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Distribution of Resonances for Open Quantum Maps

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Abstract: We analyze a simple model of quantum chaotic scattering system, namely the quantized open baker's map. This model provides a numerical confirmation of the fractal Weyl law for the semiclassical density of quantum resonances. The fractal exponent is related to the dimension of the classical repeller. We also consider a variant of this model, for which the full resonance spectrum can be rigorously computed, and satisfies the fractal Weyl law. For that model, we also compute the shot noise of the conductance through the system, and obtain a value close to the prediction of random matrix theory.

1. Introduction

1.1. Statement of the results. In this paper we analyze simple models of classical chaotic open systems and of their quantizations. They provide a numerical confirmation of the *fractal Weyl law* for the density of quantum resonances of such systems. The exponent in that law is related to the dimension of the classical repeller of the system. In a simplified model, a rigorous argument gives the full resonance spectrum, which satisfies the fractal Weyl law. Our model is similar to models recently studied in atomic and mesoscopic physics (see §2.4 below). Before stating the main result we remark that in this paper we use mathematicians' notation h for what the physicists call $\hbar$. That is partly to stress that our h is a small parameter in asymptotic analysis, not necessarily interpreted as the Planck constant.

Theorem 1. *There exist families of symplectic relations, $\widetilde{B} \subset \mathbb{T}^2 \times \mathbb{T}^2$, and of their (subunitary) quantization, $\widetilde{B}_h \in \mathcal{L}(\mathbb{C}^N)$, $N = (2\pi h)^{-1}$, such that*

$$\#\big\{\lambda \in \operatorname{Spec}(\widetilde{B}_h) \ : \ |\lambda| \geq r\big\} = c(r)\, h^{-\nu} + o(h^{-\nu}),$$
$$r > 0, \quad h = h_k = (2\pi D^k)^{-1} \to 0, \quad k \to \infty,$$
$$\nu = \dim\big(\Gamma_-(\widetilde{B}) \cap W_+(\widetilde{B})\big), \quad c(r) = (2\pi)^{-\nu} \chi_{[0, r_0(\widetilde{B})]}(r), \quad 0 < r_0(\widetilde{B}) < 1,$$

where the integer parameter $D > 1$ depends on $\widetilde{B}$. The set $\Gamma_-(\widetilde{B}) \subset \mathbb{T}^2$ is the forward trapped set of $\widetilde{B}$ and $W_+(\widetilde{B})$ is the unstable manifold of $\widetilde{B}$ at any point of $\Gamma_-(\widetilde{B})$. The eigenvalues are counted with multiplicities.

In the model discussed in detail we took $D = 3$. The asymptotics are actually much more precise and include uniform angular distribution (see Prop. 5.5). The resonances lie on a lattice, and some of this structure is also seen in numerically computed more generic situations (some numerical results have been presented in [40, 38, 39]). Each symplectic relation $\widetilde{B}$ (or "multivalued symplectic map") is defined together with the probabilities, for any point, to be mapped to each of its images: $\widetilde{B}$ thus represents a certain stochastic process. The quantizations $\widetilde{B}_h$ quantize the relations together with their jump probabilities in the precise sense given in §4.4.

In the models used in Theorem 1 we can compute the conductance and the shot noise power (or the closely related Fano factor) — see §2.4.3 and references given there for physics background and §6 for precise definitions.

Theorem 2. *Suppose that the models in Theorem 1 have the openings consisting of two "leads" of equal width* (*see* §6.1 *for a detailed description*)*, so that each lead carries the same number, $M(h) \sim h^{-1}$, of scattering channels. Then, the quantum conductance* (6.2) *between the two leads satisfies*

$$g(h) = \frac{1}{2} M(h) \big(1 + o(1)\big), \quad h = h_k \to 0. \tag{1.1}$$

The Fano factor (6.3) *is given by*

$$F(h) = \frac{11}{80} \frac{M(h)^\nu}{g(h)} \big(1 + o(1)\big), \quad h = h_k \to 0, \tag{1.2}$$

where the exponent ν is the same as in Theorem 1.

The theorem should be interpreted as follows. In (1.1) we see that for a model of scattering through a chaotic cavity, approximately one half of the scattering channels get transmitted from one lead to the other, the other half being reflected back (this is natural and well known). Asymptotics in (1.2) are more interesting. We see that the fractal Weyl law, $h^{-\nu}$, appears in the expression for the Fano factor. In the interpretation of the Fano factor in terms of "shot noise" (see §2.4.3), 11/80 gives the average "shot noise" per "nonclassical transmission channel". This number is close to the random matrix theory prediction for this quantity, namely 1/8 [26, 57]. In fact, had (1.2) come from a physical experiment rather than an asymptotic computation, it would be regarded as being in a very good agreement with random matrix theory[1].

Much of the paper is devoted to rigorous definitions of the objects appearing in the statements of the two theorems. In this section we give some general indications, with detailed references to previous works appearing below.

We consider the two-torus $\mathbb{T}^2 = [0, 1) \times [0, 1)$ as our classical phase (with coordinates $\rho = (q, p)$). Classical observables are functions on $\mathbb{T}^2$ and classical dynamics is given in terms of relations, $B \subset \mathbb{T}^2 \times \mathbb{T}^2$, which are unions of truncated graphs of symplectic (area and orientation preserving) maps $\mathbb{T}^2 \to \mathbb{T}^2$. An example is given by the baker's relation

$$(\rho'; \rho) = (q', p'; q, p) \in B \Longleftrightarrow \begin{cases} q' = 3q, & p' = p/3, \quad 0 \le q \le 1/3 \\ q' = 3q - 2, & p' = (p+2)/3, \quad 2/3 \le q < 1. \end{cases} \tag{1.3}$$

[1] We are grateful to Yan Fyodorov for this amusing comment.

This is a "rectangular horseshoe" modeling a Poincaré map of a chaotic open system: some points (here $\{\rho : 1/3 < q < 2/3\}$) are thrown out "to infinity" at each iteration.

For relations such as B we can define the *forward and backward trapped sets* (see (2.4) for the definition in the case of flows):

$$\rho \in \Gamma_- \;\Leftrightarrow\; \exists \{\rho_j\}_{j=0}^{\infty}, \quad \rho_0 = \rho, \;\; (\rho_j; \rho_{j-1}) \in B, \;\; j > 0,$$
$$\rho \in \Gamma_+ \;\Leftrightarrow\; \exists \{\rho_j\}_{j=-\infty}^{0}, \quad \rho_0 = \rho, \;\; (\rho_j; \rho_{j-1}) \in B, \;\; j \leq 0.$$

In the example (1.3), $\Gamma_- = C \times [0, 1)$, $\Gamma_+ = [0, 1) \times C$, where C is the usual $\frac{1}{3}$– Cantor set.

We also define the *trapped set* $K = \Gamma_+ \cap \Gamma_-$ and, at points of K, the stable and unstable manifolds, $W_\pm$. In the case of the above baker's relation,

$$\nu = \dim \Gamma_- \cap W_+ = \frac{1}{2} \dim K = \dim \Gamma_+ \cap W_- = \frac{\log 2}{\log 3},$$

but for general (possibly multivalued) relations these equalities do not hold.

A quantization (in the sense made rigorous in §4.5) of B is given by

$$B_h = \mathcal{F}_N^* \begin{pmatrix} \mathcal{F}_{N/3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathcal{F}_{N/3} \end{pmatrix}, \quad h = (2\pi N)^{-1}, \;\; 3|N, \tag{1.4}$$

where $\mathcal{F}_M$ is the discrete Fourier transform on $\mathbb{C}^M$.

Table 1. Number of eigenvalues of B_h in the regions $\{|\lambda| > r\}$, for $2\pi h = 1/N$, N given by powers of 3.

$N = 3^k$	$r = 0.1$	$r = 0.2$	$r = 0.3$	$r = 0.4$	$r = 0.5$	$r = 0.6$	$r = 0.7$	$r = 0.8$
$k = 1$	5	5	5	5	5	4	3	3
$k = 2$	14	14	10	9	8	8	7	6
$k = 3$	32	26	23	19	16	16	14	5
$k = 4$	63	53	45	40	33	33	30	6
$k = 5$	124	103	85	78	71	65	63	11
$k = 6$	237	196	161	150	142	131	128	12

Table 2 shows the analogies between the eigenvalues of this subunitary quantum map and the resonances of a Schrödinger operator for a scattering situation (see §2.1).

For B_h given by (1.4) we are unable to prove the fractal Weyl law presented in the last line of Table 2, but numerical results strongly support its validity [40]. A striking illustration is given by tripling N, in which case the number of eigenvalues approximately doubles, in agreement with the fractal Weyl law — see Table 1.

The family of subunitary quantum maps in the main theorem is obtained by simplifying B_h, and is described explicitly in (5.2). It is a quantization of a more complicated multivalued relation for which $\Gamma_+ = \mathbb{T}^2$, $\Gamma_- = C \times [0, 1)$, and $\Gamma_- \cap W_+ \simeq C$ — see Proposition 5.1. Theorem 1 follows from the more precise Proposition 5.4.

1.2. Organization of the paper. In §2 we present related results from recent mathematical, numerical, and physics literature. In particular, in §2.4.3 we give the physical motivation for the objects appearing in Theorem 2 above. Section 3 is devoted to the review of classical dynamics used in our models, stressing the dynamics of open baker's relations.

In §4 we first review the quantization of tori. We assume the knowledge of semiclassical quantization in $T^*\mathbb{R}^n$ (pseudodifferential operators) but otherwise the presentation is self contained. The definitions of Lagrangian states associated to smooth and singular Lagrangian submanifolds is based on the ideas of Guillemin, Hörmander, Melrose, and Uhlmann in microlocal analysis but, partly due to technical differences, we give direct proofs of the properties we need in this paper. These properties are used to analyze the quantizations of the baker's relation coming from the work of Balazs, Voros, Saraceno and Vallejos.

Numerical results for the (usual) quantization of the open baker's relation have been presented in [40], we briefly summarize them in §4.6. In §5 we discuss the toy model $\widetilde{B}_h$, with two different interpretations. That section contains the proof of Theorem 1. Finally in Section 6 we give precise definitions of objects appearing in Theorem 2 and in a lengthy computation we give its proof.

2. Motivation and Background

In this section we discuss motivating topics in mathematics and theoretical physics, and survey related results.

2.1. Schrödinger operators. The original motivation comes from the study of resonances in potential scattering. The simplest case is given by considering the following quantum Hamiltonian:

$$H = -h^2\Delta + V(q), \qquad V \in \mathcal{C}^\infty_c(\mathbb{R}^n;\mathbb{R}). \tag{2.1}$$

By assuming that the potential vanishes near infinity and that it is infinitely differentiable, we eliminate the need for technical assumptions — see [22] and [53] for more general settings, in the analytic and $\mathcal{C}^\infty$ categories respectively. For instance, as noted in [50, (c.32)–(c.33)] the theory of [22] applies to arbitrary homogeneous polynomial potentials at nondegenerate energy levels.

Table 2. Analogies between Schrödinger propagators and open quantum maps.

$h \to 0$	$N = (2\pi h)^{-1} \to \infty$
$\chi \exp\big(-it(-h^2\Delta + V)/h\big)\chi$, $t \geq 0$, χ a cut-off on the interaction region	B_h^t, $t = 0, 1, \cdots B_h$ a subunitary matrix
$e^{-itz/h}$, z a resonance of $H = -h^2\Delta + V$	λ^t, λ an eigenvalue of $B_h \in \mathcal{L}(\mathbb{C}^N)$
$z \in [E-h, E+h] - i[0, \gamma h]$	$1 \geq \lvert\lambda\rvert > r > 0$
$\#\{z \in [E-h, E+h] - i[0,\gamma h]\} \simeq C(\gamma)\, h^{-\mu_E}$	$\#\{\lambda, \lvert\lambda\rvert > r\} \simeq C(r)\, N^{\nu}$

Before discussing open systems we recall the well known results for closed systems, obtained for instance by considering H above on a bounded domain $\Omega \subset \mathbb{R}^n$ and imposing a self-adjoint boundary condition at $\partial\Omega$ (Dirichlet or Neumann). Then the spectrum, Spec(H), of H is discrete and, at a non-degenerate energy level E its density is described by the celebrated *Weyl law*:

$$\#\{\mathrm{Spec}(H)\cap[E-\delta,E+\delta]\} = \frac{1}{(2\pi h)^n}\iint_{|p^2+V(q)-E|<\delta} dq\,dp + \mathcal{O}(h^{1-n}), \tag{2.2}$$

see [14, 25] and references given there. We note that this implies a precise upper bound

$$\#\{\mathrm{Spec}(H)\cap[E-Ch,E+Ch]\} = \mathcal{O}(h^{1-n}), \tag{2.3}$$

which can be improved further by making assumptions on the classical flow of the Hamiltonian $p^2+V(q)$ on Ω, see [14, 25].

For open systems, with the simplest example given by the Hamiltonian in (2.1), real eigenvalues are replaced by complex *resonances*. The simplest definition (easily made rigorous in the case (2.1)) comes from considering the meromorphic continuation of the resolvent. Defining the Green's function $G(z,q',q)$ for $\mathrm{Im}\,z>0$ through

$$(z-H)^{-1}u(q') = \int_{\mathbb{R}^n} G(z,q',q)\,u(q)dq, \quad u\in\mathcal{C}_c^\infty(\mathbb{R}^n),$$

then $G(z,q',q)$ admits a meromorphic continuation in z across the real axis. Its poles for $\mathrm{Im}\,z<0$ (which do not depend on (q',q)) are the *quantum resonances* of H.

Counting of resonances is affected by the dynamical structure of the scatterer much more dramatically than counting of eigenvalues of closed systems. Since we are now counting points in the complex plane we need to make geometric choices dictated by dynamical and physical considerations. Here we consider scatterers and energies exhibiting a hyperbolic classical flow, and regions in the lower half-plane which lie at a distance proportional to h from the real axis. This choice is motivated as follows. Quantum mechanics interprets a resonance at $z=E-i\gamma$ in terms of a *metastable state*, which decays proportionally to $\exp(-t\gamma/h)$. Hence for $\gamma\gg h$ the decay is so rapid that the state is invisible. On the other hand, for many chaotic scatterers there are no resonances with $\gamma\ll h$. One class for which this is known rigorously consists in the Laplacian on co-compact quotients $\mathbb{H}^n/\Gamma$, $H=-h^2\Delta_{\mathbb{H}^n/\Gamma}$, when the dimension of the limit set satisfies $\delta(\Gamma)<(n-1)/2$. This follows from the work of Patterson and Sullivan — see the discussion below and [37].

After a complex deformation (see [53] and references given there) the long living quantum states should semiclassically concentrate on the set of phase space points which do not escape to infinity, that is on the *trapped set* K_E defined as follows: let Ξ_H be the Hamilton vector field of the Hamiltonian $H(q,p)=p^2/2+V(q)$:

$$\Xi_H = \sum_{j=1}^n p_j\partial_{q_j} - \partial_{q_j}V(q)\partial_{p_j}.$$

Then

$$\begin{aligned} &K_E \stackrel{\text{def}}{=} \Gamma_+(E)\cap\Gamma_-(E), \quad \text{with the forward/backward trapped sets}\\ &\Gamma_\pm(E) \stackrel{\text{def}}{=} \{\rho\in\Sigma_E\ :\ \exp t\,\Xi_H(\rho)\not\to\infty,\ \mp t\to\infty\}. \end{aligned} \tag{2.4}$$

Suppose that the flow generated by Ξ_H is hyperbolic near $K_{E'}$ for E' close to a non-degenerate energy E. That means that the field Ξ_H does not vanish on the energy surfaces $\Sigma_{E'} = \{p^2 + V(q) = E'\} \subset T^*\mathbb{R}^n$ for $E' \approx E$, and that for $\rho \in \Sigma_{E'}$ near $K_{E'}$,

$$\begin{aligned}
&T_\rho \Sigma_{E'} = \mathbb{R}\, \Xi_H(\rho) \oplus E_+(\rho) \oplus E_-(\rho), \\
&\Sigma_{E'} \ni \rho \longmapsto E_\pm(\rho) \subset T_\rho \Sigma_{E'} \ \text{ is continuous,} \\
&d(\exp t\, \Xi_H)(E_\pm(\rho)) = E_\pm(\exp t\, \Xi_H(\rho)), \\
&\|d(\exp t\, \Xi_H)(X)\| \le C e^{\pm\lambda t} \|X\|, \quad \text{for all } X \in E_\pm(\rho),\ \mp t \ge 0.
\end{aligned} \tag{2.5}$$

Weaker assumptions are possible — see [50, §5] and [53, §7].

Typically, the set K_E has a fractal structure and in the semiclassical estimates the Minkowski dimension naturally appears:

$$\begin{aligned}
\dim K_E = 2n - 1 - \sup \Big\{ c \ :\ &\limsup_{\epsilon \to 0} \epsilon^{-c} \mathrm{vol} \\
&\times \{\rho \in \Sigma_E \ :\ \mathrm{dist}(K_E, \rho) < \epsilon\} < \infty \Big\}.
\end{aligned}$$

We say that K_E is of *pure dimension* if the supremum is attained. For simplicity of the presentation we assume that this is the case.

Under these assumptions the estimate (2.3) has an analogue for chaotic open systems [53]. For $C_0 > 0$ there exists C_1 such that

$$\#\{\mathrm{Res}(H) \cap \{z :\ |z - E| < C_0 h\}\} \le C_1 h^{-\mu_E}, \quad \dim K_E = 2\mu_E + 1. \tag{2.6}$$

We notice that for a closed system the trapped set is the entire energy surface, so that in that case $\mu_E = n - 1$, hence (2.6) is consistent with (2.3). In this note we use open quantum maps to provide the first evidence that this precise estimate is optimal.

We should also mention that, as was already stressed in the work of Sjöstrand [50], the estimates involving the dimension are only reasonable when the flow is strictly hyperbolic. In the case of more complicated flows the estimates should be stated in terms of properties of *escape* or *Lyapunov* functions associated to the flow – see [50, 53]. For expository reasons the estimates involving the dimension are however most persuasive.

2.2. Survey of related results. The first indication that fractal dimensions enter into counting laws for quantum resonances of chaotic open systems appears in a result of Sjöstrand [50]:

$$\begin{gathered}
\#\big\{\mathrm{Res}(H) \cap \{z :\ |z - E| < \delta,\ \ \mathrm{Im}\, z > -\gamma\}\big\} \le C_1 \delta \left(\frac{h}{\gamma}\right)^{-n} \gamma^{-\frac{1}{2}\widetilde{m}}, \\
Ch \le \gamma \le 1/C, \ \ \max(h^{\frac{1}{2}}, h/\gamma) \le \delta \le 2/C,
\end{gathered} \tag{2.7}$$

where $\widetilde{m}$ is any number greater than the dimension of the trapped set in the shell $H^{-1}(E - 1/C_2, E + 1/C_2)$. In a homogeneous situation, such as for instance obstacle scattering, the dimension of K_E, $2\mu_E + 1$, is independent of E, so that $\widetilde{m} > 2(\mu_E + 1)$.

The improvement in [53] quoted in (2.6) lies in providing a bound for the number of resonances in a smaller region $D(E, Ch) = \{z \in \mathbb{C} \ :\ |z - E| < Ch\}$. Heuristic arguments suggesting that the estimate (2.7) should be optimal were given in [31] and [32].

Another class of Hamiltonians with chaotic classical flows and fractal trapped sets is given by Laplacians on convex co-compact quotients, $\mathbb{H}/\Gamma$. Here Γ is a discrete subgroup of isometries of the hyperbolic plane $\mathbb{H}$, such that

- All elements $\gamma \in \Gamma$ are hyperbolic, which means that their action on $\mathbb{H}$ can be represented as

$$\alpha \circ \gamma \circ \alpha^{-1}(x, y) = e^{\ell(\gamma)}(x, y), \quad (x, y) \in \mathbb{H} \simeq \mathbb{R}_+ \times \mathbb{R}, \quad \alpha \in \mathrm{Aut}(\mathbb{H}). \tag{2.8}$$

- If $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$, and $\Lambda(\Gamma) \subset \partial\mathbb{H}$ is the limit set of Γ, that is the set of limit points of $\{\gamma(z) : \gamma \in \Gamma\}$, $z \in \mathbb{H}$, then $\pi(\text{convex hull } \Lambda(\Gamma))$ is compact.

The trapped set is determined by $\Lambda(\Gamma)$: trapped trajectories are given by geodesics connecting two points of $\Lambda(\Gamma)$ at infinity, and

$$\dim K_E = 2\delta_\Gamma + 1, \quad \delta_\Gamma = \dim \Lambda(\Gamma).$$

The limit set is always of pure dimension, which coincides with its Hausdorff dimension.

A nice feature of this model is the exact correspondence between the resonances of

$$H = h^2(-\Delta_{\mathbb{H}/\Gamma} - 1/4),$$

and the zeros of the Selberg zeta function, $Z_\Gamma(s)$:[2]

$$z \in \mathrm{Res}(H) \iff Z_\Gamma(s) = 0, \quad z = h^2(s(1-s) - 1/4), \quad \mathrm{Re}\, s \leq \delta_\Gamma, \tag{2.9}$$

where the multiplicities of zeros and resonances agree. The Selberg zeta function is defined by the analytic continuation of

$$Z_\Gamma(s) = \prod_{\{\gamma\}} \prod_{k \geq 0} \left(1 - e^{-(s+k)\,\ell(\gamma)}\right), \quad \mathrm{Re}\, s > \delta_\Gamma,$$

where $\{\gamma\}$ denotes a conjugacy class of a primitive element $\gamma \in \Gamma$ (an element which is not a power of another element), and we take a product over distinct primitive conjugacy classes (each of which corresponds to a primitive closed orbit). The length $\ell(\gamma)$ of the corresponding closed orbit appears in (2.8). The exact analogue of (2.6) is given by

$$\#\{s : Z_\Gamma(s) = 0, \ \mathrm{Re}\, s > -C_0, \ r < \mathrm{Im}\, s < r + C_1\} \leq C_2\, r^{\delta_\Gamma}, \tag{2.10}$$

which is a consequence of an estimate established by Guillopé-Lin-Zworski [20] in a more general setting of convex co-compact Schottky groups in any dimension,

$$|Z_\Gamma(s)| \leq C_K\, e^{C_K\, |s|^{\delta_\Gamma}}, \quad \mathrm{Re}\, s \geq -K, \quad \text{for any } K. \tag{2.11}$$

This improved earlier estimates of [59], the proof of which was largely based on [50].

In the (non-quantum) context of rational maps on the complex plane, similar results were obtained concerning the zeros of associated zeta functions [11, 54]. Take f a uniformly expanding rational map on $\mathbb{C}$ (for instance $z \mapsto z^2 + c$, $c < -2$), and call f^n its n-fold composition. The zeta function associated with this map is given by

$$Z(s) = \exp\left(-\sum_{n=1}^{\infty} n^{-1} \sum_{f^n(z)=z} \frac{|(f^n)'(z)|^{-s}|}{1 - |(f^n)'(z)|^{-1}}\right). \tag{2.12}$$

[2] We refer to [42] for this and a general treatment. The term $\frac{1}{4}$ in the definition of the Hamiltonian H comes from requiring that the bottom of the spectrum of H is 0, so that Green's function $(H - \lambda^2)^{-1}$ is meromorphic in $\lambda \in \mathbb{C}$.

Then the number or resonances in a strip is also given by a law of the type (2.10), where δ_Γ is replaced by the dimension of the *Julia set*:

$$J = \overline{\bigcup_{n\geq 1} \{z \ : \ f^n(z) = z\}}.$$

Note that this set is also made of “trapped orbits”.

2.3. Survey of numerical results. The first model investigated numerically was perhaps the hardest to give definitive results. Lin [30, 31] studied the semiclassical Schrödinger Hamiltonian (2.1) with a potential made of 3 Gaussian “bumps”. The semiclassical resonances were computed using the method of complex scaling and were counted in boxes of type $[E-\delta, E+\delta] - i[0, h]$ with δ fixed. The purpose was to verify optimality of Sjöstrand's estimate (2.7) with these parameters. The results were encouraging but not conclusive. Since for small values of h the method of [30] required the use of large matrices to discretize the Hamiltonians, the range of h's was rather limited.

A different point of view was taken by Lu-Sridhar-Zworski [32] where resonances for the three discs scatterer in the plane were computed using the semiclassical zeta function of Eckhardt-Cvitanović, Gaspard, and others (see for instance [12, 18, 58] and references therein). The zeta function is computed using the cycle expansion method loosely based on the Ruelle theory of dynamical zeta functions. Although it is not rigorously known if the resonances computed by this method approximate resonances of the Dirichlet Laplacian in the exterior of the discs, or even if the semiclassical zeta function has an analytic continuation, proceeding this way is widely accepted in the physics literature. Resonances $z = h^2 k^2$ were counted in regions

$$\{k \in \mathbb{C} \ : \ 1 \leq \operatorname{Re} k \leq r, \ \ \operatorname{Im} k \geq -\gamma\}, \quad r \to \infty, \tag{2.13}$$

which under semiclassical rescaling correspond to counting in $[1/2, 2] - i[0, \gamma h/2]$, $h \to 0$. Let us denote the number of resonances (zeros of the semiclassical zeta function) in (2.13) by $N(r, \gamma)$. The fractal Weyl law corresponds to the claim that for γ large enough,

$$N(r, \gamma) \sim C(\gamma)\, r^{\mu+1}, \quad r \to \infty, \tag{2.14}$$

where $2\mu + 1$ is the dimension of the trapped set in the three dimensional energy shell (for such homogeneous problems, all energy shells are equivalent). In [32] the prediction (2.14) was tested by linear fitting of $\log N(r, \gamma)$ as a function of $\log r$:

$$\log N(r, \gamma) = (\alpha(\gamma) + 1) \log r + \mathcal{O}(1).$$

We found that the coefficient $\alpha(\gamma)$ was independent of γ for γ large enough, and that it agreed with μ. The counting was done for three different equilateral disc configurations, parametrized by $\rho = R/a$, where a is the radius of each disc, and R the distance between them. We also noticed that if γ_ρ is the classical rate of decay for the ρ configuration, then

$$\frac{\alpha_\rho(x\gamma_\rho/2)}{\mu_\rho}$$

is essentially independent of ρ for $1 < x < 1.5$. This corresponds to a numerical observation that for each ρ the distribution of resonance widths (imaginary parts) peaks near $\gamma = \gamma_\rho/2$.

Encouraged by the results of [32], the cycle method was used in [20] to count the zeros of the Selberg zeta function for a certain Schottky quotient, but the results were not definitive. For the dynamical zeta function (2.12) with $f(z) = z^2 + c$, $c < -2$, the resonances were computed by Strain-Zworski [54], using a different method based on the theory of the transfer operator on Hilbert spaces of holomorphic functions introduced in [20]. The zeros were counted in a region of the same type as in (2.13),

$$\{s \,:\, \operatorname{Re} s > -K,\ 0 \leq \operatorname{Im} s \leq r\},$$

where real parts and imaginary parts exchange their meaning due to different conventions[3]. By reaching very high values of r we saw a very good agreement of the log-log fit with the fractal Weyl, with μ given by the dimension of the Julia set.

In the model considered in this paper, we can verify the optimality of the fractal Weyl law on much smaller scales (see Table 1 and the numerics presented in [40, 38, 39]). That could not be seen in the other approaches.

2.4. Related models in physics. The behaviour of quantum open systems has been recently investigated in situations where the classical dynamics has chaotic features. The physical motivation can originate from nuclear or atomic physics (study the lifetime statistics of metastable states, possibly leading to ionization), mesoscopic physics (study the conductance, conductance fluctuations, shot noise in quantum dots or quantum wires), and from waveguides (optical wave propagation in an optical fiber with some dissipation, microwave propagation in an open microwave cavity).

2.4.1. Kicked rotator with absorbing boundaries. In [3, 7] a kicked rotator with absorption was used to model the process of ionization. The classical kicked rotator is Chirikov's *standard map* on the cylinder, which is a paradigmatic model for transitions from regular to chaotic motion [9]. Quantizing the map on $L^2(\mathbb{T}^1)$ results in a unitary operator U, a first instance of *quantum map*. To model the ionization process which takes place at some threshold momentum p_{ion}, the authors truncate the map U to the subspace $\mathcal{H}_{\text{ion}} = \operatorname{span}\{|p_j\rangle \,:\, |p_j| \leq p_{\text{ion}}\}$: a particle reaching that threshold is ionized, or equivalently "escapes to infinity". Here the discrete values $p_j = 2\pi hj$ are the eigenvalues of the momentum operator on $L^2(\mathbb{T}^1)$; the space $\mathcal{H}_{\text{ion}}$ is thus of dimension $N \approx p_{\text{ion}}/\pi h$. This projection leads to an *open quantum map*, namely the subunitary propagator $U_{\text{ion}} = \Pi_{\text{ion}} U$, where Π_{ion} is the orthogonal projector on $\mathcal{H}_{\text{ion}}$. For the parameters used by [3], the classical dynamics is diffusive, meaning that a particle starting from $p = 0$ will need many kicks to reach the ionization threshold.

The matrix U_{ion} was numerically diagonalized for various values of h with p_{ion} fixed, and the distribution of the N level widths $\gamma_i = -2\ln|\lambda_i|$, $\lambda_i \in \operatorname{Spec}(U_{\text{ion}})$ was found approximately independent of h, such that the number of resonances

$$n(N, \gamma) = \#\{\gamma_i \leq \gamma\}$$

scales like $C(\gamma)N$ in this case. In subsequent works [47, 60, 17], this distribution was shown to correspond to an ensemble of random subunitary random matrices, more precisely the ensemble formed by the $[\alpha N] \times [\alpha N]$ upper-left corner ($0 < \alpha < 1$ fixed) of a large $N \times N$ matrix drawn in the Circular Unitary Ensemble (that is, the set $U(N)$ equipped with Haar measure).

[3] Although frustrating, the different conventions of semiclassical, obstacle, and hyperbolic scattering show how the same phenomenon appears in historically different fields.

2.4.2. Quasi-bound states in an open quantum map. Recently, Schomerus and Tworzydło [49] have performed a similar study for the quantized kicked rotator on the torus (obtained from the map of the former section by periodizing the momentum variable). They also "opened" the map by assuming that particles reaching a certain position window $q \in L$ "escape to infinity". The quantum projector associated with these "escape windows" is denoted by Π_L, so that the remaining subunitary quantum map reads $U_{\rm op} = (I - \Pi_L)U$. The main difference with the case studied in the previous section lies in the strongly chaotic motion (as opposed to diffusive), due to a different choice of parameters. The map has a positive Lyapunov exponent λ, and a typical trajectory will escape after a few kicks: the average "dwell time", called τ_D, is of order unity.

The eigenmodes associated with eigenvalues bounded away from zero are called "quasi-bound states", as opposed to the "instantaneous decay modes" associated with very small eigenvalues. The authors provide numerical and heuristic evidence that, in the semiclassical limit, the number of quasi-bound states grows like $N_{\rm eff} = N^{1-1/(\lambda\tau_D)}$. This shows that most eigenvalues of $U_{\rm op}$ are very close to zero, while only a small fraction $N_{\rm eff}/N$ remains bounded away from zero. The authors also plot the distribution of the $\sim N_{\rm eff}$ quasi-bound eigenvalues: again, it resembles the spectrum of a random subunitary matrix obtained by keeping the upper-right corner block of size $N_{\rm eff}$ of a $[\tau_D N_{\rm eff}]$-dimensional random unitary matrix.

The quantized baker's relation we will study in §4.6–5 will be of similar nature. For the map (4.38), the fractal dimension ν given in (3.7) can be shown to be close to the formula $1 - 1/(\lambda\tau_D)$, in the limit when the dwell time τ_D is large compared to unity (limit of small opening).

2.4.3. Conductance through an open chaotic cavity. The "scattering approach to semiclassical quantization" [4, 15, 43, 41], consists in quantizing the return map on a Poincaré surface of the section of the Hamiltonian system under study. Within this approach, the *scattering matrix* of the open system can be expressed as a "multiple-scattering expansion" in terms of the quantized return map.

Using that framework, Beenakker *et al.* [57] study the quantum kicked rotator defined in the previous section, in order to understand the fluctuations of conductance through a *quantum dot*. The evolution inside the closed dot is represented by the same unitary matrix U as in the last subsection, and its opening L is split into two intervals, L_2 and L_1, which represent the two "leads" bringing in and taking out the charge carriers from the dot. The orthogonal projector corresponding to these openings reads $\Pi_L = \Pi_{L_1} \oplus \Pi_{L_2}$. The conductance can then be analyzed from the *scattering matrix* of the dot:

$$\tilde{S}(\vartheta) = \Pi_L\{e^{-i\vartheta} - U(1 - \Pi_L)\}^{-1} U \Pi_L. \tag{2.15}$$

Here $\vartheta \in [0, 2\pi)$ is called the *quasi-energy*. In terms of this parameter, the "physical half-plane" corresponds to $\operatorname{Im}\vartheta > 0$: the matrix $\tilde{S}(\vartheta)$ has no singularity in this region. On the opposite, the resonances analyzed in the previous section, which are the *poles* of $\tilde{S}(\vartheta)$, are situated in the region $\operatorname{Im}\vartheta < 0$.

While $\tilde{S}(\vartheta)$ is unitary, its subblock $t \stackrel{\rm def}{=} \Pi_{L_2}\tilde{S}(\vartheta)\Pi_{L_1}$ describes the *transmission* from the lead L_1 to the lead L_2. The dimensionless conductance (which depends on ϑ) is given by the Landauer-Büttiker formula $g = \operatorname{tr}(tt^*)$. The eigenvalues of tt^* (called "transmission eigenvalues") can be either close to 1 (corresponding to a total transmission), or close to 0 (corresponding to a total reflection), or inbetween. The last case corresponds to genuinely quantum transmission eigenmodes, which are partly transmitted, partly reflected, due to interference phenomena inside the dot. The "quantum shot noise” is due to these intermediate transmission eigenvalues. A simple measure of that noise is given by the Fano factor [6] $F = \mathrm{tr}(tt^*(1-tt^*))/\mathrm{tr}tt^*$. Using similar arguments as in the former section, the authors show that the number of intermediate transmission eigenvalues also scales like N_{eff}, and thereby estimate the Fano factor, by assuming that these eigenvalues are distributed according to the prediction of random matrix theory.

In Section 6 we will analytically compute both the conductance and the Fano factor in the case of the open quantum relation $\widetilde{B}_h$.

3. Classical Dynamics

3.1. Symplectic geometry on tori. We consider the simplest class of compact symplectic manifolds, the tori,

$$\mathbb{T}^{2n} \stackrel{\mathrm{def}}{=} \mathbb{R}^{2n}/\mathbb{Z}^{2n} \simeq (\mathbb{I}\times\mathbb{I})^n, \quad \omega = \sum_{\ell=1}^{n} dq_\ell \wedge dp_\ell, \quad (q,p)\in\mathbb{T}^{2n}.$$

Here and in what follows, we identify the interval $\mathbb{I}=[0,1)$ with the circle $\mathbb{T}^1=\mathbb{R}/\mathbb{Z}$. A Lagrangian (submanifold) $\Lambda\subset\mathbb{T}^{2n}$ is a n-dimensional embedded submanifold of $\mathbb{T}^{2n}$ such that $\omega|_\Lambda = 0$. We recall the following well known fact (see for instance [24, Theorem 21.3.2]):

Proposition 3.1. *Suppose that $\Lambda\subset\mathbb{T}^{2n}$ is a Lagrangian submanifold, and that $(q_0,p_0)\in\Lambda$. Then, after a possible permutation of indices, there exists k, $0\leq k\leq n$, and a splitting of coordinates:*

$$q=(q',q''),\quad p=(p',p''),\quad q'=(q_1,\dots q_k),\quad p''=(p_{k+1},\dots,p_n),$$

such that the map

$$\Lambda\ni(q,p)\longmapsto (q'',p')\in\mathbb{I}^{n-k}\times\mathbb{I}^k$$

is bijective from a neighbourhood V of (q_0,p_0) to a neighbourhood W of (q_0'',p_0'). Consequently there exists a function, $S=S(q'',p')$ defined on W, such that $\Lambda\cap V$ is generated by the function S, that is,

$$\Lambda\cap V=\Big\{\Big(d_{p'}S(q'',p'),q'';p',-d_{q''}S(q'',p')\Big),\ \ (q'',p')\in W\Big\}.$$

In this paper we will also consider *singular Lagrangian manifolds* obtained by taking finite unions of Lagrangians with piecewise smooth boundaries.

3.2. Symplectic relations

3.2.1. Symplectic maps. A symplectic (or “canonical”) diffeomorphism on the torus $\mathbb{T}^{2n}$ is a diffeomorphism $\kappa\ :\ \mathbb{T}^{2n}\to\mathbb{T}^{2n}$ which leaves invariant the symplectic form on $\mathbb{T}^{2n}$: $\kappa^*\omega=\omega$. An equivalent characterization of such a map is through its *graph* Γ, which is the $2n$-dimensional embedded submanifold of $\mathbb{T}^{2n}\times\mathbb{T}^{2n}$, defined as

$$\Gamma_\kappa=\Big\{(\rho';\rho)\ :\ \rho=(q,p)\in\mathbb{T}^{2n},\ \rho'=\kappa(\rho)\Big\}.$$

Using the identification $\mathbb{I}^n = \mathbb{R}^n/\mathbb{Z}^n$, we set up the reflection map $\mathbb{I}^n \ni p \mapsto -p \in \mathbb{I}^n$, and define the *twisted graph* [24, Section 25.2]

$$\Gamma'_\kappa = \{(q', q; p', -p) \ : \ (q', p'; q, p) \in \Gamma_\kappa\} \subset \mathbb{T}^{4n}. \tag{3.1}$$

Then the diffeomorphism κ is symplectic iff Γ'_κ is a Lagrangian submanifold of $\mathbb{T}^{4n}$ (equipped with the symplectic form $\sum_{j=1}^n dq'_j \wedge dp'_j + dq_j \wedge dp_j$). For this reason, we will sometimes denote Γ'_κ by Λ_κ.

The definition of the twisted graph is clearly dependent on the choice of the splitting of variables (q, p), which will be related to a choice of *polarization* in the quantization process.

More generally, one can consider invertible maps on $\mathbb{T}^{2n}$ which are smooth and symplectic except on a negligible set of singularities (say, discontinuities on a hypersurface). The twisted graph of such a map is then a singular Lagrangian submanifold of $\mathbb{T}^{4n}$.

Example. The usual "baker's map" is the following piecewise-linear transformation κ on $\mathbb{T}^2$:

$$\kappa(q, p) \stackrel{\text{def}}{=} \begin{cases} (2q, p/2) & \text{if } 0 \leq q < 1/2 \\ (2q - 1, p/2 + 1/2) & \text{if } 1/2 \leq q < 1. \end{cases} \tag{3.2}$$

The twisted graph of κ:

$$\Lambda_\kappa \stackrel{\text{def}}{=} \left\{(q', q; p', -p) \ : \ (q, p) \in \mathbb{T}^2, \ (q', p') = \kappa(q, p)\right\}$$

is a singular Lagrangian submanifold of $\mathbb{T}^4$. It can be decomposed into $\Lambda_\kappa = \Lambda_0 \cup \Lambda_1$, with the components

$$\begin{aligned}\Lambda_j &= \left\{\left(2q - j, q; \frac{p + j}{2}, -p\right) \ : \ j/2 \leq q < j/2 + 1/2, \ p \in \mathbb{I}\right\} \\ &= \left\{(2q - j, q; p', -2p' + j) \ : \ j/2 \leq q, p' < j/2 + 1/2\right\}.\end{aligned}$$

Each Λ_j is locally Lagrangian in $\mathbb{T}^4$ and, as a manifold with corners, it is diffeomorphic to a 2-dimensional square.

3.2.2. Multivalued symplectic maps. A canonical (or symplectic) *relation* is an arbitrary subset $\Gamma \subset \mathbb{T}^{2n} \times \mathbb{T}^{2n}$, such that

$$\Gamma' = \left\{(q', q; p', -p) \ : \ (q', p'; q, p) \in \Gamma\right\}$$

is a Lagrangian submanifold of $\mathbb{T}^{4n}$.

We are interested in symplectic relations coming from multivalued symplectic maps. A multivalued map is the union of finitely many components κ_j, where κ_j is a canonical diffeomorphism κ_j between an open subset $\mathcal{S}_j$ with piecewise smooth boundary of $\mathbb{T}^{2n}$ and its image $\mathcal{S}'_j = \kappa_j(\mathcal{S}_j) \in \mathbb{T}^{2n}$. A priori, the sets $\mathcal{S}_j$ (respectively $\mathcal{S}'_j$) can overlap, and their union can be a proper subset of $\mathbb{T}^{2n}$.

Each map κ_j is associated to its graph

$$\Gamma_j = \left\{(\kappa_j(\rho); \rho) \ : \ \rho \in \mathcal{S}_j\right\},$$

and the symplectic relation can now be defined through its graph

$$\Gamma = \bigcup_j \Gamma_j ,$$

or equivalently its twisted graph Γ' (defined from Γ as in (3.1)). Γ' is a singular Lagrangian in $\mathbb{T}^{4n}$.

The inverse relation can be defined by

$$\Gamma^{-1} \stackrel{\text{def}}{=} \left\{(\rho; \rho') \ : \ (\rho'; \rho) \in \Gamma\right\} = \bigcup_j \left\{(\kappa_j^{-1}(\rho); \rho) \ : \ \rho \in \mathcal{S}'_j\right\},$$

and the composition of two relations by

$$\widetilde{\Gamma} \circ \Gamma \stackrel{\text{def}}{=} \left\{(\rho''; \rho) \in \mathbb{T}^{4n} \ : \ \exists\, \rho' \in \mathbb{T}^{2n},\ (\rho'; \rho) \in \Gamma \ \text{ and } \ (\rho''; \rho') \in \widetilde{\Gamma}\right\}.$$

Following [24, Theorem 21.2.4], we note that $\widetilde{\Gamma} \circ \Gamma$ will be a (locally) smooth symplectic relation if $\widetilde{\Gamma} \times \Gamma \subset \mathbb{T}^{4n} \times \mathbb{T}^{4n}$ intersects

$$\{(\rho'', \rho', \rho', \rho) \ : \ \rho'', \rho', \rho \in \mathbb{T}^{2n}\} \subset \mathbb{T}^{4n} \times \mathbb{T}^{4n}$$

cleanly, that is the intersections of tangent spaces are the tangent spaces of intersections.

We can then iterate a relation Γ, defining a multivalued dynamical system $\{\Gamma^n : n \in \mathbb{Z}\}$ on $\mathbb{T}^{2n}$. In §3.4 we will give a stochastic interpretation to this system.

3.3. Open baker's relation. The dynamics we will consider takes place on the 2-torus phase space,

$$\mathbb{T}^2 = \{\rho = (q, p) \ : \ q,\ p \in \mathbb{I}\}.$$

On this phase space, we define two vertical strips $\mathcal{S}_j$ $(j = 1, 2)$ from the data of four real numbers $D_1,\ D_2 > 1$ and $\ell_1,\ \ell_2 \geq 0$:

$$\mathcal{S}_j = \left\{(q, p) \ : \ q \in I_j,\ p \in \mathbb{I}\right\}, \qquad \text{with} \quad I_j = \left(\frac{\ell_j}{D_j}, \frac{\ell_j + 1}{D_j}\right) \qquad j = 1, 2. \quad (3.3)$$

The strips are assumed to be disjoint, which is the case if we impose the conditions:

$$\frac{\ell_1 + 1}{D_1} \leq \frac{\ell_2}{D_2} \quad \text{and} \quad \frac{\ell_2 + 1}{D_2} \leq 1.$$

The corresponding baker's relation is made of two components B_j, $j = 1, 2$ associated with linear symplectic maps defined on the two strips:

$$B_j = \left\{(\rho'; \rho) \ : \ (q', p') = \left(D_j q - \ell_j, \frac{p + \ell_j}{D_j}\right),\ \rho = (q, p) \in \mathcal{S}_j\right\}. \qquad (3.4)$$

The baker's relation is defined as the graph $B = B_1 \cup B_2$. One clearly notices that each component map is a hyperbolic diffeomorphism, with positive stretching exponent $\log D_1$ (resp. $\log D_2$). At all points where the map is defined, the unstable (stable) direction is the horizontal (vertical) one.

Since the two strips are disjoint, each point $\rho \in \mathbb{T}^2$ has at most one image. In the notations of Proposition 3.1 (taking $q'' = q$, $q' = q'$), each Lagrangian component B'_j can be generated by the function

$$S_j(q, p') = D_j \left(q - \frac{\ell_j}{D_j}\right)\left(p' - \frac{\ell_j}{D_j}\right) \quad \text{defined on the square } \left\{(q, p') \in I_j \times I_j\right\}. \tag{3.5}$$

Let

$$\pi_L, \pi_R \ : \ \mathbb{T}^2 \times \mathbb{T}^2 \longrightarrow \mathbb{T}^2$$

be the projections on the left and right factors respectively. From the definition (3.4), the set $\pi_R(B) = \mathcal{S}_1 \cup \mathcal{S}_2$ is made of points on $\rho \in \mathbb{T}^2$ which have an image through the relation B. Hence, a point $\rho \notin \pi_R(B)$ is said to escape from the torus at time 1. Similarly, a point $\rho \notin \pi_L(B) = \pi_R(B^{-1})$ is said to escape from $\mathbb{T}^2$ at time -1. This "escape" is the reason why we call this relation an "open" relation: the system is not "closed" because it sends particles "to infinity", both in the future and in the past.

We define

$$\Gamma_\pm \stackrel{\text{def}}{=} \bigcap_{n=1}^{\infty} \pi_R\left(B^{\mp n}\right) \tag{3.6}$$

the set of points which never escape from $\mathbb{T}^2$ in the past, respectively in the future. One checks that these subsets have the form

$$\Gamma_- = C \times \mathbb{I}, \qquad \Gamma_+ = \mathbb{I} \times C,$$

where $C \subset \mathbb{I}$ is a "cookie-cutter set" in the sense of [16]: if we consider the two contracting maps on $\mathbb{I}$,

$$f_j(q) = \frac{q + \ell_j}{D_j}, \quad q \in \mathbb{I}, \quad j = 1, 2,$$

this closed set is defined as

$$C = \overline{\bigcup_{n \in \mathbb{N}} \left\{q \in \mathbb{I} \ : \ f_{j_1} \circ \cdots \circ f_{j_n}(q) = q \quad \text{for some sequence } j_m \in \{1, 2\}\right\}}.$$

The Hausdorff dimension of C (which is equal to its Minkowski and box-counting dimensions) is given by the unique $0 < \nu < 1$ solving

$$D_1^{-\nu} + D_2^{-\nu} = 1. \tag{3.7}$$

The trapped set (or set of nonwandering points) is defined as the set of points which never escape from $\mathbb{T}$:

$$K = \Gamma_+ \cap \Gamma_- = C \times C, \quad \dim K = 2\nu.$$

The baker's relation is a hyperbolic invertible map on the set K, which is a "fractal repeller". This relation is a model of Smale's horseshoe mechanism.

The simplest case consists in considering a symmetric baker's relation, with $D_1 = D_2 = D$, $\ell = \ell_1 = D - \ell_2 - 1$:

$$\begin{aligned} \frac{\ell}{D} < q < \frac{\ell+1}{D} \quad &\Longrightarrow \quad (q', p') = \left(Dq - \ell, \frac{p+\ell}{D} \right), \\ \frac{\ell}{D} < 1 - q < \frac{\ell+1}{D} \quad &\Longrightarrow \quad (q', p') = \left(D(q-1) + \ell + 1, \frac{p - \ell - 1}{D} + 1 \right). \end{aligned} \tag{3.8}$$

Now $C \subset \mathbb{I}$ is a symmetric $1/D-$Cantor set. Notice that if we take $D = 2$, $\ell_1 = 0$, $\ell_2 = 1$, we obtain the usual (closed) baker's map described in the example of Sect. 3.1, for which the trapped set ($= \mathbb{T}^2$) has dimension 2. The "3-baker" relation described in (1.3) corresponds to $D = 3$, $\ell = 0$.

For such a symmetric baker's relation, the analog of the fractal exponent of (2.6) is:

$$\mu_E \quad \longleftrightarrow \quad \nu = \frac{\log 2}{\log D}.$$

3.4. Weighted symplectic relations. To give a multivalued map Γ a physical meaning, we assign *Markovian weights* $P_j(\rho)$ to the different "jumps", $\rho \mapsto \kappa_j(\rho)$. The associated dynamical system is then stochastic, each point ρ having finitely many images with well-prescribed transition probabilities $P_j(\rho)$. The sum of all the probabilities from ρ must satisfy $0 \leq P(\rho) \stackrel{\text{def}}{=} \sum_j P_j(\rho) \leq 1$, so that $(1 - P(\rho))$ is the probability that ρ "escapes to infinity".

The weights associated with the inverse relation Γ^{-1} are the same: each point ρ' jumps back to $\kappa_j^{-1}(\rho')$ with probability $P'_j(\rho') = P_j(\kappa_j^{-1}(\rho'))$. Hence, the weights must also satisfy $0 \leq \sum_j P'_j(\rho') \leq 1$.

Such a weighted relation (in geometric optics one would speak of a "ray-splitting" map) induces a discrete-time evolution of "mass distributions", which is in general *dissipative*: the full mass decrease at each step, the system expelling part of the mass "to infinity".

In more mathematical terms, we assume that the symplectic relation $\Gamma \subset \mathbb{T}^{2n} \times \mathbb{T}^{2n}$ comes with a nonnegative measure (or weight) μ on Γ, which for any $\chi_\alpha \in \mathcal{C}^\infty(\mathbb{T}^{2n}, [0, 1])$, $\alpha = L, R$, satisfies

$$\pi_{\alpha *}(\pi_L^* \chi_L \, \pi_R^* \chi_R \, \mu) = g_\alpha^{\chi_L \chi_R} \frac{\omega^n}{n!}, \quad g_\alpha^{\chi_L \chi_R} \in \mathcal{C}^\infty(\mathbb{T}^{2n}), \quad 0 \leq g_\alpha^{\chi_L \chi_R} \leq 1, \tag{3.9}$$

where $\pi_L, \ \pi_R : \Gamma \to \mathbb{T}^{2n}$ are projections on left and right factors respectively, and ω is the symplectic form on $\mathbb{T}^{2n}$. The condition (3.9) implies that $\pi_\alpha|_\Gamma$ is a local bijection, which forces Γ to be a piecewise smooth union of graphs of symplectic transformations, as defined in §3.2.2. When Γ is singular, that is a union of smooth symplectic relations with boundaries, we demand that

$$g_\alpha^{\chi_L \chi_R} \in \mathcal{C}^\infty(\mathbb{T}^{2n}) \quad \text{if} \quad \operatorname{supp}(\pi_L^* \chi_L \, \pi_R^* \chi_R) \cap \partial\Gamma = \emptyset,$$

where $\partial\Gamma$ is the union of the boundaries of the smooth components.

The reason for introducing the measure μ is to have a quantity independent of the choice of coordinates on Γ. On $\mathbb{T}^{2n}$, an obvious intrinsic measure is given by the symplectic form, hence $g_\alpha^{\chi_L \chi_R}$ are well defined. Building an atlas of the manifold Γ we can use these functions to describe μ in local coordinates.

We denote a weighted relation by (Γ, μ). As explained above, one can invert such a relation, as well as compose them.

If $(\rho'; \rho) \in \Gamma \setminus \partial\Gamma$, the probability of a transition from ρ to $\rho' = \kappa_{j_1}(\rho)$ is obtained by letting χ_R (resp. χ_L) be supported in a sufficiently small neighbourhood of ρ (resp. of ρ'), with $\chi_R(\rho) = 1$, $\chi_L(\rho') = 1$. This probability is then given by

$$P_{j_1}(\rho) = g_R^{\chi_L\,\chi_R}(\rho) = g_L^{\chi_L\,\chi_R}(\rho') = P'_{j_1}(\rho'). \tag{3.10}$$

Examples. The simplest example is given by a graph of a symplectic transformation $\kappa : \mathbb{T}^{2n} \to \mathbb{T}^{2n}$ in which case the density μ is obtained by taking $\mu = \pi_L^*(\omega^n/n!) = \pi_R^*(\omega^n/n!)$, where the equality follows from $\kappa^*\omega = \omega$. A slightly more complicated example is given by taking a union of two non-intersecting graphs Γ_j of κ_j, $j = 1, 2$, and putting

$$\mu = (\pi_R|_{\Gamma_1})^*(g_1\,\omega^n/n!) + (\pi_R|_{\Gamma_2})^*(g_2\,\omega^n/n!),$$

where $g_j \in \mathcal{C}^\infty(\mathbb{T}^{2n}; [0, 1])$ satisfy $g_1 + g_2 \le 1$ and $g_1 \circ \kappa_1^{-1} + g_2 \circ \kappa_2^{-1} \le 1$. In this case, $g_j(\rho) = P_j(\rho)$.

In the case of an open baker B defined in §3.3, for instance the symmetric 3-baker (1.3), a natural μ comes from pulling back the Liouville measure ω to each component B_j given in (3.4). One obtains

$$\pi_{R*}\,\mu = 1\!\!1_{I_1\cup I_2}(q)\,dq\,dp, \quad \pi_{L*}\,\mu = 1\!\!1_{I_1\cup I_2}(p')\,dq'\,dp'. \tag{3.11}$$

These equations fully determine the measure μ on B.

A more interesting example, which will be relevant in §5, is given by the following multivalued generalization of the symmetric 3-baker:

$$\begin{gathered}\widetilde{B} = \bigcup_{\ell=0}^{2} \big(B + (0, \ell/3; 0, 0)\big) = \bigcup_{k=1}^{2}\bigcup_{j=0}^{2} \widetilde{B}_{kj}, \quad \text{where} \\ \widetilde{B}_{kj} = \Big\{\big(3q, \tfrac{p+j}{3}; q, p\big) \;:\; q \in I_k,\ p \in \mathbb{I}\Big\}, \quad I_1 = (0, 1/3),\ I_2 = (2/3, 1).\end{gathered} \tag{3.12}$$

Each point $\rho \in \mathcal{S}_1 \cup \mathcal{S}_2 = (I_1 \cup I_2) \times \mathbb{I}$ has 3 images, and each point $\rho' \in \mathbb{T}^2$ has two preimages.

The following measure on $\widetilde{B}$ will arise in the quantum model studied in §5. We define it explicitly on each component $\widetilde{B}_{kj}$, using the right projection on $\mathcal{S}_k$:

$$\begin{aligned}\pi_{R*}\,\tilde{\mu}|_{\widetilde{B}_{1j}} &= \frac{\sin^2(\pi p)}{9\sin^2(\pi(p+j)/3)}\,1\!\!1_{I_1}(q)\,dq\,dp, \\ \pi_{R*}\,\tilde{\mu}|_{\widetilde{B}_{2j}} &= \frac{\sin^2(\pi p)}{9\sin^2(\pi(p+j-2)/3)}\,1\!\!1_{I_2}(q)\,dq\,dp, \quad j = 0, 1, 2.\end{aligned} \tag{3.13}$$

The functions on the right-hand sides are the probabilities $P_j(\rho)$. The sum of these components reads

$$\pi_{R*}\,\tilde{\mu} = \left(\frac{1}{9}\sum_{j=0}^{2}\frac{\sin^2\pi p}{\sin^2\pi(p/3 + j/3)}\right)\,1\!\!1_{I_1\cup I_2}(q)\,dq\,dp = 1\!\!1_{I_1\cup I_2}(q)\,dq\,dp.$$

Here we used the fact[4] that $\sum_{j=0}^{D-1}\sin^2(Dx)/\sin^2(x+j\pi/D) = D^2$, with $D=3$ and $x=\pi p/3$. This right pushforward is identical to that of (3.11): in both cases, any point $\rho\in(\mathcal{S}_1\cup\mathcal{S}_2)$ has an empty escape probability, $1-P(\rho)=0$.

On the opposite, the left pushforward of $\tilde\mu$ is given by

$$\pi_{L*}\tilde\mu = \frac{\sin^2 3\pi p'}{9}\left(\frac{1}{\sin^2\pi p'} + \frac{1}{\sin^2\pi(p'-2/3)}\right) dq'\,dp'.$$

Almost any point $\rho'\in\mathbb{T}^2$ has a nonzero escape probability through $\widetilde{B}^{-1}$. This left pushforward is obviously different from that of μ.

4. Quantized Maps and Relations

Before giving the definition of the quantized baker's relation, we need to define the quantum Hilbert space corresponding to $\mathbb{T}^2$, as well as the algebra of quantum observables.

4.1. Quantized tori. The quantization of tori $\mathbb{T}^{2n}=\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ has a long tradition in mathematical physics [21, 13, 5]. It can be considered as a special case of the Berezin-Toeplitz quantization of compact symplectic Kähler manifolds — see [27] and references given there. Here we will give a self-contained presentation of the simplest case from the point of view of pseudodifferential operators.

We first recall from [14] the quantization of functions $f\in\mathcal{C}^\infty_{\rm b}(T^*\mathbb{R}^n)$,

$$\mathcal{C}^\infty_{\rm b}(T^*\mathbb{R}^n) \stackrel{\rm def}{=} \{f\in\mathcal{C}^\infty(T^*\mathbb{R}^n)\ :\ \forall\alpha,\beta\in\mathbb{N}^n,\quad \sup_{(q,p)\in T^*\mathbb{R}^n}|\partial_q^\alpha\partial_p^\beta f(q,p)|<\infty\}.$$

To any $f\in\mathcal{S}(T^*\mathbb{R}^n)$ we associate its h-Weyl quantization, that is the operator $f^w(q,hD)$ acting as follows on $\psi\in\mathcal{S}(\mathbb{R}^n)$:

$$[f^w(q,hD)\,\psi](q) \stackrel{\rm def}{=} \frac{1}{(2\pi h)^n}\int\!\!\int f\Big(\frac{q+r}{2},p\Big)\,e^{\frac{i}{h}\langle q-r,p\rangle}\,\psi(r)\,dr\,dp. \tag{4.1}$$

This operator clearly has the mapping properties

$$f^w(q,hD)\ :\ \mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}(\mathbb{R}^n),\quad f^w(q,hD)\ :\ \mathcal{S}'(\mathbb{R}^n)\longrightarrow\mathcal{S}'(\mathbb{R}^n).$$

It can be shown [14, Lemma 7.8] that $f\mapsto f^w(q,hD)$ can be extended to any $f\in\mathcal{C}^\infty_{\rm b}(T^*\mathbb{R}^n)$, and that the resulting operator has the same mapping properties. Furthermore, $f^w(q,hD)$ is a bounded operator on $L^2(\mathbb{R}^n)$.

We now introduce quantum spaces associated with the torus $\mathbb{T}^{2n}$. For that aim, we fix our notations for the semiclassical Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$:

$$\mathcal{F}_h\psi(p) \stackrel{\rm def}{=} \frac{1}{(2\pi h)^{n/2}}\int\psi(q)\,e^{-\frac{i}{h}\langle q,p\rangle}\,dq,$$

[4] The value of the sum at $x=0$ is equal to D^2, and the sum is invariant under translation $x\mapsto x+k\pi/D$. Fejér's formula for the Cesàro mean of the Fourier series shows that the sum is a trigonometric polynomial of degree $D-1$ in x, hence it is constant.

and as usual in quantum mechanics, $\mathcal{F}_h\psi(p)$ is the "momentum representation" of the state ψ. The torus quantum space is made of distributions $\psi \in \mathcal{S}'(\mathbb{R}^n)$ which are both periodic in position and momentum:

$$\psi(q+\ell) = \psi(q), \qquad \mathcal{F}_h\psi(p+\ell) = \mathcal{F}_h\psi(p). \tag{4.2}$$

Let us denote by $\mathcal{H}_h^n$ this space of distributions. We have the following elementary

Lemma 4.1. $\mathcal{H}_h^n \neq \{0\}$ *if and only if* $h = (2\pi N)^{-1}$ *for some positive integer* N*, in which case* $\dim \mathcal{H}_h^n = N^n$ *and* $\mathcal{H}_h^n$ *is generated by the following basis:*

$$\mathcal{H}_h^n = \operatorname{span}\left\{ \frac{1}{\sqrt{N^n}} \sum_{\ell\in\mathbb{Z}^n} \delta(q-\ell-j/N) \ : \ j \in (\mathbb{Z}/N\mathbb{Z})^n \right\}. \tag{4.3}$$

The distributions elements of this basis will be denoted by

$$|Q_j\rangle, \quad Q_j = \tfrac{j}{N} \in \mathbb{I}^n \text{ is the position on which that state is microlocalized.} \tag{4.4}$$

One can check that for such a value of h, the Fourier transform $\mathcal{F}_h$ maps $\mathcal{H}_h^n$ to itself. In the above basis, it is represented by the discrete Fourier transform

$$(\mathcal{F}_N)_{j\,j'} = \frac{e^{-2i\pi\langle j,j'\rangle/N}}{N^{n/2}}, \quad j, j' \in (\mathbb{Z}/N\mathbb{Z})^n. \tag{4.5}$$

It is also easy to check the following

Lemma 4.2. *Suppose that* $f \in \mathcal{C}_\mathrm{b}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ *satisfies* $f(q+\ell, p+m) = f(q,p)$ *for any* $\ell, m \in \mathbb{Z}^n$*. Then the operator* $f^w(q, hD)$ *maps* $\mathcal{H}_h^n$ *to itself.*

Identifying a function $f \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ with a periodic function on $\mathbb{R}^{2n}$, we will write $\mathrm{Op}_h(f)$ for the restriction of $f^w(q, hD)$ on $\mathcal{H}_h^n$,

$$\mathcal{C}^\infty(\mathbb{T}^{2n}) \ni f \ \longmapsto\ \mathrm{Op}_h(f) \in \mathcal{L}(\mathcal{H}_h^n).$$

We remark that $\mathrm{Op}_h(1) = \mathrm{Id}$. The vector space $\mathcal{H}_h^n$ can be equipped with a natural Hilbert structure.

Lemma 4.3. *There exists a unique (up to a multiplicative constant) Hilbert structure on* $\mathcal{H}_h^n$ *for which all* $Op_h(f) : \mathcal{H}_h^n \to \mathcal{H}_h^n$ *with* $f \in \mathcal{C}^\infty(\mathbb{T}^{2n}; \mathbb{R})$ *are self-adjoint.*

One can choose the constant such that the basis in (4.3) *is orthonormal. This implies that the Fourier transform on* $\mathcal{H}_h^n$ *(represented by the unitary matrix* (4.5)*) is unitary.*

Proof. Let $\langle \bullet, \bullet\rangle_0$ be the inner product for which the basis in (4.3) is orthonormal. We write the operator $f^w(q, hD)$ on $\mathcal{H}_h^n$ explicitly in that basis using the Fourier expansion of its symbol:

$$f(q,p) = \sum_{\ell,m\in\mathbb{Z}^n} \hat{f}(\ell, m)\, e^{2\pi i(\langle \ell,q\rangle + \langle m,p\rangle)}.$$

For that let $L_{\ell,m}(q,p) = \langle \ell, q\rangle + \langle m, p\rangle$, so that

$$f^w(q, hD) = \sum_{\ell,m\in\mathbb{Z}^n} \hat{f}(\ell,m)\ \exp(2\pi i L^w_{\ell,m}(q, hD)).$$

Applying this operator to the distributions (4.4), we get

$$\exp\left(2\pi i L^w_{\ell,m}(q,hD)\right)|Q_j\rangle = \exp\left(\frac{\pi i}{N}(2\langle j,\ell\rangle - \langle m,\ell\rangle)\right)|Q_{j-m}\rangle,$$

and consequently,

$$f^w(q,hD)\,|Q_j\rangle = \sum_{m\in\mathbb{Z}^n/(N\mathbb{Z})^n} F_{mj}\,|Q_m\rangle,$$

$$F_{mj} = \sum_{\ell,r\in\mathbb{Z}^n} \hat{f}(\ell, j-m-rN)(-1)^{\langle r,\ell\rangle}\exp\left(\frac{\pi i}{N}\langle j+m,\ell\rangle\right).$$

Since

$$\begin{aligned}\bar{F}_{jm} &= \sum_{\ell,r\in\mathbb{Z}^n} \hat{\bar{f}}(-\ell, j-m+rN)(-1)^{\langle r,\ell\rangle}\exp\left(-\frac{\pi i}{N}\langle j+m,\ell\rangle\right)\\ &= \sum_{\ell,r\in\mathbb{Z}^n} \hat{\bar{f}}(\ell, j-m-rN)(-1)^{\langle r,\ell\rangle}\exp\left(\frac{\pi i}{N}\langle j+m,\ell\rangle\right),\end{aligned}$$

we see that for real f, $f=\bar{f}$, $F_{jm}=\bar{F}_{mj}$. This means that $f^w(q,hD)$ is self-adjoint for the inner product $\langle\bullet,\bullet\rangle_0$. We also see that the map $f\mapsto (F_{jm})_{j,m\in(\mathbb{Z}/N\mathbb{Z})^n}$ is onto, from $\mathcal{C}^\infty(\mathbb{T}^{2n};\mathbb{R})$ to the space of Hermitian matrices.

Any other metric on $\mathcal{H}^n_h$ could be written as $\langle u,v\rangle = \langle Bu,v\rangle_0 = \langle u,Bv\rangle_0$. If $\langle f^w u,v\rangle = \langle u, f^w v\rangle$ for all f's, then $Bf^w = f^w B$ for all f's, and hence for all Hermitian matrices. That shows that $B = c\,\mathrm{Id}$, as claimed. $\square$

This choice of normalization $\langle\bullet,\bullet\rangle_0$ can be obtained in a natural way, if we use the following periodization operator to construct $\mathcal{H}^n_h$ from $\mathcal{S}(\mathbb{R}^n)$ [5]:

Lemma 4.4. *For any $h=(2\pi N)^{-1}$, the periodization operator $P_{\mathbb{T}^{2n}}:\mathcal{S}(\mathbb{R}^n)\to\mathcal{H}^n_h$ defined below is surjective:*

$$\forall\psi\in\mathcal{S}(\mathbb{R}^n),\qquad [P_{\mathbb{T}^{2n}}\,\psi](Q_j) \stackrel{\text{def}}{=} \frac{1}{N^{n/2}}\sum_{\nu\in\mathbb{Z}^n}\psi(Q_j-\nu),\qquad j\in(\mathbb{Z}/N\mathbb{Z})^n. \tag{4.6}$$

In the rest of this article we will always assume that $h=(2\pi N)^{-1}$ for some $N\in\mathbb{N}$, so the semiclassical limit corresponds to $N\to\infty$. The scalar product on $\mathcal{H}^n_h$ will be $\langle\bullet,\bullet\rangle_0$. From now on we will omit the subscript 0, and also often use Dirac's notation $\langle\bullet|\bullet\rangle$ for this product. For instance, the j^{th} component of a state $\psi\in\mathcal{H}^n_h$ in the basis (4.4) will be denoted by $\psi(Q_j)=\langle Q_j|\psi\rangle$. The Hilbert norm associated with $\langle\bullet,\bullet\rangle$ will simply be written $\|\bullet\|$.

4.2. Lagrangian states. We want to characterize the semiclassical localization in phase space of sequences of states of the form $\psi=\{\psi_h\in\mathcal{H}^n_h\}_{h\to 0}$. In general we will assume that each element of this sequence is normalized, $\|\psi_h\|=1$, but all definitions can be extended to sequences such that the norms satisfy $\|\psi_h\|=\mathcal{O}(h^K)$ as $h\to 0$, for some fixed $K\in\mathbb{R}$ (the sequence ψ is then said to be *tempered*).

The localization of this sequence is first characterized through its *microsupport*, or *wave front set*, which is the following subset of $\mathbb{T}^{2n}$:

$$\mathrm{WF}_h(\psi) = \complement \Big\{ \rho \in \mathbb{T}^{2n} : \exists\, f \in \mathcal{C}^\infty(\mathbb{T}^{2n}), \quad f(\rho) \neq 0, \|\mathrm{Op}_h(f)\psi_h\| = \mathcal{O}(h^\infty) \Big\}, \tag{4.7}$$

where $\complement$ stands for the set theoretical complement. It is not hard to show [44, Prop. IV-8′] that this definition is equivalent to the following: $\rho \notin \mathrm{WF}_h(\psi)$ if and only if there exists a neighbourhood W_ρ of ρ such that, for any $f \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ supported in W_ρ, $\|\mathrm{Op}_h(f)\psi_h\| = \mathcal{O}(h^\infty)$. This yields the following

Lemma 4.5. *For any function $f \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ with $f \equiv 0$ in an open neighbourhood of $\mathrm{WF}_h(\psi)$, we have $\|Op_h(f)\psi_h\| = \mathcal{O}(h^\infty)$. As a consequence, the microsupport of a sequence $\psi = \{\psi_h\}$, $\|\psi_h\| \asymp h^K$, cannot be empty.*

Proof. The (compact) support of f can be covered by finitely many W_{ρ_i}, and using a partition of unity associated with these sets we can decompose it as $f = \sum_i f_i$, with $\mathrm{supp}(f_i) \subset W_{\rho_i}$. We get the result by linearity, and using the second definition of $\mathrm{WF}_h(\psi)$. □

We also make the following observation:

Lemma 4.6. *Let $\psi = \{\psi_h \in \mathcal{H}_h^n\}_{h\to 0}$ be a tempered sequence. Considering ψ_h as a N^n-component vector in the basis* (4.3)*, we define $\bar\psi_h$ as the vector with complex conjugate components. Then*

$$\mathrm{WF}_h(\bar\psi) = \{(q, -p) \ : \ (q, p) \in \mathrm{WF}_h(\psi)\}.$$

Proof. The definition (4.1) of Weyl's quantization gives, for any function $f \in \mathcal{C}^\infty(\mathbb{T}^{2n})$,

$$\mathrm{Op}_h(f)\,\bar\psi = f^w(q, hD)\,\bar\psi = \overline{\bar f^w(q, -hD)\,\psi}.$$

The lemma follows from the definition (4.7) of the wave front set. □

Now let $\Lambda \subset \mathbb{T}^{2n}$ be a union of Lagrangian submanifolds of $\mathbb{T}^{2n}$ with piecewise smooth boundaries.

Definition 4.7. *A sequence of states $\psi = \{\psi_h \in \mathcal{H}_h^n\}$ is a* **Lagrangian state associated to** Λ*, which we denote by $\psi \in I(\Lambda)$, if for any $M \in \mathbb{N}$ and any sequence of functions,*

$$f_j \in \mathcal{C}^\infty(\mathbb{T}^{2n}), \quad 1 \le j \le M, \quad f_j|_\Lambda = 0,$$

we have

$$\|Op_h(f_M) \circ \cdots \circ Op_h(f_1)\,\psi_h\| = \mathcal{O}(h^M)\,\|\psi_h\|. \tag{4.8}$$

From the definition (4.7) of the microsupport, we obtain that, if the sequence ψ is tempered, then

$$\psi \in I(\Lambda) \implies \mathrm{WF}_h(\psi) \subset \Lambda. \tag{4.9}$$

Indeed, suppose that $\rho \notin \Lambda$. Then there exists $f \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ such that $f|_\Lambda = 0$ and $f \equiv 1$ in a neighbourhood of ρ. We can also find $a \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ such that $f = 1$ on a neighbourhood of the support of a, and $a(\rho) \neq 0$. The symbol calculus (see

[14, Chap. 7]) shows that for any M, $\mathrm{Op}_h(a)\mathrm{Op}_h(f)^M = \mathrm{Op}_h(a) + \mathcal{O}_M(h^\infty)$. On the other hand $\|\mathrm{Op}_h(f)^M \psi_h\| = \mathcal{O}(h^M \|\psi_h\|)$, and as M is arbitrary and ψ tempered, $\|\mathrm{Op}_h(a)\,\psi_h\| = \mathcal{O}(h^\infty)$. In view of (4.7), this gives (4.9).

We stress that the opposite implication in (4.9) is not true in general. To see that consider $n = 1$ and the Lagrangian $\Lambda = \{(0, p) \ : \ p \in \mathbb{I}\} \subset \mathbb{T}^2$. Let $\psi_h \in \mathcal{H}_h^1$ be the "torus coherent state at the origin":

$$\psi_h(Q_j) = \Big(\frac{2}{N}\Big)^{1/4} \sum_{r\in\mathbb{Z}} \exp\{-\pi N (Q_j - r)^2\}, \quad j = 0, \dots, N-1.$$

Then one can check that $\|\psi_h\| \xrightarrow{h\to 0} 1$, that $\mathrm{WF}_h(\psi) = \{(0, 0)\} \subset \Lambda$. On the other hand,

$$\|\mathrm{Op}_h\big(\sin(2\pi q)\big)\psi_h\| \sim \pi\sqrt{2h},$$

which shows that $\psi_h \notin I(\Lambda)$.

In the physics literature, Lagrangian states are usually called WKB states, and are introduced as Ansätze for eigenstates of integrable systems, using Bohr-Sommerfeld quantization formulae [28]. For instance, in the case $n = 1$, if Λ is generated by the function $S \in \mathcal{C}^\infty(\mathbb{I})$:

$$\Lambda_S = \big\{(q, -S'(q)),\ q \in \mathbb{I}\big\}, \tag{4.10}$$

then for any function $a(q) \in \mathcal{C}^\infty(\mathbb{I})$, the state $\psi_h \in \mathcal{H}_h^1$ defined as

$$\psi_h(Q_j) = \frac{a(Q_j)}{\sqrt{N}} \exp(-2i\pi N S(Q_j)), \quad j = 0, \dots, N-1, \tag{4.11}$$

is in $I(\Lambda_S)$. In the next proposition, we generalize this construction to any dimension.

Proposition 4.8. *Let $\Lambda \subset \mathbb{T}^{2n}$ be an embedded Lagrangian manifold. Then for any $\rho_0 \in \Lambda$ there exist Lagrangian states $\psi \in I(\Lambda)$, such that $\rho_0 \in \mathrm{WF}_h(\psi)$.*

Proof. We take $\rho_0 = (q_0, p_0) \in \Lambda$, and assume that there exists a neighbourhood V of ρ_0, and a function $S \in \mathcal{C}^\infty(\pi(V))$ (where $\pi(q, p) = q$), such that $\Lambda \cap V = \{(q; -d_q S(q)),\ q \in \pi(V)\}$. This is a particular case of Proposition 3.1. The general case of a generating function $S(q'', p')$ can be transformed to that of $S = S(q)$ using the symplectic rotation $(q', p') \mapsto (-p', q')$. On the quantum mechanical side, this rotation is performed through a partial Fourier transform in the variable q'. Our construction below can be transposed to this general case through this Fourier transform (which acts covariantly on the Weyl quantization).

We also assume that the neighbourhood V is contained in the interior of $\mathbb{I}^{2n}$, and we identify $\pi(V)$ with a subset of $\mathbb{I}^n$. We first construct a Lagrangian state in $L^2(\mathbb{R}^n)$:

$$u_h(q) = a(q)\, e^{-\frac{i}{h}S(q)}, \tag{4.12}$$

with a symbol $a \in \mathcal{C}^\infty(\mathbb{R}^n)$ compactly supported inside $\pi(V)$, and such that $a(q_0) \neq 0$. This state admits the norm $\|u_h\|_{L^2} = \|a\|_{L^2}$. For any $f \in \mathcal{C}^\infty(\mathbb{T}^{2n})$, we apply the operator $f^w(q, hD)$ to that state. Although we could do it directly using (4.1), we prefer to reduce the problem to the case of $S = 0$ by conjugation with the unitary multiplication operator

$$v(q) \longmapsto [e^{\frac{i}{h}S^w(q)}v](q) = e^{\frac{i}{h}S(q)}v(q), \tag{4.13}$$

where we can assume that $S \in C_b^\infty(\mathbb{R}^n)$. We then apply the operator

$$G^w(q, hD) \stackrel{\text{def}}{=} e^{\frac{i}{h}S^w(q)}\, f^w(q, hD)\, e^{-\frac{i}{h}S^w(q)},$$

to the function $a(q)$. The symbol calculus shows that $G(q, p)$ admits an h-expansion, with principal symbol $g(q, p) = f(q, p + d_q S(q))$: if f vanishes on Λ, then g vanishes on $\{(q, 0) \,:\, q \in \pi(V)\}$. We get

$$[f^w(q, hD)\, u_h](q) = e^{-\frac{i}{h}S^w(q)}\, G^w(q, hD)\, a(q) = e^{-\frac{i}{h}S^w(q)}\, g^w(q, hD)\, a(q) + \mathcal{O}(h).$$

The explicit integral

$$[g^w(q, hD)\, a](q) = \frac{1}{(2\pi h)^n} \int\!\!\int g\Big(\frac{q+r}{2}, p\Big)\, a(r)\, e^{\frac{i}{h}\langle q-r, p\rangle}\, dr\, dp$$

can be evaluated through the stationary phase method. The derivative of the phase vanishes at $r = q$, $p = 0$, so the integral admits the following expansion [23, Section 7.7] for $q \in \pi(V)$:

$$[g^w(q, hD)\, a](q) = L_0(g\, a)(q) + h L_1(g\, a)(q) + \mathcal{O}(h^2). \tag{4.14}$$

Here each function $L_j(g\, a)$ is obtained by applying a certain differential operator (in (r, p)) on the function $g((q+r)/2, p)\, a(r)$, taking the output at the point $(r = q,\ p = 0)$. The first term is simply $L_0(g\, a)(q) = g(q, 0)\, a(q)$. For q outside $\pi(V)$, the nonstationary phase estimates show that

$$f^w(q, hD)\, u_h(q) = \mathcal{O}\left(\Big(\frac{h}{h + \mathrm{dist}(q, \pi(V))}\Big)^\infty\right). \tag{4.15}$$

If $f(\rho_0) \neq 0$, then $L_0(g\, a)$ is nonzero in a neighbourhood W of q_0, and we obtain

$$\|f^w(q, hD)\, u_h\|_{L^2(\mathbb{R}^n)} = \|g^w(q, hD)\, a\|_{L^2(\mathbb{R}^n)} + \mathcal{O}(h) \geq \|L_0(g\, a)\|_{L^2(W)} + \mathcal{O}(h). \tag{4.16}$$

The left-hand side is thus bounded from below by a positive constant.

On the opposite, if f vanishes on Λ, then at each point $q \in \pi(V)$ we get $L_0(g\, a)(q) = 0$, which implies that $\|f^w(q, hD)\, u_h\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h)$. The same procedure can be iterated to show that, for any family of functions $f_i \in C^\infty(\mathbb{T}^{2n})$ vanishing on Λ, the function

$$u_h^{(M)}(q) \stackrel{\text{def}}{=} h^{-M}\, [f_M^w \circ \cdots \circ f_1^w\, u_h](q), \tag{4.17}$$

is uniformly bounded and smooth on $\mathbb{R}^n$, and very small outside $\pi(V)$, as in (4.15). As a result,

$$\|f_M^w(q, hD) \circ \cdots \circ f_1^w(q, hD) u_h\|_{L^2(\mathbb{R}^n)} = h^M\, \|u_h^{(M)}\| = \mathcal{O}(h^M). \tag{4.18}$$

We can now carry over the estimates (4.16,4.18) onto the state $\psi_h = P_{\mathbb{T}^{2n}}\, u_h \in \mathcal{H}_h^n$, where $P_{\mathbb{T}^{2n}}$ is the periodizing operator (4.6). Since $a(q)$ was supported inside $\pi(V) \subset \mathbb{I}^n$, this state admits the following representation, which generalizes (4.10):

$$\psi_h(Q_j) = \frac{u_h(Q_j)}{N^{n/2}} = \frac{a(Q_j)}{N^{n/2}}\, \exp(-2i\pi N S(Q_j)), \qquad j \in (\mathbb{Z}/N\mathbb{Z})^n. \tag{4.19}$$

The norm of this state is therefore the sum

$$\|\psi_h\|^2 = N^{-n} \sum_{j\in(\mathbb{Z}/N\mathbb{Z})^n} |a(Q_j)|^2 = \int dq\, |a(q)|^2 + \mathcal{O}(h^\infty),$$

where we used the smoothness of $a(q)$. Similarly, the projection on $\mathcal{H}_h^n$ of the function $u_h^{(M)}$ defined in (4.17) satisfies

$$P_{\mathbb{T}^{2n}}\, u_h^{(M)}(Q_j) = \frac{u_h^{(M)}(Q_j)}{N^{n/2}} + \mathcal{O}(h^\infty), \quad j \in (\mathbb{Z}/N\mathbb{Z})^n. \tag{4.20}$$

From the smoothness of $u_h^{(M)}$, we obtain the "projected version" of (4.18):

$$\begin{aligned}&\|\mathrm{Op}_h(f_M)\circ\cdots\circ\mathrm{Op}_h(f_1)\psi_h\|\\ &\quad = h^M\,\|P_{\mathbb{T}^{2n}}\,u_h^{(M)}\| = h^M\,\|u_h^{(M)}\|_{L^2(\mathbb{R}^n)} + \mathcal{O}(h^\infty) = \mathcal{O}(h^M).\end{aligned}$$

On the other hand, if $f(\rho_0)\neq 0$, one easily deduces from (4.16) that

$$\|\mathrm{Op}_h(f)\psi_h\| = \|f^w(q,hD)\,u_h\|_{L^2(\mathbb{R}^n)} + \mathcal{O}(h^\infty) \geq C + \mathcal{O}(h), \quad C>0.$$

These estimates show that the family $\psi\in I(\Lambda)$, and that $\rho_0\in \mathrm{WF}_h(\psi)$. □

Remark 4.1. The definition of $I(\Lambda)$ mimicks the Hörmander-Melrose definition of Lagrangian distributions [24, Def. 25.1.1] (see [1] for an adaptation to the standard semiclassical setting). The requirement that Λ is Lagrangian reflects the uncertainty principle, in the following sense. A Lagrangian submanifold is the lowest dimensional submanifold for which the conclusion of Proposition 4.8 holds, that is, for any $\rho\in\Lambda$, there exists a state ψ satisfying $\psi\in I(\Lambda)$ and $\rho\in\mathrm{WF}_h(\psi)$.

Indeed, let Λ be an embedded submanifold of $\mathbb{T}^{2n}$. Let us assume that $\psi\in I(\Lambda)$, so (4.8) must hold for any family of functions $f_j|_\Lambda = 0$. From the identity

$$\begin{aligned}&\frac{i}{h}[\mathrm{Op}_h(f_i),\mathrm{Op}_h(f_j)]\\ &\quad = \mathrm{Op}_h(\{f_i,f_j\}) + \mathcal{O}(h),\end{aligned}$$

we see that $\|\mathrm{Op}_h(\{f_i,f_j\})\,\psi_h\| = \mathcal{O}(h)$. As in the proof of (4.9), we can show that if $\{f_i,f_j\}(\rho)\neq 0$ for some $\rho\in\Lambda$, then $\rho\notin\mathrm{WF}_h(\psi)$. Hence, if we want the conclusion of Proposition 4.8 to hold for Λ, then this submanifold must satisfy

$$\forall f_i,\ f_j\in C^\infty(\mathbb{T}^{2n}), \qquad f_i|_\Lambda,\ f_j|_\Lambda = 0 \implies \{f_i,f_j\}|_\Lambda = 0.$$

This property means that Λ is *co-isotropic*, and must be of dimension $\geq n$. Lagrangian manifolds are co-isotropic manifolds of minimal dimension.

4.2.1. Singular Lagrangian states. We now give an example where Λ is a union of Lagrangians with piecewise smooth boundaries (in §3.2 we called such Λ a singular Lagrangian). Let Λ_S be given by (4.10) and ψ_h by (4.11). Let us truncate ψ_h to some proper subinterval $[Q, Q'] \subset \mathbb{I}$, that is, replace the symbol $a(q)$ by the discontinuous function $\tilde{a}(q) = a(q)\mathbb{1}_{[Q,Q']}(q)$. That gives a state $\tilde{\psi}_h \in \mathcal{H}_h^1$. One could expect $\tilde{\psi}_h$ to be a Lagrangian state in $I(\Lambda_S)$ (as is ψ_h), or rather in $I(\tilde{\Lambda}_S)$, where

$$\tilde{\Lambda}_S \stackrel{\text{def}}{=} \Lambda_S \cap ([Q, Q'] \times \mathbb{I}).$$

This is not the case: one needs to include in the Lagrangian the *singularity set*

$$\Lambda_{\text{sing}} = \{(Q, p) \ : \ p \in \mathbb{I}\} \cup \{(Q', p) \ : \ p \in \mathbb{I}\},$$

which is the "periodized" conormal bundle of the boundary $\partial\tilde{\Lambda}_S$. We will indeed prove that $\tilde{\psi}_h \in I(\tilde{\Lambda}_S \cup \Lambda_{\text{sing}})$, which can be considered as a semiclassical, discrete analogue of singular Lagrangian distributions of Guillemin-Uhlmann [19] and Melrose-Uhlmann [34]. We have the following

Lemma 4.9. *Let us truncate the state* (4.19) *to a hypercube* $H \subset \mathbb{I}^n$, $H = \prod_{\ell=1}^n [\alpha_\ell, \beta_\ell]$:

$$\tilde{\psi}_h(Q_j) = \frac{a(Q_j)\,\mathbb{1}_H(Q_j)}{N^{n/2}} \exp(-2i\pi N S(Q_j)), \quad j \in (\mathbb{Z}/N\mathbb{Z})^n. \tag{4.21}$$

Then $\tilde{\psi}_h$ *is associated with the singular Lagrangian* $\tilde{\Lambda}_S \cup \Lambda_{\text{sing}}$, *where* $\tilde{\Lambda}_S = \{(q, -d_q S(q)),\ q \in H\}$ *and*

$$\begin{aligned}\Lambda_{\text{sing}} = \bigcup_{\ell=1}^n \Big(&\big\{(q,p) : q_\ell = \alpha_\ell,\ p_\ell \in \mathbb{I}, q_m \in [\alpha_m, \beta_m]\,,\, p_m = -d_{q_m} S(q), m \neq \ell\big\}\\ &\cup \big\{(q,p) : q_\ell = \beta_\ell,\ p_\ell \in \mathbb{I},\ q_m \in [\alpha_m, \beta_m]\,,\ p_m = -d_{q_m} S(q), m \neq \ell\big\}\Big)\,.\end{aligned} \tag{4.22}$$

Remark. It would be tempting to generalize the lemma by replacing the hypercube H by an arbitrary set $\mathcal{S}$ with smooth boundaries. However, if $n = 2$, $S \equiv 0$, $a \equiv 1$, and $\partial\mathcal{S}$ does not contain a segment with rational slopes then

$$WF_h(\tilde{\psi}_h) = (\mathcal{S} \times \{0\}) \cup (\partial\mathcal{S} \times \mathbb{I}^2).$$

The second component being 3-dimensional, this set is certainly not contained in a finite union of Lagrangians.

Proof. As in the proof of Proposition 4.8, we can, by conjugation with the operator (4.13), reduce the proof to the case $S = 0$. We first consider states defined on $\mathbb{R}^n$, localized on the hypercube $H \subset \mathbb{R}^n$:

$$u_h(q) = \mathbb{1}_H(q)\,a(q), \quad a \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{4.23}$$

We use the following

Lemma 4.10. *Let* $\tilde{\Lambda}_0 = H \times \{0\}$ *and* Λ_{sing} *be as in Lemma 4.9. The ideal* $\mathcal{J}$ *of periodic functions vanishing on the singular Lagrangian* $\tilde{\Lambda}_0 \cup \Lambda_{\text{sing}}$ *is (infinitely) generated by*

$$g_j(p,q) \stackrel{\text{def}}{=} \sin\big(\pi(q_j - \alpha_j)\big)\, \sin\big(\pi(q_j - \beta_j)\big)\, \sin(\pi p_j),$$

$$g_{ij}(q,p) \stackrel{\text{def}}{=} \sin(\pi p_i)\sin(\pi p_j) \ \ i \neq j,\ 1 \leq i, j \leq n,$$

$\phi_j(q,p) = \phi(q_j, p_1, \cdots, p_{j-1}, p_{j+1}, \cdots, p_n)$, *where* $\phi(q_j, \bullet) \equiv 0$, $\alpha_j \leq q_j \leq \beta_j$,

$\psi(q)$, *where* $\psi \in \mathcal{C}^\infty(\mathbb{I}^n)$ *vanishes on* H.

Proof. We only give the proof for the following model ($n = 2$), which contains all the basic ingredients of the general case. Let us study the ideal of functions vanishing on

$$\big(\{q_1 = p_2 = 0\} \cup \{q_2 = p_1 = 0\} \cup \{p_1 = p_2 = 0\}\big) \cap \{q_1 \geq 0, q_2 \geq 0\}. \qquad (4.24)$$

The functions vanishing on the first factor in the intersection are generated by $q_1 p_1$, $q_2 p_2$, and $p_1 p_2$. Writing an arbitrary function $F(q, p)$ as

$$\begin{aligned} F(q_1, q_2, p_1, p_2) = {} & F_0(p_1, p_2) + q_1 F_1(q_1, q_2, p_2) + q_2 F_2(q_1, q_2, p_1) \\ & + q_1 p_1 F_{11}(q_1, q_2, p_1, p_2) + q_2 p_2 F_{22}(q_1, q_2, p_1, p_2), \end{aligned}$$

we need to find conditions for $q_1 F_1(q_1, q_2, p_2)$ and $q_2 F_2(q_1, q_2, p_1)$ to vanish on (4.24). We treat the first function by expanding it as

$$F_1(q_1, q_2, p_2) = F_{10}(q_1, q_2) + p_2 F_{12}(q_1, p_2) + q_2 p_2 F_{122}(q_1, q_2, p_2).$$

This forces $F_{10}(q_1, q_2)$ to vanish identically in $\{q_1, q_2 \geq 0\}$ and $F_{12}(q_1, p_2)$ to vanish identically in $\{q_1 \geq 0\}$.

The function $F_2(q_1, q_2, p_1)$ is treated identically. Hence the functions vanishing on (4.24) are generated by $q_1 p_1$, $q_2 p_2$, $p_1 p_2$, and all the smooth fuctions $\psi(q_1, q_2)$, $\phi_1(q_1, p_2)$, $\phi_2(q_2, p_1)$ vanishing on $\{q_1, q_2 \geq 0\}$. The transposition to the torus setting gives the lemma for that case. The general case can be proven similarly. $\square$

This lemma means that any $F \in \mathcal{J}$ can be decomposed as

$$F = \sum_{j \neq i} f_{ij}\, g_{ij} + \sum_j (f_{jj}\, g_j + f_j\, \phi_j + \psi),$$

where the functions $f_\bullet$ are smooth and either periodic or antiperiodic in each variable, so that $f_\bullet g_\bullet$ are periodic in all variables.

The action of each term $(f\, g)^w(q, hD)$ on the state (4.23) can be written

$$(fg)^w\, u_h = \big((f\, a)^w \circ g^w + h\, L(f, a, g)\big)\, 1\!\!1_H,$$

where $L(f, a, g)$ is a pseudodifferential operator of norm $\mathcal{O}(1)$. Therefore, we are reduced to study the action of the generators $g^w(q, hD)$, $g = g_{ij},\ g_j,\ \phi_j, \psi$, on the characteristic function $1\!\!1_H(q)$.

We first note that $\psi\ 1\!\!1_H = \phi_j^w\, 1\!\!1_H \equiv 0$, so there is nothing to prove in this case.

For each $j \in \{1, \ldots, n\}$, the generator g_j contains a factor $\sin(\pi p_j)$. Up to an error $\mathcal{O}(h)$, we first quantize this factor and apply it to $1\!\!1_H$:

$$\sin(\pi h D_j)\, 1\!\!1_H(q) = \frac{1}{2i}\big(1\!\!1_H(q_j + \pi h, q') - 1\!\!1_H(q_j - \pi h, q')\big) \stackrel{\text{def}}{=} b_j(q).$$

The function $b_j(q)$ is supported in the strips $S_j = \big\{|q_j - \alpha_j| \leq \pi h\big\} \cup \big\{|q_j - \beta_j| \leq \pi h\big\}$, where it takes values ± 1. We now apply the remaining factors of g_j: this amounts to multiplying $b_j(q)$ by the product $\sin\big(\pi(q_j - \alpha_j)\big)\, \sin\big(\pi(q_j - \beta_j)\big)$, and gives a function $\mathcal{O}(h)$. Taking the error into account, we obtain $\|g_j^w\ 1\!\!1_H\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h)$.

In the case of g_{ij}, $i \neq j$, we apply $\sin(\pi h D_i)$ to $b_j(q)$: the resulting function takes values ± 1 on its support $S_i \cap S_j$, so that $\|g_{ij}^w\ 1\!\!1_H\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h)$.

We have now proved that $\|F^w\, u_h\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h)$ for any $F \in \mathcal{J}$. The procedure can be iterated to any finite product of functions $F_i \in \mathcal{J}$, yielding the estimate (4.18).

The proof is completed by the periodization argument as in the proof of Proposition 4.8. The only slight difference lies in the fact that the analogues of the functions $u_h^{(M)}(q)$ of (4.17) may now have discontinuities near ∂D, so that $\| P_{\mathbb{T}^{2n}} u_h^{(M)} \| - \| u_h^{(M)} \|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h)$ instead of $\mathcal{O}(h^\infty)$. $\square$

4.3. Quantum relations. Suppose that $\Lambda \subset \mathbb{T}^{2n} \times \mathbb{T}^{2n}$ is a Lagrangian submanifold. The basic example is given by the twisted graph Γ'_κ of a symplectic diffeomorphism κ on $\mathbb{T}^{2n}$ (see Sect. 3.2.1):

$$\Gamma'_\kappa = \left\{ (q', q; p', -p) \;:\; (q', p') = \kappa(q, p),\ (q, p) \in \mathbb{T}^{2n} \right\}.$$

As we noticed in that section, the choice of change of sign depends on the choice of the splitting of variables (q, p), which is itself related with the choice of a *polarization* in the quantization $a \mapsto \mathrm{Op}_h(a)$ [24, §25.2]. This somewhat cumbersome convention is explained as follows.

Any state $v \in \mathcal{H}_h^n$ is naturally identified to a linear form $f_v \in (\mathcal{H}_h^n)^*$ through $f_v(w) = \langle v, w \rangle$. In our notations[5], this scalar product is antilinear in the *first* component. To make the identification linear, we choose instead

$$v \in \mathcal{H}_h^n \Longrightarrow f_v(\bullet) = \langle \bar{v}, \bullet \rangle, \tag{4.25}$$

where states v are written as vectors in the basis (4.3).

Let $\mathcal{L}(\mathcal{H}_h^n) \simeq \mathcal{H}_h^n \otimes (\mathcal{H}_h^n)^*$ be the space of linear operators on $\mathcal{H}_h^n$. The linear identification (4.25) of $\mathcal{H}_h^n$ with $(\mathcal{H}_h^n)^*$ gives the identification,

$$\mathcal{L}(\mathcal{H}_h^n) \simeq \mathcal{H}_h^{2n}, \quad \text{through} \quad (u \otimes v)(w) = u \, \langle \bar{v}, w \rangle, \quad u, v, w \in \mathcal{H}_h^n. \tag{4.26}$$

We observe that the norm on $\mathcal{H}_h^{2n}$ is the same as the Hilbert-Schmidt norm on $\mathcal{L}(\mathcal{H}_h^n)$:

$$\| T \|_{\mathcal{H}_h^{2n}} = (\mathrm{tr}_{\mathcal{H}_h^n}(T^* T))^{\frac{1}{2}}. \tag{4.27}$$

It is related to the operator norm on $\mathcal{L}(\mathcal{H}_h^n)$ as follows:

$$\| T \|_{\mathcal{L}(\mathcal{H}_h^n)} \le \| T \|_{\mathcal{H}_h^{2n}} \le N^{n/2} \| T \|_{\mathcal{L}(\mathcal{H}_h^n)}. \tag{4.28}$$

In particular, unitary operators have Hilbert-Schmidt norm $N^{n/2} = (2\pi h)^{-n/2}$.

The identification (4.26) dictates the way an operator of the type $A_1 \otimes A_2$ (with $A_i \in \mathcal{L}(\mathcal{H}_h^n)$) acts on $u \otimes v \in \mathcal{H}_h^{2n} \simeq \mathcal{L}(\mathcal{H}_h^n)$. Indeed, if we take any $w \in \mathcal{H}_h^n$, we have

$$\begin{aligned}
[(A_1 \otimes A_2)(u \otimes v)]\,(w) &= [A_1 u \otimes A_2 v]\,(w) \\
&= A_1 u \; \langle \overline{A_2 v}, w \rangle \\
&= A_1 u \; \langle \bar{v}, A_2' w \rangle \\
&= \left[(A_1 u \otimes v) \circ A_2' \right] (w).
\end{aligned}$$

Here A_2' is the transposed of the operator A_2, written as a matrix in the basis (4.3). In the case $A_1 = \mathrm{Op}_h(a)$, $A_2 = \mathrm{Op}_h(b)$ for some real functions $a, b \in \mathcal{C}^\infty(\mathbb{T}^{2n})$, one

[5] This is the physicists' convention.

checks that $A_2' = \mathrm{Op}_h(\tilde{b})$, with the same twisted function as in the proof of Lemma 4.6: $\tilde{b}(q, p) = b(q, -p)$. By linearity, for any $C_h \in \mathcal{H}_h^{2n} \simeq \mathcal{L}(\mathcal{H}_h^n)$, we have

$$\mathrm{Op}_h(a \otimes b)\, C_h = \mathrm{Op}_h(a) \circ C_h \circ \mathrm{Op}_h(\tilde{b}). \tag{4.29}$$

The sign change in the tilting $\Gamma \rightsquigarrow \Gamma'$ parallels the transformation $a(\rho')\, b(\rho) \rightsquigarrow a(\rho')\, \tilde{b}(\rho)$.

We are now in position to quantize a symplectic map, more generally a symplectic relation Γ as defined in Sect. 3.2.

Definition 4.11. *A semiclassical sequence* $U = \{U_h \in \mathcal{H}_h^{2n}\}_{h \to 0}$ *satisfying*

$$\|U_h\|_{\mathcal{H}_h^{2n}} = \mathcal{O}(h^K), \quad \text{for some fixed } K \in \mathbb{R}, \tag{4.30}$$

is a quantum relation associated with the symplectic relation Γ *if* U *is a Lagrangian state in* $I(\Gamma')$*, in the sense of Definition* 4.7.

Explicitly, for any $M \in \mathbb{N}$ and any sequence of functions

$$g_j \in \mathcal{C}^\infty(\mathbb{T}^{2n} \times \mathbb{T}^{2n}), \quad g_j|_{\Gamma'} = 0, \quad 1 \le j \le M,$$

we must have

$$\|\mathrm{Op}_h(g_M) \circ \cdots \circ \mathrm{Op}_h(g_1)\, U_h\|_{\mathcal{H}_h^{2n}} = \mathcal{O}(h^M)\, \|U_h\|_{\mathcal{H}_h^{2n}}. \tag{4.31}$$

The assumption that U_h is tempered in the sense of (4.30) (which also implies temperedness in the operator norm) is necessary to assure that composing U_h with residual ($\mathcal{O}(h^\infty)$) terms produces residual terms. That is a standard assumption in $\mathcal{C}^\infty$ semiclassical calculi — see [1, 51], and will be used in the proof of Prop. 4.12. The quantum *weighted* relations defined in §4.4 will naturally be tempered, having norms $\|U_h\|_{\mathcal{H}_h^{2n}} = \mathcal{O}(h^{-n/2})$.

If a function $g \in \mathcal{C}^\infty(\mathbb{T}^{4n})$ vanishes on Γ', then the function $\tilde{g}$ defined as $\tilde{g}(q', p'; q, p) = g(q', p'; q, -p)$ vanishes on Γ. The condition $g_j|_{\Gamma'} = 0$ can thus be written $\tilde{g}_j|_\Gamma = 0$.

We also note that (4.31) entails a version of *Egorov's theorem.* If $f_L,\ f_R \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ satisfy

$$(\rho', \rho) \in \Gamma \implies f_L(\rho') = f_R(\rho),$$

then we have

$$\|\mathrm{Op}_h(f_L)U_h - U_h\mathrm{Op}_h(f_R)\|_{\mathcal{H}_h^{2n}} = \mathcal{O}(h)\, \|U_h\|_{\mathcal{H}_h^{2n}}. \tag{4.32}$$

Indeed, the function $f \stackrel{\text{def}}{=} f_L \otimes 1 - 1 \otimes f_R$ vanishes on Γ, so that $\tilde{f}$ vanishes on Γ'. We then simply apply the definition (4.31) with $g_1 = \tilde{f}$ and use (4.29). When Γ is a graph of a symplectic transformation, f_R is the pullback of f_L, and we get a statement similar with the standard Egorov's theorem.

Remark 4.2. Following Sect. 4.2, in the case when Γ' is a Lagrangian with boundaries projecting on a hypercube, it is useful to include in the definition sequences $U = \{U_h\}$ in the (larger) space $I(\Gamma' \cup \Lambda_{\mathrm{sing}})$; the quantum baker's relation we define in the next section will belong to such an enlarged space.

Through the identification (4.26), U_h is an operator on $\mathcal{H}_h^n$. We now show that this operator "classically transports" the microsupport of a sequence $w = \{w_h \in \mathcal{H}_h^n\}$.

Proposition 4.12. *Take $U = \{U_h \in \mathcal{H}_h^{2n} \simeq \mathcal{L}(\mathcal{H}_h^n)\}$ a quantum relation $U \in I(\Gamma')$. Then for any sequence $w = \{w_h \in \mathcal{H}_h^n\}$, $\|w_h\| \asymp 1$, the microsupport of the image sequence $U(w) = \{U_h(w_h)\}$ satisfies:*

$$\begin{aligned}\mathrm{WF}_h(U(w)) &\subset \Gamma\big(\mathrm{WF}_h(w)\big)\\ &= \Big\{\rho' \in \mathbb{T}^{2n} \ : \ \exists\, \rho \in \mathrm{WF}_h(w), \quad (\rho', \rho) \in \Gamma\Big\}.\end{aligned}$$

Proof. Assume that $\rho_0' \notin \Gamma(\mathrm{WF}_h(w))$, which means that $\Gamma^{-1}(\rho_0') \cap \mathrm{WF}_h(w) = \emptyset$. Then there exists a function $f \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ with $f \equiv 1$ near ρ_0' but with $\mathrm{supp}(f)$ sufficiently small so that $\Gamma^{-1}(\mathrm{supp}(f)) \Subset \complement \mathrm{WF}_h(w)$. Consequently, there exists a function $g \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ with $g \equiv 1$ near $\mathrm{WF}_h(w)$ but $g \equiv 0$ on $\Gamma^{-1}(\mathrm{supp}(f))$. The function $f \otimes \tilde{g} \in \mathcal{C}^\infty(\mathbb{T}^{4n})$ then automatically vanishes on Γ'.

Our aim is to show that $\rho_0' \notin \mathrm{WF}_h(U(w))$. For this, we introduce one further function $a \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ such that $a(\rho_0') > 0$ and $f \equiv 1$ on $\mathrm{supp}(a)$. As in the proof of (4.9) we see that for any $M \in \mathbb{N}$, $\mathrm{Op}_h(a)\mathrm{Op}_h(f)^M = \mathrm{Op}_h(a) + \mathcal{O}(h^\infty)$. Hence

$$\begin{aligned}\|\mathrm{Op}_h(a) U_h w_h\| &= \|\mathrm{Op}_h(a)\mathrm{Op}_h(f)^M U_h w_h\| + \mathcal{O}(h^\infty)\\ &\le \|\mathrm{Op}_h(a)\mathrm{Op}_h(f)^M\, U_h \mathrm{Op}_h(g)^M\, w_h\|\\ &\quad + \|\mathrm{Op}_h(a)\mathrm{Op}_h(f)^M U_h\,(1 - \mathrm{Op}_h(g)^M)\, w_h\| + \mathcal{O}(h^\infty).\end{aligned}$$

To bound the second term on the right-hand side, we notice that the function $(1 - g^M)$ vanishes near $\mathrm{WF}_h(w)$, so from Lemma 4.5 we get $\|(1 - \mathrm{Op}_h(g)^M)\, w_h\| = \mathcal{O}(h^\infty)$; from the temperedness of U_h, the second term is thus residual.

The first term on the right-hand side is estimated using the identity

$$\begin{aligned}&\mathrm{Op}_h(f)^M\, U_h\, \mathrm{Op}_h(g)^M\\ &\quad = \mathrm{Op}_h(f \otimes \tilde{g})^M\, U_h.\end{aligned}$$

Because $f \otimes \tilde{g}$ vanishes on Γ', the Hilbert-Schmidt norm of that operator is $\mathcal{O}(h^{M+K})$, where K comes from the temperedness of U_h, (4.30). Using (4.28), we thus get $\|\mathrm{Op}_h(a)U_h\, w_h\| = \mathcal{O}(h^{M+K})$ for an arbitrary $M \in \mathbb{N}$, which shows that $\rho_0' \notin \mathrm{WF}_h(U(w))$. □

4.4. Quantized weighted relations. In Sect. 3.4 we equipped a symplectic relation Γ with a measure, or weight μ. In order to associate to the weighted relation (Γ, μ) a sequence of operators $U_h \in \mathcal{H}_h^{2n}$, we need to elaborate on Definition 4.11, thereby defining a subfamily $I(\Gamma', \mu) \subsetneq I(\Gamma')$.

In the standard microlocal context [24, Sect. 25.1], a Lagrangian state $\psi \in I(\Lambda)$ has a well defined amplitude, or *symbol*, which is a section of the Maslov half density bundle over the Lagrangian submanifold — see [24, Theorem 25.1.9]. The local aspects of this procedure have recently been adapted to the semiclassical case [1], and a similar approach can be used in the case of $\mathbb{T}^{4n}$.

Although one could characterize the operators quantizing (Γ, μ) in terms of their symbols (grossly speaking, the absolute square of the symbol should equal the weight

μ), we won't do it here, in order to avoid technical issues involved in the description of the symbol map. Instead, in the definition below we use bilinear expressions in U_h, which allows us to avoid introducing symbols.

Definition 4.13. *Let* (Γ, μ) *be a weighted piecewise smooth relation as defined in* §3.4 *and let* $U \in I(\Gamma' \cup \Lambda_{\rm sing})$*, in the sense of Definition* 4.11 *and Remark* 4.2*. For any* $\chi_\alpha \in \mathcal{C}^\infty(\mathbb{T}^{2n}; [0, 1])$*,* $\alpha = L,\ R$*, we define*

$$U_{\chi_L \chi_R} \stackrel{\rm def}{=} \mathrm{Op}_h(\chi_L)\ U_h\ \mathrm{Op}_h(\chi_R).$$

We say that U **quantizes the weighted relation** (Γ, μ) *if for all* χ_L*,* χ_R *with sufficiently small supports satisfying* $\mathrm{supp}(\chi_L \otimes \chi_R) \cap \Lambda'_{\rm sing} = \emptyset$*,*

$$\begin{aligned} U_{\chi_L \chi_R} U^*_{\chi_L \chi_R} &= \mathrm{Op}_h(g_L^{\chi_L \chi_R}) + \mathcal{O}(h) \\ U^*_{\chi_L \chi_R} U_{\chi_L \chi_R} &= \mathrm{Op}_h(g_R^{\chi_L \chi_R}) + \mathcal{O}(h), \end{aligned} \tag{4.33}$$

where $g_\alpha^{\chi_L \chi_R}$ *are the functions given in* (3.9)*, and the remainder is* $\mathcal{O}(h)$ *in the operator norm on* $\mathcal{L}(\mathcal{H}_h^n)$*. We then write*

$$U = \{U_h\} \in I(\Gamma' \cup \Lambda_{\rm sing}, \mu).$$

The conditions on the smallness of supports of χ_α guarantee that the operators appearing on the left in (4.33) are of the form $\mathrm{Op}_h(f)$, $f \in \mathcal{C}^\infty(\mathbb{T}^{2n})$. That follows from the fact that Γ is locally a graph — see §3.4.

If Γ is the graph of a symplectic diffeomorphism κ and $\mu = \pi_L^*(\omega^n/n!) = \pi_R^*(\omega^n/n!)$, then U_h is unitary to leading order:

$$U_h^* U_h = I + C_h, \quad U_h U_h^* = I + D_h, \qquad \|C_h\|_{\mathcal{L}(\mathcal{H}_h^n)} = \mathcal{O}(h),\ \|D_h\|_{\mathcal{L}(\mathcal{H}_h^n)} = \mathcal{O}(h).$$

For h small, $(I+C_h)^{-\frac{1}{2}}$, $(I+D_h)^{-\frac{1}{2}}$ exist, therefore a possibility to make the quantization strictly unitary is to replace U_h by $U_h(I + C_h)^{-\frac{1}{2}}$ or $(I + D_h)^{-\frac{1}{2}} U_h$.

The condition (4.33) can be interpreted as follows. Suppose that $\psi \in \mathcal{H}_h^n$, $\|\psi\| = 1$, is microlocalized at a single "regular" point ρ_0:

$$\mathrm{WF}_h(\psi) = \{\rho_0\} \subset \mathbb{T}^{2n} \setminus \pi_R(\Lambda'_{\rm sing}),$$

and $\Gamma(\rho_0) = \cup_{j=1}^J \rho'_j$, $\rho'_j = \kappa_j(\rho_0)$. Then,

$$\begin{aligned} U_h \psi &= \sum_{j=1}^J \psi_j + \mathcal{O}(h^\infty), \\ \|\psi_j\|^2 &= P_j(\rho_0) + \mathcal{O}(h), \qquad \mathrm{WF}_h(\psi_j) \subset \left\{\rho'_j\right\}. \end{aligned}$$

From Lemma 4.5, if $P_j(\rho_0) \neq 0$ then $\mathrm{WF}_h(\psi_j) = \left\{\rho'_j\right\}$. A similar statement holds for U_h^*.

Indeed, if for each $j = 0, \cdots, J$ we take $\chi_j \in \mathcal{C}^\infty(\mathbb{T}^{2n}; [0,1])$ supported in a small neighbourhood of ρ_0, resp. ρ'_j, and equal to 1 near that point, (3.10) shows that $g_R^{\chi_j \chi_0}(\rho_0) = P_j(\rho_0)$ for $j = 1, \ldots, J$. On the other hand, Proposition 4.12 gives

$$\begin{gathered} U_h \psi = U_h \mathrm{Op}_h(\chi_0)\psi + \mathcal{O}(h^\infty) = \sum_{j=1}^{J} U_{\chi_j \chi_0}\, \psi + \mathcal{O}(h^\infty), \\ \mathrm{WF}_h(U_{\chi_j \chi_0}\psi) \subset \left\{\rho'_j\right\}. \end{gathered} \tag{4.34}$$

If we take $\psi_j \stackrel{\text{def}}{=} U_{\chi_j \chi_0}\psi$ then

$$\|\psi_j\|^2 = \langle U^*_{\chi_j \chi_0} U_{\chi_j \chi_0}\psi, \psi\rangle = \langle \mathrm{Op}_h(g_R^{\chi_j \chi_0})\psi, \psi\rangle + \mathcal{O}(h) = P_j(\rho_0) + \mathcal{O}(h).$$

Example. We now consider a special case of quantum relations U_h, of the form

$$\langle Q_j | U_h | Q_k\rangle = N^{-n/2} a(Q_j, Q_k)\ \exp\bigl(2\pi i N S(Q_j, Q_k)\bigr), \tag{4.35}$$

where $a,\ S \in \mathcal{C}^\infty(\mathbb{T}^{2n} \times \mathbb{T}^{2n})$ and the generating function $S(q', q)$ satisfies the nondegeneracy condition $\det(\partial^2_{q' q} S) \neq 0$ near the support of $a(q', q)$. Using Definition 4.11 we see that U_h is associated to the graph Γ_S of the symplectic transformation $(q, -\partial_q S) \mapsto (q', \partial_{q'} S)$. To be more precise,

$$U_h \in I(\Gamma'_S, \mu_S), \qquad \text{for} \quad \mu_S \stackrel{\text{def}}{=} |a(q', q)|^2\, dq'\, dq, \tag{4.36}$$

where we used the coordinates (q', q) on Γ_S. Projecting this measure on the left and right tori, we get:

$$\begin{aligned} \pi_{L*}\mu_S &= \left(\sum_{q' :\, p = -\partial_q S(q', q)} |a(q', q)|^2\ |\det(\partial^2_{q' q} S)|^{-1} \right) dq\, dp, \\ \pi_{R*}\mu_S &= \left(\sum_{q \,:\, p' = \partial_{q'} S(q', q)} |a(q', q)|^2\ |\det(\partial^2_{q' q} S)|^{-1} \right) dq'\, dp'. \end{aligned} \tag{4.37}$$

The above sums are always finite. This example will be used to analyze the quantum baker's relations studied in the next sections.

4.5. Quantized baker's relation. We explicitly construct quantum relations $B_h \in \mathcal{L}(\mathcal{H}_h^1)$ associated with the "open baker's maps" described in Sect. 3.3. For simplicity, we will assume that the coefficients D_j and ℓ_j are integers. Besides, we will only consider the subsequence of Planck's constants of the form $h = (2\pi N)^{-1}$ such that $N/D_1 = M_1 \in \mathbb{N}$ and $N/D_2 = M_2 \in \mathbb{N}$ (that is, N is a multiple of $\mathrm{lcm}(D_1, D_2)$).

Restricting ourselves to this subsequence, we define the quantization of the baker's relation (3.4) as the following operators (written as $N \times N$ matrices in the bases (4.3)):

$$B_h \stackrel{\text{def}}{=} \mathcal{F}_N^* \circ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{F}_{M_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{F}_{M_2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = B_{1,h} + B_{2,h}. \tag{4.38}$$

The numbers of columns in successive blocks are respectively given by

$$\ell_1 M_1,\ M_1,\ \ell_2 M_2 - (\ell_1 + 1) M_1,\ M_2,\ (D_2 - \ell_2 - 1) M_2,$$

and $\mathcal{F}_M$ is the discrete Fourier transform given in (4.5). These matrices obviously generalize the unitary matrices associated with the closed baker's map [2].

We now check that the matrices (4.38) satisfy Definition 4.13 if we select the appropriate Lagrangian surface on $\mathbb{T}^4$, namely by adjoining a singularity set Λ_{sing} to the twisted graph B' (see Remark 4.2), and equip B with the weight μ described in (3.11). By linearity, we can separately consider the two blocks $B_{j,h}$. Let us study the left block $B_{1,h}$. Since the classical relation B_1 is generated by the function $S_1(q, p')$ of (3.5), it is natural to express the operator $B_{1,h}$ in the mixed representation (p', q), that is by a matrix from the basis $\{|Q_j\rangle\}$ to the basis $\{|P_k\rangle\}$. Since the change of basis matrix, $(|P_k\rangle\langle Q_j|)_{j,k=0,\ldots,N-1}$, equals $\mathcal{F}_N$, the operator $A_{1,h}$ defined as the matrix

$$(\langle Q_k|A_{1,h}|Q_j\rangle)_{j,k=0,\ldots,N-1} \stackrel{\mathrm{def}}{=} (\langle P_k|B_{1,h}|Q_j\rangle)_{j,k=0,\ldots,N-1} = \mathcal{F}_N \circ B_{1,h}$$

is given by the Fourier block $\mathcal{F}_{M_1}$ at the same position as in (4.38), and zeros everywhere else.

The following lemma reduces finding the (weighted) Lagrangian relation associated to $B_{1,h}$ to finding the (weighted) Lagrangian associated to $A_{1,h}$. We denote by F the following transformation of $\mathbb{T}^{2n}$: $F(q, p) = (p, -q)$. It means we rotate by $-\pi/2$ around the origin in each plane (q_i, p_i). We denote by F_L the transformation of $\mathbb{T}^{4n}$ acting through F on the left coordinates (q', p') and leaving the right coordinates unchanged.

Lemma 4.14. *Suppose that $U_h \in \mathcal{L}(\mathcal{H}_h^n) \simeq \mathcal{H}_h^{2n}$ and that $V_h \stackrel{\mathrm{def}}{=} \mathcal{F}_N \circ U_h$. Then, for any (possibly singular) Lagrangian $\mathcal{C}' \in \mathbb{T}^{4n}$,*

$$U_h \in I(\mathcal{C}') \iff V_h \in I(\mathcal{D}'),$$

where

$$\mathcal{D}' = F_L(\mathcal{C}'), \quad \textit{equivalently} \quad \mathcal{D} = F_L(\mathcal{C}) = \{(p', -q'; q, p) \, : \, (q', p'; q, p) \in \mathcal{C}\}.$$

Furthermore,

$$U_h \in I(\mathcal{C}', \mu) \iff V_h \in I(\mathcal{D}', \nu), \quad \textit{with} \quad \nu = F_{L*}\,\mu.$$

Proof. The transformation $\mathcal{C} \to \mathcal{D}$ results from a general composition formula which can be proved by mimicking the semiclassical proof in [1]. Here it follows from the covariance properties of Weyl quantization with respect to the Fourier transform: for any $a \in \mathcal{C}^\infty(\mathbb{T}^{2n})$,

$$\mathcal{F}_h^{-1} \mathrm{Op}_h(a) \circ \mathcal{F}_h = \mathrm{Op}_h(a \circ F). \tag{4.39}$$

As a result, for any $f \in \mathcal{C}^\infty(\mathbb{T}^{4n})$,

$$\mathrm{Op}_h(f)(\mathcal{F}_h \circ U_h) = \mathcal{F}_h \circ \mathrm{Op}_h(f \circ F_L)(U_h).$$

This identity proves the first assertion.

Using (4.39), we notice that for any $\chi_L,\ \chi_R \in \mathcal{C}^\infty(\mathbb{T}^{2n}; [0, 1])$, the cutoff propagator $V_{\chi_L\,\chi_R}$ satisfies

$$V^*_{\chi_L\,\chi_R}\, V_{\chi_L\,\chi_R} = U^*_{\chi_L \circ F\,\chi_R}\, U_{\chi_L \circ F\,\chi_R} = \mathrm{Op}_h(g_R^{\chi_L \circ F\,\chi_R}) + \mathcal{O}(h),$$
$$V_{\chi_L\,\chi_R}\, V^*_{\chi_L\,\chi_R} = \mathcal{F}_h\, U_{\chi_L \circ F\,\chi_R}\, U^*_{\chi_L \circ F\,\chi_R}\, \mathcal{F}_h^* = \mathrm{Op}_h(g_L^{\chi_L \circ F\,\chi_R} \circ F^{-1}) + \mathcal{O}(h).$$

Using the pushforward of functions $F_{L*}f = f \circ F_L^{-1}$ and the fact that $\pi_R \circ F_L = \pi_R$, we get

$$\begin{aligned} g_R^{\chi_L \circ F\, \chi_R} &= \pi_{R*}(\pi_L^*(F_{L*}^{-1}\chi_L)\, \pi_R^*\chi_R\, \mu) = \pi_{R*}(\pi_L^*\chi_L\, \pi_R^*\chi_R\, F_{L*}\mu), \\ g_L^{\chi_L \circ F\, \chi_R} \circ F^{-1} &= \pi_{L*}F_{L*}(\pi_L^*(F_{L*}^{-1}\chi_L)\, \pi_R^*\chi_R\, \mu) = \pi_{L*}(\pi_L^*\chi_L\, \pi_R^*\chi_R\, F_{L*}\mu). \end{aligned}$$

This proves that V_h is associated with the weight $\nu = F_{L*}\,\mu$ on $\mathcal{D}'$. $\square$

Let us now describe the weighted Lagrangian associated with the operator $A_{1,h}$. The kernel of that operator vanishes outside the square $H = I_1 \times I_1$, where $I_1 = [\ell_1/D_1, \ell_1+1/D_1]$, and on H it takes the values

$$\langle Q_k|A_{1,h}|Q_j\rangle = \langle P_k|B_{1,h}|Q_j\rangle = \sqrt{\frac{D_1}{N}}\ \mathbb{1}_H(Q_k, Q_j)\, \exp\left(-2i\pi N\, S_1(Q_k, Q_j)\right). \tag{4.40}$$

The operator $A_{1,h}$ has the same form as in (4.35), with the (obviously nondegenerate) generating function $S = -S_1$ and symbol $a(q', q) = \sqrt{D_1}\ \mathbb{1}_H(q', q)$. If we forget (for a moment) the discontinuities of the symbol, we find that $A_{1,h}$ is associated with the graph

$$\Gamma_{S_1} = \left\{\left(q', -(D_1 q - \ell_1);\ q, (D_1 q' - \ell_1)\right)\ :\ q, q' \in I_1\right\},$$

equipped with the weight

$$\mu_{S_1} = D_1\ \mathbb{1}_H(q', q)\, dq'\, dq.$$

From Lemma 4.14, the operator $B_{1,h} = \mathcal{F}_N^* \circ A_{1,h}$ is associated with the graph

$$F_L^{-1}(\Gamma_{S_1}) = \left\{\left((D_1 q - \ell_1), q';\ q, (D_1 q' - \ell_1)\right)\ :\ q, q' \in I_1\right\} = B_1$$

and the weight

$$\mu_1 \stackrel{\text{def}}{=} F_{L*}^{-1}\mu_{S_1} = D_1\ \mathbb{1}_H(p', q)\, dp'\, dq,$$

which can be expressed as

$$\pi_{R*}\,\mu_1 = \mathbb{1}_{I_1}(q)\, dq\ dp, \quad \pi_{L*}\,\mu_1 = \mathbb{1}_{I_1}(p')\, dq'\, dp'.$$

It represents the half part of the weight (3.11).

Let us now take the discontinuities of $a(q', q)$ into account. Since they occur at the boundary of the square H, they have the same consequences as in Lemma 4.9. Namely, we must add to the Lagrangian Γ'_{S_1} a "singular" Lagrangian, which is the union of 4 pieces, each piece sitting above a side of H. This Lagrangian should then be rotated through F_L^{-1} as well.

For instance, the side $\{q' \in I_1,\ q = \ell_1/D_1\}$ leads (after rotation) to the singular Lagrangian

$$\Lambda_{\text{sing},1} = \left\{\left(q' = 0,\ q = \frac{\ell_1}{D_1};\ p', p\right)\ :\ p' \in I_1,\ p \in \mathbb{I}\right\},$$

which contains the corresponding side of $\partial B_1'$. Similar Lagrangians $\Lambda_{\text{sing},i}$, $i = 2, 3, 4$, contain the other sides of $\partial B_1'$.

The same analysis applies to $B_{2,h}$ and hence we have proved the

Proposition 4.15. *The sequence of matrices* $\{B_h\}$ *given in* (4.38) *quantizes the classical baker's relation* $B = B_1 \cup B_2$ *of* (3.4)*, in the sense of Definitions* 4.11*,* 4.13*, and Remark* 4.2*:*

$$B_h \in I\left(B' \cup \bigcup_{j=1}^{8} \Lambda_{\mathrm{sing,j}},\, \mu\right),$$

where the weight μ *is given by* (3.11).

This quantization of the baker's relation is very close to the "quantum horseshoe" defined by Saraceno-Vallejos in [45]. The operator B_h is contracting, and its eigenstates can be seen as "metastable states", "decaying states" or "resonances". This contraction mirrors the decay of a classical probability density evolved through the open map B (due to the "escape" of particles to infinity). This classical decay can be analyzed in terms of a "conditionally invariant measure" on $\mathbb{T}^2$ [8], which decays according to the classical decay rate $\gamma_{\mathrm{cl}} = -\log(D_1^{-1} + D_2^{-1})$.

4.6. Numerical check of the Weyl law for the baker's relation. We have numerically computed the spectra of the quantum baker relations for various symmetric and nonsymmetric baker's relations. Results for the symmetric "3-baker" ($D = 3$, $\ell = 0$) were presented in [38] (see also Table 1), some for the "5-baker" ($D = 5$, $\ell = 1$) were given in [40], while a nonsymmetric map ($D_1 = 32$, $D_2 = 3/2$) was studied in [39]. In the symmetric cases, the trapped set is a pure Cantor set of dimension $2d = 2\frac{\log 2}{\log D}$, so that for any $1 > r > 0$, the number of resonances in the annulus $\{|\lambda| > r\}$ is expected to scale as

$$\#\{\lambda \in \mathrm{Spec}(B_h)\ :\ |\lambda| \geq r\} \sim C(r)\, N^{\frac{\log 2}{\log D}}$$

in the limit $N \to \infty$. Our numerics for both maps shows that this scaling is roughly satisfied along any sequence $N \to \infty$; much better fits are obtained for N taken along geometric sequences of the type $N = N_o\, D^k$, with N_o fixed and $k \to \infty$ (as in Table 1), which lead us to the following weaker conjecture for the symmetric maps:

$$\#\{\lambda \in \mathrm{Spec}(B_h)\ :\ |\lambda| \geq r\} \sim C(N_o, r)\, N^{\frac{\log 2}{\log D}}, \qquad (2\pi h)^{-1} = N = N_o\, D^k, \quad k \to \infty.$$

Here, the "profile function" $C(N_o, r)$ may (slightly) depend on the "root" of the geometric sequence. The special role played by geometric sequences is probably due to the strong relationship between the symmetric D-baker and the D-nary decomposition.

On the opposite, for the nonsymmetric map the fractal Weyl law seems accurate for an "arbitrary" sequence $N \to \infty$ [39], which was also the case for the nonlinear map studied by [49].

5. A Toy Model

Let us explicitly compute the matrix elements of the two vertical blocks $B_{1,h}$, $B_{2,h}$ in (4.38), for the symmetric 3-baker. Both are matrices $N \times N/3$, which we index by $0 \leq k \leq N-1$, $0 \leq l \leq N/3 - 1$:

$$
\begin{aligned}
(B_{1,h})_{k\,l} &= \begin{cases} \sqrt{3}(1-e^{2i\pi\frac{k-3l}{N}})^{-1}(1-\omega_3^k)/N & \text{if } k \neq 3l, \\ 1/\sqrt{3} & \text{if } k = 3l, \end{cases}, \\
(B_{2,h})_{k\,l} &= \omega_3^{2k}\,(B_{1,h})_{k\,l}, \qquad \text{where } \omega_3 = e^{2i\pi/3}.
\end{aligned} \tag{5.1}
$$

The largest matrix elements are near the "tilted diagonals" $k \approx 3l$, and decay as $1/|k-3l|$ away from them (see Fig. 6 in [40]).

Being unable to rigorously analyze the spectrum of B_h, we replace this matrix by the following simplified model:

$$
\begin{gathered}
\widetilde{B}_h = \widetilde{B}_N = [\widetilde{B}_{1,h}, 0, \widetilde{B}_{2,h}], \\
(\widetilde{B}_{1,h})_{k\,l} = \begin{cases} 1/\sqrt{3} & \text{if } l = \lfloor k/3 \rfloor \\ 0 & \text{if } l \neq \lfloor k/3 \rfloor \end{cases}, \qquad (\widetilde{B}_{2,h})_{k\,l} = \omega_3^{2k}\,(\widetilde{B}_{1,h})_{k\,l},
\end{gathered} \tag{5.2}
$$

where $\lfloor x \rfloor$ denotes the integer part of x. For $N = 9$, this gives

$$
\widetilde{B}_{N=9} = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & \omega_3^2 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & \omega_3 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \omega_3^2 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \omega_3 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \omega_3^2 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \omega_3
\end{pmatrix}, \ \omega_3 = e^{2\pi i/3}. \tag{5.3}
$$

The model has been obtained "by hand", by replacing "lower order" terms in the matrix B_h by 0, keeping only nonzero elements on the "tilted diagonals", and replacing $(1-e^{2\pi i(\pm 1)/3})/(N(1-e^{2i\pi(\pm 1)/N}))$ by 1.

The new matrix $\widetilde{B}_h$ retains some qualitative features of B_h but there is no immediate connection between their spectra: the "lower order" terms are not small enough for that, and B_h cannot be considered as a "small perturbation" of $\widetilde{B}_h$.

The simplicity of the matrices $\widetilde{B}_h$ will allow us to prove (in the case $N = 3^k$, $k \in \mathbb{N}$) the fractal Weyl law which we could numerically observe for B_h (see Sect. 5.2). It is interesting to notice that the simplified operator $\widetilde{B}_h$ is in fact not associated with the same classical relation as B_h:

Proposition 5.1. *In the notations of Sect.* 4.2, *the quantum relation* $\{\widetilde{B}_h\}$ *is associated with the weighted relation* $(\widetilde{B}, \tilde{\mu})$ *given by* (3.12) *and* (3.13)*:*

$$
\widetilde{B}_h \in I(\widetilde{B}' \cup \widetilde{\Lambda}_{\rm sing}, \tilde{\mu}), \qquad \textit{where}
$$

$$
\widetilde{\Lambda}_{\rm sing} = \bigcup_{j=0}^{2} \widetilde{\Lambda}_{\rm sing,j}, \qquad \widetilde{\Lambda}_{\rm sing,j} = \left\{ \left(q' = 0, q = j/3\,;\ p',\ p\right),\ \ p',\ p \in \mathbb{I} \right\}.
$$

Proof. In place of $\widetilde{B}_h$ we will consider $\widetilde{A}_h = \mathcal{F}_N \circ \widetilde{B}_h$, and apply Lemma 4.14. From the structure of $\widetilde{B}_h$, the operator $\widetilde{A}_h$ can obviously be split into $\widetilde{A}_{1,h} + \widetilde{A}_{2,h}$. We will analyze the first component in detail, the analysis for the second one being similar. The matrix $\langle Q_k|\widetilde{A}_{1,h}|Q_j\rangle$ is nonzero in the vertical strip $I_1 \times \mathbb{I}$, with $I_1 = [0, 1/3)$:

$$
\langle Q_k|\widetilde{A}_{1,h}|Q_j\rangle = \frac{1\!\!1_{I_1}(Q_j)}{\sqrt{3N}} \left(\sum_{\ell=0}^{2} e^{-2i\pi Q_k \ell} \right) \exp(-6i\pi N Q_k Q_j).
$$

Like $A_{1,h}$ (see §4.5), this operator is of the form (4.35), with generating function $S(q', q) = -S_1(q', q) = -3q'q$ and discontinuous symbol

$$a(q', q) = 1\!\!1_{I_1}(q)\,\frac{e^{-2i\pi q'}}{\sqrt{3}}\,\frac{\sin(3\pi q')}{\sin(\pi q')}.$$

Forgetting about discontinuities, $\widetilde{A}_{1,h}$ is therefore associated with the graph

$$\widetilde{\Gamma}_{S_1} = \left\{(q',\ p' = -3q; q,\ p = -3q'),\ :\ q' \in \mathbb{I},\ q \in I_1\right\},$$

and the weight

$$\mu_{S_1} = |a(q', q)|^2\, dq'\, dq = 1\!\!1_{I_1}(q)\,\frac{\sin^2(3\pi q')}{3\sin^2(\pi q')}\, dq'\, dq.$$

After applying the transformation of Lemma 4.14, this leads to the graph

$$F_L^{-1}(\widetilde{\Gamma}_{S_1}) = \left\{(q' = 3q,\ q\,;\, p',\ p = 3p'),\ :\ q \in I_1,\ p' \in \mathbb{I}\right\} = \bigcup_{j=0}^{2} \widetilde{B}_{1j},$$

and the weight

$$F_{L*}^{-1}\,\mu_{S_1} = 1\!\!1_{I_1}(q)\,\frac{\sin^2(3\pi p')}{3\sin^2(\pi p')}\, dp'\, dq.$$

Through the change of variable $(q, p') \mapsto (q, p)$, we see that this is the weight (3.13) on the component $\widetilde{B}_1$.

The discontinuities of $a(q', q)$ only occur along the two segments $\{(q' \in \mathbb{I},\ q = 0)\}$, $\{(q' \in \mathbb{I},\ q = 1/3)\}$: they generate the singular Lagrangian

$$\mathcal{D}_{\mathrm{sing},j} = \left\{\left(q' = 0,\ q = \frac{j}{3};\ p' \in \mathbb{I},\ p \in \mathbb{I}\right)\right\}, \quad j = 0, 1,$$

which transforms under F_L^{-1} into the components $\widetilde{\Lambda}_{\mathrm{sing},0}$, $\widetilde{\Lambda}_{\mathrm{sing},1}$.

Similarly, the second part of the matrix, $\widetilde{B}_{2,h}$, is associated to the twisted graph $\widetilde{B}_2'$ with weight $\tilde{\mu}_{|\widetilde{B}_2}$ and the two singular components $\widetilde{\Lambda}_{\mathrm{sing},2}$, $\widetilde{\Lambda}_{\mathrm{sing},0}$. □

As explained in Sect. 3.4, the graph $\widetilde{B}$ can be obtained by adjoining to each point $(\rho'; \rho) \in B$ the points $(\rho' + (0, 1/3); \rho)$ and $(\rho' + (0, 2/3); \rho)$. This "aliasing" is due to the diffraction created by the sharp cutoff in the matrix $\widetilde{B}_h$, as opposed to the "smooth" decay of coefficients in B_h. A similar aliasing was observed in [56] for the graph associated with the unitary matrices A_{2^k} defined in (5.9): instead of quantizing the standard 2-baker (3.2), they are associated with a multivalued map obtained from it by aliasing. This observation was obtained using the propagation of coherent states.

Both B and $\widetilde{B}$ share the same forward trapped set $\widetilde{\Gamma}_- = \Gamma_- = C \times \mathbb{I}$ (see Section 3.3), but the backwards trapped set of $\widetilde{B}$ is easily shown to be $\widetilde{\Gamma}_+ = \mathbb{T}^2$, which drastically differs from Γ_+. This asymmetry between $\widetilde{\Gamma}_-$ and $\widetilde{\Gamma}_+$ reflects the fact that, unlike B, the relation $\widetilde{B}$ is not time reversal symmetric.

The fact that $\widetilde{B}_h$ is not associated with the relation B should not bother us too much though. In the next section, we will give a more "formal" construction of the matrix $\widetilde{B}_h$, in the case where N is a power of 3 (this construction will also hold for any symmetric D-baker, for N a power of D). We will show that this matrix naturally appears through a "nonstandard" (Walsh) quantization of the open 3-baker relation B.

5.1. Walsh quantization of the baker's relation. The *Walsh model* of harmonic analysis has been originally devoted to fast signal processing [29]. It has been used recently in mathematics to obtain simpler (and provable) versions of statements of the usual harmonic analysis — see [36] for an application in scattering theory and for pointers to the recent literature. The major advantage of Walsh harmonic analysis is the possibility to completely localize a wavepacket both in position and momentum: for our problem, this has the effect of avoiding diffraction problems due to the discontinuities of the map, which spoil the usual semiclassics [46]. Closer to our context, Meenakshisundaram and Lakshminarayan recently used the Hadamard Fourier transform (which is related with the Walsh transform we give below) to analyze the multifractal structure of some eigenstates of the (unitary) quantum 2-baker B_h [33].

5.1.1. The quantum torus as a system of quantum Dits. We first fix the coefficient $D \in \mathbb{N}$ ($D \geq 2$) of the symmetric baker's relation (3.8), and will consider in this section only the inverse Planck's constants of the form $N = D^k$ for some $k \in \mathbb{N}$. In this case, integers $j \in \mathbb{Z}_{D^k} = \{0, \ldots, D^k - 1\}$ are in one-to-one correspondence with the words $\boldsymbol{\epsilon} = \epsilon_1 \epsilon_2 \cdots \epsilon_k$ made of symbols (or "*D*its") $\epsilon_\ell \in \mathbb{Z}_D$:

$$\mathbb{Z}_{D^k} \ni j = \sum_{\ell=1}^{k} \epsilon_\ell \, D^{k-\ell}. \tag{5.4}$$

The natural order for $j \in \mathbb{Z}_{D^k}$ corresponds to the lexicographic order for the symbolic words $\{\boldsymbol{\epsilon} \in (\mathbb{Z}_D)^k\}$. This way, each position eigenstate $|Q_j\rangle$ of the basis (4.3) can be associated with the unique symbolic sequence $\epsilon_1\epsilon_2 \cdots \epsilon_k$ which gives its *D*nary expansion

$$Q_j = \frac{j}{N} = 0 \cdot \epsilon_1 \epsilon_2 \cdots \epsilon_k. \tag{5.5}$$

Let us denote the canonical basis of $\mathbb{C}^D$ by $\{e_0, e_1, \ldots, e_{D-1}\}$. Then, each $|Q_j\rangle$ can be written as

$$|Q_j\rangle = e_{\epsilon_1} \otimes e_{\epsilon_2} \otimes \cdots \otimes e_{\epsilon_k}. \tag{5.6}$$

Following [48], we denote each $|Q_j\rangle$ by $|\boldsymbol{\epsilon}\rangle = |\epsilon_1 \epsilon_2 \cdots \epsilon_k\rangle$ to emphasize the above tensor product decomposition. This way, the quantum space $\mathcal{H}_h^1$ is naturally identified with the *tensor product* of k spaces $\mathbb{C}^D$:

$$\mathcal{H}_h^1 = (\mathbb{C}^D)_1 \otimes (\mathbb{C}^D)_2 \otimes \cdots \otimes (\mathbb{C}^D)_k.$$

In the quantum computating framework, each space $(\mathbb{C}^D)_\ell$ is interpreted as a "quantum *D*it", or " qu*D*it", and the basis $\{|\boldsymbol{\epsilon}\rangle\}$ is called the *computational basis* [35]. Viewed in our toral phase space, the qu*D*it $(\mathbb{C}^D)_\ell$ is associated with the *scale* $D^{-\ell}$ in the position variable, so $(\mathbb{C}^D)_1$ is called the "most significant qu*D*it".

5.1.2. Walsh Fourier transform. The discrete Fourier transform of (4.5) (with $n = 1$, $N = D^k$) is the Fourier transform (in the sense of abstract harmonic analysis) on the group $\mathbb{Z}_{D^k}$. More explicitly, each row of $\mathcal{F}_{D^k}$ corresponds to the character $j' \mapsto \exp\Big(- 2i\pi jj'/D^k \Big)$ of $\mathbb{Z}_{D^k}$. Using (5.4), the matrix elements can be factorized:

$$(\mathcal{F}_{D^k})_{jj'} = D^{-k/2} \exp\Big(- 2i\pi \frac{jj'}{D^k} \Big) = D^{-k/2} \prod_{\ell=1}^{k} \exp\Big(- 2i\pi \frac{\epsilon_\ell(jj')}{D^\ell} \Big). \tag{5.7}$$

Notice that each $\epsilon_m(jj')$ can be easily expressed in terms of the symbols of j and j':

$$\epsilon_m(jj') = \sum_{\ell+\ell'=k+m} \epsilon_\ell(j)\,\epsilon_{\ell'}(j').$$

The Walsh Fourier transform is the Fourier transform on the group $(\mathbb{Z}_D)^k$. It can be defined by keeping only the first factor on the right-hand side of (5.7): one obtains the matrix

$$(W_k)_{jj'} = D^{-k/2}\,\exp\Big(-2i\pi\frac{\epsilon_1(jj')}{D}\Big) = \prod_{\ell=1}^{k} D^{-1/2}\,\omega_D^{-\epsilon_\ell(j)\epsilon_{k+1-\ell}(j')}, \quad \omega_D = e^{2i\pi/D}. \tag{5.8}$$

Using the identification $\mathcal{H}_h^1 \simeq (\mathbb{C}^D)^{\otimes k}$, this definition can be recast as follows.

Lemma 5.2. *The Walsh Fourier transform W_k acts simply on tensor product states:*

$$W_k(v_1 \otimes \cdots \otimes v_k) = \mathcal{F}_D v_k \otimes \cdots \otimes \mathcal{F}_D v_1, \quad v_\ell \in \mathbb{C}^D,\ \ell = 1,\ldots,k.$$

Here $\mathcal{F}_D = W_1$ is the discrete Fourier transform on $\mathbb{C}^D$. As a result, W_k is a unitary tranformation on $\mathcal{H}_h^1$.

The proof consists in a straightforward algebraic check.

As opposed to the discrete Fourier transform, the Walsh Fourier transform *does not entangle* the different qu$D$its: a tensor product state is sent to another tensor product state.

Example. To illustrate this simple lemma we take $D = 2$, and consider the following $2^k \times 2^k$ matrix, $k \geq 1$:

$$A_{2^k} = \big[A_{0,2^k}, A_{1,2^k}\big], \quad (A_{j,2^k})_{0\leq n\leq 2^k-1,\,0\leq m\leq 2^{k-1}-1} = \begin{cases} (-1)^{jn}/\sqrt{2}, & m = \lfloor n/2\rfloor \\ 0, & m \neq \lfloor n/2\rfloor. \end{cases} \tag{5.9}$$

For instance when $k = 2$ we get

$$A_{2^2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

This sequence of matrices has been obtained as the "extreme" possibility among a family of different quantizations of the (closed) 2-baker's map [48][6], and its semiclassical properties were further studied in [56]. In a different context, this (unitary) matrix belongs to the family of transfer matrices associated with the de Bruijn graph with 2^k vertices [55].

The transformation A_{2^k} acts as follows on tensor product states:

$$v_1 \otimes \cdots \otimes v_k \longmapsto v_2 \otimes \cdots \otimes v_k \otimes \mathcal{F}_2 v_1.$$

[6] We thank M. Saraceno for pointing out this interpretation to us.

This implies that this matrix can be easily expressed in terms of the Walsh Fourier transform (for $D = 2$):

$$A_{2^k} = W_k \begin{pmatrix} W_{k-1} & 0 \\ 0 & W_{k-1} \end{pmatrix}, \tag{5.10}$$

where the 2×2 block structure corresponds to the most significant (leftmost) qubit. This expression exactly parallels the one defining the Balazs-Voros (unitary) quantum baker [2]. Compared to this "usual" quantum baker, A_{2^k} is thus obtained by replacing the discrete Fourier matrices $\mathcal{F}_{2^k}$, $\mathcal{F}_{2^{k-1}}$ by their Walsh analogues W_k, W_{k-1}.

The matrix A_{2^k} is unitary; as we will see in the next section, our toy model $\widetilde{B}_h$ for the quantum open 3-baker (see Eq. (5.2)) is its subunitary analogue.

5.2. Resonances for the Walsh quantization of the open baker relation. In this section we set $D = 3$, and concentrate on the symmetric 3-baker (1.3). By analogy with the example in the last section, we modify the quantization (4.38,5.1), in the case $N = 3^k$, by replacing the discrete Fourier matrices by their Walsh analogues. The resulting operator exactly coincides with the toy model (5.2) introduced in the beginning of this section:

Lemma 5.3. *In the case $N = 3^k$, the matrix $\widetilde{B}_h$ defined in* (5.2) *can be rewritten in terms of the Walsh Fourier transforms as follows:*

$$\widetilde{B}_h = W_k^* \begin{pmatrix} W_{k-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & W_{k-1} \end{pmatrix}.$$

We omit the simple algebraic proof. If we define the "truncated" inverse Fourier matrix

$$\widetilde{\mathcal{F}}_3^* \stackrel{\text{def}}{=} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & \omega_3^2 \\ 1 & 0 & \omega_3 \end{pmatrix}, \tag{5.11}$$

the toy model $\widetilde{B}_h$ acts as follows on tensor product states:

$$\widetilde{B}_h(v_1 \otimes \cdots \otimes v_k) = v_2 \otimes v_3 \otimes \cdots \otimes \widetilde{\mathcal{F}}_3^* v_1. \tag{5.12}$$

This form is particularly nice to compute the spectrum of $\widetilde{B}_h$. We start by computing the spectrum of its power $(\widetilde{B}_h)^k$, which is enough to obtain the radial distribution of resonances (that is, the distribution of *resonance widths*).

Proposition 5.4. *Let $\lambda_\pm$, $|\lambda_-| < |\lambda_+|$, be the eigenvalues of the matrix*

$$\Omega_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 1 & \omega_3 \end{pmatrix}.$$

The non-zero eigenvalues of $(\widetilde{B}_h)^k$ (for $N = (2\pi h)^{-1} = 3^k$) are given by $\lambda_+^{k-p} \lambda_-^p$, $0 \le p \le k$, each occurring with multiplicity $\binom{k}{p}$. From this we get the radial distribution of the eigenvalues of $\widetilde{B}_h$ (counted with multiplicities):

$$\begin{gathered} \forall\, r \in [0,1], \qquad \tfrac{1}{2^k} \quad \#\left\{\lambda \in Spec(\widetilde{B}_h) \,:\, |\lambda| \ge r\right\} \xrightarrow{k\to\infty} C(r), \\ C(r) = \begin{cases} 1, & r < |\det \Omega_3|^{\frac{1}{2}} \\ 0, & r > |\det \Omega_3|^{\frac{1}{2}}. \end{cases} \end{gathered} \tag{5.13}$$

Hence the nontrivial resonances accumulate near the circle of radius $r_0(\widetilde{B}) = |\det \Omega_3|^{\frac{1}{2}}$.

This proposition gives Theorem 1, where $\widetilde{B}$ is the baker's relation described in Proposition 5.1, $\widetilde{B}_h$ the matrices (5.2), and Planck's constants are taken along the sequence $\{h_k = (2\pi \times 3^k)^{-1},\ k \in \mathbb{N}\}$.

Proof. From the expression (5.12), we see that

$$(\widetilde{B}_h)^k (v_1 \otimes \cdots \otimes v_k) = \widetilde{\mathcal{F}}_3^* v_1 \otimes \cdots \otimes \widetilde{\mathcal{F}}_3^* v_k.$$

That means that $(\widetilde{B}_h)^k = (\widetilde{\mathcal{F}}_3^*)^{\otimes k}$, so one eigenbasis is obtained by taking the tensor products of eigenstates of $\widetilde{\mathcal{F}}_3^*$, and the eigenvalues of $(\widetilde{B}_h)^k$ are the corresponding products of eigenvalues of $\widetilde{\mathcal{F}}_3^*$. The nonzero eigenvalues λ_+, λ_- of $\widetilde{\mathcal{F}}_3^*$ are the eigenvalues of Ω_3, so the first part of the proposition follows. To prove the second part, notice that each eigenvalue $\lambda_+^{k-p}\lambda_-^p$ of $\widetilde{B}_h^k$ corresponds to an eigenvalue (possibly in the generalized sense) of modulus $|\lambda_+^{1-p/k}\lambda_-^{p/k}|$ of $\widetilde{B}_h$. Therefore, we are able to count eigenvalues of $\widetilde{B}_h$ (with multiplicities) in a given annulus.

Let $H(t)$ denote the Heaviside function, $H(t) = 0$ for $t < 0$, and $H(t) = 1$ otherwise. Then, for any $0 < r < 1$,

$$\begin{aligned}\#\left\{\lambda \in \mathrm{Spec}(\widetilde{B}_h) \ :\ |\lambda| \geq r\right\} &= \sum_{p=0}^{k} H(|\lambda_+|^{1-p/k}|\lambda_-|^{p/k} - r)\binom{k}{p} \\ &= \sum_{p=0}^{k} H(-p/k + 1/2 + \rho)\binom{k}{p}, \quad \rho = \frac{\log(|\lambda_-\lambda_+|^{\frac{1}{2}}/r)}{\log(|\lambda_+|/|\lambda_-|)}.\end{aligned}$$

Using Stirling's formula, one easily gets in the limit $k \to \infty$:

$$\frac{1}{2^k}\sum_{p=0}^{k} H(-p/k + 1/2 + \rho)\binom{k}{p} \sim \sqrt{\frac{2k}{\pi}}\int_{-\infty}^{\rho} e^{-2kx^2}\,dx \to H(\rho).$$

This expression shows that the distribution of resonances is semiclassically dominated by the degrees $|p - k/2| = \mathcal{O}(k^{1/2})$, and proves the second part of the proposition. $\square$

The explicit eigenvalues are $\lambda_\pm = \frac{1+i\sqrt{3}}{4\sqrt{3}} \pm \sqrt{\frac{11-i3\sqrt{3}}{24}}$, with approximate values

$$\lambda_+ \approx 0.8390 + i0.0942, \quad |\lambda_+| \approx 0.8443, \quad \lambda_- \approx -0.5504 + i4058, \quad |\lambda_-| \approx 0.6838.$$

The geometric mean of their moduli is $r_0(\widetilde{B}_h) = |\lambda_-\lambda_+|^{1/2} = \sqrt{|\det \Omega_3|} = 3^{-1/4}$.

We need to analyze the spectrum of $\widetilde{B}_h$ more precisely to show that the distribution of resonances is asymptotically uniform with respect to the angular variable.

Proposition 5.5. *Let* $h = (2\pi 3^k)^{-1}$. *As a set, the nontrivial spectrum of* $\widetilde{B}_h$ *is given by*

$$\{\lambda_+\} \cup \{\lambda_-\} \cup \bigcup_{\omega^k=1} \{\omega\lambda_+^{1-p/k}\lambda_-^{p/k} \ :\ 1 \leq p \leq k-1\}.$$

For each $p \neq 0, k$, the k eigenvalues asymptotically have the same degeneracy $\frac{1}{k}\binom{k}{p}$, which shows that their distribution is uniform in the angular variable. Therefore, for any continuous function $f \in C(D(0,1))$ we have (counting multiplicities in the LHS):

$$\frac{1}{2^k} \sum_{0 \neq \lambda \in Spec(\widetilde{B}_h)} f(\lambda) \xrightarrow{k\to\infty} \int_0^{2\pi} f(|\lambda_-\lambda_+|^{\frac{1}{2}}, \theta) \, \frac{d\theta}{2\pi}.$$

Proof. To classify the nontrivial spectrum of $\widetilde{B}_h$, we will use the eigenvectors $v_\pm$ of $\widetilde{F}_3^*$ associated with the eigenvalues $\lambda_\pm$. Call

$$\{\boldsymbol{\eta} = \eta_1\eta_2 \cdots \eta_k \;:\; \eta_\ell \in \{\pm\}\} \simeq (\mathbb{Z}_2)^k$$

the set of binary sequences of length k. The number of symbols $\eta_\ell = -$ in the sequence $\boldsymbol{\eta}$ is called the *degree* of $\boldsymbol{\eta}$. The *cyclic shift* τ acts on these sequences as $\tau(\eta_1 \cdots \eta_k) = \eta_2 \cdots \eta_k \eta_1$. The shift allows us to partition $(\mathbb{Z}_2)^k$ into periodic orbits, each orbit $\mathcal{O} = \{\boldsymbol{\eta}, \tau(\boldsymbol{\eta}), \ldots, \tau^{\ell_{\mathcal{O}}-1}(\boldsymbol{\eta})\}$ being of (primitive) period $\ell_{\mathcal{O}} = \ell_{\boldsymbol{\eta}}$. Since $\tau^k = id$, the primitive period must divide k. We call $\deg(\mathcal{O})$ the common degree of the elements of $\mathcal{O}$ and observe that

$$k \mid \ell_{\mathcal{O}} \deg(\mathcal{O}). \tag{5.14}$$

To each sequence $\boldsymbol{\eta}$ we associate the state $|\boldsymbol{\eta}\rangle \stackrel{\text{def}}{=} v_{\eta_1} \otimes v_{\eta_2} \otimes \cdots \otimes v_{\eta_k}$, which is obviously an eigenstate of $(\widetilde{B}_h)^k$, with eigenvalue $\lambda_+^{k-\deg(\boldsymbol{\eta})} \lambda_-^{\deg(\boldsymbol{\eta})}$. These 2^k states form an independent family, which span the nontrivial eigenspaces of $\widetilde{B}_h$. This operator acts very simply on these states:

$$\forall \boldsymbol{\eta} \in (\mathbb{Z}_2)^k, \qquad \widetilde{B}_h |\boldsymbol{\eta}\rangle = \lambda_{\eta_1} |\tau(\boldsymbol{\eta})\rangle.$$

Hence, for any orbit $\mathcal{O}$, $\widetilde{B}_h$ leaves invariant the $\ell_{\mathcal{O}}$-dimensional subspace $V_{\mathcal{O}} \stackrel{\text{def}}{=} \operatorname{span}\{|\boldsymbol{\eta}\rangle, \ \boldsymbol{\eta} \in \mathcal{O}\}$. To compute the spectrum of $\widetilde{B}_h|_{V_{\mathcal{O}}}$ we first observe that it is contained in the set of k^{th} roots of $\lambda_+^{k-\deg(\mathcal{O})} \lambda_-^{\deg(\mathcal{O})}$, which in view of (5.14) is equal to

$$S_{\mathcal{O}} \stackrel{\text{def}}{=} \left\{ \omega_{\ell_{\mathcal{O}}}^j \, \lambda_+^{1-\deg(\mathcal{O})/k} \lambda_-^{\deg(\mathcal{O})/k}, \ j = 0, \ldots, \ell_{\mathcal{O}} - 1 \right\}.$$

We claim that $\operatorname{Spec}(\widetilde{B}_h|_{V_{\mathcal{O}}}) = S_{\mathcal{O}}$ (clearly with no degeneracies). In fact, let $\Omega_{\mathcal{O}} : V_{\mathcal{O}} \to V_{\mathcal{O}}$ be defined by $\Omega_{\mathcal{O}} |\tau^\ell(\boldsymbol{\eta})\rangle = \omega_{\ell_{\mathcal{O}}}^{-\ell} |\tau^\ell(\boldsymbol{\eta})\rangle$, for a choice of $\boldsymbol{\eta} \in \mathcal{O}$. This operator is invertible on $V_{\mathcal{O}}$. By a verification on basis elements,

$$\widetilde{B}_h|_{V_{\mathcal{O}}} \circ \Omega_{\mathcal{O}} = \omega_{\ell_{\mathcal{O}}} \, \Omega_{\mathcal{O}} \circ \widetilde{B}_h|_{V_{\mathcal{O}}},$$

and hence if $\lambda \in \operatorname{Spec}(\widetilde{B}_h|_{V_{\mathcal{O}}})$ then $\omega_{\ell_{\mathcal{O}}}^j \lambda \in \operatorname{Spec}(\widetilde{B}_h|_{V_{\mathcal{O}}})$ for any j.

Since $\mathcal{O} \neq \mathcal{O}' \Longrightarrow V_{\mathcal{O}} \cap V_{\mathcal{O}'} = \{\underline{0}\}$, enumerating the orbits decomposition of $(\mathbb{Z}_2)^k$ yields the full nontrivial spectrum of $\widetilde{B}_h$. In spite of the large degeneracies, this nontrivial spectrum does not contain any Jordan block.

The degree $p = 0$ corresponds to the unique orbit $\mathcal{O} = \{\boldsymbol{\eta} = ++\cdots+\}$, so the eigenvalue λ_+ is simple. Similarly, the degree $p = k$ leads to the simple eigenvalue λ_-.

For any degree $1 \leq p \leq k-1$, call $g = \gcd(k, p)$. The sequences of degree p will take all possible periods $\ell_{\boldsymbol{\eta}} = k/\ell$, where $\ell \in \mathbb{N}$, $\ell | g$. We show below that, in the semiclassical limit, the huge majority of the sequences of any degree $p \neq 0, k$ have primitive period k.

Lemma 5.6. *There exists $C > 0$, $K > 0$ s.t., for any $k \geq K$ and any degree $1 \leq p \leq k-1$,*

$$\frac{\#\left\{\boldsymbol{\eta} \in (\mathbb{Z}_2)^k \;:\; \deg(\boldsymbol{\eta}) = p, \ell_{\boldsymbol{\eta}} < k\right\}}{\#\left\{\boldsymbol{\eta} \in (\mathbb{Z}_2)^k \;:\; \deg(\boldsymbol{\eta}) = p\right\}} \leq C\,\frac{\log k}{k}.$$

Proof. We still use $g = \gcd(k, p)$.

If $g = 1$, then all orbits of degree p are of primitive period k.

If $g > 1$, there exists $\ell > 1$, $\ell | g$. For any P prime divisor of ℓ, any sequence of primitive period $\ell_{\boldsymbol{\eta}} = k/\ell$ is also of (nonnecessarily primitive) period k/P. Any sequence of degree p and (nonnecessarily primitive) period k/P can be seen as the P repetitions of a sequence of k/P bits, among which p/P take the value $(-)$. Therefore, the number of such sequences is exactly $\binom{k/P}{p/P}$. As a consequence, we have

$$\frac{\#\left\{\boldsymbol{\eta} \in (\mathbb{Z}_2)^k \;:\; \deg(\boldsymbol{\eta}) = p, \ell_{\boldsymbol{\eta}} < k\right\}}{\#\left\{\boldsymbol{\eta} \in (\mathbb{Z}_2)^k \;:\; \deg(\boldsymbol{\eta}) = p\right\}} \leq \sum_{P \text{ prime},\; P|g} \frac{\binom{k/P}{p/P}}{\binom{k}{p}}. \tag{5.15}$$

We will now estimate each term in the above sum. From the symmetry $\binom{k}{p} = \binom{k}{k-p}$, we can assume $p \leq k/2$. Expanding the coefficient $\binom{k}{p}$ into

$$\binom{k}{p} = \frac{k(k-1)\cdots(k-p+1)}{p(p-1)\cdots 1},$$

we notice that both the numerator and the denominator contain exactly p/P factors which are multiples of P. Their ratio gives $\binom{k/P}{p/P}$, while the ratio of the remaining factors is

$$\begin{aligned}\frac{\binom{k}{p}}{\binom{k/P}{p/P}} &= \frac{(k-1)\cdots(k-P+1)(k-P-1)\cdots(k-p+1)}{(p-1)\cdots(p-P+1)(p-P-1)\cdots 1}\\ &\geq \frac{k-p+1}{1} \geq \frac{k}{2} + 1.\end{aligned}$$

Here we used the fact that each factor $(k-m)/(p-m) > 1, 0 \leq m \leq p-2$, and only kept explicit the last factor. The last inequality comes from $p \leq k/2$.

We have obtained a uniform upper bound for each term in the sum of (5.15). By standard arguments, there exists K, $\widetilde{C} > 0$ s.t. the number of prime factors of any $k \geq K$ is $\leq \widetilde{C} \log k$, so the number of terms in the sum is $\leq \widetilde{C} \log k$. As a result, (5.15) is bounded from above by $\widetilde{C} \log k/(k+2)$, which proves the lemma. □

This lemma shows that the orbits of period $\ell_{\mathcal{O}} < k$ have a negligible contribution to the asymptotic density of resonances. We can therefore act as if, for any $1 \leq p \leq k-1$, each orbit of degree p had period k, leading to the k eigenvalues $\left\{\omega_k^j \, \lambda_+^{1-p/k} \lambda_-^{p/k},\; j = 0, \ldots, k-1\right\}$. In the semiclassical limit, these k eigenvalues are uniformly distributed on the circle of radius $|\lambda_+^{1-p/k} \lambda_-^{p/k}|$, and each of them has multiplicity $\frac{1}{k}\binom{k}{p}$. This shows that the asymptotic resonance distribution is circular-symmetric, with the radial distribution described in Proposition 5.4. □

Remark 5.1. Several features of the (nontrivial) spectrum of $\widetilde{B}_h$ are very different from what one expects for a random subunitary matrix of size $2^k \times 2^k$: the (logarithms of the) resonances form a regular lattice, most eigenvalues are highly degenerate, and the radial density is a delta function at $r_0(\widetilde{B})$. Actually, the only generic feature seems to be the global fractal scaling of the Weyl law, and the uniform angular distribution.

Remark 5.2. The radial density of resonances is governed by $r_0(\widetilde{B}) = \sqrt{|\det \Omega_3|}$, which seems to depend on the subtleties of the quantization. As an example of this fact, in Section 6 we will consider the open baker's map with $D = 4$, which we call B, obtained by keeping only the second and third strips. It has Lyapounov exponent $\log 4$, and the Cantor set C (see §3.3) has dimension $\nu = \log 2/\log 4 = 1/2$. The open map B' obtained by removing the first and third strips has the same characteristics. However, if we Walsh-quantize these two maps, the spectra of $\widetilde{B}_h$ and $\widetilde{B}'_h$ are very different. These spectra are obtained from the eigenvalues of different 2×2 blocks of the inverse Fourier matrix $\mathcal{F}_4^*$. The first map leads to the block

$$\Omega_4 = \frac{1}{2}\begin{pmatrix} i & -1 \\ -1 & 1 \end{pmatrix},$$

with two nonzero eigenvalues $\lambda_\pm$ of different moduli, so the spectrum of $\widetilde{B}_h$ will satisfy the fractal Weyl law, and be concentrated around the circle of radius

$$r_0(\widetilde{B}) = \sqrt{|\det \Omega_4|} = 2^{-3/4}.$$

In an opposite way, the second map leads to the singular block

$$\Omega'_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The nontrivial spectrum of $\widetilde{B}'_h$ then reduces to a simple eigenvalue $\lambda_+ = 1$. In that case, the Weyl law is singular, corresponding to the profile function $C(r) \equiv 0$. This qualitative difference between both spectra cannot be explained from the classical maps, but is due to the phases in the matrix elements of the quantum maps.

6. Conductance in the Walsh Model

6.1. Quantum transport. In this section, we consider open baker's relations for which the "opening" consists in two disjoint intervals, which are supposed to represent two "leads" connecting a quantum dot to the outside world. We will prove Theorem 2 in this setting: (1.1) in §6.2 and (1.2) in §6.3.

The baker's relations defined in §3.3 can all be seen as truncations of invertible maps on $\mathbb{T}^2$. More precisely there exists an invertible baker's map, $\kappa : \mathbb{T}^2 \to \mathbb{T}^2$, such that the graph $B = B_1 \cup B_2$ of the open baker's map is

$$B = \Gamma_\kappa \cap \{(q, p) \, : \, q \in I_1 \cup I_2 = I, \; p \in \mathbb{I}\}.$$

For admissible values of N, one can quantize the closed map κ into a unitary transformation $U_h = U_{\kappa,h}$ on $\mathcal{H}_h^1$ by straightforwardly generalizing the method of [2]. Multiplying this unitary operator by the quantum projector $\Pi_I = \sum_{Q_j \in I} |Q_j\rangle\langle Q_j|$, we obtain the quantum open baker's map (4.38)

$$B_h = U_{\kappa,h} \, \Pi_I.$$

To obtain an agreement with the notations of §2.4.3, we can interpret the set $I = I_1 \cup I_2$ as the "wall" of the quantum dot, while the complementary interval $L = \mathbb{I} \setminus I$ represents the "openings" of the dot, perfectly connected with the "leads". In the previous sections, we studied the resonances, that is, the eigenvalues of $B_h = U_{\kappa,h}\Pi_I$ (or of $\widetilde{B}_h$ when choosing the Walsh quantization). Now, we want to study the "transport" through the dot, using the formalism presented in §2.4.3. We assume that the opening L splits into two disjoint "leads" $L = L_1 \cup L_2$, and we study the transmission matrix from lead L_1 to lead L_2 (for simplicity, both leads will have the same width). This matrix is obtained by decomposing the scattering matrix (2.15)

$$\tilde{S}(\vartheta) = \Pi_L \sum_{n\geq 0} \left(e^{i\vartheta}\, U_h\, \Pi_I\right)^n e^{i\vartheta} U_h\, \Pi_L$$

into 4 blocks, according to the decomposition $\Pi_L = \Pi_{L_1} \oplus \Pi_{L_2}$. The transmission matrix from L_1 to L_2 is defined as the block

$$t(\vartheta) = \sum_{n\geq 1} e^{in\vartheta}\, \Pi_{L_2}\, U_h\, (\Pi_I\, U_h)^{n-1}\, \Pi_{L_1} \stackrel{\text{def}}{=} \sum_{n\geq 1} e^{in\vartheta}\, t_n. \tag{6.1}$$

Because Π_{L_1} and Π_{L_2} have the same rank $M = N|L_1|$, $t(\vartheta)$ is a square matrix of size M. According to Landauer's theory of coherent transport, each eigenvalue $T_i(\vartheta)$ of the matrix $t^*(\vartheta)t(\vartheta)$ corresponds to a "transmission channel". The dimensionless conductance of the system is then given by the sum over these transmission eigenvalues:

$$g(\vartheta) = \operatorname{tr}\big(t^*(\vartheta)t(\vartheta)\big). \tag{6.2}$$

A transmission channel is "classical" if the eigenvalue T_i is very close to unity (perfect transmission) or close to zero (perfect reflection). The intermediate values correspond to the "nonclassical channels", the importance of which is reflected in the noise power

$$P(\vartheta) = \operatorname{tr}\Big(t^*t(\vartheta)\big(Id - t^*t(\vartheta)\big)\Big), \quad \text{or the Fano factor} \quad F(\vartheta) = \frac{P(\vartheta)}{g(\vartheta)}. \tag{6.3}$$

In general it may be necessary to perform the "ensemble averaging" (averaging over ϑ) to obtain significant results [57]. However, for the model we will study below, both conductance and noise power will depend very little on ϑ, so this averaging will not be necessary. To alleviate notations we will suppress the dependence in ϑ in the transmission matrix t.

Our model. In the remainder of this section, we will compute the quantities characterizing the transport through the "dot" when U_h is a Walsh-quantized baker's map similar to the operator (5.9), but with $D = 4$ instead of $D = 2$. The sequence of values of h consequently is given by

$$2\pi h_k = 4^{-k}, \quad k = 1, 2, \cdots .$$

We will choose the two leads $L_1 = [0, 1/4]$ and $L_2 = [3/4, 1]$: this way, the projectors Π_{L_i} and $\Pi_I = Id - \Pi_{L_1} - \Pi_{L_2}$ can be represented as tensor product operators:

$$\begin{aligned}
\Pi_{L_1} &= \pi_0 \otimes Id_4 \otimes \cdots \otimes Id_4,\\
\Pi_{L_2} &= \pi_3 \otimes Id_4 \otimes \cdots \otimes Id_4,\\
\Pi_I &= \pi_I \otimes Id_4 \otimes \cdots \otimes Id_4,
\end{aligned}$$

where $\pi_i = |e_i\rangle\langle e_i|$ is a rank-1 orthogonal projector acting on $\mathbb{C}^4$, and we note $\pi_I = \pi_1 \oplus \pi_2$.

The "inside" propagator for this model, namely $\widetilde{B}_h = U_h \Pi_I$, is the first one among the two quantum maps mentioned in Remark 5.2: its nontrivial spectrum satisfies the fractal Weyl law with exponent $\nu = \frac{1}{2}$, and is concentrated near the radius $r_0(\widetilde{B}) = 2^{-3/4}$.

The number of *scattering channels* in each lead is the rank of Π_{L_1} (equal to that of Π_{L_2}). It is given by $\frac{1}{4}$ of the total dimension, and we denote it by

$$M(h) = \frac{1}{4}(2\pi h)^{-1} = 4^{k-1}, \qquad h \in \{h_k\}. \tag{6.4}$$

The number of channels is "macroscopic", and each channel is "fully coupled" to the leads. We are therefore in a very nonperturbative régime, where resonances have no memory at all of the eigenvalues of the closed (unitary) system.

6.2. Conductance. We will crucially use the fact that all operators under consideration act nicely on the tensor product structure $\mathcal{H}_h^1 = (\mathbb{C}^4)^{\otimes k}$, that is, they do not entangle the qu*D*its. It is then suitable to compute the trace of t^*t in a basis adapted to this tensor product, and we naturally choose the computational (or position) basis. The conductance is then given by

$$\operatorname{tr}(t^*t) = \sum_{Q_j \in L_1} \langle Q_j | t^* t | Q_j \rangle = \sum_{j=0}^{4^{k-1}-1} \| t | Q_j \rangle \|^2.$$

Let us consider an arbitrary $j = \epsilon_1 \epsilon_2 \cdots \epsilon_k$ with $\epsilon_1 = 0$, so that $0 \leq j \leq 4^{k-1} - 1$. Using (6.1) we write

$$t|Q_j\rangle = \sum_{n \geq 1} e^{in\vartheta}\, \Pi_{L_2} U_h\, (\Pi_I U_h)^{n-1}\, \Pi_{L_1} |Q_j\rangle = \sum_{n \geq 1} e^{in\vartheta}\, t_n |Q_j\rangle, \tag{6.5}$$

so that

$$\| t|Q_j\rangle \|^2 = \sum_{m,n \geq 0} e^{i(n-m)\vartheta}\, \langle Q_j | t_m^* t_n | Q_j \rangle.$$

From now on, we replace the notation $|Q_j\rangle$, $j \in [0, 4^{k-1} - 1]$, by the symbolic notation $|\boldsymbol{\epsilon}\rangle$, where the sequence $\boldsymbol{\epsilon} = 0\,\epsilon_2 \cdots \epsilon_k$ corresponds to j.

6.2.1. Classical transmission channels. To understand the action of t_n on $|\boldsymbol{\epsilon}\rangle$ we notice that $\Pi_I U_h$ acts on tensor products as

$$\Pi_I U_h(v_1 \otimes \cdots \otimes v_k) = \pi_I v_2 \otimes \cdots \otimes v_k \otimes \mathcal{F}_4^* v_1.$$

If $n < k$, we obtain

$$t_n |\boldsymbol{\epsilon}\rangle = \pi_3 e_{\epsilon_{n+1}} \otimes e_{\epsilon_{n+2}} \otimes \cdots e_k \otimes \mathcal{F}_4^* e_0 \otimes \mathcal{F}_4^* \pi_I e_{\epsilon_2} \otimes \cdots \otimes \mathcal{F}_4^* \pi_I e_{\epsilon_n}, \tag{6.6}$$

frow which we draw the

Lemma 6.1. *Consider a sequence* $\boldsymbol{\epsilon} = 0\,\epsilon_2\epsilon_3\cdots\epsilon_k$, *and assume that there exists an index* $2 \leq \ell \leq k$ *such that* $\epsilon_\ell \in \{0, 3\}$. *Let* ℓ_0 *be the smallest such index. Then*

$$\|t|\boldsymbol{\epsilon}\rangle\| = \begin{cases} 0 & \text{if } \epsilon_{\ell_0} = 0, \\ 1 & \text{if } \epsilon_{\ell_0} = 3. \end{cases}$$

This shows that $|\boldsymbol{\epsilon}\rangle$ *is a classical transmission channel.*

Proof. For any $1 \leq n \leq \ell_0 - 2$, $\epsilon_{n+1} \in \{1, 2\}$ by assumption. Hence the first qu$D$it on the right-hand side of (6.6) vanishes and $t_n|\boldsymbol{\epsilon}\rangle = 0$. Furthermore, the state $(\Pi_I U_h)^{\ell_0-1}|\boldsymbol{\epsilon}\rangle$ admits as first qu$D$it $\pi_I\, e_{\epsilon_{\ell_0}} = 0$, so that $t_n|\boldsymbol{\epsilon}\rangle = 0$ for any $n \geq \ell_0$. The only remaining term in (6.5) is $t_{\ell_0-1}|\boldsymbol{\epsilon}\rangle$:

- if $\epsilon_{\ell_0} = 0$, the first qu$D$it of that term is $\pi_3 e_{\epsilon_{\ell_0}} = 0$, so $t_{\ell_0-1}|\boldsymbol{\epsilon}\rangle = t|\boldsymbol{\epsilon}\rangle = 0$.
- if $\epsilon_\ell = 3$, $t_{\ell_0-1}|\boldsymbol{\epsilon}\rangle = e_{\epsilon_3} \otimes e_{\epsilon_{\ell_0+1}} \otimes \cdots \otimes \mathcal{F}_4^* e_0 \otimes \mathcal{F}_4^* e_{\epsilon_2} \otimes \cdots \mathcal{F}_4^* e_{\epsilon_{\ell_0-1}}$. Since $\mathcal{F}_4^*$ is unitary, $\|t_{\ell_0-1}|\boldsymbol{\epsilon}\rangle\| = \|t|\boldsymbol{\epsilon}\rangle\| = 1$. $\square$

The number of classical channels discussed in Lemma 6.1 is easy to compute: it is obtained by removing from the set $[0, 4^{k-1} - 1] \equiv \left\{\epsilon_2 \cdots \epsilon_k \in (\mathbb{Z}_4)^{k-1}\right\}$ the sequences such that $\epsilon_\ell \in \{1, 2\}$ for all $2 \leq \ell \leq k$ (these will be called "nonclassical sequences"). The number of the latter is 2^{k-1}, so the number of classical channels is $4^{k-1} - 2^{k-1}$. Among them, half are fully reflected, $t|\boldsymbol{\epsilon}\rangle = 0$, and half are fully transmitted, $\|t|\boldsymbol{\epsilon}\rangle\| = 1$. Hence the conductance through these classical channels is

$$\mathrm{tr}_{\mathrm{cl}}(t^* t) = \frac{4^{k-1} - 2^{k-1}}{2}.$$

Remark 6.1. Such classical channels are mentioned in the analysis of [57] for the transmission through an open kicked rotator. They sit in the phase space regions above the lead L_1 which are either sent back to L_1, or sent to L_2 through the classical dynamics, in a time smaller than the Ehrenfest time $T_{\mathrm{Ehr}} = \log N/(\log 4) = k$. For our baker's map B, these regions are vertical strips of widths $4^{-\ell}$, $\ell = 2, \ldots, k$ which exit to L_1 or L_2 at time ℓ. The particularity of the Walsh quantization is the *exact* full transmission (or reflection) through these channels.

6.2.2. Nonclassical transmission channels. The nonclassical channels are necessarily combinations of the position states $|\boldsymbol{\epsilon}\rangle$ with $\epsilon_\ell \in \{1, 2\}$ for all $2 \leq \ell \leq k$ ("nonclassical" sequences or states). The associated positions $4Q_j = 0 \cdot \epsilon_2\epsilon_3 \cdots \epsilon_k$ lie close to the Cantor set C, such that $\Gamma_- = C \times \mathbb{I}$ is the set of points never escaping through B or $\widetilde{B}$ (see Eq. (3.6)).

For such a state $|\boldsymbol{\epsilon}\rangle$, the term (6.6) vanishes for all $n < k$, due to the first qu$D$it $\pi_I e_{\epsilon_{n+1}} = 0$. That state therefore accomplishes k "unitary bounces" inside the cavity, before it starts to decay out of it. We first consider the terms $t_{k+m}|\boldsymbol{\epsilon}\rangle$ for $0 \leq m < k$:

$$t_k|\boldsymbol{\epsilon}\rangle = (e_3/2) \otimes \mathcal{F}_4^* e_{\epsilon_2} \otimes \mathcal{F}_4^* e_{\epsilon_3} \cdots\ \mathcal{F}_4^* e_{\epsilon_k}, \quad \text{while for } 0 < m < k,$$
$$t_{k+m}|\boldsymbol{\epsilon}\rangle = \pi_3 \mathcal{F}_4^* e_{\epsilon_{m+1}} \otimes \mathcal{F}_4^* e_{\epsilon_{m+2}} \cdots\ \mathcal{F}_4^* e_{\epsilon_k} \otimes \mathcal{F}_4^* (e_3/2) \otimes \mathcal{F}_4^* \pi_I \mathcal{F}_4^* e_{\epsilon_2} \cdots \mathcal{F}_4^* \pi_I \mathcal{F}_4^* e_{\epsilon_m}. \tag{6.7}$$

An explicit computation shows that

$$\|\pi_3 \mathcal{F}_4^* e_j\|^2 = \frac{1}{4}, \qquad \|\pi_I \mathcal{F}_4^* e_j\|^2 = \frac{1}{2}, \quad j = 0, \cdots, 3,$$

so that

$$\|t_{k+m}|\boldsymbol{\epsilon}\rangle\|^2 = \frac{1}{4\times 2^m},\quad 0 \le m \le k-1. \tag{6.8}$$

For larger times $n = pk + m$, $p > 1$, $m \in [0, k-1]$, the state $t_n|\boldsymbol{\epsilon}\rangle$ is obtained from (6.7) by inserting the operator $(\pi_I \mathcal{F}_4^*)^{p-1}$ in front of each qu*D*it e_{ϵ_ℓ}. Since $\pi_I \mathcal{F}_4^*$ has a spectral radius $|\lambda_+| < 1$, the norms of these states will decay exponentially fast as $n \to \infty$. This argument gives the following

Lemma 6.2. *For any* $0 < \Theta < 1$*, there exists* $C > 0$ *such that, for any* $k \ge 1$ *and any nonclassical state* $|\boldsymbol{\epsilon}\rangle$*, we have*

$$\sum_{m > \lfloor \Theta k \rfloor} \|t_{k+m}|\boldsymbol{\epsilon}\rangle\| \le C\, 2^{-\Theta k/2}.$$

Neglecting errors of order $\mathcal{O}(2^{-\Theta k/2})$, we just need to compute $\| \sum_{m=0}^{\lfloor \Theta k \rfloor} t_{k+m}|\boldsymbol{\epsilon}\rangle\|^2$. From (6.8) we already know the diagonal terms:

$$\sum_{m=0}^{\lfloor \Theta k \rfloor} \|t_{k+m}|\boldsymbol{\epsilon}\rangle\|^2 = \frac{1}{2} + \mathcal{O}(2^{-\Theta k}). \tag{6.9}$$

In the next lemma we will show that the contribution to the conductance of the nondiagonal terms is negligible in the semiclassical limit.

Lemma 6.3. *Let* $0 < \Theta \le 1/5$*. There exists* $C = C(\Theta) > 0$ *such that for any* $k \ge 1$*,*

$$\frac{\#\left\{\text{nonclassical } \boldsymbol{\epsilon},\ \exists m, m' \in [0, \Theta k],\ \ \langle \boldsymbol{\epsilon}| t_{k+m}^* t_{k+m'} |\boldsymbol{\epsilon}\rangle \neq 0\right\}}{\#\left\{\text{nonclassical } \boldsymbol{\epsilon}\right\}} \le C\, 2^{-k/2}.$$

In other words, in the semiclassical limit, a "generic" nonclassical state $|\boldsymbol{\epsilon}\rangle$ will satisfy

$$\forall m, m' \in [0, \Theta k],\quad m \neq m' \Longrightarrow \langle \boldsymbol{\epsilon}| t_{k+m}^* t_{k+m'} |\boldsymbol{\epsilon}\rangle = 0.$$

Proof. Take an arbitrary nonclassical state $|\boldsymbol{\epsilon}\rangle$, and any $m, m' \in [0, \Theta k]$, $m > m'$. From (6.7), the first $(k-m)$ qu*D*its of the states $t_{k+m}|\boldsymbol{\epsilon}\rangle$ and $t_{k+m'}|\boldsymbol{\epsilon}\rangle$) are respectively

$$\pi_3 \mathcal{F}_4^* e_{\epsilon_{m+1}} \otimes \mathcal{F}_4^* e_{\epsilon_{m+2}} \otimes \cdots \otimes \mathcal{F}_4^* e_{\epsilon_k},$$
$$\pi_3 \mathcal{F}_4^* e_{\epsilon_{m'+1}} \otimes \mathcal{F}_4^* e_{\epsilon_{m'+2}} \otimes \cdots \otimes \mathcal{F}_4^* e_{\epsilon_{k+m'-m}}.$$

Due to the unitarity of $\mathcal{F}_4^*$ and the fact that the e_i form an orthonormal basis of $\mathbb{C}^4$, the two states $t_{k+m}|\boldsymbol{\epsilon}\rangle$, $t_{k+m'}|\boldsymbol{\epsilon}\rangle$ will be orthogonal if the sequences $\epsilon_{m+2}\cdots\epsilon_k$ and $\epsilon_{m'+2}\cdots\epsilon_{k+m'-m}$ are not equal. Since we took $m < \Theta k$, these two sequences are subsequences of length $(k-m-1) \ge (1-\Theta)k$ of the sequence $\boldsymbol{\epsilon}$, shifted from one another by $(m-m')$ steps.

If the two subsequences are equal, then all subsequences

$$\epsilon_{k-(p+1)\Delta+1}\cdots\epsilon_{k-p\Delta},\quad p = 0, \cdots, R,$$
$$\Delta \stackrel{\text{def}}{=} (m-m'),\quad R \stackrel{\text{def}}{=} \left\lceil \frac{k-m-1}{\Delta} \right\rceil,$$

have to be equal to each other. Hence $\boldsymbol{\epsilon}$ contains a subsequence of length $(R+1)\Delta$ which is periodic with period Δ.

Let us count the number of such sequences $\boldsymbol{\epsilon}$. Once we have fixed the Δ bits $\epsilon_{k-\Delta+1}\cdots\epsilon_k$, the remaining free bits are $\epsilon_2\cdots\epsilon_{k-(R+1)\Delta}$. The number $\#(m,m')$ of such sequences is therefore $2^{\Delta}\times 2^{k-(R+1)\Delta-1}$. From the definition of R, we get

$$\#(m,m') < 2^{2m-m'} \leq 2^{2\Theta k}.$$

Taking into account all possible pairs (m,m'), we obtain the following bound for the number of nongeneric nonclassical channels:

$$\#\left\{\text{nonclassical } \boldsymbol{\epsilon},\ \exists\, m,m',\ 0\leq m'<m\leq \Theta k,\ \langle \boldsymbol{\epsilon}|t^*_{k+m}t_{k+m'}|\boldsymbol{\epsilon}\rangle \neq 0\right\} \leq (\Theta k)^2\, 2^{2\Theta k}.$$

Since $\#\{\text{nonclassical } \boldsymbol{\epsilon}\} = 2^{k-1}$ and $2\Theta \leq 2/5$, we have proven the lemma. $\square$

From Lemma 6.2 and Eq. (6.9), a generic nonclassical sequence $\boldsymbol{\epsilon}$ will satisfy $\|t|\boldsymbol{\epsilon}\rangle\|^2 = \frac{1}{2} + \mathcal{O}(2^{-\Theta k/2})$. For a nongeneric nonclassical sequence $\boldsymbol{\epsilon}$, we use the simple bound $\|t|\boldsymbol{\epsilon}\rangle\|^2 \leq 1$. As a result, we get the following estimate for the "nonclassical conductance":

$$\operatorname{tr}_{\text{noncl}}(t^*t) = \sum_{\substack{\text{nonclassical}\\ \text{generic}}} \|t|\boldsymbol{\epsilon}\rangle\|^2 + \sum_{\substack{\text{nonclassical}\\ \text{nongeneric}}} \|t|\boldsymbol{\epsilon}\rangle\|^2 = \frac{2^{k-1}}{2}\left(1+\mathcal{O}(2^{-\Theta k/2})\right). \quad (6.10)$$

Adding this to the "classical conductance", we get the full conductance

$$g(\vartheta) = \operatorname{tr}(t^*t(\vartheta)) = \frac{4^{k-1}}{2} + \mathcal{O}(2^{(1-\Theta/2)k}). \quad (6.11)$$

The implied constant is independent of $\vartheta\in[0,2\pi)$ and $0<\Theta\leq 1/5$. The number of scattering channels in our model is given by $M(h) = 4^{k-1}$, see (6.4), so we have proven (1.1) in Theorem 2.

6.3. Noise power. The conductance corresponds to the first moment of the distribution of transmission eigenvalues. It can not distinguish between a purely classical transport ($T_i\in\{0,1\}$) and a quantum one (some T_i take intermediate values). To do so, we need to compute the second moment of these eigenvalues, that is, the trace

$$\operatorname{tr}((t^*t)^2) = \sum_{Q_j\in L_1} \|t^*t|Q_j\rangle\|^2,$$

or equivalently the noise power (6.3). As in the previous section, we split the sum on the right-hand side between the classical and nonclassical states $|Q_j\rangle = |\boldsymbol{\epsilon}\rangle$.

Lemma 6.1 shows that half the classical states are in the kernel of t^*t, half in the eigenspace of t^*t associated with the eigenvalue 1 (full transmission). As a consequence, the trace over the classical states takes the value

$$\operatorname{tr}_{\text{cl}}((t^*t)^2) = \frac{4^{k-1}-2^{k-1}}{2}.$$

Obviously, the classical states are noiseless.

The contribution of the nonclassical states is more delicate. According to Lemma 6.2, for any nonclassical state $|\boldsymbol{\epsilon}\rangle$ we have (for any $0 < \Theta < 1$)

$$t|\boldsymbol{\epsilon}\rangle = \sum_{m=0}^{\lfloor \Theta k \rfloor} e^{in\vartheta}\, t_{k+m}|\boldsymbol{\epsilon}\rangle + \mathcal{O}(2^{-\Theta k/2}).$$

We now apply to each state $t_{k+m}|\boldsymbol{\epsilon}\rangle$, $m \leq \Theta k$, the adjoint operator

$$t^* = \sum_{n \geq 0} e^{-in\vartheta}\, t_n^*. \tag{6.12}$$

According to (6.7), the state $t_{k+m}|\boldsymbol{\epsilon}\rangle$ has the form

$$t_{k+m}|\boldsymbol{\epsilon}\rangle = e_3 \otimes \mathcal{F}_4^* \pi_I w_2 \otimes \cdots \otimes \mathcal{F}_4^* \pi_I w_k,$$

for some explicit set of qu*D*its $w_\ell \in \mathbb{C}^4$, $2 \leq \ell \leq k$. From the expressions

$$\begin{aligned} t_n^* &= \Pi_{L_1}\, U_h^*\, (\Pi_I\, U_h^*)^{n-1}\, \Pi_{L_2}, \\ \Pi_I U_h^* (v_1 \otimes \cdots \otimes v_k) &= \pi_I \mathcal{F}_4 v_k \otimes v_1 \otimes \cdots \otimes v_{k-1}, \end{aligned}$$

we can easily write the action of the operators t_n^* on $t_{k+m}|\boldsymbol{\epsilon}\rangle$:

$$\text{if } n < k, \text{ then} \quad t_n^*\, t_{k+m}|\boldsymbol{\epsilon}\rangle = \pi_0 \pi_I w_{k-n+1} \otimes \ldots = 0.$$

The first non-trivial case of $n = k$ is given by

$$t_k^* t_{k+m}|\boldsymbol{\epsilon}\rangle = \pi_0 \mathcal{F}_4 e_3 \otimes \pi_I w_2 \otimes \cdots \pi_I w_k,$$

while for any $1 \leq m' \leq k-1$, we have

$$\begin{aligned} |\psi_{m',m}(\boldsymbol{\epsilon})\rangle &\stackrel{\text{def}}{=} t_{k+m'}^* t_{k+m}|\boldsymbol{\epsilon}\rangle \\ &= \pi_0 \mathcal{F}_4 \pi_I w_{k-m'+1} \otimes \pi_I \mathcal{F}_4 \pi_I w_{k-m'+2} \otimes \cdots \pi_I \mathcal{F}_4 \pi_I w_k \otimes \pi_I \mathcal{F}_4 e_3 \\ &\quad \otimes \pi_I w_2 \otimes \cdots \pi_I w_{k-m'}. \end{aligned}$$

The above state is obtained by first evolving $|\boldsymbol{\epsilon}\rangle$ $k+m$ times through the "inside propagator" $U_h \Pi_I$, then projecting on the "output lead" L_2, then evolving backwards (through $\Pi_I U_h^*$) $k + m'$ times, and finally projecting on the "input lead" L_1.

As for the case of the operator t, we see that by increasing m' we increase the number of qu*D*its on which we apply the operator $\pi_I \mathcal{F}_4$. Therefore, for any index m, the norm of $|\psi_{m',m}(\boldsymbol{\epsilon})\rangle$ will decrease exponentially fast with m'. As in Lemma 6.2, we truncate the expansion (6.12) to the range $m' \leq \Theta k$, thereby omitting a remainder $\mathcal{O}(2^{-\Theta k/2})$.

We now replace the qu*D*its w_ℓ by their explicit values, which depend on the index m. We introduce the following notations for operators on $\mathbb{C}^4$:

$$\mathcal{P}_{\alpha\beta} \stackrel{\text{def}}{=} \pi_\alpha \mathcal{F}_4 \pi_\beta \mathcal{F}_4^*, \qquad \text{with} \quad \alpha, \beta \in \{0, I, 3\}.$$

The explicit decomposition of $|\psi_{m',m}(\boldsymbol{\epsilon})\rangle$ depends on the sign of $\Delta \stackrel{\text{def}}{=} m - m'$, and on whether $m, m' = 0$ or not (we will often omit to indicate the dependence in $\boldsymbol{\epsilon}$):

- for $m' = m$,

$$
\begin{aligned}
|\psi_{0,0}\rangle &= \mathcal{P}_{03} e_0 \otimes e_{\epsilon_2} \otimes \cdots e_{\epsilon_k}, \\
|\psi_{m,m}\rangle &= \mathcal{P}_{0I} e_0 \otimes \mathcal{P}_{II} e_{\epsilon_2} \otimes \cdots \mathcal{P}_{II} e_{\epsilon_m} \otimes \mathcal{P}_{I3} e_{\epsilon_{m+1}} \otimes e_{\epsilon_{m+2}} \otimes \cdots e_{\epsilon_k}.
\end{aligned} \tag{6.13}
$$

- for $m = m' + \Delta$, $\Delta > 0$,

$$
\begin{aligned}
|\psi_{0,\Delta}\rangle &= \mathcal{P}_{03} e_{\epsilon_{\Delta+1}} \otimes e_{\epsilon_{\Delta+2}} \otimes \cdots\cdots e_{\epsilon_k} \otimes \pi_I \mathcal{F}_4^* e_0 \otimes \pi_I \mathcal{F}_4^* e_{\epsilon_2} \otimes \cdots \pi_I \mathcal{F}_4^* e_{\epsilon_\Delta}, \\
|\psi_{m',m'+\Delta}\rangle &= \mathcal{P}_{0I} e_{\epsilon_{\Delta+1}} \otimes \mathcal{P}_{II} e_{\epsilon_{\Delta+2}} \otimes \cdots \mathcal{P}_{II} e_{\epsilon_{\Delta+m'}} \otimes \mathcal{P}_{I3} e_{\epsilon_{\Delta+m'+1}} \\
&\quad \otimes e_{\epsilon_{\Delta+m'+2}} \otimes \cdots e_{\epsilon_k} \otimes \pi_I \mathcal{F}_4^* e_0 \otimes \pi_I \mathcal{F}_4^* e_{\epsilon_2} \otimes \cdots \pi_I \mathcal{F}_4^* e_{\epsilon_\Delta}.
\end{aligned} \tag{6.14}
$$

- for $m = m' + \Delta$, $\Delta < 0$,

$$
\begin{aligned}
|\psi_{|\Delta|,0}\rangle &= \pi_0 \mathcal{F}_4 e_{\epsilon_{k-|\Delta|+1}} \otimes \pi_I \mathcal{F}_4 e_{\epsilon_{k-|\Delta|+2}} \otimes \cdots \pi_I \mathcal{F}_4 e_{\epsilon_k} \otimes \mathcal{P}_{I3} e_0 \otimes e_{\epsilon_2} \otimes \cdots e_{\epsilon_{k-|\Delta|}}, \\
|\psi_{m+|\Delta|,m}\rangle &= \pi_0 \mathcal{F}_4 e_{\epsilon_{k-|\Delta|+1}} \otimes \pi_I \mathcal{F}_4 e_{\epsilon_{k-|\Delta|+2}} \otimes \cdots \pi_I \mathcal{F}_4 e_{\epsilon_k} \otimes \mathcal{P}_{II} e_0 \\
&\quad \otimes \mathcal{P}_{II} e_{\epsilon_2} \otimes \cdots \otimes \mathcal{P}_{II} e_{\epsilon_m} \otimes \mathcal{P}_{I3} e_{\epsilon_{m+1}} \otimes e_{\epsilon_{m+2}} \otimes \cdots e_{\epsilon_{k-|\Delta|}}.
\end{aligned} \tag{6.15}
$$

We notice that each of these states contains subfactors $e_{\epsilon_{m+2}} \otimes \cdots \otimes e_{\epsilon_k}$ if $m \geq m'$, and $e_{\epsilon_{m+2}} \otimes \cdots \otimes e_{\epsilon_{k+m-m'}}$ if $m < m'$. Compared to its position in the tensor product expansion of $|\boldsymbol{\epsilon}\rangle$, this subfactor is shifted by $m' - m = -\Delta$ steps. From this remark, and using similar methods as for Lemma 6.3, we can easily prove the following

Lemma 6.4. *Let* $0 < \Theta < 1/6$ *and for any pair of indices* (m, m')*, denote* $\Delta = m - m'$. *There exists* $C = C(\Theta) > 0$ *such that*

$$
\frac{\#\Big\{\text{nonclass.}\boldsymbol{\epsilon} : \exists\, m_1, m_1', m_2, m_2' \in [0, \Theta k],\ \Delta_1 \neq \Delta_2, \langle \psi_{m_1',m_1} | \psi_{m_2',m_2} \rangle \neq 0 \Big\}}{\#\{\text{nonclass}\boldsymbol{\epsilon}\}} \leq C 2^{-Ck}.
$$

In other words, for a generic nonclassical sequence $\boldsymbol{\epsilon}$, any two states $|\psi_{m_1',m_1}(\boldsymbol{\epsilon})\rangle$, $|\psi_{m_2',m_2}(\boldsymbol{\epsilon})\rangle$ with $m_i,\ m_i' \leq \Theta k$ will be orthogonal to each other if $\Delta_1 \neq \Delta_2$, that is, if the shifts between their respective forward and backward evolution times are different.

From now on we assume that $\boldsymbol{\epsilon}$ is a generic nonclassical sequence in the sense of the above lemma. If we group the states $|\psi_{m+\Delta,m}(\boldsymbol{\epsilon})\rangle$ into

$$
|\Psi_\Delta(\boldsymbol{\epsilon})\rangle \stackrel{\text{def}}{=} \sum_{\substack{0 \leq m, m' \leq \Theta k \\ m = m' + \Delta}} |\psi_{m+\Delta,m}(\boldsymbol{\epsilon})\rangle,
$$

then genericity implies that $\langle \Psi_\Delta(\boldsymbol{\epsilon}) | \Psi_{\Delta'}(\boldsymbol{\epsilon}) \rangle = 0$ if $\Delta \neq \Delta'$. The square-norm of $t^* t |\boldsymbol{\epsilon}\rangle$ is then given by

$$
\| t^* t |\boldsymbol{\epsilon}\rangle \|^2 = \sum_{|\Delta| \leq \Theta k} \| \Psi_\Delta(\boldsymbol{\epsilon}) \|^2 + \mathcal{O}(2^{-\Theta k/2}). \tag{6.16}
$$

As we will see, no further simplification occurs in this expression, meaning that two different states $|\psi_{m',m}\rangle$ with the same Δ will generally interfere with each other. Our remaining task consists in computing each square norm on the right-hand side of (6.16). We will use the explicit tensor decompositions (6.13-6.15), and notice that the overlap between two states $|\psi_{m',m}\rangle$ is the product of the overlaps of their tensor factors. We split the lengthy, yet straightforward computation according to the value of Δ.

6.3.1. Norm of Ψ_0. We have

$$\|\Psi_0\|^2 = \sum_{m\le\Theta k} \|\psi_{m,m}\|^2 + 2 \sum_{0\le m<n\le\Theta k} \mathrm{Re}\langle\psi_{m,m}|\psi_{n,n}\rangle. \tag{6.17}$$

The successive diagonal terms take the values

$$\|\psi_{0,0}\|^2 = \|\mathcal{P}_{03}e_0\|^2 = \frac{1}{16}, \quad \text{while for } m \ge 1,$$
$$\|\psi_{m,m}\|^2 = \|\mathcal{P}_{0I}e_0\|^2 \left(\prod_{\ell=2}^{m} \|\mathcal{P}_{II}e_{\epsilon_\ell}\|^2\right) \|\mathcal{P}_{I3}e_{\epsilon_{m+1}}\|^2 = \frac{1}{4}\left(\frac{3}{8}\right)^{m-1}\frac{1}{8}.$$

The sum over the diagonal terms is therefore

$$\sum_{m\le\Theta k} \|\psi_{m,m}\|^2 = \frac{9}{80} + \mathcal{O}\big((3/8)^{\Theta k}\big).$$

The nondiagonal terms read, for any $0 < n \le \Theta k$,

$$\langle\psi_{0,0}|\psi_{n,n}\rangle = \langle\mathcal{P}_{03}e_0, \mathcal{P}_{0I}e_0\rangle \left(\prod_{\ell=2}^{n} \langle e_{\epsilon_\ell}, \mathcal{P}_{II}e_{\epsilon_\ell}\rangle\right) \langle e_{\epsilon_{n+1}}, \mathcal{P}_{I3}e_{\epsilon_{n+1}}\rangle = \frac{1}{8}\left(\frac{1}{2}\right)^{n-1}\frac{1}{4},$$

and for $0 < m < n \le \Theta k$, one similarly gets

$$\langle\psi_{m,m}|\psi_{n,n}\rangle = \frac{1}{4}\left(\frac{3}{8}\right)^{m-1}\frac{1\pm i}{16}\left(\frac{1}{2}\right)^{n-m-1}\frac{1}{4}.$$

The sign is + if $\epsilon_{m+1} = 1$, and $-$ if $\epsilon_{m+1} = 2$. Adding up the real parts of these off-diagonal terms, we obtain

$$2 \sum_{0\le m<n\le\Theta k} \mathrm{Re}\langle\psi_{m,m}|\psi_{n,n}\rangle = \frac{3}{20} + \mathcal{O}(2^{-\Theta k}).$$

We notice that this contribution is of the same order as the diagonal one. Summing the diagonal and nondiagonal parts yields the norm

$$\|\Psi_0\|^2 = \frac{21}{80} + \mathcal{O}(2^{-\Theta k}). \tag{6.18}$$

6.3.2. Norm of Ψ_Δ *with* $\Delta > 0$. From Eq. (6.14) we notice that all states $|\psi_{m,m+\Delta}\rangle$, $0 \le m \le \Theta k - \Delta$ share the same Δ last qu$D$its, which results in a common factor

$$\prod_{\ell=1}^{\Delta} \|\pi_I \mathcal{F}_4^* e_{\epsilon_\ell}\|^2 = \frac{1}{2^\Delta} \quad \text{in the norm } \|\Psi_\Delta\|^2.$$

To avoid taking this factor into account at all steps, we rather consider the states $|\psi'_{m,m+\Delta}\rangle$ obtained by removing these last Δ qu$D$its from $|\psi_{m,m+\Delta}\rangle$.

We first compute the square-norm $\|\psi'_{0,\Delta}\|^2 = \frac{1}{16}$.

For all $m > 0$, the first qu$D$it of $|\psi'_{m,m+\Delta}\rangle$ is $\mathcal{P}_{0I}e_{\epsilon_{\Delta+1}}$. From the explicit expression of $\mathcal{P}_{0I}$, this qu$D$it vanishes if $\epsilon_{\Delta+1} = 2$, so that

$$\text{if } \epsilon_{\Delta+1} = 2, \qquad \|\Psi_\Delta(\boldsymbol{\epsilon})\|^2 = \frac{1}{16}\,\frac{1}{2^\Delta} \quad \text{for all } 1 \le \Delta \le \Theta k. \tag{6.19}$$

In the opposite case $\epsilon_{\Delta+1} = 1$, the states $|\psi'_{m,m+\Delta}\rangle$ are nontrivial:

$$\text{for any } 0 < m \le \Theta k - \Delta, \qquad \|\psi'_{m,m+\Delta}\|^2 = \frac{1}{8}\,(\frac{3}{8})^{m-1}\,\frac{1}{8},$$
$$\text{so that} \qquad \sum_{m=0}^{[\Theta k]-\Delta} \|\psi'_{m,m+\Delta}\|^2 = \frac{7}{80} + \mathcal{O}\big((3/8)^{\Theta k-\Delta}\big). \tag{6.20}$$

We then compute the off-diagonal terms:

$$\langle\psi'_{0,\Delta}|\psi'_{m,m+\Delta}\rangle = \frac{-1-i}{16}\,\frac{1}{2^{m-1}}\,\frac{1}{4} \qquad \text{for } 0 < m \le \Theta k - |\Delta|,$$
$$\langle\psi'_{m,m+\Delta}|\psi'_{n,n+\Delta}\rangle = \frac{1}{8}\,(\frac{3}{8})^{m-1}\,\frac{1\pm i}{16}\,\frac{1}{2^{n-m-1}}\,\frac{1}{4} \qquad \text{for } 1 \le m < n \le \Theta k - |\Delta|$$

(the sign $\pm$ in the last line depends on $\epsilon_{\Delta+m+1}$). Summing up the real parts yields:

$$2 \sum_{0\le m<n\le\Theta k-\Delta} \mathrm{Re}\langle\psi'_{m,m+\Delta}|\psi'_{n,n+\Delta}\rangle = -\frac{1}{20} + \mathcal{O}(2^{-\Theta k+\Delta}).$$

Adding this to the diagonal terms (6.20), restoring the factor $2^{-\Delta}$, and using (6.19) results in the following norm (which explicitly depends on $\boldsymbol{\epsilon}$):

$$\|\Psi_\Delta(\boldsymbol{\epsilon})\|^2 = \frac{1}{2^\Delta}\big(\frac{3}{80}\,\delta_{\epsilon_{\Delta+1}=1} + \frac{1}{16}\,\delta_{\epsilon_{\Delta+1}=2}\big) + \mathcal{O}(2^{-\Theta k}) \quad \text{for any } 1 \le \Delta \le \Theta k. \tag{6.21}$$

6.3.3. Norm of Ψ_Δ *with* $\Delta < 0$. As in the previous case, we notice from (6.15) that all components of Ψ_Δ share the same $|\Delta|$ first qu$D$its, which contribute a factor

$$\|\pi_0\mathcal{F}_4 e_{\epsilon_{k-|\Delta|+1}}\|^2 \left(\prod_{\ell=2}^{|\Delta|} \|\pi_I\mathcal{F}_4 e_{\epsilon_{k-|\Delta|+\ell}}\|^2\right) = \frac{1}{4}\,\frac{1}{2^{|\Delta|-1}}. \tag{6.22}$$

We call $\psi'_{m+|\Delta|,m}$ the states with these $|\Delta|$ qu$D$its removed. They have the norms

$$\|\psi'_{|\Delta|,0}\|^2 = \frac{1}{8}, \quad \text{and for } m \ge 1, \quad \|\psi'_{m+|\Delta|,m}\|^2 = \frac{1}{8}\left(\frac{3}{8}\right)^{m-1}\frac{1}{8}.$$

Hence, the diagonal contribution reads

$$\sum_{m=0}^{\lfloor\Theta k\rfloor-|\Delta|} \|\psi'_{m+|\Delta|,m}\|^2 = \frac{3}{20} + \mathcal{O}\left((3/8)^{\Theta k-|\Delta|}\right).$$

The nondiagonal terms take the values

$$\langle\psi'_{|\Delta|,0}|\psi'_{n+|\Delta|,n}\rangle = \frac{-1+i}{16}\,\frac{1}{2^{n-1}}\,\frac{1}{4} \qquad \text{for } 0 < n \le \Theta k - |\Delta|,$$

$$\langle\psi'_{m+|\Delta|,m}|\psi'_{n+|\Delta|,n}\rangle = \frac{1}{8}\left(\frac{3}{8}\right)^{m-1}\frac{1\pm i}{16}\,\frac{1}{2^{n-m-1}}\,\frac{1}{4} \qquad \text{for } 1 \le m < n \le \Theta k - |\Delta|.$$

These contributions sum up to

$$2\sum_{0\le m<n\le\Theta k-|\Delta|}\mathrm{Re}\langle\psi'_{m+|\Delta|,m}|\psi'_{n+|\Delta|,n}\rangle = -\frac{1}{20} + \mathcal{O}(2^{-\Theta k+|\Delta|}).$$

Putting together this with the diagonal contributions and restoring the factor (6.22), yields

$$\|\Psi_\Delta\|^2 = \frac{1}{20}\,\frac{1}{2^{|\Delta|}} + \mathcal{O}(2^{-\Theta k}), \quad -\Theta k \le \Delta \le -1. \tag{6.23}$$

6.3.4. Summing up. We can now sum over all shifts Δ, $|\Delta| \le \Theta k$ for a given generic nonclassical sequence $\boldsymbol{\epsilon}$. The sum over the shifts $\Delta \le 0$ is simple, and independent on the sequence $\boldsymbol{\epsilon}$:

$$\sum_{-\Theta k\le\Delta\le 0}\|\Psi_\Delta(\boldsymbol{\epsilon})\|^2 = \frac{25}{80} + \mathcal{O}(k\,2^{-\Theta k}).$$

The sum over the shifts $\Delta > 0$ is slightly more delicate, since the norm of $|\Psi_\Delta(\boldsymbol{\epsilon})\rangle$ depends on $\boldsymbol{\epsilon}$ — see Eq. (6.21). However, we notice that the set of generic nonclassical sequences can be partitioned into "mirror pairs" $(\boldsymbol{\epsilon}, \overline{\boldsymbol{\epsilon}})$ such that

$$\forall \ell \in [2,k], \quad \overline{\epsilon}_\ell = 1 \Longleftrightarrow \epsilon_\ell = 2.$$

Summing the norms over a "mirror pair" is easy:

$$\text{for } 1 \le \Delta \le \Theta k, \qquad \|\Psi_\Delta(\boldsymbol{\epsilon})\|^2 + \|\Psi_\Delta(\overline{\boldsymbol{\epsilon}})\|^2 = \frac{1}{2^\Delta}\,\frac{1}{10} + \mathcal{O}(2^{-\Theta k}).$$

This contribution is identical (up to the remainder) with $\|\Psi_{-\Delta}(\boldsymbol{\epsilon})\|^2 + \|\Psi_{-\Delta}(\overline{\boldsymbol{\epsilon}})\|^2$, which shows a sort of symmetry between positive and negative shifts. Summing over all $|\Delta| \le \Theta k$, we get, for any mirror pair $(\boldsymbol{\epsilon}, \overline{\boldsymbol{\epsilon}})$ of generic nonclassical sequences:

$$\|t^*t|\boldsymbol{\epsilon}\rangle\|^2 + \|t^*t|\overline{\boldsymbol{\epsilon}}\rangle\|^2 = \frac{58}{80} + \mathcal{O}(2^{-\Theta k/2}).$$

Using Lemma 6.4, we obtain the trace over the nonclassical states:

$$\mathrm{tr}_{\mathrm{noncl}}((t^*t)^2) = 2^{k-2}\left(\frac{58}{80} + \mathcal{O}(2^{-Ck}) + \mathcal{O}(2^{-\Theta k/2})\right).$$

Subtracting this expression from the "nonclassical conductance" (6.10), and calling $\widetilde{C} = \min(C, \frac{1}{2}\Theta)$, we finally obtain the noise power:

$$\begin{aligned} P &= \mathrm{tr}(t^*t - (t^*t)^2) = \mathrm{tr}_{\mathrm{noncl}}(t^*t - (t^*t)^2) \\ &= 2^{k-1}\left(\frac{11}{80} + \mathcal{O}(2^{-\widetilde{C}k})\right). \end{aligned}$$

This proves (1.2) in Theorem 2. As remarked in §1.1, the factor 11/80 is close to the random-matrix prediction for this quantity, namely 1/8 [26, 57]. This is in contrast with our remark 5.1 that the semiclassical resonance spectrum of the propagator inside the dot, $\widetilde{B}_h = U_h \Pi_I$, is quite different from that of a random subunitary matrix. Somehow, the matrix t, obtained by summing iterates of $\widetilde{B}_h$, has acquired some "genericity", as far as the distribution of its singular values is concerned. It would be interesting (but quite cumbersome) to compute the higher moments of that distribution.

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