x,\lambda r))}. \tag{2}$$

Analogously, by *interior Sobolev inequalities*, we mean that, if ω is supported in $B(x,r)$, then there exists ϕ supported in $B(x, \lambda r)$ such that $d\phi = \omega$ and

$$\|\phi\|_{L^q(B(x,\lambda r))} \le C\,\|\omega\|_{L^p(B(x,r))}. \tag{3}$$

Here we use the word *interior* to stress the fact that inequality (2) provides no information on the behaviour of differential forms near the boundary of their domain of definition.

It turns out that in several situations, the loss on domain is harmless. This is for instance the case of $L^{q,p}$-cohomological applications, see [28].

Relying on these weaker notions, we have been able to cover also the case $p = 1$ (see [5]). The other endpoint result $p = N$, $q = \infty$, is more delicate. Indeed, it is well known that the interior $\mathrm{Poincaré}_{N,\infty}(1)$ fails to hold in $\mathbb{R}^N$ (see e.g. [38], p. 484), and has to be replaced by the so-called Trudinger exponential estimate (see [36]) or by the more precise Adams-Trudinger inequality ([38], Theorem 15.30).

However, rather surprisingly, in [14] Bourgain & Brezis proved that a global $\mathrm{Poincaré}_{N,\infty}(h)$ holds for $1 < h < N-1$.

1.2. **Heisenberg groups and Rumin's complex.** In the last few years, the authors of the present paper have attacked the study of Poincaré and Sobolev inequalities in sub-Riemannian manifolds endowed with a "suitable" complex of differential forms (we remind that the data of a smooth manifold M and of a sub-bundle $H \subset TM$ equipped with a scalar product g is called a *sub-Riemannian* manifold). See, e.g., [23], [27].

More precisely, we have considered differential forms of the so-called Rumin complex of Heisenberg groups: see [3], [4], [8], [6], [7]. The Heisenberg group $\mathbb{H}^n$, $n \geq 1$, is the connected, simply connected Lie group whose Lie algebra is the central extensions

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2, \quad \text{with } \mathfrak{h}_2 = \mathbb{R} = Z(\mathfrak{h}), \tag{4}$$

with bracket $\mathfrak{h}_1 \otimes \mathfrak{h}_1 \to \mathfrak{h}_2 = \mathbb{R}$ being a non-degenerate skew-symmetric 2-form. Due to its stratification (4), the Heisenberg Lie algebra admits a one parameter group of automorphisms δ_t,

$$\delta_t = t \text{ on } \mathfrak{h}_1, \quad \delta_t = t^2 \text{ on } \mathfrak{h}_2,$$

which are counterparts of the usual Euclidean dilations in $\mathbb{R}^N$. Through exponential coordinates, $\mathbb{H}^n$ can be identified with the Euclidean space $\mathbb{R}^{2n+1}$, endowed with the non-commutative product induced by the Campbell-Hausdorff formula. In this system of coordinates, the identity element $e \in \mathbb{H}^n$ is the zero of the vector space $\mathbb{R}^{2n+1}$, and $p^{-1} = -p$. In addition, in this system of coordinates, the Haar measure of the group is the $(2n+1)$-dimensional Lebesgue measure $\mathcal{L}^{2n+1}$.

Heisenberg groups are the simplest nontrivial (i.e. non-commutative) instance of the so called *Carnot groups*, connected, simply connected and stratified Lie groups. Heisenberg groups can be viewed as sub-Riemannian spaces, where the sub-Riemannian structure is obtained by left-translating $\mathfrak{h}_1$ (we remind that the Lie algebra of $\mathbb{H}^n$ can be identified with the tangent space to $\mathbb{H}^n$ at e). In addition, Heisenberg groups are the local models of contact manifolds, since, according to a theorem by Darboux, every $2n+1$-dimensional contact manifold is locally contactomorphic to $\mathbb{H}^n$.

For a general review on Heisenberg groups and their properties, we refer for instance to [34], [23], [13], [37]. The main properties of $\mathbb{H}^n$ that we shall need in this paper will be presented below in Section 2. Here we limit ourselves to reminding that Heisenberg groups carry natural left-invariant metrics, either Carnot-Carathéodory distances as sub-Riemannian manifolds or, equivalently, Cygan-Korányi norms ρ (see (9) below). Throughout this paper we use systematically the Cygan-Korányi distance $d(p,q) := \rho(p^{-1} \cdot q)$. The distance d is homogeneous of degree one with respect to group dilations δ_t, so that, if we denote by $B(p,r)$ the Cygan-Korányi ball of radius $r > 0$ centered at $p \in \mathbb{H}^n$, then $\mathcal{L}^{2n+1}(B(p,r)) = cr^{2n+2}$. In particular, this implies that the Hausdorff dimension of $\mathbb{H}^n$ with respect to d equals $Q := 2n+2$.

As a consequence of the stratification (4), the differential forms on $\mathfrak{h}$ split into 2 eigenspaces under δ_t, therefore de Rham complex lacks scale invariance under these anisotropic dilations. A substitute for de Rham's complex, that recovers scale invariance under δ_t has been defined by M. Rumin, [30].

Let $h = 0, \dots, 2n+1$. Rumin's substitute for smooth differential forms of degree h are the smooth sections of a left-invariant vector bundle E_0^h. If $h \leq n$, E_0^h is a subbundle of

$\Lambda^h H^*$. If $h \geq n$, E_0^h is a subbundle of $\Lambda^h H \otimes (TM/H)$. Rumin's substitute for de Rham's exterior differential is a linear differential operator d_c from sections of E_0^h to sections of E_0^{h+1} such that $d_c^2 = 0$. Further details about Rumin's complex are contained in Section 2.2 below. We refer to also to [30], [9] [10] and [11] for details of the construction.

We stress that the operator d_c is a left-invariant differential operator of order 2 when acting on forms of degree n and of order 1 otherwise.

This phenomenon will be a major issue in our results and will affect the proofs (think for instance of Leibniz formula) as well as the choice of the exponents p, q in our inequalities.

1.3. **Poincaré and Sobolev inequalities: precise definitions.** We can state now the notions of (global and interior) Poincaré and Sobolev inequalities in the setting of Rumin's complex.

Definition 1.1. *If $1 \leq h \leq 2n+1$ and $1 \leq p \leq q \leq \infty$, we say that the global $\mathbb{H}\text{-Poincaré}_{p,q}$ inequality holds in E_0^h if there exists a constant C such that, for every d_c-exact differential h-form ω in $L^p(\mathbb{H}^n; E_0^h)$ there exists a differential $(h-1)$-form ϕ in $L^q(\mathbb{H}^n, E_0^{h-1})$ such that $d_c\phi = \omega$ and*

$$\|\phi\|_{L^q(\mathbb{H}^n, E_0^{h-1})} \leq C\,\|\omega\|_{L^p(\mathbb{H}^n, E_0^h)} \qquad \text{global}\,\mathbb{H}\text{-Poincaré}_{p,q}(h). \tag{5}$$

Definition 1.2. *Let $B := B(e,1)$ and $B_\lambda := B(e,\lambda)$. Given $1 \leq h \leq 2n+1$ and $1 \leq p \leq q \leq \infty$, we say that the interior $\mathbb{H}\text{-Poincaré}_{p,q}$ inequality holds in E_0^h if there exist constants $\lambda > 1$ and C such that, for every d_c-exact differential h-form ω in $L^p(B_\lambda; E_0^h)$ there exists a differential $(h-1)$-form ϕ in $L^q(B, E_0^{h-1})$ such that $d_c\phi = \omega$ and*

$$\|\phi\|_{L^q(B, E_0^{h-1})} \leq C\,\|\omega\|_{L^p(B_\lambda, E_0^h)} \qquad \text{interior}\,\mathbb{H}\text{-Poincaré}_{p,q}(h). \tag{6}$$

Definition 1.3. *If $1 \leq h \leq 2n+1$, $1 \leq p \leq q \leq \infty$, we say that the global $\mathbb{H}\text{-Sobolev}_{p,q}(h)$ inequality holds if there exists a constant C such that for every compactly supported d_c-exact differential h-form ω in $L^p(\mathbb{H}^n; E_0^h)$ there exists a compactly supported differential $(h-1)$-form ϕ in $L^q(\mathbb{H}^n, E_0^{h-1})$ such that $d_c\phi = \omega$ and*

$$\|\phi\|_{L^q(\mathbb{H}^n, E_0^{h-1})} \leq C\,\|\omega\|_{L^p(\mathbb{H}^n, E_0^h)} \qquad \text{global}\,\mathbb{H}\text{-Sobolev}_{p,q}(h). \tag{7}$$

Definition 1.4. *Let $B := B(e,1)$ and $B_\lambda := B(e,\lambda)$. Given $1 \leq h \leq 2n$, $1 \leq p \leq q \leq \infty$, we say that the interior $\mathbb{H}\text{-Sobolev}_{p,q}(h)$ inequality holds if there exist constants $\lambda > 1$ and C such that for every compactly supported d_c-exact differential h-form ω in $L^p(B; E_0^h)$ there exists a compactly supported differential $(h-1)$-form ϕ in $L^q(B_\lambda, E_0^{h-1})$ such that $d_c\phi = \omega$ in B_λ and*

$$\|\phi\|_{L^q(B_\lambda, E_0^{h-1})} \leq C\,\|\omega\|_{L^p(B, E_0^h)} \qquad \text{interior}\,\mathbb{H}\text{-Sobolev}_{p,q}(h). \tag{8}$$

Here we have extended ω by 0 to all of B_λ.

Remark 1.5. *As in* [8]*, Corollary 5.21, an elementary scaling argument shows that, if $h \neq n$, $1 \leq p < Q$ and $q = pQ/(Q-p)$, or $h = n$, $1 \leq p < Q/2$ and $q = pQ/(Q-2p)$ then the interior $\mathbb{H}\text{-Sobolev}_{p,q}(h)$ implies the global $\mathbb{H}\text{-Sobolev}_{p,q}(h)$ inequality.*

Suppose $1 < h < 2n$. If $h \neq n$, $p = Q$ take $q = \infty$, and, $h = n$, $p = Q/2$ take $q = \infty$. We shall see later that (unlike in the case $h = 1$ or $h = 2n$), interior $\mathbb{H}\text{-Sobolev}_{Q,\infty}(h)$ hold. Then, again the corresponding global inequalities hold, thanks to the same scaling argument.

In the sequel, we shall refer to the exponents $q = pQ/(Q-p)$ or $q = pQ/(Q-2p)$ according to the degree of the forms as to the sharp Sobolev exponent*.*

1.4. **State of the art.** In [8] and [6] the following Poincaré and Sobolev inequalities have been proven. More precisely, [8] deals with the case $p > 1$, whereas [6] covers the case $p = 1$.

Theorem 1.6 (Poincaré inequality)**.** *If* $1 \le h \le 2n+1$*, we have:*

i) *if* $h \neq n+1, 2n+1$ *and* $1 \le p < Q$*, then the interior* $\mathbb{H}$-Poincaré$_{p,pQ/(Q-p)}(h)$ *holds;*
ii) *if* $h = n+1$ *and* $1 \le p < Q/2$*, then the interior* $\mathbb{H}$-Poincaré$_{p,pQ/(Q-2p)}(n+1)$ *holds;*
iii) *if* $h = 2n+1$ *and* $1 < p < Q$*, then the interior* $\mathbb{H}$-Poincaré$_{p,pQ/(Q-p)}(h)$ *holds.*

Analogous statements hold for global Poincaré inequalities on $\mathbb{H}^n$.

Theorem 1.7 (Sobolev inequality)**.** *If* $1 \le h \le 2n+1$*, we have:*

i) *if* $h \neq n+1, 2n+1$ *and* $1 \le p < Q$*, then the interior* $\mathbb{H}$-Sobolev$_{p,pQ/(Q-p)}(h)$ *holds;*
ii) *if* $h = n+1$ *and* $1 \le p < Q/2$*, then the interior* $\mathbb{H}$-Sobolev$_{p,pQ/(Q-2p)}(n+1)$ *holds;*
iii) *if* $h = 2n+1$ *and* $1 < p < Q$*, then the interior* $\mathbb{H}$-Sobolev$_{p,pQ/(Q-p)}(h)$ *holds.*

Analogous statements hold for global Sobolev inequalities on $\mathbb{H}^n$.

1.5. **Main results and sketch of the proofs.** The aim of the present paper is to complete the results gathered in Section 1.4, by covering (when possible) the endpoints $p = Q$ or $p = Q/2$ according to the degree of the forms.

Thus, the core of the present paper consists of the following theorems:

Theorem 1.8. *If* $2 \le h \le 2n+1$*, we have:*

i) *if* $h \neq n+1$*, then the interior* $\mathbb{H}$-Poincaré$_{Q,\infty}(h)$ *holds;*
ii) *if* $h = n+1$*, then the interior* $\mathbb{H}$-Poincaré$_{Q/2,\infty}(n)$ *holds.*

Analogous statements hold for global Poincaré inequalities on $\mathbb{H}^n$.

Theorem 1.9. *If* $2 \le h \le 2n+1$*, we have:*

i) *if* $h \neq n+1$*, then the interior* $\mathbb{H}$-Sobolev$_{Q,\infty}(h)$ *holds;*
ii) *if* $h = n+1$*, then the interior* $\mathbb{H}$-Sobolev$_{Q/2,\infty}(n)$ *holds.*

Analogous statements hold for global Sobolev inequalities on $\mathbb{H}^n$.

Remark 1.10. *In Euclidean space* $\mathbb{R}^N$ *it is well known that the interior* Poincaré$_{N,\infty}(1)$ *fails to hold (see e.g.* [38]*, p. 484), and has to be replaced with Trudinger's exponential estimate (see* [36]*) or the more precise Adams-Trudinger inequality (*[38]*, Theorem 15.30). Analogous estimates in Heisenberg groups can be found e.g. in* [12], [15] *(see also* [7]*).*

On the contrary, the statement of Theorem 1.8 states the $\mathbb{H}$-Poincaré$_{p,\infty}(h)$ *holds for* $h \ge 2$ *with sharp exponent* $p = Q$ *or* $p = Q/2$*, according to the degree of the forms.*

We refer to [14]*, Theorem 5, for related statements in Euclidean spaces.*

Remark 1.11. *By the way, the proofs presented here can be carried out (with obvious simplifications) also in the Euclidean setting. In particular, we can obtain interior estimates that are, at least partially, the interior counterparts of the results of* [14].

Let us give a sketch of the paper. Section 2 gathers the basic notions about Heisenberg groups. In particular, in Section 2.1 we state several properties of the convolution kernels in $\mathbb{H}^n$. These results are more or less known, but they have to be handled carefully because of the presence of L^∞-spaces, precluding density arguments. Subsequently, Section 2.2 contains a minimal presentation of Rumin's complex, with the aim of making the paper as self-contained as possible. In particular, Lemma 2.11 deals with Leibniz formula when the exterior differential d_c of the complex is a differential operator of order 2. A more extensive presentation of the complex is contained in the Appendix (Section 8). Finally, Section 2.3 contains some basic properties of Rumin's Laplacian in $(E_0^\bullet, d_c)$ and of its fundamental solution.

The core of the present paper is contained in Sections 3 – 7 where the proofs of Theorems 1.8 and 1.9 are carried out.

More precisely, Section 3 produces our proof of Poincaré inequalities (Theorem 1.8) which relies both on the formulation by duality of the Poincaré inequalites $\mathbb{H}\text{-Poincaré}_{p,\infty}(h)$. Therein we prove a relationship between $\mathbb{H}\text{-Sobolev}_{1,p'}(2n+2-h)$ and $\mathbb{H}\text{-Poincaré}_{p,\infty}(h)$ when $1 < p, p' < \infty$ are dual exponents, and eventually we combine this relationship with the previous duality argument for Poincaré inequalites.

Contrary to what happens when $1 \le p < Q$ (or $1 \le p < Q/2$), where the proofs of Poincaré and Sobolev inequalities proceeded on parallel tracks, here the proof of Sobolev inequalities require more delicate arguments: first we prove a L^∞-homotopy formula (see Section 4) and then we derive interior $\mathbb{H}\text{-Poincaré}_{\infty,\infty}$ and $\mathbb{H}\text{-Sobolev}_{\infty,\infty}$ inequalities. Then the subsequent step (Section 6) consists in proving that the $L^{\infty,\infty}$ cohomology of Rumin's forms vanishes on a suitable family of (Cygan-Korányi) annuli. Finally, Section 7 contains L^∞-estimates associated with Leibniz formula and hence ends with the proof of Sobolev inequalities stated in Theorem 1.9.

2. Heisenberg groups: definitions and preliminary results

We denote by $\mathbb{H}^n$ the n-dimensional Heisenberg group, identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted by $p = (x, y, t)$, with both $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. If p and $p' \in \mathbb{H}^n$, the group operation is defined by

$$p \cdot p' = (x + x', y + y', t + t' + \frac{1}{2}\sum_{j=1}^{n}(x_j y'_j - y_j x'_j)).$$

The unit element of $\mathbb{H}^n$ is the origin, that will be denote by e. For any $q \in \mathbb{H}^n$, the *(left) translation* $\tau_q : \mathbb{H}^n \to \mathbb{H}^n$ is defined as

$$p \mapsto \tau_q p := q \cdot p.$$

The Lebesgue measure in $\mathbb{R}^{2n+1}$ is a Haar measure in $\mathbb{H}^n$.

For a general review on Heisenberg groups and their properties, we refer to [34], [23] and to [37]. We limit ourselves to fix some notations, following [19].

The Heisenberg group $\mathbb{H}^n$ can be endowed with the homogeneous norm (Cygan-Korányi norm)

$$\varrho(p) = \big(|p'|^4 + 16\, p_{2n+1}^2\big)^{1/4}, \tag{9}$$

and we define the gauge distance (a true distance, see [34], p. 638), that is left invariant i.e. $d(\tau_q p, \tau_q p') = d(p, p')$ for all $p, p' \in \mathbb{H}^n$) as

$$d(p, q) := \varrho(p^{-1} \cdot q). \tag{10}$$

Finally, the balls for the metric d are le so-called Korányi balls

$$B(p,r) := \{q \in \mathbb{H}^n;\ d(p,q) < r\}. \tag{11}$$

Notice that Korányi balls are star-shaped with respect to the origin and convex smooth sets.

A straightforward computation shows that there exists $c_0 > 1$ such that

$$c_0^{-2}|p| \le \rho(p) \le |p|^{1/2}, \tag{12}$$

provided p is close to e. In particular, for $r > 0$ small, if we denote by $B_{\mathrm{Euc}}(e,r)$ the Euclidean ball centred at e of radius r,

$$B_{\mathrm{Euc}}(e,r^2) \subset B(e,r) \subset B_{\mathrm{Euc}}(e,c_0^2 r). \tag{13}$$

We denote by $\mathfrak{h}$ the Lie algebra of the left invariant vector fields of $\mathbb{H}^n$. The standard basis of $\mathfrak{h}$ is given, for $i = 1,\dots,n$, by

$$X_i := \partial_{x_i} - \frac{1}{2}y_i\partial_t, \quad Y_i := \partial_{y_i} + \frac{1}{2}x_i\partial_t, \quad T := \partial_t.$$

The only non-trivial commutation relations are $[X_i, Y_i] = T$, for $i = 1,\dots,n$. The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}$ spanned by $X_1,\dots,X_n$ and $Y_1,\dots,Y_n$: $\mathfrak{h}_1 := \mathrm{span}\,\{X_1,\dots,X_n,Y_1,\dots,Y_n\}$.
Coherently, from now on, we refer to $X_1,\dots,X_n,Y_1,\dots,Y_n$ (identified with first order differential operators) as the *horizontal derivatives*. Denoting by $\mathfrak{h}_2$ the linear span of T, the 2-step stratification of $\mathfrak{h}$ is expressed by

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

The stratification of the Lie algebra $\mathfrak{h}$ induces a family of non-isotropic dilations $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$, $\lambda > 0$ as follows: if $p = (x,y,t) \in \mathbb{H}^n$, then

$$\delta_\lambda(x,y,t) = (\lambda x, \lambda y, \lambda^2 t). \tag{14}$$

Notice that the gauge norm (9) is positively δ_λ-homogenous, so that the Lebesgue measure of the ball $B(x,r)$ is r^{2n+2} up to a geometric constant (the Lebesgue measure of $B(e,1)$).

Thus, the *homogeneous dimension* of $\mathbb{H}^n$ with respect to δ_λ, $\lambda > 0$, equals

$$Q := 2n+2.$$

It is well known that the topological dimension of $\mathbb{H}^n$ is $2n+1$, since as a smooth manifold it coincides with $\mathbb{R}^{2n+1}$, whereas the Hausdorff dimension of $(\mathbb{H}^n, d)$ is Q.

The vector space $\mathfrak{h}$ can be endowed with an inner product, indicated by $\langle\cdot,\cdot\rangle$, making $X_1,\dots,X_n,Y_1,\dots,Y_n$ and T orthonormal.

Throughout this paper, we write also

$$W_i := X_i, \quad W_{i+n} := Y_i \quad \text{and} \quad W_{2n+1} := T, \quad \text{for } i = 1,\dots,n. \tag{15}$$

As in [17], we also adopt the following multi-index notation for higher-order derivatives. If $I = (i_1,\dots,i_{2n+1})$ is a multi–index, we set

$$W^I = W_1^{i_1}\cdots W_{2n}^{i_{2n}}\, T^{i_{2n+1}}. \tag{16}$$

By the Poincaré–Birkhoff–Witt theorem, the differential operators W^I form a basis for the algebra of left invariant differential operators in $\mathbb{H}^n$. Furthermore, we set

$$|I| := i_1 + \cdots + i_{2n} + i_{2n+1}$$

the order of the differential operator W^I, and

$$d(I) := i_1 + \cdots + i_{2n} + 2i_{2n+1}$$

its degree of homogeneity with respect to group dilations.

The dual space of $\mathfrak{h}$ is denoted by $\bigwedge^1 \mathfrak{h}$. The basis of $\bigwedge^1 \mathfrak{h}$, dual to the basis $\{X_1, \dots, Y_n, T\}$, is the family of covectors $\{dx_1, \dots, dx_n, dy_1, \dots, dy_n, \theta\}$ where

$$\theta := dt - \frac{1}{2}\sum_{j=1}^{n}(x_j dy_j - y_j dx_j)$$

is the *contact form* in $\mathbb{H}^n$. We denote by $\langle \cdot, \cdot \rangle$ the inner product in $\bigwedge^1 \mathfrak{h}$ that makes $(dx_1, \dots, dy_n, \theta)$ an orthonormal basis.

2.1. **Sobolev spaces, distributions and kernels in $\mathbb{H}^n$.** Let $U \subset \mathbb{H}^n$ be an open set. We shall use the following classical notations: $\mathcal{E}(U)$ is the space of all smooth function on U, and $\mathcal{D}(U)$ is the space of all compactly supported smooth functions on U, endowed with the standard topologies (see e.g. [35]). The spaces $\mathcal{E}'(U)$ and $\mathcal{D}'(U)$ are their dual spaces of distributions.

Let $1 \le p \le \infty$ and $m \in \mathbb{N}$, $W^{m,p}_{\mathrm{Euc}}(U)$ denotes the usual Sobolev space.

We remind also the notion of (integer order) Folland-Stein Sobolev space (for a general presentation, see e.g. [16] and [17]).

Definition 2.1. *If $U \subset \mathbb{H}^n$ is an open set, $1 \le p \le \infty$ and $m \in \mathbb{N}$, then the space $W^{m,p}(U)$ is the space of all $u \in L^p(U)$ such that, with the notation of* (16),

$$W^I u \in L^p(U) \quad \textit{for all multi-indices } I \textit{ with } d(I) \le m,$$

endowed with the natural norm

$$\| u\|_{W^{k,p}(U)} := \sum_{d(I)\le m} \|W^I u\|_{L^p(U)}.$$

Folland-Stein Sobolev spaces enjoy the following properties akin to those of the usual Euclidean Sobolev spaces (see [16], and, e.g. [18]).

Theorem 2.2. *If $U \subset \mathbb{H}^n$, $1 \le p \le \infty$, and $k \in \mathbb{N}$, then*

i) *$W^{k,p}(U)$ is a Banach space.*

In addition, if $p < \infty$,

ii) *$W^{k,p}(U) \cap C^\infty(U)$ is dense in $W^{k,p}(U)$;*
iii) *if $U = \mathbb{H}^n$, then $\mathcal{D}(\mathbb{H}^n)$ is dense in $W^{k,p}(U)$;*
iv) *if $1 < p < \infty$, then $W^{k,p}(U)$ is reflexive.*

Theorem 2.3. *[see* [16]*, Theorem 5.15] If $p > Q$, then*

$$W^{1,p}(\mathbb{H}^n) \subset L^\infty(\mathbb{H}^n)$$

algebraically and topologically.

Definition 2.4. *If $U \subset \mathbb{H}^n$ is open and if $1 \le p < \infty$, we denote by $\overset{\circ}{W}{}^{k,p}(U)$ the completion of $\mathcal{D}(U)$ in $W^{k,p}(U)$.*

Remark 2.5. *If $U \subset \mathbb{H}^n$ is bounded, by (iterated) Poincaré inequality (see e.g.* [25]*), it follows that the norms*

$$\|u\|_{W^{k,p}(U)} \quad \textit{and} \quad \sum_{d(I)=k} \|W^I u\|_{L^p(U)}$$

are equivalent on $\overset{\circ}{W}{}^{k,p}(U)$ when $1 \le p < \infty$.

Again as in [17] it is possible to associate with the group structure a convolution (still denoted by $*$): if, for instance, $f \in \mathcal{D}(\mathbb{H}^n)$ and $g \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$, we set

$$f * g(p) := \int f(q) g(q^{-1}p)\, dq \quad \text{for } q \in \mathbb{H}^n. \tag{17}$$

We remind that, if (say) g is a smooth function and L is a left invariant differential operator, then $L(f*g) = f * Lg$. We remind also that the convolution is again well defined when $f, g \in \mathcal{D}'(\mathbb{H}^n)$, provided at least one of them has compact support (as customary, we denote by $\mathcal{E}'(\mathbb{H}^n)$ the class of compactly supported distributions in $\mathbb{H}^n$ identified with $\mathbb{R}^{2n+1}$). In this case the following identities hold

$$\langle f * g|\phi\rangle = \langle g|{}^{\mathrm{v}}f * \phi\rangle \quad \text{and} \quad \langle f * g|\phi\rangle = \langle f|\phi * {}^{\mathrm{v}}g\rangle \tag{18}$$

for any test function ϕ (if f is a real function defined in $\mathbb{H}^n$, we denote by ${}^{\mathrm{v}}f$ the function defined by ${}^{\mathrm{v}}f(p) := f(p^{-1})$ and, if $T \in \mathcal{D}'(\mathbb{H}^n)$, then ${}^{\mathrm{v}}T$ is the distribution defined by $\langle {}^{\mathrm{v}}T|\phi\rangle := \langle T|{}^{\mathrm{v}}\phi\rangle$ for any test function ϕ).

Suppose now $f \in \mathcal{E}'(\mathbb{H}^n)$ and $g \in \mathcal{D}'(\mathbb{H}^n)$. Then, if $\psi \in \mathcal{D}(\mathbb{H}^n)$, we have

$$\begin{aligned} \langle (W^I f) * g|\psi\rangle &= \langle W^I f|\psi * {}^{\mathrm{v}}g\rangle = (-1)^{|I|}\langle f|\psi * (W^I\, {}^{\mathrm{v}}g)\rangle \\ &= (-1)^{|I|}\langle f * {}^{\mathrm{v}}W^I\, {}^{\mathrm{v}}g|\psi\rangle. \end{aligned} \tag{19}$$

Proposition 2.6. *We have:*

1. *if $\phi \in \mathcal{D}(\mathbb{H}^n)$ and $T \in \mathcal{D}'(\mathbb{H}^n)$, then $\phi * T \in \mathcal{E}(\mathbb{H}^n)$ (see* [35]*, Theorem 7.23);*
2. *the convolution maps $\mathcal{E}(\mathbb{H}^n) \times \mathcal{E}'(\mathbb{H}^n)$ into $\mathcal{E}(\mathbb{H}^n)$ (see* [35]*, p. 288);*
3. *the map $(S,T) \to S * T$ defined by*
$$\langle S * T|\phi\rangle_{\mathcal{D}',\mathcal{D}} =: \langle S|\phi * {}^{\mathrm{v}}T\rangle_{\mathcal{E}',\mathcal{E}}$$
is a separately continuous bilinear map from $\mathcal{E}'(\mathbb{H}^n) \times \mathcal{D}'(\mathbb{H}^n)$ to $\mathcal{D}'(\mathbb{H}^n)$;
4. *if $T \in \mathcal{E}'(\mathbb{H}^n)$, and P is a differential operator, then $PT \in \mathcal{E}'(\mathbb{H}^n)$ and $\mathrm{supp}\, PT \subset \mathrm{supp}\, T$ (see* [35]*, 24.3);*
5. *let $U, U' \subset \mathbb{H}^n$ be open sets, $U \subset U'$. If $T \in \mathcal{D}'(U')$, we define its restriction*
$$T\big|_U \in \mathcal{D}'(U)$$
in the sense of [35]*, Example II pag. 245, i.e. for all $\phi \in \mathcal{D}(U) \subset \mathcal{D}(U')$ we set*
$$\langle T\big|_U|\phi\rangle := \langle T|\phi\rangle.$$
6. *let $U, U' \subset \mathbb{H}^n$ be open sets, $U \Subset U'$. Let $\beta, \hat\beta \in \mathcal{E}'(\mathbb{H}^n)$ be such that $\hat\beta\big|_{U'} = \beta\big|_{U'}$. If $k \in L^1(\mathbb{H}^n)$ and $\mathrm{supp}\, k \subset B(e,R)$ with $R > 0$ small enough, then*
$$\big(\beta * k\big)\big|_U = \big(\hat\beta * k\big)\big|_U.$$

Proof. Let us prove (6). Take $\phi \in \mathcal{D}(U)$ and assume $R < \mathrm{dist}\,(U, \partial U')$. Then

$$\begin{aligned} \langle \beta * k|\phi\rangle &= \langle \beta|\phi * {}^{\mathrm{v}}k\rangle \\ &= \langle \hat\beta|\phi * {}^{\mathrm{v}}k\rangle \qquad \text{(since supp } \phi * {}^{\mathrm{v}}k \text{ is contained in a } R\text{-neighborhood of } U) \\ &= \langle \hat\beta * k|\phi\rangle. \end{aligned}$$

□

Theorem 2.7. *We have:*

i) *Hausdorff-Young's inequality holds, i.e., if* $f \in L^p(\mathbb{H}^n)$, $g \in L^q(\mathbb{H}^n)$, $1 \le p, q, r \le \infty$ *and* $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$, *then* $f * g \in L^r(\mathbb{H}^n)$ *(see* [17], *Proposition 1.18) .*

ii) *If* K *is a kernel of type* 0, $1 < p < \infty$, $s \ge 0$, *then the mapping* $T : u \to u * K$ *defined for* $u \in \mathcal{D}(\mathbb{H}^n)$ *extends to a bounded operator on* $W^{s,p}(\mathbb{H}^n)$ *(see* [16], *Theorem 4.9).*

iii) *Suppose* $0 < \mu < Q$, $1 < p < Q/\mu$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\mu}{Q}$. *Let* K *be a kernel of type* μ. *If* $u \in L^p(\mathbb{H}^n)$ *the convolutions* $u * K$ *and* $K * u$ *exists a.e. and are in* $L^q(\mathbb{H}^n)$ *and there is a constant* $C_p > 0$ *such that*

$$\|u * K\|_q \le C_p \|u\|_p \quad \text{and} \quad \|K * u\|_q \le C_p \|u\|_p$$

(see [16], *Proposition 1.11).*

iv) *Suppose* $s \ge 1$, $1 < p < Q$, *and let* U *be a bounded open set. If* K *is a kernel of type* 1 *and* $u \in W^{s-1,p}(\mathbb{H}^n)$ *with* $\operatorname{supp} u \subset U$, *then*

$$\|u * K\|_{W^{s,p}(\mathbb{H}^n)} \le C_U \|u\|_{W^{s-1,p}}(\mathbb{H}^n).$$

Proof. The proof of iv) can be carried out relying on Theorems 4.10, 4.9 and Proposition 1.11 of [16], keeping into account that $L^{pQ/(Q-p)}(U) \subset L^p$ and ii) above. Indeed

$$\begin{aligned}
\|u * K\|_{W^{s,p}(\mathbb{H}^n)} &\le C\big\{\|u * K\|_{L^p(\mathbb{H}^n)} + \sum_{\ell=1}^{m} \|u * W_\ell K\|_{W^{s-1,p}(\mathbb{H}^n)}\big\} \\
&\le C\big\{\|u * K\|_{L^p(\mathbb{H}^n)} + \|u\|_{W^{s-1,p}(\mathbb{H}^n)}\big\} \\
&\le C\big\{\|u\|_{L^{pQ/(Q-p}(\mathbb{H}^n)} + \|u\|_{W^{s-1,p}(\mathbb{H}^n)}\big\} \le C_U \|u\|_{W^{s-1,p}}(\mathbb{H}^n).
\end{aligned}$$

□

We state now a few properties related to the convolution in L^∞ that will be used in the sequel and - as far as we know - are not explicitly stated in the literature.

We stress that we have to proceed carefully and we cannot use the corresponding results in [8], [6] because of the presence of the L^∞-space. The proofs follow verbatim those of analogous statements in the Euclidean setting, keeping in mind that the group convolution is not commutative.

We remind first that, if $a \in L^\infty(\mathbb{H}^n) \subset L^1_{\mathrm{loc}}$, then the map $\phi \mapsto \int a(x)\phi(x)\,dx$ defines a distribution in $\mathcal{D}'(U)$ for all open sets $U \subset \mathbb{H}^n$.

Lemma 2.8. *Let* $a \in \mathcal{E}'(\mathbb{H}^n)$ *(i.e.* $a \in \mathcal{D}'(\mathbb{H}^n)$ *and* a *has compact support, see* [35] *Theorem 24.2). If* $\phi \in \mathcal{D}(\mathbb{H}^n)$ *and* K *is a kernel in* L^1_{loc}, *we notice first that* $\phi * {}^{\mathrm{v}}K \in \mathcal{E}(\mathbb{H}^n)$ *so that the map*

$$\phi \mapsto \langle a | \phi * {}^{\mathrm{v}}K \rangle_{\mathcal{E}',\mathcal{E}} =: \langle a * K | \phi \rangle_{\mathcal{D}',\mathcal{D}}$$

belongs to $\mathcal{D}'$.

Moreover, if W *is a horizontal derivative, then the convolution* $a * WK$ *is well defined since* a *is compactly supported. In addition,*

$$W(a * K) = a * WK. \tag{20}$$

Proof. The first statement follows from [35], Definition 27.3 and Theorem 27.6.

As for the last statement, consider a test function $\phi \in \mathcal{D}(\mathbb{H}^n)$. We claim that

$$\langle W(a * K) | \phi \rangle = \langle a * K | W\phi \rangle = \langle a * WK | \phi \rangle.$$

Indeed $\langle a * K | W\phi\rangle = \langle a | W\phi * {}^{\rm v}K\rangle$. Since

$$W\phi * {}^{\rm v}K(x) = \int W\phi(y)K(x^{-1}y)\,dy = \int \phi(y)(WK)(x^{-1}y)\,dy = \phi * {}^{\rm v}(WK)(x),$$

we can conclude since $\langle a | W\phi * {}^{\rm v}K\rangle = \langle a | \phi * {}^{\rm v}(WK)\rangle = \langle a * WK | \phi\rangle$.

□

Proposition 2.9. *Let U be a bounded open subset of $\mathbb{H}^n$, and suppose $a \in L^\infty(U)$ is compactly supported. If K is a kernel in $L^1_{\rm loc}$, then the convolution $a * K$ defined in Lemma 2.8 belongs to $L^\infty(U)$, and*

$$\|a * K\|_{L^\infty(U)} \le C(K, U, \operatorname{supp} a)\|a\|_{L^\infty(U)}.$$

Proof. Take $\phi \in \mathcal{D}(U)$. We note first that $\phi * {}^{\rm v}K$ belongs to $L^1(\operatorname{supp} a)$ and

$$\|\phi * {}^{\rm v}K\|_{L^1(\operatorname{supp} a)} \le C(K)\|\phi\|_{L^1(U)}. \tag{21}$$

Indeed, if $x \in \operatorname{supp} a$,

$$|(\phi * {}^{\rm v}K)(x)| \le \int_{d(z,e)\le R} |\phi(xz)|\,|K(z)|\,dz,$$

since

$$d(z,e) \le d(z^{-1}, x) + d(x,e) \le \operatorname{diam}(U) + d(\operatorname{supp} a, e) =: R.$$

Thus

$$\|\phi * {}^{\rm v}K\|_{L^1(\operatorname{supp} a)} \le \||\phi| * (|K|\chi_{B(e,R)}\|_{L^1(\mathbb{H}^n)} \le C(K,R)\|\phi\|_{L^1(U)}.$$

Thus, by definition,

$$|\langle a * K | \phi\rangle_{\mathcal{D}',\mathcal{D}}| \le C(K, U, \operatorname{supp} a)\|\phi\|_{L^1(U)}\|a\|_{L^\infty(U)},$$

and the assertion follows by duality since $L^\infty = (L^1)^*$. □

2.2. **Multilinear algebra in $\mathbb{H}^n$ and Rumin's complex.** Unfortunately, when dealing with differential forms in $\mathbb{H}^n$, the de Rham complex lacks scale invariance under anisotropic dilations (see (14)). Thus, a substitute for de Rham's complex, that recovers scale invariance under δ_t has been defined by M. Rumin, [30]. In turn, this notion makes sense for arbitrary contact manifolds. We refer to [30] and [9], [8] for details of the construction. In the present paper, we shall merely need the following list of formal properties (for the sake of completeness, in an Appendix we describe in more detail the construction of Rumin's complex).

Throughout this paper, $\bigwedge^h \mathfrak{h}$ denotes the h-th exterior power of the Lie algebra $\mathfrak{h}$. Keeping in mind that the Lie algebra $\mathfrak{h}$ can be identified with the tangent space to $\mathbb{H}^n$ at $x = e$ (see, e.g. [21], Proposition 1.72), starting from $\bigwedge^h \mathfrak{h}$ we can define by left translation a fiber bundle over $\mathbb{H}^n$ that we can still denote by $\bigwedge^h \mathfrak{h} \simeq \bigwedge^h T^*\mathbb{H}^n$. Moreover, a scalar product in $\mathfrak{h}$ induces a scalar product and a norm on $\bigwedge^h \mathfrak{h}$.

We can think of h-forms as sections of $\bigwedge^h \mathfrak{h}$ and we denote by Ω^h the vector space of all smooth h-forms.

- For $h = 0, \dots, 2n+1$, the space of Rumin h-forms, E_0^h, is the space of smooth sections of a left-invariant subbundle of $\bigwedge^h \mathfrak{h}$ (that we still denote by E_0^h). Hence it inherits the inner product and the norm of $\bigwedge^h \mathfrak{h}$.

- A differential operator $d_c : E_0^h \to E_0^{h+1}$ is defined. It is left-invariant, homogeneous with respect to group dilations. It is a first order homogeneous operator in the horizontal derivatives in degree $\neq n$, whereas **it is a second order homogeneous horizontal operator in degree** n.
- Altogether, operators d_c form a complex: $d_c \circ d_c = 0$.
- This complex is homotopic to de Rham's complex $(\Omega^\bullet, d)$. More precisely there exist a sub-complex (E, d) of the de Rham complex and a suitable "projection" $\Pi_E : \Omega^\bullet \to E^\bullet$ such that Π_E is a differential operator of order ≤ 1 in the horizontal derivatives.
- Π_E is a chain map, i.e.
$$d\Pi_E = \Pi_E d.$$
- Let Π_{E_0} be the orthogonal projection on $E_0^\bullet$. Then
$$\Pi_{E_0}\Pi_E\Pi_{E_0} = \Pi_{E_0} \qquad \text{and} \qquad \Pi_E\Pi_{E_0}\Pi_E = \Pi_E.$$
(we stress that Π_{E_0} is an algebraic operator).
- The exterior differential d_c can be written as
$$d_c = \Pi_{E_0} d\Pi_E\Pi_{E_0}.$$

Let us list a bunch of notations for vector-valued function spaces (for the scalar case, we refer to Section 2.1).

Definition 2.10. *If $U \subset \mathbb{H}^n$ is an open set, $0 \leq h \leq 2n+1$, $1 \leq p \leq \infty$ and $m \geq 0$, we denote by $L^p(U, \bigwedge^h \mathfrak{h})$, $\mathcal{E}(U, \bigwedge^h \mathfrak{h})$, $\mathcal{D}(U, \bigwedge^h \mathfrak{h})$, $W^{m,p}(U, \bigwedge^h \mathfrak{h})$ (by $\overset{\circ}{W}{}^{m,p}(U, \bigwedge^h \mathfrak{h})$) the space of all sections of $\bigwedge^h \mathfrak{h}$ such that their components with respect to a given left-invariant frame belong to the corresponding scalar spaces.*

The spaces $L^p(U, E_0^h)$, $\mathcal{E}(U, E_0^h)$, $\mathcal{D}(U, \mathbb{E}_0^h)$, $W^{m,p}(U, E_0^h)$ and $\overset{\circ}{W}{}^{m,p}(U, E_0^h)$ are defined in the same way.

Finally, the spaces $W^{m,p}_{\mathrm{Euc}}(U, \bigwedge^h \mathfrak{h})$, $\overset{\circ}{W}{}^{m,p}_{\mathrm{Euc}}(U, \bigwedge^h \mathfrak{h})$, $W^{m,p}_{\mathrm{Euc}}(U, E_0^h)$ and $\overset{\circ}{W}{}^{m,p}_{\mathrm{Euc}}(U, E_0^h)$ are defined replacing Folland-Stein Sobolev spaces by usual Sobolev spaces.

Clearly, all these definitions are independent of the choice of frame.

When d_c is second order (when acting on forms of degree n), $(E_0^\bullet, d_c)$ stops behaving like a differential module. This is the source of many complications. In particular, the classical Leibniz formula for the de Rham complex $d(\alpha\wedge\beta) = d\alpha\wedge\beta \pm \alpha\wedge d\beta$ in general fails to hold (see [10]-Proposition A.7). This causes several technical difficulties when we want to localize our estimates by means of cut-off functions.

Lemma 2.11 (see also [6], Lemma 4.1)**.** *If ζ is a smooth real function, then the following formulae hold in the sense of distributions:*

i) *if $h \neq n$, then on E_0^h we have*
$$[d_c, \zeta] = P_0^h(W\zeta),$$
where $P_0^h(W\zeta) : E_0^h \to E_0^{h+1}$ is a linear homogeneous differential operator of order zero with coefficients depending only on the horizontal derivatives of ζ. If $h \neq n+1$, an analogous statement holds if we replace d_c in degree h with d_c^* in degree $h+1$;

ii) *if $h=n$, then on E_0^n we have*

$$[d_c,\zeta] = P_1^n(W\zeta) + P_0^n(W^2\zeta),$$

where $P_1^n(W\zeta) : E_0^n \to E_0^{n+1}$ is a linear homogeneous differential operator of order 1 (and therefore horizontal) with coefficients depending only on the horizontal derivatives of ζ, and where $P_0^h(W^2\zeta) : E_0^n \to E_0^{n+1}$ is a linear homogeneous differential operator in the horizontal derivatives of order 0 with coefficients depending only on second order horizontal derivatives of ζ. If $h=n+1$, an analogous statement holds if we replace d_c in degree n with d_c^* in degree $n+1$.

Remark 2.12. *On forms of degree $h>n$, Lemma 2.11 i) takes the following simpler form. If $\alpha \in L^1_{\mathrm{loc}}(\mathbb{H}^n, E_0^h)$ with $h>n$ and $\psi \in \mathcal{E}(\mathbb{H}^n)$, then*

$$d_c(\psi\alpha) = d(\psi\alpha) = d\psi\wedge\alpha + \psi d\alpha = d_c\psi\wedge\alpha + \psi d_c\alpha,$$

This follows from Theorem 8.6, viii), since α is a multiple of θ.

Leibniz formula has the following quantitative form.

Remark 2.13. *Denote by $B = B(e,1)$ the unit ball in $\mathbb{H}^n$. If $\lambda>1$, let ζ be a smooth function on $\mathbb{H}^n$ that is supported outside of a neighborhood of B, such that $W_i\zeta$ is compactly supported in $B_\lambda = B(e,\lambda)$ for $i=1,\dots,2n$.*

i) *If $h\neq n$, let $\sigma \in L^\infty\left(B_\lambda, E_0^h\right)\cap d^{-1}L^\infty\left(B_\lambda, E_0^h\right)$, then*

$$\|d_c(\zeta\sigma)\|_{L^\infty(B_\lambda,E_0^h)} \le C_\zeta\Big(\|\sigma\|_{L^\infty(B_\lambda,E_0^h)} + \|d_c\sigma\|_{L^\infty(B_\lambda,E_0^{h+1})}\Big). \tag{22}$$

ii) *If $h=n$ let $\sigma\in W^{1,\infty}(B_\lambda,E_0^n)\cap d^{-1}L^\infty(B_\lambda,E_0^n)$ and, then*

$$\|d_c(\zeta\sigma)\|_{L^\infty(B_\lambda,E_0^{n+1})} \le C_\zeta\Big(\|\sigma\|_{W^{1,\infty}(B_\lambda,E_0^n)} + \|d_c\sigma\|_{L^\infty(S_\zeta,E_0^{n+1})}\Big), \tag{23}$$

where S_ζ is a neighborhood in B_λ of $\operatorname{supp}\zeta\cap B_\lambda$ (contained in $B_\lambda\setminus\overline{B}$).

The following generalizes Remark 2.16 of [9]. The proof uses a notation from the Appendix, Theorem 8.6.

Lemma 2.14. *Let $U\subset\mathbb{H}^n$ be an open set.*

i) *Let ψ be an h-form in $L^1_{\mathrm{loc}}(U,E_0^h)$ and $\alpha\in\mathcal{D}(U,E_0^{2n-h})$. Then*

$$\int_U (d_c\phi)\wedge\alpha = (-1)^{h+1}\int_U \phi\wedge d_c\alpha,$$

where the left-hand side is understood in distribution sense.

Assume further that U is contractible. Let ω and ψ be d_c-closed Rumin forms on U of complementary degrees h and $2n+1-h$, with $1\le h\le 2n$. Then

$$\int_U \omega\wedge\psi = 0$$

in the following cases:

ii) $\omega\in L^1_{\mathrm{loc}}(U,E_0^h)$ *and* $\psi\in\mathcal{D}(U,E_0^{2n+1-h})$.

iii) $1<p<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$, $\omega\in L^p_{\mathrm{loc}}(U,E_0^h)$ *and* $\psi\in L^{p'}(U,E_0^{2n+1-h})$ *is compactly supported in U.*

Proof. Assume first that ϕ is smooth. Since $\phi \wedge \alpha$ has degree $2n \geq n+1$, $d(\phi \wedge \alpha) = d_c(\phi \wedge \alpha)$. If $h \neq n$, the formula

$$d_c(\phi \wedge \alpha) = (d_c\phi) \wedge \alpha + (-1)^h \phi \wedge d_c\alpha$$

is established in [29], Prop. 4.2. Let us assume that both ϕ and α have degree n. Then, by definition, $d_c\phi = d\Pi_E\phi$ where $\Pi_E\phi - \phi$ has weight $n+1$ (see Theorem 8.6 ix)). Since $d_c\alpha$ has weight $n+2$, $(\Pi_E\phi - \phi) \wedge d_c\alpha = 0$. Symmetrically, $(\Pi_E\alpha - \alpha) \wedge d_c\phi = 0$. It follows that

$$d(\Pi_E\phi \wedge \Pi_E\alpha) = (d_c\phi) \wedge \alpha + (-1)^n \phi \wedge d_c\alpha.$$

In all cases, we have come up with a compactly supported primitive of $(d_c\phi)\wedge\alpha+(-1)^h\phi\wedge d_c\alpha$, hence

$$\int_U ((d_c\phi) \wedge \alpha + (-1)^h \phi \wedge d_c\alpha) = 0.$$

Formula i) extends to Rumin forms ϕ with distributional coefficients, and in particular to forms in $L^1_{\mathrm{loc}}(U, E_0^h)$.

Assume first that ω and ψ are smooth. Since Rumin's complex is homotopic to de Rham's, ω admits a smooth primitive ϕ on U, $d_c\phi = \omega$. Then i) implies that

$$\int_U \omega \wedge \psi = \int_U (d_c\phi) \wedge \psi = \pm \int_U \phi \wedge d_c\psi = 0.$$

ii) By right convolution (which commutes with the left-invariant operator d_c), closed forms are dense in L^1_{loc} d_c-closed forms, so the identity extends to the case where $\omega \in L^1_{\mathrm{loc}}(U, E_0^h)$.

iii) Again by right convolution, smooth d_c-closed forms are dense in $L^p_{\mathrm{loc}}(U, E_0^h)$ d_c-closed forms and smooth compactly supported d_c-closed forms are dense in compactly supported $L^{p'}(U, E_0^{2n+1-h})$ d_c-closed forms.

□

2.3. **Rumin's Laplacian.**

Definition 2.15. *In $\mathbb{H}^n$, following* [30]*, we define the operators $\Delta_{\mathbb{H},h}$ on E_0^h by setting*

$$\Delta_{\mathbb{H},h} = \begin{cases} d_c d_c^* + d_c^* d_c & \text{if} \quad h \neq n, n+1; \\ (d_c d_c^*)^2 + d_c^* d_c & \text{if} \quad h = n; \\ d_c d_c^* + (d_c^* d_c)^2 & \text{if} \quad h = n+1. \end{cases}$$

Notice that $-\Delta_{\mathbb{H},0} = \sum_{j=1}^{2n}(W_j^2)$ is the usual sub-Laplacian of $\mathbb{H}^n$.

For sake of simplicity, once a basis of E_0^h is fixed, the operator $\Delta_{\mathbb{H},h}$ can be identified with a matrix-valued map, still denoted by $\Delta_{\mathbb{H},h}$,

$$\Delta_{\mathbb{H},h} = (\Delta_{\mathbb{H},h}^{ij})_{i,j=1,\dots,N_h} : \mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h}) \to \mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h}), \tag{24}$$

where $\mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h})$ is the space of vector-valued distributions on $\mathbb{H}^n$, and N_h is the dimension of E_0^h (see [2]).

This identification makes possible to avoid the notion of currents: we refer to [9] for a more elegant presentation.

Combining [30], Section 3, and [11], Theorems 3.1 and 4.1, we obtain the following result.

Theorem 2.16 (see [11], Theorem 3.1)**.** *If $0 \leq h \leq 2n+1$, then the differential operator $\Delta_{\mathbb{H},h}$ is homogeneous of degree μ with respect to group dilations, where $\mu = 2$ if $h \neq n, n+1$ and $\mu = 4$ if $h = n, n+1$. It follows that*

i) *for* $j = 1, \dots, N_h$ *there exists*

$$K_j = \big(K_{1j}, \dots, K_{N_h j}\big), \quad j = 1, \dots N_h \tag{25}$$

with $K_{ij} \in \mathcal{D}'(\mathbb{H}^n) \cap \mathcal{E}(\mathbb{H}^n \setminus \{0\})$, $i, j = 1, \dots, N_h$*;*

ii) *if* $\mu < Q$*, then the* K_{ij}*'s are kernels of type* μ *for* $i, j = 1, \dots, N_h$

If $\mu = Q$*, then the* K_{ij}*'s satisfy the logarithmic estimate* $|K_{ij}(p)| \le C(1 + |\ln \rho(p)|)$ *and hence belong to* $L^1_{\mathrm{loc}}(\mathbb{H}^n)$*. Moreover, their horizontal derivatives* $W_\ell K_{ij}$*,* $\ell = 1, \dots, 2n$*, are kernels of type* $Q - 1$*;*

iii) *when* $\alpha \in \mathcal{D}(\mathbb{H}^n, \mathbb{R}^{N_h})$*, if we set*

$$\Delta^{-1}_{\mathbb{H},h} \alpha := \big(\sum_j \alpha_j * K_{1j}, \dots, \sum_j \alpha_j * K_{N_h j}\big), \tag{26}$$

then $\Delta_h \Delta^{-1}_{\mathbb{H},h} \alpha = \alpha$*. Moreover, if* $\mu < Q$*, also* $\Delta^{-1}_{\mathbb{H},h} \Delta_h \alpha = \alpha$*.*

iv) *if* $\mu = Q$*, then for any* $\alpha \in \mathcal{D}(\mathbb{H}^n, \mathbb{R}^{N_h})$ *there exists* $\beta_\alpha := (\beta_1, \dots, \beta_{N_h}) \in \mathbb{R}^{N_h}$*, such that*

$$\Delta^{-1}_{\mathbb{H},h} \Delta_h \alpha - \alpha = \beta_\alpha.$$

iv) $\Delta^{-1}_{\mathbb{H},h} : \mathcal{D}(\mathbb{H}^n, \mathbb{R}^{N_h}) \to \mathcal{E}(\mathbb{H}^n, \mathbb{R}^{N_h})$ *and* $\Delta^{-1}_{\mathbb{H},h} : \mathcal{E}'(\mathbb{H}^n, \mathbb{R}^{N_h}) \to \mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h})$.

Remark 2.17. *If* $\mu < Q$*,* $\Delta_{\mathbb{H},h}(\Delta^{-1}_{\mathbb{H},h} - {}^{\mathrm{v}}\Delta^{-1}_{\mathbb{H},h}) = 0$ *and hence* $\Delta^{-1}_{\mathbb{H},h} = {}^{\mathrm{v}}\Delta^{-1}_{\mathbb{H},h}$*, by the Liouville-type theorem of* [11]*, Proposition 3.2.*

Remark 2.18. *From now on, if there are no possible misunderstandings, we identify* $\Delta^{-1}_{\mathbb{H},h}$ *with its kernel.*

Lemma 2.19 (see [8], Lemma 4.11)**.** *If* $\phi \in \mathcal{D}(\mathbb{H}^n, E_0^h)$ *and* $n \ge 1$*, then*

i) $d_c \Delta^{-1}_{\mathbb{H},h} \phi = \Delta^{-1}_{\mathbb{H},h+1} d_c \phi$, $\quad h = 0, 1, \dots, 2n$, $\quad h \ne n - 1, n + 1$.
ii) $d_c \Delta^{-1}_{\mathbb{H},n-1} \phi = d_c d_c^* \Delta^{-1}_{\mathbb{H},n} d_c \phi \quad (h = n - 1)$.
iii) $d_c d_c^* d_c \Delta^{-1}_{\mathbb{H},n+1} \phi = \Delta^{-1}_{\mathbb{H},n+2} d_c \phi, \quad (h = n + 1)$.
iv) $d_c^* \Delta^{-1}_{\mathbb{H},h} \phi = \Delta^{-1}_{\mathbb{H},h-1} d_c^* \phi \quad h = 1, \dots, 2n + 1, \quad h \ne n, n + 2$.
v) $d_c^* \Delta^{-1}_{\mathbb{H},n+2} \phi = d_c^* d_c \Delta^{-1}_{\mathbb{H},n+1} d_c^* \phi \quad (h = n + 2)$.
vi) $d_c^* d_c d_c^* \Delta^{-1}_{\mathbb{H},n} \phi = \Delta^{-1}_{\mathbb{H},n-1} d_c^* \phi, \quad (h = n)$.

3. Dual formulations and proof of Theorem 1.8

The interior Poincaré inequality of Theorem 1.8 relies on three tools:

i) the formulation by duality of the interior Poincaré inequality $\mathbb{H}$-Poincaré$_{p,\infty}(h)$ (see Proposition 3.1) below;
ii) the integration by parts formula of Lemma 2.14;
iii) the relationship between $\mathbb{H}$-Sobolev$_{1,p'}(2n+2-h)$ and $\mathbb{H}$-Poincaré$_{p,\infty}(h)$, when $1 < p < \infty$ and p' is the dual exponent of p.

Proposition 3.1. *Assume that* $1 \le h < 2n + 1$*. Let* B*,* B_λ *be concentric balls as in Definition 1.2. Take* $1 < p < \infty$*. Then the* $\mathbb{H}$-Poincaré$_{p,\infty}(h)$ *inequality in* E_0^h *holds if and only if there exists a constant* C *such that for every* d_c*-closed differential* h*-form* ω *on* $L^p(B_\lambda, E_0^h)$ *and every smooth differential* $(2n + 1 - h)$*-form* α *with compact support in* B*,*

$$\Big|\int_B \omega \wedge \alpha\Big| \le C\, \|\omega\|_{L^p(B_\lambda, E_0^h)} \|d_c \alpha\|_{L^1(B, E_0^{2n+2-h})}. \tag{27}$$

An analogous statement holds for the global Poincaré inequality on $\mathbb{H}^n$*.*

Proof. Suppose ω is a h-form satisfying Definition 1.2. If $\omega_{|B} = d_c\phi$ with $\phi \in L^\infty(\mathbb{H}^n, E_0^{h-1})$ as in Definition 1.2, and $\alpha \in \mathcal{D}(B, E_0^{2n+1-h})$, then, according to Lemma 2.14 i),

$$\begin{aligned}|\int_B \omega \wedge \alpha| &= |\int_B \phi \wedge d_c\alpha| \\ &\le \|\phi\|_{L^\infty(B,E^{h-1})} \|d_c\alpha\|_{L^1(B,E_0^{2n+2-h})}.\end{aligned}$$

Since $\|\phi\|_{L^\infty(B,E_0^{h-1})}$ can be estimated by $\|\omega\|_{L^p(B_\lambda,E_0^h)}$, inequality (27) follows.

Conversely, assume that for all forms $\alpha \in \mathcal{D}(B; E_0^{2n+1-h})$,

$$|\int_B \omega \wedge \alpha| \le C\, \|\omega\|_{L^p(B_\lambda;E_0^h)} \|d_c\alpha\|_{L^1(B;E_0^{2n+2-h})}.$$

Define a linear functional η on differentials of smooth $(2n+1-k)$-forms with compact support in B as follows. If $\beta = d_c\alpha$, $\alpha \in \mathcal{D}(B; E_0^{2n+1-h})$, set

$$\eta(\beta) = \int_B \omega \wedge \alpha.$$

Then η is well defined since $\omega \in L^1_{\rm loc}$, and is continuous in $L^1(B; E_0^{2n+2-h})$-norm, by (27), i.e.

$$|\eta(\beta)| \le C\, \|\omega\|_{L^p(B_\lambda;E_0^h)} \|\beta\|_{L^1(B;E_0^{2n+2-h})}.$$

By the Hahn-Banach theorem, η extends to a linear functional on all of $L^1(B; E_0^{2n+2-h})$, with the same norm. Such a functional is represented by a differential form $\phi \in L^\infty(B; E_0^{h-1})$ as follows,

$$\eta(\beta) = \int_B \phi \wedge \beta.$$

The L^∞-norm of ϕ is at most $C\, \|\omega\|_{L^p(B_\lambda;E_0^h)}$. Since, for all forms $\alpha \in \mathcal{D}(B; E_0^{2n+1-h})$,

$$\int_B \phi \wedge d_c\alpha = \eta(d_c\alpha) = \int_B \omega \wedge \alpha,$$

one concludes that $d_c\phi = \omega$ on B in the distributional sense.

The proof of the statement for the global Poincaré inequality can be carried out repeating verbatim the same arguments. □

Proposition 3.2. *If $2 \le h \le 2n+1$, $1 < p < \infty$ and p' is the dual exponent of p, then the interior $\mathbb{H}$-Sobolev$_{1,p'}(2n+2-h)$ inequality implies the interior $\mathbb{H}$-Poincaré$_{p,\infty}(h)$ inequality.*

An analogous statement holds for global inequalities on $\mathbb{H}^n$.

Proof. To prove the assertion, we argue by duality relying on Proposition 3.1. Let B and B_λ be concentric balls, such that $\mathbb{H}$-Sobolev$_{1,p'}(2n+2-h)$ inequality holds in B, B_λ. Take a d_c-closed h-form ω in $L^p(B_\lambda, E_0^h)$ and an arbitrary smooth differential $(2n+1-h)$-form α with compact support in B. By Sobolev inequality, there exists a compactly supported differential $(2n+1-h)$-form $\beta \in L^{p'}(B_\lambda, E_0^{2n+1-h})$ such that $d_c\beta = d_c\alpha$ in B and

$$\|\beta\|_{L^{p'}(B_\lambda,E_0^{2n+1-h})} \le C\, \|d_c\alpha\|_{L^1(B,E_0^{2n+2-h})}. \tag{28}$$

If $h = 2n+1$, α and β are compactly supported functions and $d_c(\beta-\alpha) = 0$, hence $\beta = \alpha$. Otherwise, since $\psi = \beta - \alpha \in L^{p'}(B_\lambda, E_0^{2n+1-h})$ is d_c-closed, Lemma 2.14 iii) implies that

$$\int_{B_\lambda} \omega \wedge (\beta - \alpha) = 0.$$

Therefore in both cases, by Hölder inequality and by (28),

$$|\int_B \omega\wedge\alpha| = |\int_{B_\lambda} \omega\wedge\alpha| = |\int_{B_\lambda} \omega\wedge\beta| \le C\,\|\omega\|_{L^p(B_\lambda,E_0^h)}\|d_c\alpha\|_{L^1(B,E_0^{2n+2-h})}.$$

By Proposition 3.1, this implies $\mathbb{H}$-Poincaré$_{p,\infty}(h)$.

The global Sobolev inequality implies the global Poincaré inequality in the same manner.

□

Proof of Theorem 1.8. Theorem 1.8 follows straightforwardly, combining the L^1 Sobolev inequality proven in [6], Corollary 6.5, and previous Proposition 3.2.

□

4. Homotopy formulæ in L^∞

The following global homotopy formula has been proven in $\mathcal{D}(\mathbb{H}^n, E_0^\bullet)$ in [8], Proposition 6.9. However we have to stress that here we deal with L^∞ forms, hence we have to adapt the proof since we cannot rely on a density argument in L^∞.

Proposition 4.1. *If $\alpha \in L^\infty(\mathbb{H}^n, E_0^h)$ is compactly supported, then the following homotopy formulas hold: there exist operators K_1, K_2 such that*

i) *if $h \neq n, n+1$, then $\alpha = d_cK_1\alpha + K_1d_c\alpha$ in the sense of distributions, where K_1 is associated with a kernel k_1 of type 1;*
ii) *if $h = n$, then $\alpha = d_cK_1\alpha + K_2d_c\alpha$ in the sense of distributions, where K_1 and K_2 are associated with kernels k_1, k_2 of type 1 and 2, respectively;*
iii) *if $h = n+1$, then $\alpha = d_cK_2\alpha + K_1d_c\alpha$ in the sense of distributions, where K_2 and K_1 are associated with kernels k_2, k_1 of type 2 and 1, respectively.*

Proof. The proof can be carried out by duality. Consider for instance the case i), and let $\phi \in \mathcal{D}(\mathbb{H}^n, E_0^h)$ be a test form. By Theorem 2.16,

$$\begin{aligned}
\langle\alpha|\phi\rangle_{\mathcal{D}',\mathcal{D}} &= \langle\alpha|(d_cd_c^* + d_c^*d_c)\Delta_{\mathbb{H}}^{-1}\phi\rangle_{\mathcal{D}',\mathcal{D}} = \langle\alpha|(d_cd_c^* + d_c^*d_c)\Delta_{\mathbb{H}}^{-1}\phi\rangle_{\mathcal{E}',\mathcal{E}}\\
&= \langle\alpha|\Delta_{\mathbb{H}}^{-1}d_cd_c^*\phi\rangle_{\mathcal{E}',\mathcal{E}} + \langle\alpha|d_c^*\Delta_{\mathbb{H}}^{-1}d_c\phi\rangle_{\mathcal{E}',\mathcal{E}} \qquad \text{(by Lemma 2.19)}\\
&= \langle\Delta_{\mathbb{H}}^{-1}\alpha|d_cd_c^*\phi\rangle_{\mathcal{D}',\mathcal{D}} + \langle d_c^*\Delta_{\mathbb{H}}^{-1}d_c\alpha|\phi\rangle_{\mathcal{D}',\mathcal{D}} \qquad \text{(since } d_c\alpha \in \mathcal{E}')\\
&= \langle d_cd_c^*\Delta_{\mathbb{H}}^{-1}\alpha|\phi\rangle_{\mathcal{D}',\mathcal{D}} + \langle d_c^*\Delta_{\mathbb{H}}^{-1}d_c\alpha|\phi\rangle_{\mathcal{D}',\mathcal{D}}\\
&=: \langle d_cK_1\alpha + K_1d_c\alpha|\phi\rangle_{\mathcal{D}',\mathcal{D}},
\end{aligned}$$

and the assertion follows since $K_1 := d_c^*\Delta_{\mathbb{H}}^{-1}$ is a kernel of type 1 by Theorem 2.16.

The proofs of ii) and iii) can be carried out through similar duality arguments keeping in mind [8], Proposition 6.9.

□

Remark 4.2. *To avoid cumbersome notations, from now on we denote by K_0 one of the convolution operators K_1, K_2, so that the homotopy formulas of Proposition 4.1 can be* *written concisely as follows: if $\alpha \in L^\infty(\mathbb{H}^n, E_0^h)$ is compactly supported then $K_0\alpha$ and $K_0 d_c\alpha$ are well defined distributions by Lemma 2.8 and*

$$\alpha = d_c K_0\alpha + K_0 d_c\alpha, \tag{29}$$

where K_0 is associated with a kernel of type 1 or 2, depending on the degree of α. Notice that in any case, K_0 belongs to L^1_{loc}.

Proposition 4.3. *Let $U \Subset U'$ be bounded open sets in $\mathbb{H}^n$. For $h = 1, \dots, 2n$, for every $s \in \mathbb{N}$, there exist a bounded operator*

$$T : L^\infty(U', E_0^\bullet) \to L^\infty(U, E_0^{\bullet-1}) \tag{30}$$

and a smoothing operator

$$S : L^\infty(U', E_0^\bullet) \to \mathcal{E}(U, E_0^{\bullet-1}) \tag{31}$$

such that, in addition,

$$S \in \mathcal{L}\big(L^\infty(U', E_0^\bullet), W^{s,\infty}(U, E_0^{\bullet-1})\big) \tag{32}$$

so that for any h-forms α in $L^\infty(U', E_0^\bullet)$ such that $d_c\alpha \in L^\infty(U', E_0^{\bullet+1})$ the following approximate homotopy formula holds in the sense of distributions

$$\alpha = d_c T\alpha + T d_c\alpha + S\alpha \qquad \text{on } U. \tag{33}$$

In addition, on forms of degree $n+1$, T is bounded

$$T : L^\infty(U', E_0^{n+1}) \to W^{1,\infty}(U, E_0^n). \tag{34}$$

Finally, if $\alpha \in L^\infty(U', E_0^\bullet) \cap d_c^{-1}(L^\infty(U', E_0^{\bullet+1}))$ we notice that, by difference,

$$d_c T\alpha \in L^\infty(U, E_0^h).$$

Proof. If $\alpha \in L^\infty(U', E_0^h)$, we set α_0 to be α continued by zero outside U', the so-called *trivial extension* of α. Obviously, α_0 belongs to $L^\infty(\mathbb{H}^n, E_0^h)$ and is compactly supported, hence belongs to $\mathcal{E}'(\mathbb{H}^n, E_0^h)$. The trivial extension defines a continuous linear map from $\mathcal{D}(U')$ to $\mathcal{D}(U)$.

Denote by k_0 the kernel associated with K_0 as defined in Remark 4.2. We consider a cut-off function ψ_R supported in a R-neighborhood of the origin, such that $\psi_R \equiv 1$ near the origin. Then we have $k_0 = k_0\psi_R + (1-\psi_R)k_0$ Let us denote by $K_{0,R}$ the convolution operator associated with $k_{0,R} := \psi_R k_0$ and by $K'_{0,R} = K_0 - K_{0,R}$ the convolution operator associated with the kernel $k'_{0,R} := k_0 - k_{0,R}$.

The kernel $\psi_R k_0 \in L^1(\mathbb{H}^n)$, so that, by Theorem 2.7, i), $K_{0,R}$ maps L^∞ to L^∞.

Let us apply Proposition 4.1 using the decomposition $K_0 = K_{0,R} + K'_{0,R}$: for $\phi \in \mathcal{D}(\mathbb{H}^n, E_0^h)$,

$$\langle \alpha_0 | \phi\rangle_{\mathcal{D}',\mathcal{D}} = \langle d_c K_{0,R}\alpha_0 + K_{0,R} d_c\alpha_0 + d_c K'_{0,R}\alpha_0 + K'_{0,R} d_c\alpha_0 | \phi\rangle_{\mathcal{D}',\mathcal{D}},$$

i.e.

$$\alpha_0 = d_c K_{0,R}\alpha_0 + K_{0,R} d_c\alpha_0 + d_c K'_{0,R}\alpha_0 + K'_{0,R} d_c\alpha_0$$

in the sense of distributions. Taking the restriction to U, we get

$$\alpha = \big(d_c K_{0,R}\alpha_0\big)\Big|_U + \big(K_{0,R} d_c\alpha_0\big)\Big|_U + \big(d_c K'_{0,R}\alpha_0 + K'_{0,R} d_c\alpha_0\big)\Big|_U, \tag{35}$$

where the restriction has to be meant as restriction of a distribution as in Proposition 2.6, (5). First we notice that, by Proposition 2.6, ii) and iv), we have

$$S\alpha := \big(d_c K'_{0,R}\alpha_0 + K'_{0,R} d_c\alpha_0\big)\Big|_U \in \mathcal{E}(U),$$

yielding (31). Since derivatives commute with restriction, if $R > 0$ is small enough, we have

$$\big(d_c K_{0,R}\alpha_0\big)\Big|_U = d_c\big(K_{0,R}\alpha_0\big)\Big|_U ,$$

and (35) reads

$$\alpha\Big|_U = d_c\big(K_{0,R}\alpha_0\big)\Big|_U + \big(K_{0,R}d_c\alpha_0\big)\Big|_U + S\alpha. \tag{36}$$

If now $\beta \in L^\infty(U', E_0^\bullet)$, we set

$$T\beta := K_{0,R}\beta_0\Big|_U .$$

Thus, in (36), we have

$$d_c\big(K_{0,R}\alpha_0\big)\Big|_U = d_c T\alpha.$$

Consider now in (36) the term $\big(K_{0,R}d_c\alpha_0\big)\Big|_U$. We observe preliminarily that

$$(d_c\alpha_0)\Big|_{U'} = \big((d_c\alpha)_0\big)\Big|_{U'} , \tag{37}$$

where, as above, $(d_c\alpha)_0$ is the trivial extension of $d_c\alpha$. Indeed, if $\phi \in \mathcal{D}(U', E_0^{\bullet+1})$ then

$$\begin{aligned}\langle d_c\alpha_0|\phi\rangle_{\mathcal{D}',\mathcal{D}} &= \langle \alpha_0|d_c^*\phi\rangle_{\mathcal{D}',\mathcal{D}} = \int_{U'}\langle \alpha, d_c^*\phi\rangle\, dx = \int_{U'}\langle d_c\alpha, \phi\rangle\, dx \\ &= \int_{\mathbb{H}^n}\langle (d_c\alpha)_0, \phi\rangle\, dx = \langle (d_c\alpha)_0|\phi\rangle_{\mathcal{D}',\mathcal{D}}.\end{aligned}$$

This proves (37). Thus, we can apply Proposition 2.6, (6) and we get, for R small enough,

$$\begin{aligned}\big(K_{0,R}d_c\alpha_0\big)\Big|_U &= \big((d_c\alpha_0) * k_{0,R}\big)\Big|_U \\ &= \big((d_c\alpha)_0 * k_{0,R}\big)\Big|_U = \big(K_{0,R}(d_c\alpha)_0\big)\Big|_U = T(d_c\alpha).\end{aligned}$$

Eventually, identity (36) becomes

$$\alpha = d_cT\alpha + Td_c\alpha + S\alpha \qquad \text{in } U.$$

This proves the homotopy formula (33).

Since the kernel $k_{0,R}$ belongs to $L^1(\mathbb{H}^n)$, by Hausdorff-Young inequality (see Theorem 2.7, i),

$$T : L^\infty(U', E_0^\bullet) \to L^\infty(U, E_0^{\bullet-1}),$$

and this proves (30).

Let us prove the continuity estimates (32) for the operator S. Consider first the term

$$\big(d_c(K'_{0,R}\alpha_0)\big)\Big|_U = \big(d_c(\alpha_0 * k'_{0,R})\big)\Big|_U .$$

If $1 \le h \le 2n$, let $(\xi_1^h, \dots, \xi^h_{\dim E_0^h})$ be a basis of E_0^h. Then $\alpha = \sum_j \alpha_j\xi_j^h$ with $\alpha_j \in L^\infty(U')$, $j = 1, \dots, \dim E_0^h$. Obviously, $\alpha_0 = \sum_j(\alpha_j)_0\xi_j^h$, and $d_c(\alpha_0 * k'_{0,R})$ can be written as sum of terms of the form

$$W^I\big((\alpha_j)_0 * \kappa\big) = (\alpha_j)_0 * W^I\kappa,$$

where κ is a smooth kernel and $d(I) = 1$ or $d(I) = 2$, according to the degree h. Thus, in order to prove (32) we have to estimate the L^∞-norms in U of a sum of terms of the form

$$(\alpha_j)_0 * W^J\kappa,$$

with $d(J) = s+1$ or $d(J) = s+2$, according to the degree h. Then the assertion follows by Proposition 2.9, since the smooth kernel $W^J\kappa$ belongs to $L^1_{\mathrm{loc}}(\mathbb{H}^n)$ (notice that $\|\alpha_j\|_{L^\infty(U)} = \|(\alpha_j)_0\|_{L^\infty(\mathbb{H}^n)}$).

Analogously, if we aim to estimate the term $K'_{0,R} d_c\alpha_0 = (d_c\alpha_0) * k'_{0,R}$, we have to estimate in $L^\infty(U)$ a sum of terms of the form (keep in mind (19))

$$(W^I(\alpha_j)_0) * W^J\kappa = (\alpha_j)_0) * {}^{\mathrm{v}}W^{I\,\mathrm{v}}W^J\kappa,$$

where κ is a smooth kernel, $d(J) = s$ and $d(I) = 1$ or $d(I) = 2$ according to the degree h. Since ${}^{\mathrm{v}}W^{I\,\mathrm{v}}W^J\kappa$ is still a smooth kernel, the estimate can be carried out as for the first term.

Thus we are left with the proof of (34). To this end, we notice first that on forms of degree $h = n+1$, the kernel of $K_{0,R}$ is obtained by truncation near the origin of a kernel of type 2. Therefore, on forms of degree $h = n+1$ all the horizontal derivatives $WK_{0,R}$ belongs to L^1 and, if $\alpha \in L^\infty(U', E_0^{n+1})$, then

$$\begin{aligned}\|WT\alpha\|_{L^\infty(U,E_0^n)} &= \|W(\alpha * k_{0,R})\|_{L^\infty(U,E_0^n)}\\ &= \|\alpha * Wk_{0,R}\|_{L^\infty(U,E_0^n)},\end{aligned}$$

and the proof can be carried out again by Proposition 2.9.

$\square$

Remark 4.4. *Since, in U', $W^{2s,p} \subset W^{s,p}_{\mathrm{Euc}}$ for $1 \le p \le \infty$, then* (32) *can be equivalently stated as*

$$S \in \mathcal{L}\big(L^\infty(U', E_0^\bullet), W^{s,\infty}_{\mathrm{Euc}}(U, E_0^{\bullet-1})\big). \tag{38}$$

5. Intermediate tools: interior $\mathbb{H}$-Poincaré$_{\infty,\infty}$ and $\mathbb{H}$-Sobolev$_{\infty,\infty}$ inequalities

In [24], starting from Cartan's homotopy formula, the authors proved that, if $D \subset \mathbb{R}^N$ is a convex set, $1 < p < \infty$, $1 \le h \le N$, then there exists a bounded linear map:

$$K_{\mathrm{Euc},h} : L^p(D, \textstyle\bigwedge^h) \to W^{1,p}_{\mathrm{Euc}}(D, \textstyle\bigwedge^{h-1})$$

that is a homotopy operator, i.e.

$$\omega = dK_{\mathrm{Euc},h}\omega + K_{\mathrm{Euc},h+1}d\omega \qquad \text{for all } \omega \in C^\infty(D, \textstyle\bigwedge^h). \tag{39}$$

(see Proposition 4.1 and Lemma 4.2 in [24]). More precisely, $K_{\mathrm{Euc},h}$ has the form

$$K_{\mathrm{Euc},h}\omega(x) = \int_D \psi(y) K_y\omega(x)\,dy, \tag{40}$$

where $\psi \in \mathcal{D}(D)$, $\int_D \psi(y)\,dy = 1$, and

$$\begin{aligned}&\langle K_y\omega(x) | \xi_1 \wedge \cdots \wedge \xi_{h-1})\rangle :=\\ &\qquad \int_0^1 t^{h-1}\langle \omega(y + t(x-y)) | (x-y) \wedge \xi_1 \wedge \cdots \wedge \xi_{h-1})\rangle.\end{aligned} \tag{41}$$

The definition (41) can be written as

$$K_y\omega(x) = \int_0^1 t^{\ell-1}\iota_{x-y}\omega(y_t)\,dt,$$

where $y_t = y + t(x-y)$. Here, ι denotes the interior product of a differential form with a vector field, i.e. $\iota : \bigwedge^{h+1} \to \bigwedge^h$ and is defined by

$$\langle \iota_Y\omega | v_1 \wedge \cdots \wedge v_h\rangle := \langle \omega | Y \wedge v_1 \wedge \cdots \wedge v_h\rangle.$$

Let us remind the following identity that follows straightforwardly from the relationship between the Lie derivative $\mathcal{L}_X$ along a vector field X of a differential form and the interior product of a vector field Y and a differential form:

$$[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}. \tag{42}$$

The following theorem provides a continuity result in $W^{k,p}$ of Iwaniec & Lutoborski's kernel $K_{\mathrm{Euc},\bullet}$, though with a loss on domain.

Theorem 5.1. *Let $B = B(0,1)$ and $B' = B(0,2)$ be concentric Euclidean balls in $\mathbb{R}^N$. Then for $k \in \mathbb{N}$ and $p \in [1,\infty]$, Iwaniec-Lutoborski's homotopy $K_{\mathrm{Euc},h}$ is a bounded operator*

$$K_{\mathrm{Euc},h} : W^{k,p}_{\mathrm{Euc}}(B', {\textstyle\bigwedge}^\bullet) \to W^{k,p}_{\mathrm{Euc}}(B, {\textstyle\bigwedge}^{\bullet-1})$$

Proof. For the sake of simplicity, from now on we omit the degree h of the form and we write simply K_{Euc}. We show that for every k-th order partial derivative D^k there exist matrix valued kernels M_1 and M_2 on the ball of radius 2 such that for every differential form ω on the unit ball,

$$D^k K_{\mathrm{Euc}}\omega = M_1 * (D^k\omega) + M_2 * (RD^{k-1}\omega),$$

where RD^{k-1} is a constant coefficient $(k-1)$-order differential operator and for all $h \in \mathbb{R}^n$, $|h| < 2$, $i = 1, 2$,

$$|M_i(h)| \le C\,|h|^{1-N}.$$

We set $y_t = y + t(x-y)$. Iterating (42), we obtain

$$D^k(\iota_{x-y}\omega(y_t)) = t^k \iota_{x-y} D^k\omega(y_t) + t^{k-1} RD^{k-1}\omega(y_t),$$

where RD^{k-1} denotes the following $(k-1)$-order differential operator from ℓ-forms to $\ell-1$ forms. If, for sake of simplicity, we take D^k of the form $D^k = D_1 \cdots D_k$,

$$RD^{k-1}\omega = \sum_{i=1}^{k} \iota_{D_i}(D_1 \cdots D_{i-1} D_{i+1} \cdots D_k\omega).$$

Therefore

$$\begin{aligned} D^k K_{\mathrm{Euc}}\omega(x) &= \int_0^1 t^{\ell-1} \int_B \phi(y) D^k(\iota_{x-y}\omega(y_t))\,dy\,dt \\ &= \int_B \int_0^1 t^{\ell-1}\phi(y)(t^k \iota_{x-y} D^k\omega(y_t) + t^{k-1} RD^{k-1}\omega(y_t))\,dy\,dt. \end{aligned}$$

Let us perform a change of variables $z = y_t$ and denote by $h = x - z$. T hen $y = \frac{1}{1-t}z - \frac{t}{1-t}x = z - \frac{t}{1-t}h$, $x - y = (1-t)^{-1}h$, $dy = (1-t)^{-n}dz$, whence

$$\begin{aligned} D^k K_{\mathrm{Euc}}\omega(x) = \int_B \int_0^1 & t^{\ell-1}\phi(z - sh)(t^k(1-t)^{-1}\iota_h D^k\omega(z) \\ &+ t^{k-1} RD^{k-1}\omega(z))\,(1-t)^{-N}\,dt\,dz. \end{aligned}$$

We treat both terms separately. The first one is

$$\begin{aligned}\int_B \int_0^1 & t^{\ell-1}\phi(z-\frac{t}{1-t}h)t^k(1-t)^{-1}\iota_h D^k\omega(z)\,(1-t)^{-N}\,dt\,dz\\ &= \int_B \int_0^1 t^{k+\ell-1}(1-t)^{-N-1}\phi(z-\frac{t}{1-t}h)\iota_h D^k\omega(z)\,dt\,dz\\ &= \int_B \langle \int_0^\infty (\frac{s}{1+s})^{k+\ell-1}(1+s)^{N-1}\phi(z-sh)\iota_h\,ds, D^k\omega(z)\rangle\,dz,\end{aligned}$$

where we have made the change of variables $s=\frac{t}{1-t}$. One recognizes the convolution of the Λ^ℓ-valued function $D^k\omega$ with the matrix valued kernel

$$M_1(z,h) := \int_0^\infty (\frac{s}{1+s})^{k+\ell-1}(1+s)^{N-1}\phi(z-sh)\iota_h\,ds.$$

The second term is

$$\begin{aligned}\int_B \int_0^1 & t^{\ell-1}\phi(z-\frac{t}{1-t}h)t^{k-1}RD^{k-1}\omega(z)\,(1-t)^{-N}\,dt\,dz\\ &= \int_B \int_0^1 t^{k+\ell-2}(1-t)^{-N}\phi(z-\frac{t}{1-t}h)RD^{k-1}\omega(z)\,dt\,dz\\ &= \int_B \langle \int_0^\infty (\frac{s}{1+s})^{k+\ell-2}(1+s)^{N}\phi(z-sh)\,ds, RD^{k-1}\omega(z)\rangle\,dz.\end{aligned}$$

Again, this is the convolution of the $\Lambda^{\ell-1}$-valued function $SW^{k-1}\omega$ with the scalar kernel

$$M_2(z,h) := \int_0^\infty (\frac{s}{1+s})^{k+\ell-2}(1+s)^{N-2}\phi(z-sh)\,ds.$$

Since ϕ has compact support in B, in both cases, the integral stops no later that $2|h|^{-1}$, thus

$$\begin{aligned}|M_1(z,h)| &\le C\,|h|\int_0^{2|h|^{-1}}(1+s)^{N-1}\,ds \le C\,|h|^{1-N},\\ |M_2(z,h)| &\le C\int_0^{2|h|^{-1}}(1+s)^{N-2}\,ds \le C\,|h|^{1-N}.\end{aligned}$$

With Young's inequality, this implies that for all $p\in[1,\infty]$,

$$\|D^kK_{\mathrm{Euc}}\omega\|_{L^p(B,\bigwedge^{\bullet-1})} \le C\left(\|\nabla^k\omega\|_{L^p(B',\bigwedge^\bullet)} + \|\nabla^{k-1}\omega\|_{L^p(B',\bigwedge^\bullet)}\right).$$

Since this holds for every k-th order partial derivative,

$$\|K_{\mathrm{Euc}}\omega\|_{W^{k,p}_{\mathrm{Euc}}(B,\bigwedge^{\bullet-1})} \le C\,\|\omega\|_{W^{k,p}_{\mathrm{Euc}}(B',\bigwedge^\bullet)}.$$

□

Starting from [24], in [26], Theorem 4.1, the authors define a compact homotopy operator $J_{\mathrm{Euc},h}$ in Lipschitz star-shaped domains in Euclidean space $\mathbb{R}^N$, providing an explicit representation formula for $J_{\mathrm{Euc},h}$, together with continuity properties among Sobolev spaces. More precisely:

Theorem 5.2. *[(see [26], formula (167))] if $D\subset\mathbb{R}^N$ is a star-shaped Lipschitz domain and $1\le h\le N$, then there exists*

$$J_{\mathrm{Euc},h} : L^p(D,\bigwedge^h) \to W^{1,p}_{0,\mathrm{Euc}}(D,\bigwedge^{h-1})$$

such that

$$\omega = dJ_{\mathrm{Euc},h}\omega + J_{\mathrm{Euc},h+1}d\omega \qquad \textit{for all } \omega \in \mathcal{D}(D, \textstyle\bigwedge^h)$$

and for $1 < p < \infty$ *and* $k \in \mathbb{N} \cup \{0\}$

$$J_{\mathrm{Euc},h} : W^{k,p}_{0,\mathrm{Euc}}(D, \textstyle\bigwedge^h) \to W^{k+1,p}_{0,\mathrm{Euc}}(D, \textstyle\bigwedge^{h-1}).$$

Furthermore, $J_{\mathrm{Euc},h}$ *maps smooth compactly supported forms to smooth compactly supported forms.*

We need now construct a homotopy operator, fitting the intrinsic group structure, that can invert Rumin's differential d_c. To this aim take $D = B(e,1) =: B$ and $N = 2n+1$. If $\omega \in C^\infty(B, E_0^h)$, then we set

$$K = \Pi_{E_0} \circ \Pi_E \circ K_{\mathrm{Euc}} \circ \Pi_E \tag{43}$$

(for the sake of simplicity, from now on we drop the index h - the degree of the form - writing, e.g., K_{Euc} instead of $K_{\mathrm{Euc},h}$).

Analogously, we can define

$$J = \Pi_{E_0} \circ \Pi_E \circ J_{\mathrm{Euc}} \circ \Pi_E. \tag{44}$$

Then K and J invert Rumin's differential d_c on closed forms of the same degree. More precisely, we have:

Lemma 5.3. *If* ω *is a smooth* d_c*-exact differential form, then*

$$\omega = d_c K\omega \quad \textit{if } 1 \le h \le 2n+1 \qquad \textit{and} \qquad \omega = d_c J\omega \quad \textit{if } 1 \le h \le 2n+1. \tag{45}$$

In addition, if ω *is compactly supported in* B*, then* $J\omega$ *is still compactly supported in* B*.*

For the proof of the lemma above we refer to Lemma 5.7 in [8].

Imitating [8], we are now able to prove interior Poincaré inequality and Sobolev inequality for Rumin forms in the sense of Definitions 1.2 and 1.4.

Theorem 5.4. *Take* $\lambda > 1$ *and set* $B = B(e,1)$ *and* $B_\lambda = B(e,\lambda)$*. If* $1 \le h \le 2n+1$ *then*

- i) *an interior* $\mathbb{H}\text{-Poincaré}_{\infty,\infty}(h)$ *inequality holds with respect to the balls* B *and* B_λ*;*
- ii) *in addition, an interior* $\mathbb{H}\text{-Sobolev}_{\infty,\infty}(h)$ *inequality holds for* $1 \le h \le 2n+1$*.*

Proof. Consider the balls $B := B(e,1) \Subset B(e,\lambda/2) \Subset B(e,\lambda) =: B_\lambda$, so that Proposition 4.3 and Theorem 5.1 can be applied to the couple $B(e,1), B(e,\lambda/2)$ and can be applied also to the couple $B(e,\lambda/2), B(e,\lambda)$. Put $B_1 := B(e,\lambda/2)$.

i) Interior $\mathbb{H}\text{-Poincaré}_{\infty,\infty}(h)$ inequality: let $\omega \in L^\infty(B_\lambda, E_0^h)$ be d_c-closed. By (33), if we take therein $U := B$ and $U' := B_\lambda$, we can write

$$\omega = d_c T\omega + S\omega \qquad \text{in } B. \tag{46}$$

By (31) $S\omega \in \mathcal{C}^\infty(B, E_0^h)$ and $d_c S\omega = 0$ since $d_c\omega = d_c^2 T\omega + d_c S\omega$ in B and $d_c\omega = 0$ (by assumption).

Thus we can apply (45) to $S\omega$ and we get $S\omega = d_c KS\omega$, where K is defined in (43). In B, put now

$$\phi := (KS + T)\omega.$$

Trivially,

$$d_c\phi = d_c KS\omega + d_c T\omega = S\omega + d_c T\omega = \omega, \tag{47}$$

by (46).

On the other hand,

$$\|\phi\|_{L^\infty(B,E_0^{h-1})} \le \|KS\omega\|_{L^\infty(B,E_0^{h-1})} + \|T\omega\|_{L^\infty(B,E_0^{h-1})}. \tag{48}$$

First of all, by (30),

$$\|T\omega\|_{L^\infty(B,E_0^{h-1})} \le C\|\omega\|_{L^\infty(B,E_0^{h-1})}. \tag{49}$$

Take now $q > 2n+1$. By [1], Theorem 4.12, keeping in mind that Π_E is an operator of order 0 or 1, depending on the degree of the form, we have:

$$\begin{aligned}
\|KS\omega\|_{L^\infty(B,E_0^{h-1})} &\le C\|KS\omega\|_{W^{1,q}_{\mathrm{Euc}}(B,E_0^{h-1})} \\
&= C\|(\Pi_{E_0}\circ\Pi_E\circ K_{\mathrm{Euc}}\circ\Pi_E)S\omega\|_{W^{1,q}_{\mathrm{Euc}}(B,E_0^{h-1})} \\
&\le C\|(\Pi_E\circ K_{\mathrm{Euc}}\circ\Pi_E)S\omega\|_{W^{1,q}_{\mathrm{Euc}}(B,\bigwedge^{h-1})} \\
&\le C\|(K_{\mathrm{Euc}}\circ\Pi_E)S\omega\|_{W^{2,q}_{\mathrm{Euc}}(B,\bigwedge^{h-1})} \\
&\le C\|\Pi_E S\omega\|_{W^{2,q}_{\mathrm{Euc}}(B_1,\bigwedge^{h-1})} \qquad \text{(by Theorem 5.1)} \\
&\le C\|S\omega\|_{W^{3,q}_{\mathrm{Euc}}(B_1,E_0^{h-1})} \le C\|S\omega\|_{W^{3,\infty}_{\mathrm{Euc}}(B_1,E_0^{h-1})} \\
&\le C\|\omega\|_{L^\infty(B_\lambda,E_0^{h-1})} \qquad \text{(by (32))}.
\end{aligned} \tag{50}$$

Combining (49) and (50) it follows from (48) that

$$\|\phi\|_{L^\infty(B,E_0^{h-1})} \le C\|\omega\|_{L^\infty(B_\lambda,E_0^{h-1})}, \tag{51}$$

i.e. (keeping in mind (47)), interior $\mathbb{H}$-Poincaré$_{\infty,\infty}(h)$ inequality holds.

ii) Interior $\mathbb{H}$-Sobolev$_{\infty,\infty}(h)$ inequality: let $\omega \in L^\infty(B_\lambda, E_0^h)$ be d_c-closed and compactly supported. By (33), if we take therein $U := B_1$ and $U' := B_\lambda$, we can write

$$\omega = d_c T\omega + S\omega \qquad \text{in } B_1.$$

By (31) $S\omega \in \mathcal{C}^\infty(B_1, E_0^h)$ and $d_c S\omega = 0$ since $d_c\omega = d_c^2 T\omega + d_c S\omega$ in B_1 and $d_c\omega = 0$ (by assumption).

We notice now that $T\omega$ is supported in B_1 provided $R > 0$ is small enough, so that, by (31), also $S\omega$ is supported in B_1. Thus, arguing as above, we can apply (45) to $S\omega$ and we get $S\omega = d_c JS\omega$, where J is defined in (44). In B_1, put now

$$\phi := (JS + T)\omega.$$

We stress that, again by Lemma 5.3, $JS\omega$ is compactly supported in B_1. Again as above,

$$d_c\phi = d_c JS\omega + d_c T\omega = S\omega + d_c T\omega = \omega. \tag{52}$$

We can repeat now the arguments yielding the estimates (50) and (51), replacing Theorem 5.1 by Theorem 5.2. Thus interior $\mathbb{H}$-Sobolev$_{\infty,\infty}(h)$ inequality follows. □

6. COHOMOLOGY FOR ANNULI

The proof of $\mathbb{H}$-Poincaré$_{Q,\infty}(h)$ (Theorem 1.8) given in Section 3 relies basically on a duality argument and the dual inequality of [6] (see Theorem 1.7).

On the contrary, the proof of $\mathbb{H}$-Sobolev$_{Q,\infty}(h)$ (Theorem 1.9) requires a more sophisticated argument based on localization on Korányi annuli. The present section is precisely devoted to prove that the $L^{\infty,\infty}$ cohomology of Rumin's closed forms vanishes on Korányi annuli. To this end, we prove first that de Rham $L^{\infty,\infty}$ cohomology of closed forms vanishes on Euclidean annuli. It follows that the same statement holds for suitable Korányi annuli (see Corollary 6.6) end eventually the assertion is proven.

Let us start with the following definition.

Definition 6.1. *Let $D \subset \mathbb{R}^{2n+1}$ be an open set. Let $s \in \mathbb{N}$ and $1 \le p,q \le \infty$. If $1 \le h \le 2n+1$, we define cohomology spaces*

$$H^{s,q,p,h}_{\mathrm{de\ Rham}}(D) = (W^{s,p}_{\mathrm{Euc}}(D, \bigwedge^h \mathfrak{h}) \cap \ker d)/dW^{s,q}_{\mathrm{Euc}}(D, \bigwedge^{h-1} \mathfrak{h}),$$

and we denote by

$$EH^{s,q,p,h}_{\mathrm{de\ Rham}}(D) = \ker(H^{s,q,p,h}_{\mathrm{de\ Rham}}(D) \to H^h(D))$$

the cohomology of exact differential forms. Similar definitions hold with d replaced with d_c, yielding the corresponding spaces

$$H^{s,q,p,h}_{E_0} \quad \text{and} \quad EH^{s,q,p,h}_{E_0}$$

for Rumin's differential forms.

If $s=0$ we shall write $H^{q,p,h}_{E_0} for H^{0,q,p,h}_{E_0}$.

Notation 6.2. *If $0 < s_1 < s_2$ we denote by $A^{\mathrm{Euc}}_{s_1,s_2}$ the (Euclidean) annulus*

$$A^{\mathrm{Euc}}_{s_1,s_2} = B_{\mathrm{Euc}}(e,s_2) \setminus \overline{B_{\mathrm{Euc}}(e,s_1)}.$$

Analogously, if $0 < r_1 < r_2$, we denote by A_{r_1,r_2} the (Korányi) annulus

$$A_{r_1,r_2} = B(e,r_2) \setminus \overline{B(e,r_1)}.$$

Given $0 < r_1 < r_2$, let A_{r_1,r_2} be the (Korányi) annulus in $\mathbb{H}^n$. Put $\partial^+ A_{r_1.r_2} := \partial B(e,r_2)$ and $\partial^- A_{r_1.r_2} := \partial B(e,r_1)$. The meaning, in the Euclidean case, of $\partial^\pm A^{\mathrm{Euc}}_{r_1.r_2}$ is analogous. Set

$$V := A_{1,2}$$

and $\sigma_1 := \frac{1}{2}\min_{\partial^- V}|x| > 0$ and $\sigma_2 := 2\max_{\partial^+ V}|x|$. It turns out that $V \Subset A^{\mathrm{Euc}}_{\sigma_1,\sigma_2} =: \tilde V$. Put now $\tilde V' := A^{\mathrm{Euc}}_{\frac12\sigma_1, 2\sigma_2}$, obviously $\tilde V \Subset \tilde V'$. Finally set $\tau_1 := \frac12 \min_{\partial^- \tilde V'} \rho(x) > 0$ and we fix (once for all) $\tau_2 > \max_{\partial^+ \tilde V'} \rho(x)$. Then the Korány annulus $V' := A_{\frac12 \tau_1, 2\tau_2}$ satisfies

$$V \Subset \tilde V \Subset \tilde V' \Subset V'. \tag{53}$$

Notice that $\sigma_1 = \frac12 \min_{B(e,1)^c}|x|$, $\tau_1 = \frac12 \min_{B_{\mathrm{Euc}}(e,\frac12\sigma_1)^c}\rho(x)$ and $\sigma_2 = 2\max_{B(e,2)}|x|$, $\tau_2 = 2\max_{B_{\mathrm{Euc}}(e,2\sigma_2)}\rho(x)$.

Definition 6.3. *With the notation introduced above, let $U = A_{s_1,s_2}$ and $U' = A_{r_1,r_2}$ be concentric Korányi annuli in $\mathbb{H}^n$, $U \subset U'$. We say that the couple (U,U') is* **annulus-admissible** *if, with the notations of* (53)*, there exists $t > 0$ such that*

$$V' \subset \delta_t U' \quad \text{and} \quad \delta_t U \subset V.$$

Remark 6.4. *A straightforward computation shows that the previous definition make sense.*

Indeed, if $0 < r_1 < r_2 < \frac{\tau_2}{\tau_1} r_1$ there exist $0 < s_1 < s_2$ such that the couple

$$U := A_{s_1,s_2} \quad \text{and} \quad U' := A_{r_1,r_2}$$

is annulus admissible. More precisely the assertion holds provided

$$\frac{r_2}{\tau_2} < s_1 < \frac{r_1}{\tau_1} \quad \text{or} \quad \frac{2r_2}{\tau_2} < s_2 < \frac{2r_1}{\tau_1}.$$

Proposition 6.5. *Let $1 \le p \le \infty$. Let $A^{\mathrm{Euc}}_{s_1,s_2} \Subset A^{\mathrm{Euc}}_{r_1,r_2}$ be concentric Euclidean annuli in $\mathbb{R}^{2n+1}$. Then the map $EH^{s,p,p,*}_{\mathrm{de\ Rham}}(A^{\mathrm{Euc}}_{r_1,r_2}) \to EH^{s,p,p,*}_{\mathrm{de\ Rham}}(A^{\mathrm{Euc}}_{s_1,s_2})$ induced by the inclusion $A^{\mathrm{Euc}}_{s_1,s_2} \subset A^{\mathrm{Euc}}_{r_1,r_2}$ vanishes.*

Proof. We use a diffeomorphism of U' to $(-2,2)\times\Sigma$ mapping U to $(-1,1)\times\Sigma$, where Σ denotes the $2n$-sphere. Then we use Poincaré's homotopy formula in order to relate the cohomology of $(-2,2)\times\Sigma$ to the cohomology of Σ. Forms on the product can be written

$$\omega = a_t + dt\wedge b_t,$$

where a_t and b_t are forms on Σ. Then

$$d\omega = da_t + dt\wedge(\frac{\partial a_t}{\partial t} - db_t),$$

where the right-hand side d is the exterior differential on Σ. Assuming that $d\omega = 0$, i.e. $da_t = 0$ and $\frac{\partial a_t}{\partial t} = db_t$ for all $t\in(-2,2)$, set, for $\sigma\in\Sigma$ and $x\in(-1,1)$,

$$\gamma_x(t,\sigma) = \int_x^t b_u\,du.$$

We observe that for all $p\geq 1$,

$$\|\frac{1}{2}\int_{-1}^1 \gamma_x\,dx\|_{W^{s,p}_{\mathrm{Euc}}} \leq \|\omega\|_{W^{s,p}_{\mathrm{Euc}}}$$

By construction,

$$d\gamma_x = dt\wedge b_t + \int_x^t db_u\,du = \omega - a_x.$$

Now assume that ω is exact, $\omega = d(e_t + dt\wedge f_t)$. Then $de_t = a_t$ for all $t\in(-2,2)$. Set

$$\gamma = \frac{1}{2}\int_{-1}^1 (e_x+\gamma_x)\,dx, \quad \text{so that} \quad d\gamma = \omega.$$

If $\omega\in W^{s,p}_{\mathrm{Euc}}$, so is each γ_x. On Σ, use the coexact primitive $e_x = \delta\Delta^{-1}a_x$ (see, e.g. [22], Section 2.5). Here Δ is the usual Hodge Laplacian on de Rham's differential d). Then, if $p<\infty$,

$$\|\frac{1}{2}\int_{-1}^1 e_x\,dx\|_{W^{s+1,p}_{\mathrm{Euc}}} \leq C\,\|\frac{1}{2}\int_{-1}^1 a_x\,dx\|_{W^{s,p}_{\mathrm{Euc}}} \leq C'\,\|\omega\|_{W^{s,p}_{\mathrm{Euc}}}.$$

If $p=\infty$, one picks $p>2n+1$, so that the Sobolev embedding theorem applies,

$$\|\frac{1}{2}\int_{-1}^1 e_x\,dx\|_{W^{s,\infty}_{\mathrm{Euc}}} \leq C\,\|\frac{1}{2}\int_{-1}^1 e_x\,dx\|_{W^{s+1,p}_{\mathrm{Euc}}}.$$

Obviously, $\|\omega\|_{W^{s,p}_{\mathrm{Euc}}} \leq C\,\|\omega\|_{W^{s,\infty}_{\mathrm{Euc}}}$. Hence the primitive γ is bounded by ω in $W^{s,p}_{\mathrm{Euc}}$ norm in all cases. This shows that the cohomology class of ω in $EH^{s,p,p,*}_{\mathrm{de\ Rham}}(U)$ vanishes. □

As a consequence of the previous result and keeping in mind Definition 6.3, we can prove the following corollary.

Corollary 6.6. *Let U, U' be concentric Korányi annuli in $\mathbb{H}^n$, $U\subset U'$ such that the couple (U,U') is* annulus-admissible. *Then the map $EH^{s,p,p,\bullet}_{\mathrm{de\ Rham}}(U')\to EH^{s,p,p,\bullet}_{\mathrm{de\ Rham}}(U)$ induced by the inclusion $U\subset U'$ vanishes for $1\leq p\leq\infty$.*

Proof. Suppose $U'\subset\delta_t V'$ and $\delta_t V\subset U$, where V,V' and $t>0$ are as in Definition 6.3. By (53) and Proposition 6.5 we can conclude straightforwardly that the map $EH^{s,p,p,*}_{\mathrm{de\ Rham}}(V')\to EH^{s,p,p,*}_{\mathrm{de\ Rham}}(V)$ induced by the inclusion $V\subset V'$ vanishes, so that the map $EH^{s,p,p,*}_{\mathrm{de\ Rham}}(\delta_t U)\to EH^{s,p,p,*}_{\mathrm{de\ Rham}}(\delta_t U')$ vanishes. The assertion follows by a pull-back argument.

□

Proposition 6.7. *Let $U = A_{s_1,s_2}$ and $U'' = A_{r_1,r_2}$ be concentric Korány annuli in $\mathbb{H}^n$. Assume (U, U'') are annulus-admissible (see Definition 6.3). Then the map*

$$EH_{E_0}^{\infty,\infty,\bullet}(A_{r_1,r_2}) \to EH_{E_0}^{\infty,\infty,\bullet}(A_{s_1,s_2})$$

induced by inclusion $U \subset U''$ vanishes.

Proof. Let the annulus U' be such that $U \subset U' \subset U''$ and such that the couple (U, U') is still annulus-admissible as in Definition 6.3 (this is possible by Remark 6.4).

Let ω be a d_c-exact Rumin form on U'', which belongs to $L^\infty(U'', E_0^\bullet)$.

Apply formula (32) of Proposition 4.3 with $s = 5$. Then, if we set $S\omega =: \omega' \in W^{5,\infty}(U', E_0^\bullet)$, we have

$$\omega = \omega' + d_c\alpha \quad \text{on } U', \qquad \text{where} \qquad \alpha = T\omega \in L^\infty(U', E_0^{\bullet-1}). \tag{54}$$

Consider $\omega'' := \Pi_E\omega'$. Obviously, $\omega'' = \Pi_E\omega''$. Moreover, by Theorem 8.6-iv), $\Pi_{E_0}\Pi_E\Pi_{E_0} = \Pi_{E_0}$, and $\Pi_{E_0}\omega'$ since ω' is a Rumin form. Therefore

$$\Pi_{E_0}\omega'' = \Pi_{E_0}\Pi_E\omega' = \Pi_{E_0}\Pi_E\Pi_{E_0}\omega' = \omega'.$$

Notice that $d\omega'' = 0$ in U'. Indeed, since ω is d_c-exact, then $0 = d_c\omega$ and hence $d_c\omega' = 0$ in U'. Therefore $0 = d_c\omega' = \Pi_{E_0}\Pi_E d\omega'$, so that $0 = \Pi_E\Pi_{E_0}\Pi_E d\omega' = \Pi_E d\omega' = d\Pi_e\omega' = d\omega''$ in U' (keep in mind $d\Pi_E = \Pi_E d$ by Theorem 8.6).

In addition, $\omega'' \in W^{4,\infty}(U', \Omega^\bullet) \subset W^{2,\infty}_{\mathrm{Euc}}(U', \Omega^\bullet)$.

According to Corollary 6.6, there exists a differential form $\gamma \in W^{2,\infty}_{\mathrm{Euc}}(U)$ such that $\omega'' = d\gamma$ on U. Hence

$$\omega'' = \Pi_E\omega'' = \Pi_E d\gamma = d\Pi_E\gamma.$$

If we set $\eta = \Pi_{E_0}\Pi_E\gamma$, then in particular $\eta \in L^\infty(U, E_0^{\bullet-1})$ and it follows that

$$\begin{aligned} d_c\eta &= \Pi_{E_0} d\Pi_E\Pi_{E_0}\Pi_E\gamma = \Pi_{E_0} d\Pi_E\gamma \\ &= \Pi_{E_0}\Pi_E d\gamma = \Pi_{E_0}\Pi_E\omega'' = \Pi_{E_0}\omega'' = \omega'. \end{aligned}$$

Hence, by (54),

$$\omega = d_c(\eta + \alpha) \qquad \text{in } U.$$

This shows that the cohomology class of the restriction of ω to U vanishes in $EH_{E_0}^{\infty,\infty,\bullet}(U)$. $\square$

Remark 6.8. *Repeating verbatim the proof of the previous theorem and keeping into account* (34) *in Proposition 4.3, when dealing with $(n+1)$-forms the previous result guarantees the existence of a $W^{1,\infty}$-primitive, i.e. the map*

$$EH_{E_0}^{\infty,\infty,n+1}(A_{r_1,r_2}) \to EH_{E_0}^{1,\infty,\infty,n+1}(A_{s_1,s_2})$$

induced by the inclusion $U \subset U''$ vanishes.

7. Proof of Theorem 1.9

We are now able to prove the Sobolev inequality as stated in Theorem 1.9.

The proof will be carried out starting from the corresponding Poincaré inequality by means of localizations of our estimates on a family of annuli via a suitable cut-off. Then a problem arises since the differential d_c may have order 1 or 2 according to the degree of the forms on which it acts. Keeping in mind Remark 2.13, for technical reasons, during the proof we are led to distinguish the case $h \neq n+2$ from the case $h = n+2$.

Proof of Theorem 1.9. We set

$$q = q(h) := \begin{cases} Q & \text{if } h \neq n; \\ Q/2 & \text{if } h = n. \end{cases}$$

Take $B := B(e,1)$ and $B_\lambda := B(e,\lambda)$ with $1+\epsilon < \lambda < \frac{\tau_2}{\tau_1}(1+\epsilon)$, where $\epsilon > 0$ and τ_1, τ_2 are the geometric constants introduced in Definition 6.3.

By Remark 6.4 there exist $0 < s_1 < s_2$ such that for two concentric annuli $A_{1+\epsilon,\lambda}$ and A_{s_1,s_2}, the couple $(A_{1+\epsilon,\lambda}\,,\, A_{s_1,s_2})$ is annulus-admissible in the sense of Definition 6.3 and

$$A_{s_1,s_2} \subset A_{1+\epsilon,\lambda} \subset B_\lambda \setminus B.$$

If we set $U' := A_{1+\epsilon,\lambda}$ and $U := A_{s_1,s_2}$, the inclusion above reads as

$$U \subset U' \subset B_\lambda \setminus B.$$

Let $\alpha \in L^q(B, E_0^h)$ be a compactly supported d_c-exact h-form on the unit ball B. If $h = 2n+1$, this implies that $\int_B \alpha = 0$. Otherwise, this simply means that $d_c\alpha = 0$ in B. We continue α by zero on $\mathbb{H}^n \setminus B$.

We apply $\mathbb{H}$-Poincaré$_{q,\infty}(h)$ in $2B_\lambda$ (see Theorem 1.8), and we find $\gamma \in L^\infty(B_\lambda, E_0^{h-1})$ such that

(55)

$$d_c\gamma = \alpha \quad \text{in } B_\lambda \qquad \text{and} \qquad \|\gamma\|_{L^\infty(B_\lambda, E_0^{h-1})} \le C\,\|\alpha\|_{L^q(2B_\lambda, E_0^h)} = \|\alpha\|_{L^q(B, E_0^h)}.$$

We emphasize here that the exponent q in (55) equals Q if $h \neq n$ and $Q/2$ if $h = n$.

As announced above, we have to distinguish two cases: $h \neq n+2$ and $h = n+2$. Since in $U' \subset B_\lambda \setminus B$ we have $d_c\gamma = 0$ in U'. Furthermore, if $h = 2n+1$,

$$\int_{\partial B} \gamma = \int_B \alpha = 0,$$

which implies that γ is exact on $B_\lambda \setminus B$. Hence by Proposition 6.7, if $h-2 \neq n$, there exists a $(h-2)$-form γ' on U such that

$$d_c\gamma' = \gamma \qquad \text{in } U \text{ and} \qquad \|\gamma'\|_{L^\infty(U, E_0^{h-2})} \le C\,\|\gamma\|_{L^\infty(U', E_0^{h-1})}. \tag{56}$$

On the other hand, if $h-2 = n$ then, by Remark 6.8, there exists a $\gamma' \in W^{1,\infty}(U, E_0^n)$ such that $d_c\gamma' = \gamma$ in U and

$$\|\gamma'\|_{W^{1,\infty}(U, E_0^n)} \le C\,\|\gamma\|_{L^\infty(U', E_0^{n+1})}. \tag{57}$$

Let ζ be a smooth function in B_λ vanishing in c_1B with $s_1 < c_1 < s_2$, such that $\zeta \equiv 1$ outside of c_2B, where $c_1 < c_2 < s_2$. We stress that γ' is defined on U, and $\zeta\gamma'$ is supported outside of a neighborhood of $\overline{c_1B}$ and therefore can be continued by 0 on all the ball c_1B and then is defined on all of s_2B.

We set

$$\beta := \gamma - d_c(\zeta\gamma') \tag{58}$$

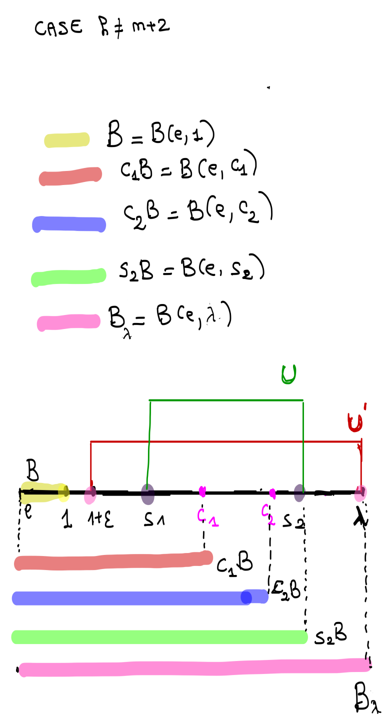
(that is still defined on all of s_2B). Now on $s_2B \setminus \overline{c_2B} = U \setminus \overline{c_2B}$ we have

$$\beta = \gamma - d_c\gamma' \equiv 0,$$

so that β is compactly supported in s_2B and can be continued by 0 to a compactly supported form in B_λ.

In addition, by (55),

$$d_c\beta = d_c\gamma = \alpha \qquad \text{in } B_\lambda.$$



By Remark 2.13, keeping into account that $\zeta\gamma' \in E_0^{h-2}$, if $h-2 \neq n$, we have

$$\begin{aligned}
\|\beta\|_{L^\infty(B_\lambda, E_0^{h-1})} &= \|\beta\|_{L^\infty(s_2B, E_0^{h-1}))} \le \|\gamma\|_{L^\infty(B_\lambda, E_0^{h-1}))} + \|d_c(\zeta\gamma')\|_{L^\infty(s_2B, E_0^{h-1})} \\
&= \|\gamma\|_{L^\infty(B_\lambda, E_0^{h-1})} + \|d_c(\zeta\gamma')\|_{L^\infty(U, E_0^{h-1})} \\
&\le \|\gamma\|_{L^\infty(B_\lambda, E_0^{h-1})} + \|d_c\gamma'\|_{L^\infty(U, E_0^{h-1})} + C\|\gamma'\|_{L^\infty(U, E_0^{h-2})} \\
&\le \|\gamma\|_{L^\infty(B_\lambda, E_0^{h-1})} + \|\gamma\|_{L^\infty(U, E_0^{h-1})} + C\|\gamma\|_{L^\infty(U', E_0^{h-1})} \qquad \text{(by (56))} \\
&\le C\|\gamma\|_{L^\infty(B_\lambda, E_0^{h-1})} \le C\|\alpha\|_{L^q(B, E_0^h)} \qquad \text{(by (55))}.
\end{aligned}$$

Thus, $\mathbb{H}$-Sobolev$_{q,\infty}(h)$ holds for $h \neq n+2$.

On the other hand, when $h-2 = n$, keeping into account Remark 2.13-ii), we have

$$\begin{aligned}
\|\beta\|_{L^\infty(B_\lambda, E_0^{h-1})} &= \|\beta\|_{L^\infty(s_2B, E_0^{h-1}))} \le \|\gamma\|_{L^\infty(B_\lambda, E_0^{h-1}))} + \|d_c(\zeta\gamma')\|_{L^\infty(s_2B, E_0^{h-1})} \\
&= \|\gamma\|_{L^\infty(B_\lambda, E_0^{h-1})} + \|d_c(\zeta\gamma')\|_{L^\infty(U, E_0^{h-1})} \\
&\le \|\gamma\|_{L^\infty(B_\lambda, E_0^{h-1})} + \|d_c\gamma'\|_{L^\infty(U, E_0^{h-1})} + C\|\gamma'\|_{W^{1,\infty}(U, E_0^{h-2})} \\
&\le \|\gamma\|_{L^\infty(B_\lambda, E_0^{h-1})} + \|\gamma\|_{L^\infty(U, E_0^{h-1})} + C\|\gamma\|_{L^\infty(U', E_0^{h-1})} \qquad \text{(by (57))} \\
&\le C\|\gamma\|_{L^\infty(B_\lambda, E_0^{h-1})} \le C\|\alpha\|_{L^q(B, E_0^h)} \qquad \text{(by (55))}.
\end{aligned}$$

Thus, by Definition 1.4, $\mathbb{H}$-Sobolev$_{q,\infty}(n+2)$ holds. $\square$

8. Appendix: Rumin's complex

Coherently with the notations introduced through the paper, we set (see (15))

$$\omega_i := dx_i, \quad \omega_{i+n} := dy_i \quad \text{and} \quad \omega_{2n+1} := \theta, \quad \text{for } i = 1, \dots, n$$

we denote by $\langle \cdot, \cdot \rangle$ the inner product in $\bigwedge^1 \mathfrak{h}$ that makes $(dx_1, \dots, dy_n, \theta)$ an orthonormal basis.

We put $\bigwedge_0 \mathfrak{h} := \bigwedge^0 \mathfrak{h} = \mathbb{R}$ and, for $1 \le h \le 2n+1$,

$$\bigwedge^h \mathfrak{h} := \mathrm{span}\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_h} : 1 \le i_1 < \cdots < i_h \le 2n+1\}.$$

In the sequel we shall denote by Θ^h the basis of $\bigwedge^h \mathfrak{h}$ defined by

$$\Theta^h := \{\omega_{i_1} \wedge \cdots \wedge \omega_{i_h} : 1 \le i_1 < \cdots < i_h \le 2n+1\}.$$

To avoid cumbersome notations, if $I := (i_1, \dots, i_h)$, we write

$$\omega_I := \omega_{i_1} \wedge \cdots \wedge \omega_{i_h}.$$

The inner product $\langle \cdot, \cdot \rangle$ on $\bigwedge^1 \mathfrak{h}$ yields naturally a inner product $\langle \cdot, \cdot \rangle$ on $\bigwedge^h \mathfrak{h}$ making Θ^h an orthonormal basis.

The volume $(2n+1)$-form $\theta_1 \wedge \cdots \wedge \theta_{2n+1}$ will be also written as dV.

Throughout this paper, the elements of $\bigwedge^h \mathfrak{h}$ are identified with *left invariant* differential forms of degree h on $\mathbb{H}^n$.

Definition 8.1. *A h-form α on $\mathbb{H}^n$ is said left invariant if*

$$\tau_q^{\#} \alpha = \alpha \qquad \text{for any } q \in \mathbb{H}^n.$$

Here $\tau_q^{\#} \alpha$ denotes the pull-back of α through the left translation τ_q.

The same construction can be performed starting from the vector subspace $\mathfrak{h}_1 \subset \mathfrak{h}$, obtaining the *horizontal h-covectors*

$$\bigwedge^h \mathfrak{h}_1 := \mathrm{span}\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_h} : 1 \le i_1 < \cdots < i_h \le 2n\}.$$

It is easy to see that

$$\Theta_0^h := \Theta^h \cap \bigwedge^h \mathfrak{h}_1$$

provides an orthonormal basis of $\bigwedge^h \mathfrak{h}_1$.

Keeping in mind that the Lie algebra $\mathfrak{h}$ can be identified with the tangent space to $\mathbb{H}^n$ at $x = e$ (see, e.g. [21], Proposition 1.72), starting from $\bigwedge^h \mathfrak{h}$ we can define by left translation a fiber bundle over $\mathbb{H}^n$ that we can still denote by $\bigwedge^h \mathfrak{h}$. We can think of h-forms as sections of $\bigwedge^h \mathfrak{h}$. We denote by Ω^h the vector space of all smooth h-forms.

We already pointed out in Section 2 that the stratification of the Lie algebra $\mathfrak{h}$ yields a lack of homogeneity of de Rham's exterior differential with respect to group dilations δ_λ. Thus, to keep into account the different degrees of homogeneity of the covectors when they vanish on different layers of the stratification, we introduce the notion of *weight* of a covector as follows.

Definition 8.2. *If $\eta \neq 0$, $\eta \in \bigwedge^1 \mathfrak{h}_1$, we say that η has* weight 1*, and we write $w(\eta) = 1$. If $\eta = \theta$, we say $w(\eta) = 2$. More generally, if $\eta \in \bigwedge^h \mathfrak{h}$, $\eta \neq 0$, we say that η has* pure weight *p if η is a linear combination of covectors $\omega_{i_1} \wedge \cdots \wedge \omega_{i_h}$ with $w(\omega_{i_1}) + \cdots + w(\omega_{i_h}) = p$.*

Notice that, if $\eta, \zeta \in \bigwedge^h \mathfrak{h}$ and $w(\eta) \neq w(\zeta)$, then $\langle \eta, \zeta \rangle = 0$ (see [9], Remark 2.4). We notice also that $w(d\theta) = w(\theta)$.

We stress that generic covectors may fail to have a pure weight: it is enough to consider $\mathbb{H}^1$ and the covector $dx_1 + \theta \in \bigwedge^1 \mathfrak{h}$. However, the following result holds (see [9], formula (16)):

$$\bigwedge^h \mathfrak{h} = \bigwedge^{h,h} \mathfrak{h} \oplus \bigwedge^{h,h+1} \mathfrak{h} = \bigwedge^h \mathfrak{h}_1 \oplus \Big(\bigwedge^{h-1} \mathfrak{h}_1\Big) \wedge \theta, \tag{59}$$

where $\bigwedge^{h,p} \mathfrak{h}$ denotes the linear span of the h-covectors of weight p. By our previous remark, the decomposition (59) is orthogonal. In addition, since the elements of the basis Θ^h have pure weights, a basis of $\bigwedge^{h,p} \mathfrak{h}$ is given by $\Theta^{h,p} := \Theta^h \cap \bigwedge^{h,p} \mathfrak{h}$ (such a basis is usually called an adapted basis).

As above, starting from $\bigwedge^{h,p} \mathfrak{h}$, we can define by left translation a fiber bundle over $\mathbb{H}^n$ that we can still denote by $\bigwedge^{h,p} \mathfrak{h}$. Thus, if we denote by $\Omega^{h,p}$ the vector space of all smooth h–forms in $\mathbb{H}^n$ of weight p, i.e. the space of all smooth sections of $\bigwedge^{h,p} \mathfrak{h}$, we have

$$\Omega^h = \Omega^{h,h} \oplus \Omega^{h,h+1}. \tag{60}$$

Definition of Rumin's complex

Let us give a short introduction to Rumin's complex. For a more detailed presentation we refer to Rumin's papers [33] following verbatim the presentation of [8]. Here we follow the presentation of [9].

The exterior differential d does not preserve weights. It splits into

$$d = d_0 + d_1 + d_2$$

where d_0 preserves weight, d_1 increases weight by 1 unit and d_2 increases weight by 2 units.

More explicitly, let $\alpha \in \Omega^h$ be a (say) smooth form of pure weight h. We can write

$$\alpha = \sum_{\omega_I \in \Theta_0^h} \alpha_I \, \omega_I, \quad \text{with } \alpha_I \in \mathcal{C}^\infty(\mathbb{H}^n).$$

Then

$$d\alpha = \sum_{\omega_I \in \Theta_0^h} \sum_{j=1}^{2n} (W_j \alpha_I) \, \omega_j \wedge \omega_I + \sum_{\omega_I \in \Theta_0^h} (T\alpha_I) \, \theta \wedge \omega_I = d_1\alpha + d_2\alpha,$$

and $d_0\alpha = 0$. On the other hand, if $\alpha \in \Omega^{h,h+1}$ has pure weight $h+1$, then

$$\alpha = \sum_{\omega_J \in \Theta_0^{h-1}} \alpha_J \, \theta \wedge \omega_J,$$

and

$$d\alpha = \sum_{\omega_J \in \Theta_0^h} \alpha_J \, d\theta \wedge \omega_J + \sum_{\omega_J \in \Theta_0^h} \sum_{j=1}^{2n} (W_j \alpha_J) \, \omega_j \wedge \theta \wedge \omega_I = d_0\alpha + d_1\alpha,$$

and $d_2\alpha = 0$.

It is crucial to notice that d_0 is an algebraic operator, in the sense that for any real-valued $f \in \mathcal{C}^\infty(\mathbb{H}^n)$ we have

$$d_0(f\alpha) = f d_0\alpha,$$

so that its action can be identified at any point with the action of a linear operator from $\bigwedge^h \mathfrak{h}$ to $\bigwedge^{h+1} \mathfrak{h}$ (that we denote again by d_0).

Following M. Rumin ([33], [31]) we give the following definition:

Definition 8.3. *If $0 \le h \le 2n+1$, keeping in mind that $\bigwedge^h \mathfrak{h}$ is endowed with a canonical inner product, we set*

$$E_0^h := \ker d_0 \cap (\mathrm{Im}\ d_0)^\perp.$$

Straightforwardly, E_0^h inherits from $\bigwedge^h \mathfrak{h}$ the inner product.

As above, $E_0^\bullet$ defines by left translation a fibre bundle over $\mathbb{H}^n$, that we still denote by $E_0^\bullet$. To avoid cumbersome notations, we denote also by $E_0^\bullet$ the space of sections of this fibre bundle.

Let $L : \bigwedge^h \mathfrak{h} \to \bigwedge^{h+2} \mathfrak{h}$ the Lefschetz operator defined by

$$L\,\xi = d\theta \wedge \xi. \tag{61}$$

Then the spaces $E_0^\bullet$ can be defined explicitly as follows:

Theorem 8.4 (see [30], [32])**.** *We have:*

i) $E_0^1 = \bigwedge^1 \mathfrak{h}_1$;
ii) *if $2 \le h \le n$, then $E_0^h = \bigwedge^h \mathfrak{h}_1 \cap \left(\bigwedge^{h-2} \mathfrak{h}_1 \wedge d\theta\right)^\perp$ (i.e. E_0^h is the space of the so-called* primitive covectors *of $\bigwedge^h \mathfrak{h}_1$);*
iii) *if $n < h \le 2n+1$, then $E_0^h = \{\alpha = \beta \wedge \theta,\ \beta \in \bigwedge^{h-1} \mathfrak{h}_1,\ \gamma \wedge d\theta = 0\} = \theta \wedge \ker L$;*
iv) *if $1 < h \le n$, then $N_h := \dim E_0^h = \binom{2n}{h} - \binom{2n}{h-2}$;*
v) *if $*$ denotes the Hodge duality associated with the inner product in $\bigwedge^\bullet \mathfrak{h}$ and the volume form dV, then $*E_0^h = E_0^{2n+1-h}$.*

Notice that all forms in E_0^h have weight h if $1 \le h \le n$ and weight $h+1$ if $n < h \le 2n+1$.

A further geometric interpretation (in terms of decomposition of $\mathfrak{h}$ and of graphs within $\mathbb{H}^n$) can be found in [20].

Notice that there exists a left invariant orthonormal basis

$$\Xi_0^h = \{\xi_1^h, \dots, \xi_{\dim E_0^h}^h\} \tag{62}$$

of E_0^h that is adapted to the filtration (59). Such a basis is explicitly constructed by induction in [2].

The core of Rumin's theory consists in the construction of a suitable "exterior differential" $d_c : E_0^h \to E_0^{h+1}$ making $\mathcal{E}_0 := (E_0^\bullet, d_c)$ a complex homotopic to the de Rham complex.

Let us sketch Rumin's construction: first the next result (see [9], Lemma 2.11 for a proof) allows us to define a (pseudo) inverse of d_0 :

Lemma 8.5. *If $1 \le h \le n$, then $\ker d_0 = \bigwedge^h \mathfrak{h}_1$. Moreover, if $\beta \in \bigwedge^{h+1} \mathfrak{h}$, then there exists a unique $\gamma \in \bigwedge^h \mathfrak{h} \cap (\ker d_0)^\perp$ such that*

$$d_0\gamma - \beta \in \mathcal{R}(d_0)^\perp.$$

With the notations of the previous lemma, we set

$$\gamma := d_0^{-1}\beta.$$

We notice that d_0^{-1} preserves the weights.

The following theorem summarizes the construction of the intrinsic differential d_c (for details, see [33] and [9], Section 2) .

Theorem 8.6. *The de Rham complex $(\Omega^\bullet, d)$ splits into the direct sum of two sub-complexes $(E^\bullet, d)$ and $(F^\bullet, d)$, with*

$$E := \ker d_0^{-1} \cap \ker(d_0^{-1} d) \quad \text{and} \quad F := \mathcal{R}(d_0^{-1}) + \mathcal{R}(d d_0^{-1}).$$

Let Π_E be the projection on E along F (that is not an orthogonal projection). We have

i) *If $\gamma \in E_0^h$, then*
   - $\Pi_E \gamma = \gamma - d_0^{-1} d_1 \gamma$ *if* $1 \le h \le n$;
   - $\Pi_E \gamma = \gamma$ *if* $h > n$.

ii) Π_E *is a chain map, i.e.*

$$d\Pi_E = \Pi_E d.$$

iii) *Let* Π_{E_0} *be the orthogonal projection from* $\bigwedge^* \mathfrak{h}$ *on* $E_0^\bullet$, *then*

$$\Pi_{E_0} = Id - d_0^{-1} d_0 - d_0 d_0^{-1}, \quad \Pi_{E_0^\perp} = d_0^{-1} d_0 + d_0 d_0^{-1}. \tag{63}$$

iv) $\Pi_{E_0} \Pi_E \Pi_{E_0} = \Pi_{E_0}$ *and* $\Pi_E \Pi_{E_0} \Pi_E = \Pi_E$.

Set now

$$d_c = \Pi_{E_0}\, d\, \Pi_E : E_0^h \to E_0^{h+1}, \quad h = 0, \dots, 2n.$$

We have:

v) $d_c^2 = 0$;

vi) *the complex $E_0 := (E_0^\bullet, d_c)$ is homotopic to the de Rham complex;*

vii) $d_c : E_0^h \to E_0^{h+1}$ *is a homogeneous differential operator in the horizontal derivatives of order 1 if $h \neq n$, whereas $d_c : E_0^n \to E_0^{n+1}$ is an homogeneous differential operator in the horizontal derivatives of order 2;*

viii) *on forms of degree $h > n$ we have $d_c = d$. Indeed, if $\gamma \in E_0^h$ with $h > n$, then, by i) and iv)*

$$d_c \gamma = \Pi_{E_0} \Pi_E d\gamma = \Pi_E \Pi_{E_0} \Pi_E d\gamma = \Pi_E d\gamma = d\Pi_E \gamma = d\gamma,$$

(see also [10]*).*

ix) *on forms of degree $h = n$, $\Pi_E - Id_{E_0^n} = -d_0^{-1} d_1$ raises weight by one unit, i.e. it maps $E_0^n \subset \bigwedge^{n,n}$ to $\bigwedge^{n,n+1}$.*

The next remarkable property of Rumin's complex is its invariance under contact transformations. In particular,

Proposition 8.7. *If we write a form $\alpha = \sum_j \alpha_j \xi_j^h$ in coordinates with respect to a left-invariant basis of E_0^h (see* (62)*) we have:*

$$\tau_q^\# \alpha = \sum_j (\alpha_j \circ \tau_q) \xi_j^h \tag{64}$$

for all $q \in \mathbb{H}^n$. In addition, for $t > 0$,

$$\delta_t^\# \alpha = t^h \sum_j (\alpha_j \circ \delta_t) \xi_j^h \qquad \text{if } 1 \le h \le n \tag{65}$$

and

$$\delta_t^\# \alpha = t^{h+1} \sum_j (\alpha_j \circ \delta_t) \xi_j^h \qquad \text{if } n+1 \le h \le 2n+1\,. \tag{66}$$

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