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Resonant eigenstates for a quantized chaotic system

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Abstract

We study the spectrum of quantized open maps as a model for the resonance spectrum of quantum scattering systems. We are particularly interested in open maps admitting a fractal repeller. Using the 'open baker's map' as an example, we numerically investigate the exponent appearing in the fractal Weyl law for the density of resonances; we show that this exponent is not related with the 'information dimension', but rather the Hausdorff dimension of the repeller. We then consider the *semiclassical measures* associated with the eigenstates: we prove that these measures are conditionally invariant with respect to the classical dynamics. We then address the problem of classifying semiclassical measures among conditionally invariant ones. For a solvable model, the 'Walsh-quantized' open baker's map, we manage to exhibit a family of semiclassical measures with simple self-similar properties.

Mathematics Subject Classification: 35B34, 37D20, 81Q50, 81U05

(Some figures in this article are in colour only in the electronic version)

1. Introduction

1.1. Quantum scattering on $\mathbb{R}^D$ and resonances

In a typical scattering system, particles of positive energy come from infinity, interact with a localized potential $V(q)$ and then leave to infinity. The corresponding quantum Hamiltonian $H_\hbar = -\hbar^2\Delta + V(q)$ has an absolutely continuous spectrum on the positive axis. Yet, Green's function $G(z; q', q) = \langle q'|(H_\hbar - z)^{-1}|q\rangle$ admits a meromorphic continuation from the upper half-plane $\{\Im z > 0\}$ to (some part of) the lower half-plane $\{\Im z < 0\}$. This continuation generally has *poles* $z_j = E_j - \mathrm{i}\Gamma_j/2$, $\Gamma_j > 0$, which are called *resonances* of the scattering system.

The probability density of the corresponding 'eigenfunction' $\varphi_j(q)$ decays in time like $\mathrm{e}^{-t\Gamma_j/\hbar}$, so physically φ_j represents a metastable state with a decay rate $\Gamma_j/\hbar$ or lifetime

$\tau_j = \hbar/\Gamma_j$. In the semiclassical limit $\hbar \to 0$, we will call 'long-living' the resonances z_j such that $\Gamma_j = \mathcal{O}(\hbar)$, equivalently with lifetimes bounded away from zero.

The eigenfunction $\varphi_j(q)$ is meaningful only near the interaction region, while its behaviour outside that region (exponentially increasing outgoing waves) is clearly unphysical. As a result, one practical method to compute resonances (at least approximately) consists of adding a *smooth absorbing potential* $-\mathrm{i}W(q)$ to the Hamiltonian $\hat{H}$, thereby obtaining a nonself-adjoint operator $H_{W,\hbar} = H_\hbar - \mathrm{i}W(q)$. The potential $W(q)$ is supposed to vanish in the interaction region but is positive outside: its effect is to *absorb* outgoing waves, as opposed to a real positive potential which would reflect the waves back into the interaction region. Equivalently, the (nonunitary) propagator $\mathrm{e}^{-\mathrm{i}H_{W,\hbar}/\hbar}$ kills wavepackets localized oustide the interaction region.

The spectrum of $H_{W,\hbar}$ in some neighbourhood of the positive axis is then made up of discrete eigenvalues $\tilde{z}_j$ associated with square-integrable eigenfunctions $\tilde{\varphi}_j$. Absorbing Hamiltonians of the type of $H_{W,\hbar}$ have been widely used in quantum chemistry to study reaction or dissociation dynamics [23, 40]; in these works it is implicitly assumed that eigenvalues $\tilde{z}_j$ close to the real axis are small perturbations of the resonances z_j and that the corresponding eigenfunctions $\varphi_j(q)$, $\tilde{\varphi}_j(q)$ are close to one another in the interaction region. Very close to the real axis (namely, for $|\Im\tilde{z}_j| = \mathcal{O}(\hbar^n)$ with n sufficiently large), one can prove that this is indeed the case [43]. Such very long-living resonances are possible when the classical dynamics admits a trapped region of positive Liouville volume. In that case, resonances and the associated eigenfunctions can be approximated by quasimodes of an associated closed system [44].

1.2. Resonances in chaotic scattering

We are interested in a different situation, where the set of trapped trajectories has volume zero and is a fractal hyperbolic repeller. This case encompasses the famous 3-disc scatterer in two dimensions [13] or its smoothing, namely, the 3-bump potential introduced in [41] and numerically studied in [24]. Resonances then lie deeper below the real line (typically, $\Gamma_j \gtrsim \hbar$) and are not perturbations of an associated real spectrum. Previous studies have focused on counting the number of resonances in small discs around the energy E, in the semiclassical régime. Based on the seminal work of Sjöstrand [41], several authors have conjectured the following Weyl-type law:

$$\#\left\{z_j \in \mathrm{Res}(H_\hbar) \,:\, |E - z_j| \leqslant \gamma\hbar\right\} \sim C(\gamma)\hbar^{-d}\,. \tag{1.1}$$

Here the exponent d is related to the trapped set at energy E: the latter has the (Minkowski) dimension $2d + 1$. This asymptotics was numerically checked by Lin and Zworski for the 3-bump potential [24, 26] and by Guillopé, Lin and Zworski for scattering on a hyperbolic surface [15]. However, only the upper bounds for the number of resonances could be rigorously proven [15, 41, 42, 49].

To avoid the complexity of 'realistic' scattering systems, one can study simpler models, namely, quantized open maps on a compact phase space, for instance, the quantized open baker's map studied in [20, 30, 31] (see section 2.1). Such a model is meant to mimic the propagator of the nonself-adjoint Hamiltonian $H_{W,\hbar}$ in the case where the classical flow at energy E is chaotic in the interacting region. The above fractal Weyl law has a direct counterpart in this setting; such a fractal scaling was checked for the open kicked rotator in [38] and for the 'symmetric' baker's map in [30]. In section 4.1 we numerically check this fractal law for an asymmetric version of the open baker's map; apart from extending the results of [30], this model allows us to specify more precisely the dimension d appearing in the scaling law. To our knowledge, so far the only system for which the asymptotics corresponding to (1.1) could be rigorously proven is the 'Walsh-quantization' of the symmetric open baker's map [31], which will be described in section 6.

After counting resonances, the next step consists of studying the long-living resonant eigenstates φ_j or $\tilde{\varphi}_j$. Some results on this matter have been announced by Zworski and the first author in [32]. Interesting numerics were performed by Lebental and coworkers for a model of an open stadium billiard, relevant for describing an experimental micro-laser cavity [22]. In the framework of quantum open maps, eigenstates of the open Chirikov map have been numerically studied by Casati *et al* [5]; the authors showed that, in the semiclassical limit, the long-living eigenstates concentrate on the classical hyperbolic repeller. They also found that the phase space structure of the eigenstates is much correlated with their decay rate. In this paper we will consider general 'quantizable' open maps and formalize the above observations into rigorous statements. Our main result is theorem 1 (see section 5.1), which shows that the *semiclassical measure* associated with a sequence of quantum eigenstates is (up to subtleties due to discontinuities) necessarily an *eigenmeasure* of the classical open map. Such eigenmeasures are necessarily supported on the *backward trapped set*, which, in the case of a chaotic dynamics, is a fractal subset of the phase space: this motivated the denomination of 'quantum fractal eigenstates' used in [5]. Let us mention that eigenstates of quantized open maps have been studied in parallel by Keating and coworkers [20]. Our theorem 1 provides a rigorous version of statements contained in their work.

Inspired by our experience with closed chaotic systems, in section 5.2 we attempt to classify semiclassical measures among all possible eigenmeasures, in particular for the open baker's map. In the case of the 'standard' quantized open baker, the classification remains open. In section 6 we consider a solvable model, the Walsh-quantization of the open baker's map, introduced in [30, 31]. For that model, one can explicitly construct some semiclassical measures and partially answer the above questions. A further study of semiclassical measures for the Walsh model will appear in a joint publication with Keating, Novaes and Sieber.

2. The open baker's map

2.1. Closed and open symplectic maps

Although many of the results we will present deal with a particular family of maps on the 2-torus, namely the family of baker's maps, we start by some general considerations on open maps defined on a compact metric space $\hat{M}$, equipped with a probability measure μ_L. We borrow some ideas and notation from the recent review of Demers and Young [9]. We start with an invertible map $\hat{T} : \hat{M} \to \hat{M}$, which we assume to be piecewise smooth and to preserve the measure μ_L (when $\hat{M}$ is a symplectic manifold, μ_L is the Liouville measure). We then *dig a hole* in $\hat{M}$, which is a certain subset $H \subset \hat{M}$, and decide that points falling in the hole are no more iterated but rather 'disappear' or 'go to infinity'. The hole is assumed to be a Borel subset of $\hat{M}$.

Taking $M \stackrel{\text{def}}{=} \hat{M} \setminus H$, we are thus lead to consider the *open map* $T = \hat{T}_{|M} : M \to \hat{M}$ or, equivalently say that T sends points in the hole to infinity. By iterating T, we see that any point $x \in M$ has a certain time of escape $n(x)$, which is the smallest integer such that $\hat{T}^n(x) \in H$ (this time can be infinite). For each $n \in \mathbb{N}^*$ we consider

$$M^n = \{x \in M,\ n(x) \geqslant n\} = \bigcap_{j=0}^{n-1} \hat{T}^{-j}(M)\,. \tag{2.1}$$

This is the domain of definition of the iterated map T^n. The *forward trapped set* for the open map T is made up of the points which will never escape in the future:

$$\Gamma_- = \bigcap_{n\geqslant 1} M^n = \bigcap_{j=0}^{\infty} \hat{T}^{-j}(M)\,. \tag{2.2}$$

These definitions allow us to split the full phase space into a disjoint union

$$\hat{M} = \left(\bigsqcup_{n=1}^{\infty} M_n\right) \sqcup \Gamma_-, \tag{2.3}$$

where $M_1 \stackrel{\text{def}}{=} H$, and for each $n \geqslant 2$, $M_n \stackrel{\text{def}}{=} M^{n-1} \setminus M^n$ is the set of points escaping exactly at time n.

We also consider the backward evolution given by the inverse map $\hat{T}^{-1}$. The hole for this backwards map is the set $H^{-1} = \hat{T}(H)$, and we call T^{-1} the restriction of $\hat{T}^{-1}$ to $M^{-1} = \hat{T}(M)$ (the 'backwards open map'). We also define $M^{-n} = \bigcap_{j=1}^{n} \hat{T}^j(M)$, $M_{-1} = H^{-1}$, $M_{-n} = M^{-n+1} \setminus M^{-n}$ ($n \geqslant 2$). This leads to the the *backward trapped set*

$$\Gamma_+ = \bigcap_{j=1}^{\infty} \hat{T}^j(M) \qquad \text{and the } \textit{trapped set } K = \Gamma_- \cap \Gamma_+\,. \tag{2.4}$$

We also have a 'backward partition' of the phase space:

$$\hat{M} = \left(\bigsqcup_{n=1}^{\infty} M_{-n}\right) \sqcup \Gamma_+. \tag{2.5}$$

The dynamics of the open map T is interesting if the trapped sets are not empty. This is the case for the open baker's map we will study more explicitly (see section 2.2.2 and figure 2).

In order to quantize the maps $\hat{T}$ and T, one needs further assumptions. In general, $\hat{M}$ is a *symplectic manifold*, μ_L its Liouville measure and $\hat{T}$ a canonical transformation on $\hat{M}$. Also, we will assume that T is sufficiently regular (see section 3.1).

Yet, one may also consider 'quantizations' on more general phase spaces, such as in the axiomatic framework of Marklof and O'Keefe [28]. As we explain in section 6.1.2, the 'Walsh-quantization' of the baker's map is easier to analyse if we consider it as the quantization of an open map on a certain symbolic space, which is not a symplectic manifold. By extension, we also call 'Liouville' the measure μ_L for this case.

2.2. The open baker's map and its symbolic dynamics

We present the closed and open maps which will be our central examples: the baker's maps and their associated symbolic shifts.

2.2.1. The closed baker.

The phase space of the baker's map is the two-dimensional torus $\hat{M} = \mathbb{T}^2 \simeq [0, 1) \times [0, 1)$. A point on $\mathbb{T}^2$ is described with the coordinates $x = (q, p)$, which we call, respectively, the position (horizontal) and the momentum (vertical), to insist on the symplectic structure $dq \wedge dp$. We split $\mathbb{T}^2$ into three vertical rectangles R_i with widths $(r_0, r_1, r_2) \stackrel{\text{def}}{=} \mathbf{r}$ (such that $r_0 + r_1 + r_2 = 1$, with all $r_\epsilon > 0$) and first define a closed baker's

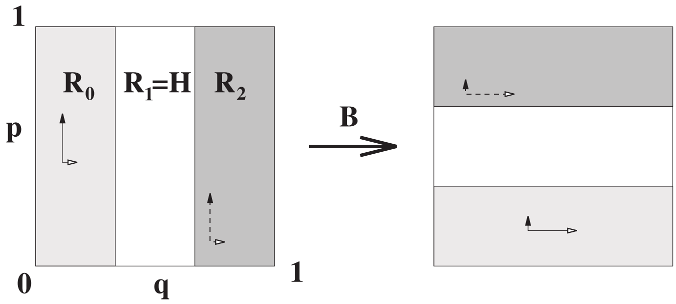

Figure 1. Sketch of the closed baker's map $\hat{B}_{\boldsymbol{r}}$ and its open counterpart $B_{\boldsymbol{r}}$, for the case $\boldsymbol{r} = \boldsymbol{r}_{\rm sym} = (1/3, 1/3, 1/3)$. The three rectangles form a Markov partition.

map on $\mathbb{T}^2$ (see figure 1):

$$(q, p) \mapsto \hat{B}_{\boldsymbol{r}}(q, p) \stackrel{\rm def}{=} (q', p') = \begin{cases} \left(\dfrac{q}{r_0},\ p\, r_0\right) & \text{if } 0 \leqslant q < r_0, \\ \left(\dfrac{q - r_0}{r_1},\ p\, r_1 + r_0\right) & \text{if } r_0 \leqslant q < r_0 + r_1, \\ \left(\dfrac{q - r_0 - r_1}{r_2},\ p\, r_2 + r_1 + r_0\right) & \text{if } 1 - r_2 \leqslant q < 1\,. \end{cases} \tag{2.6}$$

This map is invertible and symplectic on $\mathbb{T}^2$. It is discontinuous on the boundaries of the rectangles R_i but smooth (actually, affine) inside them. We now recall how $\hat{B}_{\boldsymbol{r}}$ can be conjugated with a symbolic dynamics. We introduce the right shift $\hat{\sigma}$ on the *symbolic space* $\Sigma = \{0, 1, 2\}^{\mathbb{Z}}$

$$\boldsymbol{\epsilon} = \ldots \epsilon_{-2}\epsilon_{-1} \cdot \epsilon_0\epsilon_1 \ldots \in \Sigma \longmapsto \hat{\sigma}(\boldsymbol{\epsilon}) = \ldots \epsilon_{-2}\epsilon_{-1}\epsilon_0 \cdot \epsilon_1 \ldots .$$

The symbolic space Σ can be mapped to the 2-torus as follows: every bi-infinite sequence $\boldsymbol{\epsilon} \in \Sigma$ is mapped to the point $x = J_{\boldsymbol{r}}(\boldsymbol{\epsilon})$ with coordinates

$$q(\boldsymbol{\epsilon}) = \sum_{k=0}^{\infty} r_{\epsilon_0} r_{\epsilon_1} \cdots r_{\epsilon_{k-1}}\, \alpha_{\epsilon_k}\,, \qquad p(\boldsymbol{\epsilon}) = \sum_{k=1}^{\infty} r_{\epsilon_{-1}} r_{\epsilon_{-2}} \cdots r_{\epsilon_{-k+1}}\, \alpha_{\epsilon_{-k}}\,, \tag{2.7}$$

where we have set $\alpha_0 = 0$, $\alpha_1 = r_0$, $\alpha_2 = r_0 + r_1$. The position coordinate (unstable direction) depends on symbols on the right of the comma, while the momentum coordinate (stable direction) depends on symbols on the left. The map $J_{\boldsymbol{r}} : \Sigma \to \mathbb{T}^2$ is surjective but not injective; for instance, the sequences $\ldots \epsilon_{n-1}\epsilon_n 000\ldots$ (with $\epsilon_n \neq 0$) and $\ldots \epsilon_{n-1}(\epsilon_n - 1)222\ldots$ have the same image on $\mathbb{T}^2$. For this reason, it is convenient to restrict $J_{\boldsymbol{r}}$ to the subset $\Sigma' \subset \Sigma$ obtained by removing from Σ the sequences ending by $\ldots 2222\ldots$ on the left or the right. Σ' is invariant through the shift $\hat{\sigma}$. The map $J_{\boldsymbol{r}|\Sigma'} : \Sigma' \to \mathbb{T}^2$ is now bijective, and it conjugates $\hat{\sigma}_{|\Sigma'}$ with the baker's map $\hat{B}_{\boldsymbol{r}}$ on $\mathbb{T}^2$:

$$\hat{B}_{\boldsymbol{r}} = J_{\boldsymbol{r}|\Sigma'} \circ \hat{\sigma} \circ (J_{\boldsymbol{r}|\Sigma'})^{-1}\,. \tag{2.8}$$

Any finite sequence $\boldsymbol{\epsilon} = \epsilon_{-m} \ldots \epsilon_{-1} \cdot \epsilon_0 \ldots \epsilon_{n-1}$ represents a *cylinder* $[\boldsymbol{\epsilon}] \subset \Sigma$, which consists of the sequences sharing the same symbols between indices $-m$ and $n - 1$. We call $J_{\boldsymbol{r}}([\boldsymbol{\epsilon}])$ a *rectangle* on the torus (sometimes we will note it as $[\boldsymbol{\epsilon}]$). This rectangle has sides parallel to the two axes; it has width $r_{\epsilon_0} r_{\epsilon_1} \cdots r_{\epsilon_{n-1}}$ and height $r_{\epsilon_{-1}} \cdots r_{\epsilon_{-m}}$.

For any triple, the Liouville measure on $\mathbb{T}^2$ is the push-forward through $J_{\boldsymbol{r}}$ of a certain Bernoulli measure on Σ, namely, $\mu_L^{\Sigma} = \nu_{\boldsymbol{r}}^{\Sigma_-} \times \nu_{\boldsymbol{r}}^{\Sigma_+}$ (see section 2.3.5).

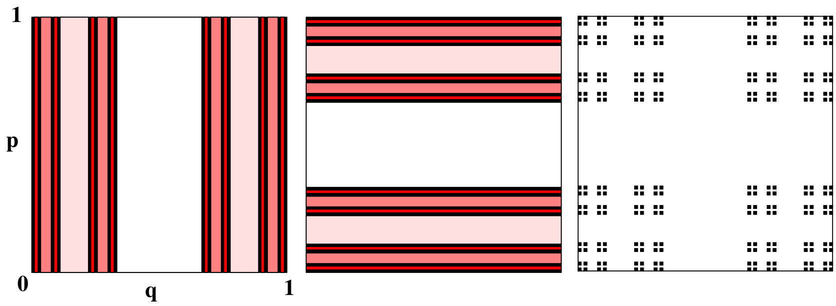

Figure 2. In black we approximate the forward (left), backward (centre) and joint (right) trapped sets for the symmetric open baker $B_{r_{\rm sym}}$. On the left (respectively, centre), red/grey scales (from white to dark) correspond to points escaping at successive times in the future (respectively, past), that is, to sets M_n (respectively, M_{-n}), for $n = 1, 2, 3, 4$.

2.2.2. The open baker. We choose to take for the hole the middle rectangle $H = R_1 = \{r_0 \leqslant q < 1 - r_2\}$. We thus obtain an 'open baker's map' $B_{\boldsymbol{r}}$ defined on $M = \mathbb{T}^2 \setminus H$:

$$(q, p) \mapsto B_{\boldsymbol{r}}(q, p) \stackrel{\rm def}{=} (q', p') = \begin{cases} \left(\dfrac{q}{r_0},\ p\, r_0\right) & \text{if } 0 \leqslant q < r_0, \\ \infty & \text{if } r_0 \leqslant q < 1 - r_2, \\ \left(\dfrac{q - 1 + r_2}{r_2},\ p\, r_2 + 1 - r_2\right) & \text{if } 1 - r_2 \leqslant q < 1\,. \end{cases}$$

The 'inverse map' $B_{\boldsymbol{r}}^{-1}$ is defined on the set $M^{-1} = B_{\boldsymbol{r}}(M)$, which is outside the backward hole $H^{-1} = B_{\boldsymbol{r}}(H) = \{r_0 \leqslant p < 1 - r_2\}$ (see figure 1, right).

Due to the choice of the hole, this open map is still easy to analyse through symbolic dynamics: the hole R_1 is the image through $J_{\boldsymbol{r}}$ of the set $\{\epsilon_0 = 1\} \cap \Sigma'$. Let us define as follows the *open shift* σ on Σ:

$$\boldsymbol{\epsilon} \in \Sigma \longmapsto \sigma(\boldsymbol{\epsilon}) \stackrel{\rm def}{=} \begin{cases} \infty & \text{if} \quad \epsilon_0 = 1, \\ \hat{\sigma}(\boldsymbol{\epsilon}) & \text{if} \quad \epsilon_0 \in \{0, 2\}\,. \end{cases} \tag{2.9}$$

$J_{\boldsymbol{r}|\Sigma'}$ conjugates the open shift $\sigma_{|\Sigma'}$ with $B_{\boldsymbol{r}}$ as in (2.8). Similarly, the backward open shift σ^{-1}, which kills the sequences s.t. $\epsilon_{-1} = 1$ and otherwise moves the comma to the left, is conjugated with $B_{\boldsymbol{r}}^{-1}$.

These conjugations allow us to easily characterize the various trapped sets of $B_{\boldsymbol{r}}$. On the symbolic space Σ, the forward (respectively, backward) trapped set of the open shift σ is given by the sequences $\boldsymbol{\epsilon}$ such that $\epsilon_n \in \{0, 2\}$ for all $n \geqslant 0$ (respectively, for all $n < 0$). To obtain the trapped sets for the baker's map, we restrict ourselves on Σ' and conjugate by $J_{\boldsymbol{r}|\Sigma'}$. The image sets are given by the direct products $\Gamma_- = \mathcal{C}_{\boldsymbol{r}} \times [0, 1)$, $\Gamma_+ = [0, 1) \times \mathcal{C}_{\boldsymbol{r}}$ and $K = \mathcal{C}_{\boldsymbol{r}} \times \mathcal{C}_{\boldsymbol{r}}$, where $\mathcal{C}_{\boldsymbol{r}}$ is (up to a countable set) the Cantor set on $[0, 1)$ adapted to the partition $\boldsymbol{r}$ (see figure 2).

For future use, we define the subset $\Sigma'' \subset \Sigma$ by

$$\Sigma'' \stackrel{\rm def}{=} \Sigma \setminus (\{\ldots 222 \cdot 0\epsilon_1\epsilon_2 \ldots\} \cup \{\ldots 222 \cdot 2\epsilon_1\epsilon_2 \ldots\} \cup \{\ldots \epsilon_{-2}\epsilon_{-1} \cdot \epsilon_0 222 \ldots\}s)\,. \tag{2.10}$$

We note that $\Sigma'' \supsetneq \Sigma'$ and that $J_{\boldsymbol{r}}(\Sigma \setminus \Sigma'')$ is a subset of the discontinuity set of the map $B_{\boldsymbol{r}}$ (see section 3.2.2 and figure 4). The map $J_{\boldsymbol{r}}$ realizes a kind of *semiconjugacy* between $\sigma_{|\Sigma''}$ and B:

$$\forall \boldsymbol{\epsilon} \in \Sigma'', \qquad J_{\boldsymbol{r}} \circ \sigma(\boldsymbol{\epsilon}) = B_{\boldsymbol{r}} \circ J_{\boldsymbol{r}}(\boldsymbol{\epsilon}). \tag{2.11}$$

It is understood above that both sides are 'sent to infinity' if $\epsilon_0 = 1$.

2.3. *Eigenmeasures of open maps*

Before defining eigenmeasures of open maps, we briefly recall how invariant measures emerge in the study of the quantized *closed* maps.

2.3.1. Quantum ergodicity for closed chaotic systems. The quantum-classical correspondence between a closed symplectic map $\hat{T}$ and its quantization $\hat{T}_N$ (see section 3.1) has one important consequence: in the semiclassical limit $N \to \infty$, stationary states of the quantum system (that is, eigenstates of $\hat{T}_N$) should reflect the stationary properties of the classical map, namely, its invariant measures. To be more precise, with any sequence of eigenstates $(\psi_N)_{N\to\infty}$ of the quantum map, one can associate at least one *semiclassical measure* (see section 3.1.2). Omitting problems due to the discontinuities of $\hat{T}$, the quantum-classical correspondence implies that a semiclassical measure μ must be invariant w.r.t. $\hat{T}$:

$$\hat{T}^* \mu = \mu \quad \Longleftrightarrow \quad \text{for any Borel set } S \subset \mathbb{T}^2, \qquad \mu(\hat{T}^{-1}(S)) = \mu(S)\,. \tag{2.12}$$

μ is then also invariant w.r.t. the inverse map $\hat{T}^{-1}$.

If the map $\hat{T}$ is ergodic w.r.t. the Liouville measure μ_L on $\hat{M}$, the quantum ergodicity theorem (or Schnirelman's theorem [37]) states that, for 'almost any' sequence $(\psi_N)_{N\to\infty}$, there is a unique associated semiclassical measure, which is μ_L itself.

Such a theorem was first proven for eigenstates of the Laplacian on compact Riemannian manifolds with ergodic geodesic flow [7, 46] and then for more general Hamiltonians [17], billiards [14, 48] and maps [4, 47]. Quantum ergodicity for piecewise smooth maps was proven in a general setting in [28], and the particular case of the baker's map was treated in [8]. Finally, in [1] it was shown that the (closed) Walsh–baker's map, seen as the quantization of the right shift $\hat{\sigma}$ on (Σ, μ_L), also satisfies quantum ergodicity with respect to μ_L.

It is generally unknown whether there exist 'exceptional sequences' of eigenstates, converging to a different invariant measure. The absence of such sequences is expressed by the quantum unique ergodicity conjecture [33], which has been proven only for systems with arithmetic properties [21, 25]. This conjecture has been disproved for some specific systems enjoying large spectral degeneracies at the quantum level, allowing for sufficient freedom to build up partially localized eigenstates [1, 12, 18]. Some special eigenstates of the standard quantum baker with interesting multifractal properties have been numerically identified [29], but their persistence in the semiclassical limit remains unclear.

2.3.2. Eigenmeasures of open maps. We now dig the hole $H = \hat{M} \backslash M$ and consider the open map $T = \hat{T}_{|M}$. Eigenmeasures (also called conditionally invariant measures) of maps 'with holes' have been less studied than their invariant counterparts. The recent paper of Demers and Young [9] summarizes most of the properties of these measures, for maps enjoying various dynamical properties. Below we describe some of these properties for general maps, before being more specific in the case of the baker's map.

A probability measure μ on $\hat{M}$ which is invariant through T *up to a multiplicative factor* will be called an eigenmeasure of T:

$$T^* \mu = \Lambda_\mu \, \mu \quad \Longleftrightarrow \quad \text{for any Borel set } S \subset \hat{M}, \qquad \mu(T^{-1}(S)) = \Lambda_\mu \, \mu(S)\,. \tag{2.13}$$

Here $\Lambda_\mu \in [0, 1]$ (or rather $\gamma_\mu = -\log \Lambda_\mu$) is called the 'escape rate' or the 'decay rate' of the eigenmeasure μ and is given by $\Lambda_\mu = \mu(M)$. It corresponds to the fact that a fraction of the particles in the support of μ escape at each step. Our definition slightly differs from the one in [9]: their measures are supported on M and normalized there, while we choose the normalization $\mu(\hat{M}) = 1$.

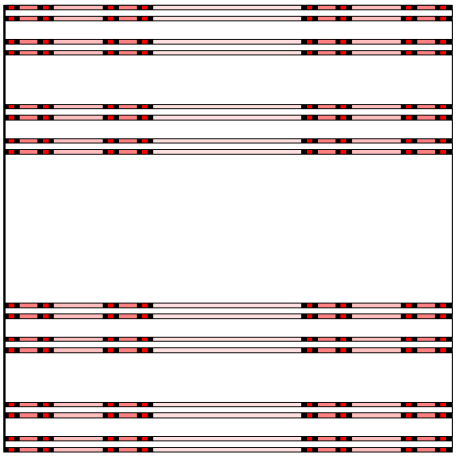

Figure 3. Various components $\Gamma_+^{(n)}$ of the forward trapped set for the open baker $B_{r_{\rm sym}}$. Various red/grey scales (from light to dark) correspond to $n = 1, 2, 3, 4$.

Here are some simple properties.

Proposition 1. *Let μ be an eigenmeasure of T with the decay rate Λ_μ.*
If $\Lambda_\mu = 0$, μ is supported in the hole H.
If $\Lambda_\mu = 1$, μ is supported in the trapped set K (invariant measure).
If $0 < \Lambda_\mu < 1$, μ is supported on the set $\Gamma_+ \setminus K$.

Following the decomposition (2.3), the set $\Gamma_+ \setminus K$ can be split into a disjoint union (see figure 3):

$$\Gamma_+ \setminus K = \bigsqcup_{n \geqslant 1} \Gamma_+^{(n)}, \qquad \text{where} \qquad \Gamma_+^{(n)} \stackrel{\rm def}{=} \Gamma_+ \cap M_n = T^{-n+1} \Gamma_+^{(1)}, \quad n \geqslant 1\,.$$

Following [9, theorem. 3.1], any eigenmeasure can be constructed as follows.

Proposition 2. *Take some $\Lambda \in [0, 1)$ and ν an arbitrary Borel probability measure on $\Gamma_+^{(1)} = \Gamma_+ \cap H$. Define the probability measure μ on $\mathbb{T}^2$ as follows:*

$$\mu = (1 - \Lambda) \sum_{n \geqslant 0} \Lambda^n \,(T^*)^n\, \nu\,. \tag{2.14}$$

Then $T^* \mu = \Lambda \mu$. All Λ-eigenmeasures of T can be written this way.

This construction shows that, as long as $\Gamma_+^{(1)}$ is not empty (that is, there exists at least a point x_0 trapped in the past but escaping in the future), there are plenty of eigenmeasures.

The case where the map $\hat{T}$ is uniformly hyperbolic has been studied in detail by Chernov and Markarian [6, 27]. The set $\Gamma_+^{(1)}$ is then uncountable, and it is foliated by the unstable foliation. These authors focused on eigenmeasures of the open map T which are absolutely continuous along the unstable direction. Even with this condition, one has plenty of eigenmeasures (we recall that for a closed hyperbolic map, unstable absolute continuity is satisfied for a unique invariant measure, namely, the SRB measure).

2.3.3. Pure point eigenmeasures. Applying the recipe of proposition 2 to the Dirac measure δ_{x_0} on an arbitrary point $x_0 \in \Gamma_+^{(1)}$, we obtain a simple, pure point Λ-eigenmeasure supported on the backward trajectory $(x_{-n} = T^{-n}(x_0))_{n\geqslant 0}$. For any $\Lambda \in (0, 1)$, we call this eigenmeasure

$$\mu_{x_0,\Lambda} \stackrel{\text{def}}{=} (1-\Lambda)\sum_{n\geqslant 0} \Lambda^n \, \delta_{x_{-n}} . \tag{2.15}$$

We recall that the pure point *invariant* measures of T are localized on periodic orbits of T on K (which, for the case of a horseshoe map $T_{|K}$, form a countable family). In contrast, for each $\Lambda \in [0, 1)$ the family of pure point eigenmeasures $\mu_{x_0,\Lambda}$ is labelled by all $x_0 \in \Gamma_+^{(1)}$ (an uncountable set).

2.3.4. Natural eigenmeasure. As discussed in [9], one may search for the definition of a *natural* eigenmeasure of T. The simplest ('ideal') definition reads as follows: for any initial measure ρ absolutely continuous w.r.t. μ_L, and such that $\rho(\Gamma_-) > 0$,

$$\mu_{nat} = \lim_{n\to\infty} \frac{T^{*n}\,\rho}{\|T^{*n}\,\rho\|}, \qquad \text{where } \|\mu\| \stackrel{\text{def}}{=} \mu(\hat{M}) . \tag{2.16}$$

It was shown in [6, 27] that, for a hyperbolic open map, the limit exists and is independent of ρ. Besides, the natural measure is then absolutely continuous along the unstable direction. Yet, for a general open map the above limit does not necessarily exist or it may depend on the initial distribution ρ [9].

In case (2.16) holds, the eigenvalue $\Lambda_{\text{nat}} = \mu_{\text{nat}}(M)$ associated with this measure is generally called 'the decay rate of the system' by physicists. For a *closed* chaotic map $\hat{T}$, the natural measure is the Liouville measure μ_L (indeed, $\hat{T}^{*n}\rho \to \mu_L$ is equivalent to the fact that $\hat{T}$ is *mixing* with respect to μ_L). As explained in section 2.3.1, the quantum ergodicity theorem shows that this particular invariant measure is 'favoured' by quantum mechanics. One interesting question we will address is the relevance of μ_{nat} with respect to the quantized open map T_N.

2.3.5. Bernoulli eigenmeasures of the open baker. We now focus on the open baker $B_{\boldsymbol{r}}$ and the open shift σ it is conjugated with and construct a family of eigenmeasures called Bernoulli eigenmeasures. Some of these measures will appear in section 6 as semiclassical measures for the Walsh-quantized open baker.

The first equality in (2.7) maps the set Σ_+ of one-sided sequences $\cdot\epsilon_0\epsilon_1\ldots$ to the position interval [0, 1]. By a slight abuse, we also call this mapping $J_{\boldsymbol{r}}$. We will first construct Bernoulli measures on the symbol spaces Σ_+ and Σ and then push them on the interval or the torus using $J_{\boldsymbol{r}}$.

Let us recall the definition of Bernoulli measures on Σ_+. Choose a *weight distribution* $\mathbf{P} = (P_0, P_1, P_2)$, where $P_\epsilon \in [0, 1]$ and $P_0 + P_1 + P_2 = 1$. We define the measure $\nu_{\boldsymbol{P}}^{\Sigma_+}$ on Σ_+ as follows: for any n-sequence $\boldsymbol{\epsilon} = \cdot\epsilon_0\epsilon_1\cdots\epsilon_{n-1}$, the weight of the cylinder $[\boldsymbol{\epsilon}]$ is given by

$$\nu_{\boldsymbol{P}}^{\Sigma_+}([\boldsymbol{\epsilon}]) = P_{\epsilon_0}\cdots P_{\epsilon_{n-1}} .$$

The push-forward on [0, 1] of this measure will also be called a Bernoulli measure and called $\nu_{\boldsymbol{r},\boldsymbol{P}} = J_{\boldsymbol{r}}^*\,\nu_{\boldsymbol{P}}^{\Sigma_+}$. If we take $P_\epsilon = r_\epsilon$ for $\epsilon = 0, 1, 2$, we recover $\nu_{\boldsymbol{r},\boldsymbol{r}} = \nu_{\text{Leb}}$ as the Lebesgue measure on [0, 1]. If for some $\epsilon \in \{0, 1, 2\}$ we take $P_\epsilon = 1$, we get for $\nu_{\boldsymbol{r},\boldsymbol{P}}$ the Dirac measure at the point $q(\cdot\epsilon\epsilon\epsilon\ldots)$, which takes the respective values 0, $r_0/(1-r_1)$ and 1. For any other distribution $\boldsymbol{P}$, the Bernoulli measure $\nu_{\boldsymbol{r},\boldsymbol{P}}$ is purely singular continuous w.r.t. the Lebesgue measure. Fractal properties of the measures $\nu_{\boldsymbol{r},\boldsymbol{P}}$ were studied in [16]. If the weight P_ϵ vanishes for a single ϵ (e.g. $P_1 = 0$), $\nu_{\boldsymbol{r},\mathbf{P}}$ is supported on a Cantor set (e.g. $\mathcal{C}_{\boldsymbol{r}}$).

A Bernoulli measure on $\nu_{\boldsymbol{P}}^{\Sigma_-}$ can be defined similarly. By taking products of two Bernoulli measures, one easily constructs eigenmeasures of the open shift σ or the open baker $B_{\boldsymbol{r}}$.

Proposition 3. *Take any weight distribution $\boldsymbol{P}$ such that $P_1 < 1$. Then the following hold.*

(i) *There is a unique auxiliary distribution, namely, $\boldsymbol{P}^* = \{P_0/(P_0 + P_2), 0, P_2/(P_0 + P_2)\}$, such that the product measure*

$$\mu_{\boldsymbol{P}}^{\Sigma} \stackrel{\text{def}}{=} \nu_{\boldsymbol{P}}^{\Sigma_+} \times \nu_{\boldsymbol{P}^*}^{\Sigma_-}$$

is an eigenmeasure of the open shift σ. The corresponding decay rate is

$$\Lambda_{\boldsymbol{P}} = 1 - P_1 = P_0 + P_2\,. \tag{2.17}$$

(ii) *For any $\boldsymbol{r}$, the push-forward $\mu_{\boldsymbol{r},\boldsymbol{P}} = J_{\boldsymbol{r}}^* \,\mu_{\boldsymbol{P}}^{\Sigma}$ is an eigenmeasure of the open baker $B_{\boldsymbol{r}}$.*

By definition, the product measure has the following weight on a cylinder $[\boldsymbol{\epsilon}] = [\epsilon_{-m} \dots \epsilon_{n-1}]$

$$\mu_{\boldsymbol{P}}^{\Sigma}([\boldsymbol{\epsilon}]) = P_{\epsilon_{-m}}^* \dots P_{\epsilon_{-1}}^* \, P_{\epsilon_0} \cdots P_{\epsilon_{n-1}}\,.$$

The proof of the first statement is straightforward. There is a slight subtlety concerning push-forwards of σ-eigenmeasures, which is resolved in the following proposition.

Proposition 4. *Let μ^Σ be an eigenmeasure of the open shift $\sigma : \Sigma \to \Sigma$. If μ^Σ does not charge the subset $\Sigma\setminus\Sigma''$ described in (2.10), that is if $\mu^\Sigma(\Sigma\setminus\Sigma'') = 0$, then its push-forward $\mu = J_{\boldsymbol{r}}^*\mu^\Sigma$ on $\mathbb{T}^2$ is an eigenmeasure of $B_{\boldsymbol{r}}$.*

Proof. Take any Borel set $S \in \mathbb{T}^2$. Then the following identities hold:

$$\mu(B_{\boldsymbol{r}}^{-1}(S)) \stackrel{\text{def}}{=} \mu^\Sigma(J_{\boldsymbol{r}}^{-1} \circ B_{\boldsymbol{r}}^{-1}(S)) = \mu^\Sigma(\sigma^{-1} \circ J_{\boldsymbol{r}}^{-1}(S)) = \Lambda\, \mu^\Sigma(J_{\boldsymbol{r}}^{-1}(S)) \stackrel{\text{def}}{=} \Lambda\, \mu(S)\,.$$

The second equality is a consequence of the semiconjugacy (2.11), which implies the fact that the symmetric difference of the sets $J_{\boldsymbol{r}}^{-1} \circ B_{\boldsymbol{r}}^{-1}(S)$ and $\sigma^{-1} \circ J_{\boldsymbol{r}}^{-1}(S)$ is necessarily a subset of $\Sigma\setminus\Sigma''$.

The condition $\mu^\Sigma(\Sigma\setminus\Sigma'') = 0$ in the proposition cannot be removed. Indeed, by applying the construction of proposition 2 to an initial 'seed' supported on $\{\dots 222 \cdot 1\epsilon_1 \dots\}$, one obtains an eigenmeasure of σ charging Σ'' and such that its push-forward is not an eigenmeasure of $B_{\boldsymbol{r}}$.

To prove the second point of the proposition, we remark that the only Bernoulli eigenmeasure $\mu_{\boldsymbol{P}}^{\Sigma}$ charging $\Sigma\setminus\Sigma''$ is the Dirac measure on the sequence $\dots 2222 \dots$, which is pushed forward to the delta measure at the origin of $\mathbb{T}^2$. □

If we take $\boldsymbol{P} = \boldsymbol{r}$, the push-forward is the natural eigenmeasure $\mu_{\boldsymbol{r},\boldsymbol{r}} = \mu_{\text{nat}}$ of the open baker $B_{\boldsymbol{r}}$. If $\boldsymbol{P} \neq \boldsymbol{P}'$, the Bernoulli eigenmeasures $\mu_{\boldsymbol{P}}^{\Sigma}$ and $\mu_{\boldsymbol{P}'}^{\Sigma}$ are *mutually singular* (there exists disjoint Borel subsets A, A' of Σ such that $\nu_{\boldsymbol{r},\boldsymbol{P}}(A) = \nu_{\boldsymbol{r},\boldsymbol{P}'}(A') = 1$), even though they may share the same decay rate. Except in the case $\boldsymbol{P} = (1, 0, 0)$, $\boldsymbol{P}' = (0, 0, 1)$, the push-forwards $\mu_{\boldsymbol{r},\boldsymbol{P}}$ and $\mu_{\boldsymbol{r},\boldsymbol{P}'}$ are also mutually singular.

3. Quantized open maps

3.1. 'Axioms' of quantization

For appropriate 2D-dimensional compact symplectic manifolds $\hat{M}$, one may define a sequence of 'quantum' Hilbert spaces $(\mathcal{H}_N)_{N\to\infty}$ of finite dimensions N, which are related to Planck's constant by $N \sim \hbar^{-D}$. Quantum states are normalized vectors in $\mathcal{H}_N$. One also wants to quantize observables, that is, functions $a \in C^\infty(\hat{M})$, into operators $\text{Op}_N(a)$ on $\mathcal{H}_N$. An invertible (respectively, open) map $\hat{T}$ (respectively, T) is quantized into a family of *unitary* (respectively, contracting) operators $\hat{T}_N$ (respectively, T_N), which satisfies certain properties when $N \to \infty$ (see below).

For the example that we will treat explicitly (the open baker's map), the propagators of the closed and open maps are related by $T_N = \hat{T}_N \circ \Pi_{M,N}$, where $\Pi_{M,N}$ is a projector associated with the subset M: it kills the quantum states microlocalized in the hole, while keeping unchanged the states microlocalized inside M [35]. Yet, in general the propagator T_N can be defined without having to construct $\hat{T}_N$ beforehand.

The proof of our main result, theorem 1, only uses some 'minimal' properties of the quantized observables and maps. These properties were presented as 'quantization axioms' by Marklof and O'Keefe in the case of closed maps [28]. We adopt the same approach, namely, define quantization through these 'minimal axioms' and state our result in this general framework. Afterwards, we will check that these axioms are satisfied for the quantized open baker.

3.1.1. Axioms on observables. All axioms will describe properties of the quantum operators in the semiclassical limit $N \to \infty$. With a slight abuse of notation, we write $A_N \sim B_N$ when two families of operators (A_N), (B_N) on $\mathcal{H}_N$ satisfy

$$\|A_N - B_N\|_{\mathcal{L}(\mathcal{H}_N)} \stackrel{N\to\infty}{\longrightarrow} 0\,.$$

The axioms concerning the quantization of observables read as follows [28, axiom 2.1].

Axioms 1. *For any $a \in C^\infty(\hat{M})$, the operators $\mathrm{Op}_N(a)$ on $\mathcal{H}_N$ must satisfy, in the limit $N \to \infty$:*

$$\begin{aligned}
&\mathrm{Op}_N(\bar{a}) \sim \mathrm{Op}_N(a)^\dagger && \text{(asymptotic hermiticity)},\\
&\mathrm{Op}_N(a)\,\mathrm{Op}_N(b) \sim \mathrm{Op}_N(ab) && \text{(0th order symbolic calculus)},\\
&\lim_{N\to\infty} N^{-1}\,\mathrm{Tr}\,\mathrm{Op}_N(a) = \int_{\hat{M}} a\,\mathrm{d}\mu_L && \text{(normalization)}\,.
\end{aligned} \tag{3.1}$$

These axioms are satisfied by all standard quantization recipes (e.g. geometric or Toeplitz quantization on Kähler manifolds [47]). Note that they do not involve the symplectic structure on $\hat{M}$ and can thus be extended to more general phase spaces.

3.1.2. Semiclassical measures. From this, we may define semiclassical measures associated with sequences of (normalized) quantum states $(\psi_N \in \mathcal{H}_N)_{N\to\infty}$. Such a sequence is said to converge to the distribution μ on $\hat{M}$ iff, for any observable $a \in C^\infty(\hat{M})$,

$$\langle \psi_N, \mathrm{Op}_N(a)\psi_N\rangle \stackrel{N\to\infty}{\longrightarrow} \int_{\hat{M}} a\,\mathrm{d}\mu\,. \tag{3.2}$$

From the above axioms, one can show that μ is necessarily a probability measure on $\hat{M}$, which is called the semiclassical measure associated with $(\psi_N \in \mathcal{H}_N)$. By weak compactness, from any sequence $(\psi_N \in \mathcal{H}_N)$ one can always extract a subsequence (ψ_{N_k}) converging in the above sense to a certain measure. The latter is called *a* semiclassical measure of the sequence (ψ_N).

3.1.3. Axioms on maps. In the axiomatic framework of [28], the conditions satisfied by the unitary propagators $(\hat{T}_N)$ quantizing a closed map $\hat{T}$ consist of some form of quantum-classical correspondence (Egorov's theorem), when evolving quantum observables $\mathrm{Op}_N(a)$ through these propagators. However, if $\hat{T}$ is discontinuous (or nonsmooth) at some points, its propagator $\hat{T}$ will exhibit diffraction phenomena around these 'singular' points, which alter the propagation properties. At the classical level, a smooth observable $a \in C^\infty(\hat{M})$ is transformed into a smooth observable $a \circ \hat{T} \in C^\infty(\hat{M})$ only if a vanishes near the singular points of $\hat{T}^{-1}$. As a result, the quantum-classical correspondence is a reasonable axiom only if the observable a is supported 'far away' from the singular set of $\hat{T}^{-1}$.

We now specify our assumptions on the open map $T : M \to \hat{M}$. Firstly, the hole H will be a 'nice' set, that is, a set with nonempty interior and such that ∂H has Minkowski content zero (i.e. the volume of its ϵ-neighbourhood vanishes when $\epsilon \to 0$). We also assume that M, the domain of definition of T, can be decomposed using finitely or countably many open connected sets O_i: $\bar{M} = \overline{\sqcup_{i\geqslant 1} O_i}$, with $O_i \cap O_j = \emptyset$, and for each i, $T_{|O_i} : O_i \to T(O_i)$ is a smooth canonical diffeomorphism, with all derivatives uniformly bounded. Let us split the hole into $H = \mathring{H} \sqcup DH$. The *continuity set* (respectively, *discontinuity set*) of the map T is defined as

$$C(T) = \sqcup_{i\geqslant 1} O_i \subset \mathring{M}, \qquad \text{respectively,} \quad D(T) \stackrel{\text{def}}{=} (M \setminus C(T)) \sqcup DH\,.$$

We assume that $D(T)$ has Minkowski content zero. We have the decomposition $\hat{M} = C(T) \sqcup \mathring{H} \sqcup D(T)$. A similar decomposition holds for the inverse map:

$$\hat{M} = C(T^{-1}) \sqcup \mathring{H}^{-1} \sqcup D(T^{-1}), \qquad \text{with} \quad C(T^{-1}) = T(C(T))\,. \tag{3.3}$$

Adapting the axioms of [28] to the case of open maps, we set as follows the characteristic property of the operators $(T_N)_{N\to\infty}$ quantizing T.

Axioms 2. *We say that the operators $(T_N \in \mathcal{L}(\mathcal{H}_N))_{N\to\infty}$ quantize the open map T iff*

- *for N large enough, $\|T_N\|_{\mathcal{L}(\mathcal{H}_N)} \leqslant 1$,*
- *for any observable $a \in C_c^\infty(C(T^{-1}) \sqcup \mathring{H}^{-1})$, we have in the limit $N \to \infty$*

$$T_N^\dagger \,\mathrm{Op}_N(a)\, T_N \sim \mathrm{Op}_N \mathbb{1}_M \times (a \circ T))\,. \tag{3.4}$$

Here $C_c^\infty(S)$ indicates the smooth functions compactly supported inside S and $\mathbb{1}_M$ is the characteristic function on M.

Note that, if a is supported inside $C(T^{-1}) \sqcup \mathring{H}^{-1}$, the function $\mathbb{1}_M \times (a \circ T)$ is well defined, smooth and supported inside $C(T)$. The factor $\mathbb{1}_M$ ensures that an observable supported inside H^{-1} is 'semiclassically killed' by the evolution through T_N. The condition (3.4) reminds us of the definition of a 'quantized weighted relation' introduced in [31], but it is less precise (it only describes the lowest order in $\hbar$).

Remark 1. Going back to the problem of potential scattering mentioned in the introduction, we expect the operators T_N to share some spectral properties with the propagator $\exp(-\mathrm{i}H_{W,\hbar}/\hbar)$ of the 'absorbing Hamiltonian' in the semiclassical limit. The eigenvalues $\{\lambda_j\}$ of T_N should be compared with $\{\mathrm{e}^{-\mathrm{i}\tilde{z}_j/\hbar}\}$ or with $\{\mathrm{e}^{-\mathrm{i}z_j/\hbar}\}$, where $\{\tilde{z}_j\}$ (respectively, $\{z_j\}$) are the eigenvalues of $H_{W,\hbar}$ (respectively, the resonances of $H_\hbar$). Similarly, the eigenstates of T_N should share some microlocal properties with the eigenfunctions $\tilde{\varphi}_j$ of $H_{W,\hbar}$ (respectively, the resonant states φ_j of $H_\hbar$) inside the interaction region. Accordingly, we will sometimes call 'resonances' and 'resonant eigenstates' the eigenvalues/states of quantized open maps.

3.2. The quantum open baker's map

For any dimension $N = (2\pi\hbar)^{-1}$, the quantum Hilbert space $\mathcal{H}_N$ adapted to the torus phase space is spanned by the orthonormal *position basis* $\{\boldsymbol{q}_j,\ j = 0, \ldots, N-1\}$, localized at the discrete positions $q_j = j/N$.

There are several standard ways to quantize observables on $\mathbb{T}^2$: Weyl quantization, Toeplitz (or anti-Wick) quantizations [4] or Walsh-quantization [1]. Weyl and anti-Wick quantizations are equivalent to each other in the semiclassical limit, in the sense that $\mathrm{Op}_N^W(a) \sim \mathrm{Op}_N^{AW}(a)$ for any smooth observable. In contrast, the Walsh-quantization (see section 6.1) is not equivalent to the previous ones.

After recalling the definition of the anti-Wick quantization on $\mathbb{T}^2$ and the 'standard' quantization of the baker's map, following the original approach of Balazs–Voros, Saraceno, Saraceno–Vallejos [2, 34, 35], we check that the latter satisfies axiom 2 with respect to the anti-Wick quantization.

3.2.1. Anti-Wick quantization of observables. We recall the definition and properties of coherent states on $\mathbb{T}^2$, which we use to construct the anti-Wick quantization of observables and by duality the Husimi representation of quantum states [4]. The Gaussian coherent state in $L^2(\mathbb{R})$, localized at the phase space point $x = (q_0, p_0) \in \mathbb{R}^2$ and with squeezing parameter $s > 0$, is defined by the normalized wavefunction

$$\Psi_{x,s}(q) \stackrel{\text{def}}{=} \left(\frac{s}{\pi\hbar}\right)^{1/4} \mathrm{e}^{-\mathrm{i}\frac{p_0 q_0}{2\hbar}}\, \mathrm{e}^{\mathrm{i}\frac{p_0 q}{\hbar}}\, \mathrm{e}^{-s\frac{(q-q_0)^2}{2\hbar}} . \tag{3.5}$$

When $\hbar = (2\pi N)^{-1}$, this state can be periodized on the torus, to yield the torus coherent state $\psi_{x,s} \in \mathcal{H}_N$ with the following components on the basis of $\{\boldsymbol{q}_j\}$:

$$\langle \boldsymbol{q}_j, \psi_{x,s}\rangle = \frac{1}{\sqrt{N}} \sum_{\nu\in\mathbb{Z}} \Psi_{x,s}(j/N + \nu), \qquad j = 0, \ldots, N-1 . \tag{3.6}$$

For $s > 0$ fixed, these states are asymptotically normalized when $N \to \infty$.

With any squeezing $s > 0$ and inverse Planck's constant $N \in \mathbb{N}$ we associate the anti-Wick (or Toeplitz) quantization

$$f \in C^\infty(\mathbb{T}^2) \longmapsto \mathrm{Op}_N^{\mathrm{AW},s}(f) \stackrel{\text{def}}{=} \int_{\mathbb{T}^2} |\psi_{x,s}\rangle\langle\psi_{x,s}|\; f(x)\; N\,\mathrm{d}x\,, \tag{3.7}$$

which satisfies axiom 1 [4]. By duality, this quantization defines, for any state $\psi \in \mathcal{H}_N$, a Husimi distribution H_ψ^s:

$$\forall f \in C^\infty(\mathbb{T}^2), \qquad H_\psi^s(f) \stackrel{\text{def}}{=} \langle \psi, \mathrm{Op}_N^{\mathrm{AW},s}(f)\,\psi\rangle\,.$$

For $\|\psi\| = 1$, this distribution is a probability measure, with density given by the (smooth, nonnegative) Husimi function

$$H_\psi^s(x) = N\,|\langle\psi_{x,s}, \psi\rangle|^2\,, \qquad x \in \mathbb{T}^2\,. \tag{3.8}$$

Applying definition (3.2) to the present framework, a sequence of states $(\psi_N \in \mathcal{H}_N)_{N\to\infty}$ converges to the measure μ on $\mathbb{T}^2$ iff, for any given $s > 0$, the Husimi measures $(H_{\psi_N}^s)_{N\to\infty}$ weak-$*$ converge to the measure μ.

Following Schubert's work [39], one can also consider anti-Wick quantizations (and dual Husimi measures) in which the squeezing parameter s in integral (3.7) depends on the phase space point. Adapting the proofs of [39] to the torus setting, one shows that all these quantizations are equivalent to one another.

Proposition 5. *Choose two functions $s_1, s_2 \in C^\infty(\mathbb{T}^2, (0,\infty))$. Then the two associated anti-Wick quantizations become close to one another when $N \to \infty$:*

$$\forall f \in C^\infty(\mathbb{T}^2), \qquad \|\mathrm{Op}_N^{\mathrm{AW},s_1}(f) - \mathrm{Op}_N^{\mathrm{AW},s_2}(f)\| = \mathcal{O}(N^{-1})\,.$$

3.2.2. Standard quantization of the baker's map. Strictly speaking, the quantization of the closed baker's map $\hat{B}_{\boldsymbol{r}}$ is well defined only if the coefficients $\mathbf{r}$ are such that

$$N\,r_i = N_i \in \mathbb{N}, \qquad i = 0, 1, 2\,. \tag{3.9}$$

Yet, in the semiclassical limit $N \to \infty$ one can, if necessary, slightly modify the r_i by amounts $\leqslant 1/N$ in order to satisfy this condition: such a modification is irrelevant for the classical dynamics. Assuming (3.9), the quantization of $\hat{B}_{\boldsymbol{r}}$ on $\mathcal{H}_N$ is given by the following unitary matrix on the position basis [2]:

$$\hat{B}_{\boldsymbol{r},N} = F_N^{-1} \begin{pmatrix} F_{N_0} & & \\ & F_{N_1} & \\ & & F_{N_2} \end{pmatrix}. \tag{3.10}$$

Here F_{N_i} denotes the N_i-dimensional discrete Fourier transform

$$(F_{N_i})_{jk} = N_i{}^{-1/2}\,\mathrm{e}^{-2\mathrm{i}\pi jk/N_i}, \qquad j, k = 0, \ldots N_i - 1. \tag{3.11}$$

Since we already have a quantization for $\hat{B}_{\boldsymbol{r}}$ and the hole is a rectangle $H = R_1 = \{r_0 \leqslant q < 1 - r_2\}$, a natural choice to quantize $B_{\boldsymbol{r}}$ is to project on the positions $q \in [0, 1) \setminus [r_0, 1 - r_2)$ and then apply $\hat{B}_{\boldsymbol{r},N}$. One gets the following open propagator on the position basis [35]:

$$B_{\boldsymbol{r},N} = F_N^{-1} \begin{pmatrix} F_{N_0} & & \\ & 0 & \\ & & F_{N_2} \end{pmatrix}. \tag{3.12}$$

This is the 'standard' quantization of the open baker's map $B_{\boldsymbol{r}}$. These matrices obviously contract on $\mathcal{H}_N$. The semiclassical connection between the matrices $\hat{B}_{\boldsymbol{r},N}$ (respectively, $B_{\boldsymbol{r},N}$) and the classical map map $\hat{B}_{\boldsymbol{r}}$ (respectively, $B_{\boldsymbol{r}}$) has been analysed in detail in [31]; this analysis implies property (3.4) with respect to the Weyl quantization. Below we give an alternative proof of property (3.4) for the open propagator $B_{\boldsymbol{r},N}$, with respect to the anti-Wick quantization. The proof uses the semiclassical propagation of coherent states analysed in [8].

To start with, we precisely give the continuity set of $B_{\boldsymbol{r}}^{-1}$:

$$C(B_{\boldsymbol{r}}^{-1}) = \tilde{R}_0 \sqcup \tilde{R}_2\,,$$

where $\tilde{R}_0 = \{q \in (0, 1),\; p \in (0, r_0)\} \subset B_{\boldsymbol{r}}(R_0)$ and similarly for $\tilde{R}_2$: in the case of the symmetric baker, these are the open grey rectangles in the right plot of figure 1. On the other hand, the hole $H^{-1} = B_{\boldsymbol{r}}(R_1) = \{q \in [0, 1),\; p \in [r_0, 1 - r_2)\}$ (this is the white strip, including the vertical side). The discontinuity set $D(B_{\boldsymbol{r}_{\rm sym}}^{-1})$ is shown in figure 4 (pink/light grey lines in the left plot).

Any observable $a \in C_c^\infty(C(B_{\boldsymbol{r}}^{-1}) \sqcup \mathring{H}^{-1})$ is supported at a distance $\geqslant \delta > 0$ from the discontinuity set $D(B_{\boldsymbol{r}}^{-1})$. Let us select some smooth $s \in C^\infty(\mathbb{T}^2, (0, \infty))$ and consider the corresponding anti-Wick quantization (see (3.7)). From definition (3.7), the operator $\mathrm{Op}^{\mathrm{AW},s}$ only involves coherent states $\psi_{x,s(x)}$ located at a distance $\geqslant \delta > 0$ from $D(B_{\boldsymbol{r}}^{-1})$. Adapting the proof of [8, proposition 5] to the general open baker $B_{\boldsymbol{r}}$, one can show the following propagation for such states:

$$B_{\boldsymbol{r},N}^\dagger\;\psi_{x,s(x)} = \mathbb{1}_{M^{-1}}(x)\,\mathrm{e}^{\mathrm{i}\theta(x,N)}\,\psi_{x',s'(x')} + \mathcal{O}_{\mathcal{H}_N}(\mathrm{e}^{-N\,C(s,\delta)})\,, \qquad N \to \infty\,. \tag{3.13}$$

Here $x' = B_{\boldsymbol{r}}^{-1}(x)$, $s'(x') = s(x)/r_\epsilon^2$ if x' lies inside the rectangle R_ϵ, and $\theta(x, N)$ is a phase which can be explicitly computed.

Through the symplectic change in variable $y = B_{\boldsymbol{r}}^{-1}(x)$ for $y \in M$, one gets

$$B_{\boldsymbol{r},N}^\dagger\,\mathrm{Op}^{\mathrm{AW},s}(a)\,B_{\boldsymbol{r},N} = \mathrm{Op}^{\mathrm{AW},s'}\mathbb{1}_M \times (a \circ B_{\boldsymbol{r}})) + \mathcal{O}_{\mathcal{B}(\mathcal{H}_N)}(\mathrm{e}^{-N\,C'(s,\delta)})\,.$$

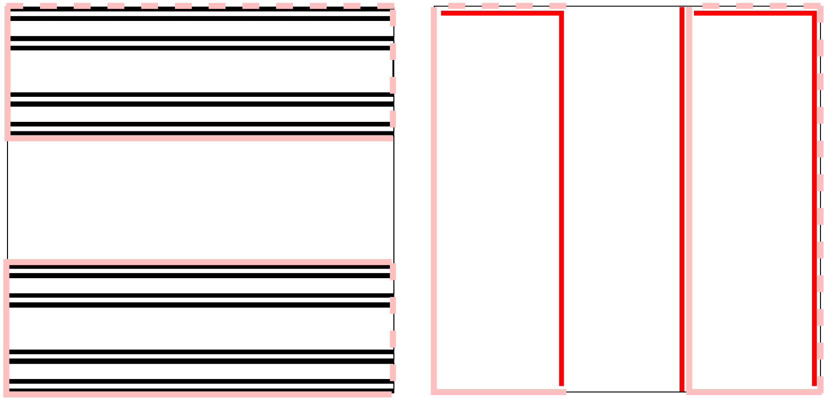

Figure 4. On the left, we show the backward trapped set Γ_+ for the symmetric open baker $B_{\boldsymbol{r}_{\rm sym}}$ (black) and the discontinuity set $D(B_{\boldsymbol{r}_{\rm sym}}^{-1})$ (pink/light grey). Their union gives $\Gamma_+ \sqcup D_{-\infty}$, which contains the support of semiclassical measures (see theorem 1, (i)). The dotted lines indicate identical points on $\mathbb{T}^2$. On the right, we show the backward image $B_{\boldsymbol{r}_{\rm sym}}^{-1}(D(B_{\boldsymbol{r}_{\rm sym}}^{-1}))$ involved in theorem 1, (iii) (pink/light grey), and the projection on $\mathbb{T}^2$ of the set $\Sigma \setminus \Sigma''$ defined in (2.10) (red/dark grey).

The function s' is obtained by taking $s'(\boldsymbol{y}) = s(B_{\boldsymbol{r}}(\boldsymbol{y}))/r_\epsilon^2$ for $\boldsymbol{y} \in B_{\boldsymbol{r}}^{-1}(\operatorname{supp} a \cap \tilde{R}_\epsilon)$ and smoothly extending the function to $s' \in C^\infty(\mathbb{T}^2, (0,\infty))$. Since the quantizations with parameters s and s' are equivalent (proposition 5), we have proven property (3.4) for the family $(B_{\boldsymbol{r},N})$. □

4. Fractal Weyl law for the quantized open baker

In this section, which mainly presents numerical results, we exclusively consider the open baker's maps $B_{\boldsymbol{r}}$ and their quantizations (3.12). Our aim is to investigate the precise notion of dimension entering the fractal Weyl law conjectured below through a numerical study which complements the one performed in [30]. Still, we believe that most statements should hold as well if we replace $B_{\boldsymbol{r}}$ by a map T obtained by restricting an Anosov diffeomorphism $\hat{T}$ outside some 'small hole', as described in [6].

From the explicit formula (3.12), the subspace of $\mathcal{H}_N$ spanned by $\{\boldsymbol{q}_j,\ N_0 \leqslant j < N - N_2\}$ is in the kernel of $B_{\boldsymbol{r},N}$. We call the spectrum of $B_{\boldsymbol{r},N}$ on the complementary subspace 'nontrivial'. This spectrum is situated inside the unit disc. In the semiclassical limit $N \to \infty$, most of it accumulates near the origin, which corresponds to 'short-living' eigenvalues [38]. We rather focus on 'long-living' eigenvalues, situated in some annulus away from the origin. By analogy with the case of potential scattering (1.1), a fractal Weyl law for the semiclassical density of 'long-living' eigenvalues was conjectured in [31].

Conjecture 1 (fractal Weyl law). *Let $B_{\boldsymbol{r},N}$ be the quantized open baker's map described in section 3.2. Then, for any radius $0 < r < 1$, there exists $C_r \geqslant 0$ such that*

$$n(N,r) \stackrel{\text{def}}{=} \#\{\lambda \in \operatorname{Spec}(B_{\boldsymbol{r},N}),\ :\ |\lambda| \geqslant r\} = C_r\, N^d + o(N^d), \qquad N \to \infty. \tag{4.1}$$

The eigenvalues are counted with multiplicities, and $2d$ is an appropriate fractal dimension of the trapped set K.

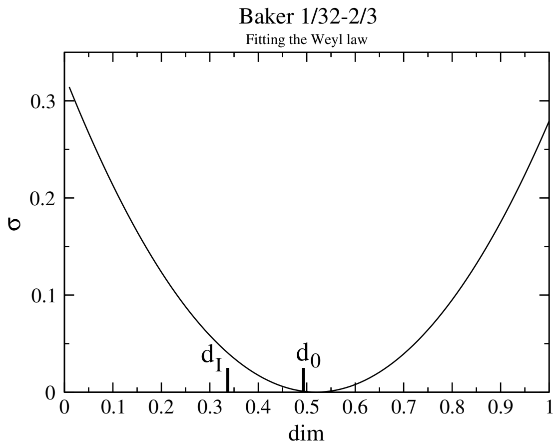

Figure 5. Standard deviations when fitting the Weyl law (4.1) to various dimensions, integrated on $0.1 \leqslant r \leqslant 1$. The two marks on the horizontal axis indicate the theoretical values d_I and d_0.

4.1. Which dimension plays a role?

In the proofs for upper bounds of the Weyl law (1.1), the exponent d is defined in terms of the *upper Minkowski dimension* of the trapped set K [15,41,42,49]. In the case of the open baker $B_{\boldsymbol{r}}$, we therefore expect that the exponent d appearing in the conjecture (respectively, $d+1$) is given by the Minkowski dimension of the Cantor set $\mathcal{C}_{\boldsymbol{r}}$ (respectively, Γ_+), which is equal to its box, Hausdorff and packing dimensions. We call this theoretical value d_0.

For the symmetric baker $B_{\boldsymbol{r}_{\mathrm{sym}}}$, the Hausdorff dimension $d_H(\Gamma_+) = d_0 + 1$ happens to be equal to the Hausdorff dimension of the natural measure μ_{nat}, defined by

$$d_H(\mu_{\mathrm{nat}}) = \inf_{A\subset\mathbb{T}^2,\ \mu_{\mathrm{nat}}(A)=1} d_H(A)\,.$$

For a nonsymmetric baker's map $B_{\boldsymbol{r}}$, these two Hausdorff dimensions only satisfy the inequality $d_H(\mu_{\mathrm{nat}}) \leqslant d_H(\Gamma_+)$ (see the explicit expressions below). We want to investigate a possible 'role' of the natural eigenmeasure μ_{nat} regarding the structure of the quantum spectrum. It is therefore legitimate to ask the following question.

Question 1. *Is the correct exponent in the Weyl law (4.1) given by* $d_I = d_H(\mu_{\mathrm{nat}}) - 1$ *instead of* $d_0 = d_H(\Gamma_+) - 1$*?*

Here the suffix I indicates that d_I is sometimes called the *information dimension* [16]. As mentioned above, both dimensions are equal for the symmetric baker $B_{\boldsymbol{r}_{\mathrm{sym}}}$, for which the Weyl law (1) has been numerically tested in [30]. They are also equal in the case of a closed map on $\mathbb{T}^2$: in this case, the Weyl law has exponent 1 (the whole spectrum lies on the unit circle), and we have $d_H(\mathbb{T}^2) = d_H(\mu_L) = 2$.

For a nonsymmetric baker $B_{\boldsymbol{r}}$, the two dimensions take different values:

$$d_0 \quad \text{is the solution of} \quad r_0^d + r_2^d = 1, \qquad \text{while} \quad d_I = \frac{r_0 \log p_0 + r_2 \log p_2}{r_0 \log r_0 + r_2 \log r_2}\,.$$

To answer the above question, we considered a very asymmetric baker, taking $\boldsymbol{r}_{\mathrm{asym}}$ with $r_0 = 1/32$, $r_2 = 2/3$. The two dimensions then take the values $d_0 \approx 0.493$, $d_I \approx 0.337$. We computed the counting function $n(N, r)$ for several radii $0.1 \leqslant r \leqslant 1$ and several values of N. We then tried to fit the Weyl law (4.1) with an exponent d varying in a certain range and computed the standard deviations (see figure 5). The numerical result is unambiguous: the

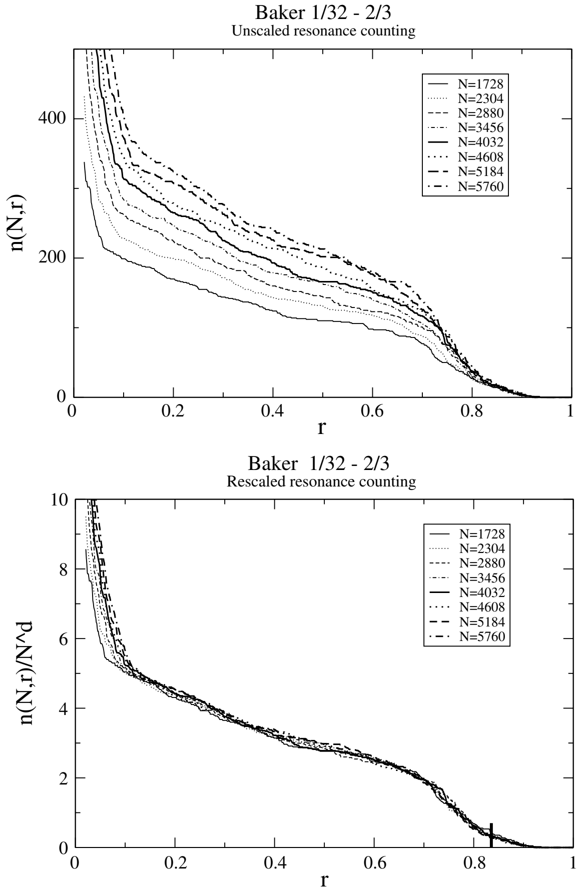

Figure 6. Top: spectral counting function for the asymmetric baker $B_{\boldsymbol{r}_{\mathrm{asym}}}$, for various values of Planck's constant N. Bottom: same curves vertically rescaled by N^{-d_0}. The thick tick mark indicates the radius $\sqrt{\Lambda_{\mathrm{nat}}}$ corresponding to the natural measure.

best fit clearly occurs away from d_I, but it is close to d_0. This numerical test rules out the possibility that d_I provides the correct exponent of the Weyl law and suggests to indeed take $d = d_0$.

To further illustrate the Weyl law for the asymmetric baker $B_{\boldsymbol{r}_{\mathrm{asym}}}$, we plot in figure 6 (top) the counting functions $n(N, r)$ as a function of $r \in (0, 1)$, for several values of N. On the bottom plot, we rescale $n(N, r)$ by the power N^{-d_0}: the rescaled curves almost perfectly overlap, indicating that scaling (4.1) is correct.

Remark 2. In figure 6 (right) the rescaled counting function seems to converge to a function which is strictly decreasing on an interval $[\lambda_{\min}, \lambda_{\max}]$, where $\lambda_{\min} \approx 0.1$, $\lambda_{\max} \approx 0.9$. This implies that the spectrum of $B_{\boldsymbol{r}_{\mathrm{asym}},N}$ becomes dense in the whole annulus $\{\lambda_{\min} \leqslant |\lambda| \leqslant \lambda_{\max}\}$, when $N \to \infty$. Therefore, at this heuristic level, for any $\lambda \in [\lambda_{\min}, \lambda_{\max}]$ one may consider sequences of eigenvalues $(\lambda_N)_{N \geqslant 1}$ with the property $|\lambda_N| \xrightarrow{N\to\infty} \lambda$. In particular, we may consider sequences converging to $\sqrt{\Lambda_{\mathrm{nat}}}$.

5. Localization of resonant eigenstates

We now return to the general framework of an open map T on a subset M of a compact phase space $\hat{M}$, which is quantized into a sequence of operators $(T_N)_{N\to\infty}$ according to axioms 2. We want to study the semiclassical measures associated with the long-living eigenstates of T_N.

To start with, we fix some $\lambda_m \in (0,1)$, such that, for N large enough, $\mathrm{Spec} T_N \cap \{|\lambda| \geqslant \lambda_m\} \neq \emptyset$. We can then consider sequences of eigenstates $(\psi_N)_{N\to\infty}$ such that, for N large enough,

$$T_N\,\psi_N = \lambda_N\,\psi_N\,, \qquad \|\psi_N\| = 1\,, \qquad |\lambda_N| \geqslant \lambda_m\,. \tag{5.1}$$

The role of the (quite arbitrary) lower bound $\lambda_m > 0$ is to ensure that the eigenstates we consider are 'long-living'. Up to extracting a subsequence, we can assume that (ψ_N) converges to a certain semiclassical measure μ (see section 3.1.2).

To state our result, we first need to analyse the continuity sets of the backward iterates of T. For any $n \geqslant 1$, the map T^{-n} is defined on the set M^{-n}. Its continuity set $C(T^{-n})$ can be obtained iteratively through

$$C(T^{-n}) = T(C(T^{-n+1}) \cap C(T))\,, \qquad n \geqslant 2\,.$$

The set M_{-n} of points escaping exactly at time $(-n)$ can also be split between its continuity subset

$$CM_{-n} \stackrel{\mathrm{def}}{=} C(T^{-n+1}) \cap \mathring{M}_{-n} = T^{n-1}(C(T^{n-1}) \cap \mathring{H}^{-1})\,,$$

which is a union of open connected sets, and its discontinuity subset $DM_{-n} = M_{-n} \setminus CM_{-n}$, which has Minkowski content zero (in the case $n = 1$, we take $CM_{-1} = \mathring{H}^{-1}$). Using this splitting of the M_{-n}, decomposition (2.5) can be recast into

$$\hat{M} = C_{-\infty} \sqcup D_{-\infty} \sqcup \Gamma_+, \qquad \text{where} \quad C_{-\infty} = \left(\bigsqcup_{n=1}^{\infty} CM_{-n}\right), \quad D_{-\infty} = \left(\bigsqcup_{n=1}^{\infty} DM_{-n}\right). \tag{5.2}$$

The set $C_{-\infty}$ consists of points which eventually fall in the hole when evolved through T^{-1} and remain at a finite distance from $D(T^{-1})$ all along their transient trajectory.

We can now state our main result concerning semiclassical measures.

Theorem 1. *Assume that a sequence of eigenstates $(\psi_N)_{N\to\infty}$ of the open quantum map T_N, with eigenvalues $|\lambda_N| \geqslant \lambda_m > 0$, converges to the semiclassical measure μ on $\hat{M}$. Then the following hold.*

(*i*) *The support of μ is a subset of $\Gamma_+ \sqcup D_{-\infty}$.*

(*ii*) *If $\mu(C(T^{-1})) > 0$, there exists $\Lambda \in [\lambda_m^2, 1]$ such that the eigenvalues $(\lambda_N)_{N\to\infty}$ satisfy*

$$|\lambda_N|^2 \stackrel{N\to\infty}{\longrightarrow} \Lambda\,.$$

For any Borel subset S not intersecting $D(T^{-1})$, one has $\mu(T^{-1}(S)) = \Lambda\,\mu(S)$.

(*iii*) *If $\mu(D(T^{-1})) = \mu(T^{-1}(D(T^{-1}))) = 0$, then μ is an eigenmeasure of T, with the decay rate Λ.*

Proof. To prove the first statement, let us choose some $n \geqslant 1$ and take an observable $a \in C_c^\infty(CM_{-n})$. Every point $x \in \mathrm{supp}\, a$ has the property that for any $0 \leqslant j < n-1$, $T^{-j}(x) \in C(T^{-1})$, while $T^{-n+1}(x) \in \mathring{H}^{-1}$. Applying iteratively property (3.4), one finds that in the semiclassical limit

$$(T_N^\dagger)^n\,\mathrm{Op}_N(a)\,(T_N)^n \sim 0\,.$$

We take into account the fact that ψ_N is a right eigenstate of T_N:

$$\langle\psi_N,\ \mathrm{Op}_N(a)\,\psi_N\rangle = |\lambda_N|^{-2n}\,\langle\psi_N,\,(T_N^\dagger)^n\,\mathrm{Op}_N(a)\,(T_N)^n\,\psi_N\rangle\,.$$

Using $|\lambda_N| \geqslant \lambda_m$, these two expressions imply $\mu(a) \stackrel{\text{def}}{=} \int a\,\mathrm{d}\mu = 0$. Since n and a are arbitrary, this shows $\mu(C_{-\infty}) = 0$, which is the first statement.

From the assumption in (ii), we may select $a \in C_c^\infty(C(T^{-1}))$ such that $\int a\,\mathrm{d}\mu > 0$. The first iterate $a \circ T$ is supported in $C(T) \subset M$. Applying (3.4), we get

$$|\lambda_N|^2\,\langle\psi_N,\ \mathrm{Op}_N(a)\,\psi_N\rangle \sim \langle\psi_N,\ \mathrm{Op}_N(a\circ T)\,\psi_N\rangle\,. \tag{5.3}$$

For N large enough, the matrix element $\langle\psi_N,\ \mathrm{Op}_N(a)\,\psi_N\rangle > \mu(a)/2$, so we may divide the above equation by this element and obtain

$$|\lambda_N|^2 \stackrel{N\to\infty}{\longrightarrow} \frac{\mu(a\circ T)}{\mu(a)}\,.$$

We call $\Lambda \geqslant |\lambda_m|^2$ this limit, which is obviously independent of the choice of a. For any Borel set $S \subset C(T^{-1})$, one can approximate $\mathbb{1}_S$ by smooth functions supported in $C(T^{-1})$ to prove that

$$\mu(T^{-1}(S)) = \Lambda\,\mu(S)\,.$$

If $S \subset \mathring{H}^{-1}$, we know that $\mu(S) = 0$ and $T^{-1}(S) = \emptyset$, so the above equality still makes sense. This proves (ii).

To obtain (iii), we split any Borel set S into $S = (S \cap D(T^{-1})) \sqcup CS$. From (ii), we have $\mu(T^{-1}(CS)) = \Lambda\,\mu(CS)$. By assumption, $\mu(S \cap D(T^{-1})) = \mu(T^{-1}(S \cap D(T^{-1}))) = 0$, so we get $\mu(T^{-1}(S)) = \Lambda\,\mu(S)$. □

5.1. Semiclassical measures of the open baker's map

In this section we apply the above theorem to the case of some open baker's map $B_{\boldsymbol{r}}$, quantized as in section 3.2.2. We also numerically compute the Husimi measures $H^1_{\psi_N}$ associated with some quantum eigenstates (we choose the isotropic squeezing $s = 1$ for convenience): although we cannot really go to the semiclassical limit, we hope that for $N \sim 1000$ these Husimi measures already give some idea of the semiclassical measures.

5.1.1. Applying theorem 1. To simplify the presentation we will restrict the discussion to the symmetric baker $B_{\boldsymbol{r}_{\mathrm{sym}}}$, which will be denoted by B in short. The discontinuity set $D(B^{-1})$, its backward image and the trapped sets were described in section 2.2.2 and section 3.2.2 and plotted in figure 4. The sets M_{-n} and their continuity subsets are also simple to describe using symbolic dynamics. For any $n = 1$, we know that $CM_{-1} = \mathring{H}^{-1}$ is the open rectangle $\{q \in [0, 1),\ p \in (1/3, 2/3)\}$. For $n \geqslant 2$, CM_{-n} is the union of 2^{n-1} horizontal rectangles, indexed by the n-sequences $\epsilon = \epsilon_{-1}\ldots\epsilon_{-n}$ such that $\epsilon_{-i} \in \{0, 2\}$, $i = 1, \ldots, n-1$, while $\epsilon_{-n} = 1$. Each such rectangle is of the form $\{q \in (0, 1),\ p \in (p(\epsilon), p(\epsilon) + 3^{-n})\}$, where $p(\epsilon) \equiv \cdot\,\epsilon_{-1}\ldots\epsilon_n$ in ternary decomposition. Some of those rectangles are shown in figure 2 (centre). The union of all these rectangles (for $n \geqslant 1$) makes up $C_{-\infty}$. Its complement $\Gamma_+ \sqcup D_{-\infty}$ is given by

$$\Gamma_+ \sqcup D_{-\infty} = ([0, 1) \times \mathcal{C}_{\boldsymbol{r}_{\mathrm{sym}}}) \cup D(B^{-1})\,.$$

This set is shown in figure 4 (left).

In figure 7, we plot the Husimi densities $H^1_{\psi_N}$ of some (right) eigenstates of $B_{\boldsymbol{r},N}$, for the symmetric baker and an asymmetric one, $\boldsymbol{r}_2 = (1/9, 5/9, 1/3)$.

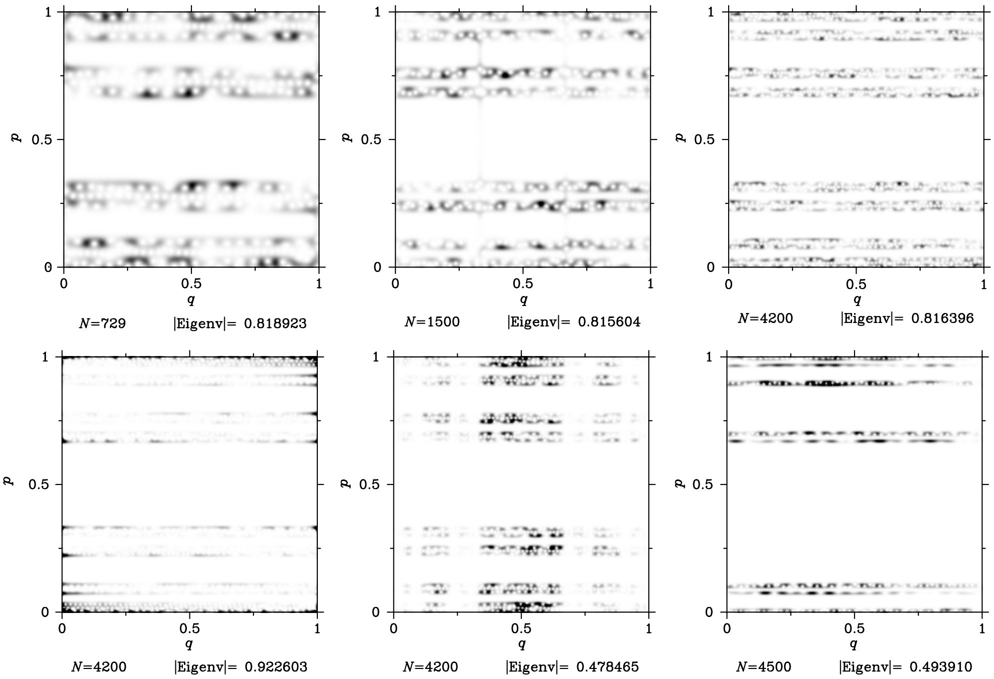

Figure 7. Husimi densities of (right) eigenstates of $B_{\boldsymbol{r},N}$ (black=large values, white=0). Top: 3 eigenstates of $B_{\boldsymbol{r}_{\rm sym},N}$ with $|\lambda_N| \approx \sqrt{\Lambda_{\rm nat}}$. Bottom left, centre: two eigenstates of $B_{\boldsymbol{r}_{\rm sym},N}$ with different $|\lambda_N|$. Bottom right: one eigenstate of $B_{\boldsymbol{r}_2,N}$.

Remark 3. We note that all Husimi functions are indeed very small in the horizontal rectangles M_{-n} for $n = 1, 2$ (in the case $N \leqslant 1500$) and also $n = 3$ for $N = 4200$. Using (3.13), one can refine the proof of theorem 1 to show that $H^1_{\psi_N}(x) = \mathcal{O}(\mathrm{e}^{-cN})$ for $x \in C_{-\infty}$, $N \to \infty$.

Remark 4. Although this is not proven in our theorem, all the Husimi functions that we have computed are very small on the set $D(B^{-1})\backslash\Gamma_+ \subset \{q = 0\}$ (see the left plot in figure 4). On the other hand, some of these Husimi functions are large on $D(B^{-1}) \cap \Gamma_+$, so we cannot rule out the possibility that some semiclassical measures μ charge this set. For instance, in some of the Husimi plots (e.g. the bottom left in figure 7), we clearly see a strong peak at the origin and lower peaks on other points of $D(B^{-1}) \cap \Gamma_+$. We call the components of an eigenstate localized on $D_{-\infty}$ 'diffractive'.

From these observations, we state the following conjecture.

Conjecture 2. *Let $B_{\boldsymbol{r}}$ be an open baker's map, quantized by section 3.2.2. Then*

- *all long-living semiclassical measures are supported on $\overline{\Gamma_+}$,*
- *'almost all' long-living eigenstates are nondiffractive, that is, their weights on $D(B^{-1})$ and $B^{-1}(D(B^{-1}))$ are negligible in the semiclassical limit. The corresponding semiclassical measures are then eigenmeasures of $B_{\boldsymbol{r}}$.*

In the next section we further comment on the above Husimi plots.

5.2. Abundance of semiclassical measures

Let us draw some consequences from theorem 1 and the following remarks. Statement (ii) of the theorem strongly constrains the converging sequences of eigenstates: a sequence (ψ_N) can converge to some measure μ (with $\mu(C(B^{-1})) > 0$) only if the corresponding eigenvalues (λ_N) asymptotically approach the circle of radius $\sqrt{\Lambda}$.

We take for granted the density argument in remark 2 and assume that a 'dense interval' $[\lambda_{\min}, \lambda_{\max}] \subset (0, 1]$ exists for any open baker's map $B_{\boldsymbol{r}}$. Hence, for any $\Lambda \in [\lambda_{\min}^2, \lambda_{\max}^2]$ there exist many sequences of eigenstates of $B_{\boldsymbol{r},N}$, such that $|\lambda_N|^2 \xrightarrow{N\to\infty} \Lambda$. From our conjecture 2, almost any semiclassical measure associated with such a sequence will be an eigenmeasure with the decay rate Λ. Therefore, two converging sequences (ψ_N), (ψ'_N) associated with limiting decays $\Lambda \neq \Lambda'$ will necessarily converge to different eigenmeasures. This already shows that the semiclassical eigenmeasures generated by all possible sequences in the annulus $\{\lambda_{\min} \leqslant |\lambda| \leqslant \lambda_{\max}\}$ form an uncountable family.

According to section 2.3.2, for each decay rate $\Lambda \in (0, 1)$, there exist uncountably many eigenmeasures. A natural question thus concerns the variety of semiclassical measures associated with a given $\Lambda \in [\lambda_{\min}^2, \lambda_{\max}^2]$.

Question 2.

For a given $\Lambda \in [\lambda_{\min}^2, \lambda_{\max}^2]$, what are the semiclassical measures of $B_{\boldsymbol{r}}$ of the decay rate Λ?

- *Is there a unique such measure?*
- *Otherwise, is some limit measure 'favoured', in the sense that 'almost all' sequences (ψ_N) with $|\lambda_N|^2 \to \Lambda$ converge to μ?*
- *Can the natural measure μ_{nat} be obtained as a semiclassical measure?*

The same type of questions were asked by Keating and coworkers in [20]. At present we are unable to answer them rigorously for the quantum open baker $B_{\boldsymbol{r},N}$.

From a heuristic point of view we notice the following features in the plots of figure 7. The three top plots correspond to eigenvalues $|\lambda_N|$ close to the value $\sqrt{\Lambda_{\mathrm{nat}}} \approx 0.8165$. However, the Husimi measures on the left and centre seem very different from the natural measure (the latter is approximated by the black rectangles in figure 4). These two Husimi functions seem to charge different parts of Γ_+. Only the rightmost state seems compatible with a convergence to μ_{nat}.

As commented in remark 4, the bottom-left state has strong concentrations on $D(B^{-1}) \cap \Gamma_+$. For the various values of N that we have investigated, this concentration seems characteristic of the eigenstate with the largest $|\lambda_N|$. The discontinuities of $B_{\boldsymbol{r}}$ manage to 'trap' these quantum states better than periodic orbits on $C(B^{-1}) \cap \Gamma_+$. Such states seem 'very diffractive'.

Finally, the bottom-centre state, with a smaller eigenvalue, clearly shows a self-similar structure along the horizontal direction, with the probability inside the hole being higher than for the top eigenstates. This feature had already been noticed in [20] for averages over Husimi functions with comparable $|\lambda_N|$.

In section 6 we will address question 2 for a different quantization of the symmetric open baker, namely, the Walsh-quantized open baker, where one can compute some semiclassical measures explicitly.

Before that, we explain how to construct approximate eigenstates (*pseudomodes*) for the quantum baker $B_{\boldsymbol{r}_{\mathrm{sym}},N}$.

5.3. Pseudomodes and pseudospectrum

From the explicit representation of eigenmeasures given in proposition 2, it is possible to construct approximate eigenstates of T_N by backward propagating wavepackets localized on the set $\Gamma_+^{(1)} = \Gamma_+ \cap H$.

We call these approximate eigenstates *pseudomodes*, by analogy with the recent literature on nonself-adjoint semiclassical operators (see, e.g. [3, 10]). These papers deal with pseudodifferential operators obtained by quantizing complex-valued observables, and they construct pseudo-modes of error $\mathcal{O}(\hbar^\infty)$, microlocalized at a single phase space point.

Our pseudomodes will be less precise and will be microlocalized on a countable set. We will restrict ourselves to the case of the open symmetric baker $B = B_{\boldsymbol{r}_{\rm sym}}$, but our construction works for any $B_{\boldsymbol{r}}$ and can probably be extended to other maps or phase spaces.

Inspired by the pointwise eigenmeasures (2.15), we construct approximate eigenstates by backward evolving coherent states localized in $\Gamma_+^{(1)}$. Precisely, for any $x_0 \in \Gamma_+^{(1)}$, $s > 0$ and $\lambda \in \mathbb{C}$, $|\lambda| < 1$, $N \in \mathbb{N}$, we define the following quantum state:

$$\Psi^s_{\lambda,x_0} \stackrel{\rm def}{=} \sqrt{1-|\lambda|^2}\, \sum_{n\geqslant 0} \lambda^n \, B_N^{\dagger n}\, \psi_{x_0,s}\,, \tag{5.4}$$

where $\psi_{x_0,s}$ is the coherent state defined in section 3.2.1 and B_N is the standard quantization of B.

Proposition 6. *Consider the symmetric open baker $B = B_{\boldsymbol{r}_{\rm sym}}$ and its quantization (B_N). Fix $\lambda \in \mathbb{C}$ with $|\lambda| < 1$.*

(*i*) *Choose $\varepsilon > 0$ small and call $\alpha = (1-\varepsilon)(\log 1/|\lambda|/\log 3)$. For any $N \in \mathbb{N}^*$, one can choose a point $x_0(N) \in \Gamma_+^{(1)}$ and a squeezing parameter $s = s(N) > 0$ such that the states $\left(\Psi_N \stackrel{\rm def}{=} \Psi^{s(N)}_{\lambda,x_0(N)}\right)_{N\to\infty}$ defined in (5.4) satisfy*

$$\|\Psi_N\| = 1 + \mathcal{O}(N^{-\alpha}), \qquad \|(B_N-\lambda)\Psi_N\| = \mathcal{O}(N^{-\alpha})\,, \qquad N\to\infty.$$

(*ii*) *For any $x_0 \in \Gamma_+^{(1)}$, one can select the points $(x_0(N))$ such that $x_0(N) \stackrel{N\to\infty}{\longrightarrow} x_0$. In this case, the sequence $(\Psi_N)_{N\to\infty}$ converges to the eigenmeasure $\mu_{x_0,\Lambda}$ described in (2.15), with $\Lambda = |\lambda|^2$.*

This proposition shows that, for any $\alpha > 0$ and N large enough, the $N^{-\alpha}$-pseudospectrum of B_N contains the disc $\{|\lambda| \leqslant 3^{-\alpha(1+\varepsilon)}\}$. We note that the errors are not very small and increase when $|\lambda| \to 1$. We should emphasize that, in our numerical trials, we never found eigenstates of B_N with Husimi measures looking like $\mu_{x_0,\Lambda}$.

Proof of the proposition. To control series (5.4), we would like to ensure that the evolved coherent state $B_N^{\dagger j}\,\psi_{x_0(N),s(N)}$ remains close to an approximate coherent state $\psi_{x_{-j},s_{-j}}$ (as in (3.13)) up to large times j. For this we need the points $x_{-j} = B^{-j}(x_0(N))$ to stay 'far' from the discontinuity set $D(B^{-1})$, and we also need all $\psi_{x_{-j},s_{-j}}$ to be microlocalized in a small neighbourhood of x_{-j}. Due to the hyperbolicity of B^{-1}, the second condition constrains the times j to be smaller than the *Ehrenfest time* [8]

$$n_E = (1-\varepsilon)\frac{\log N}{\log 3}\,. \tag{5.5}$$

Here $\varepsilon > 0$ is the small parameter in the statement of the proposition.

To identify a good 'starting point' $x_0(N)$, we set $n = [n_E]$ and consider the following subset of $\mathbb{T}^2$, for $\delta, \gamma > 0$:

$$\mathcal{D}_{n,\delta,\gamma} \stackrel{\rm def}{=} \{(q,p)\in\mathbb{T}^2,\ q\in(1/3+\delta, 2/3-\delta),\ \forall k\in\mathbb{Z},\ |p-\frac{k}{3^n}| > \gamma\} \cap \Gamma_+^{(1)}\,.$$

We first show that this set is nonempty if $\delta < 1/9$ and $\gamma = 3^{-n}\gamma'$, $\gamma' < 1/9$. Take $q \in (1/3+\delta, 2/3-\delta)$ arbitrary and select $p = p(\epsilon)$ with the following properties: take all indices $\epsilon_{-j} \in \{0, 2\}$, $j \geqslant 1$, so that $p \in \mathcal{C}_{\boldsymbol{r}_{\rm sym}}$, and require furthermore that the word $\epsilon_{-n-1}\epsilon_{-n-2} \in \{02, 20\}$. The point (q, p) is then in $\mathcal{D}_{n,\delta,\gamma}$.

Let us take any point $x_0(N)$ in that set. It automatically lies in $C(B^{-n})$, so its backward iterates stay away from $D(B^{-1})$ at least until the time n. Let us select the squeezing parameter $s(N) = s_0 = N^{-1+\varepsilon}$. A simple adaptation of [8, proposition 5] implies the following estimate:

$$\exists c, C > 0, \qquad \forall j, \quad 0 \leqslant j \leqslant n_E, \qquad \| B_N^{\dagger j}\, \psi_{x_0,s_0} - \mathrm{e}^{\mathrm{i}\theta_j}\, \psi_{x_{-j},s_{-j}} \| \leqslant C\, \mathrm{e}^{-c N^{\varepsilon}}. \tag{5.6}$$

Here we can take $c = \min(\delta^2, \gamma'^2)$ and C is uniform w.r.t. the initial point x_0. From the values of x_0, s_0, one checks that the components $\langle \boldsymbol{q}_k, \psi_{x_0,s_0}\rangle$ for $k/N \notin [1/3, 2/3]$ are of order $\mathcal{O}(\mathrm{e}^{-cN^{\varepsilon}})$: this state is (very) localized in the hole $H = R_1$, so

$$B_N\, \psi_{x_0,s_0} = \mathcal{O}(\mathrm{e}^{-c N^{\varepsilon}}).$$

Similarly, for any $j \leqslant n_E$, the state $\psi_{x_{-j},s_{-j}}$ is localized inside a certain connected component of M_{j+1} (one of the pink/grey rectangles in figure 2, left). In particular, the components of $\langle \boldsymbol{q}_k, \psi_{x_{-j+1},s_{-j+1}}\rangle$ are exponentially small in H. Therefore,

$$\forall j, \quad 0 \leqslant j \leqslant n_E, \qquad B_N\, B_N^{\dagger} \psi_{x_{-j},s_{-j}} = \psi_{x_{-j},s_{-j}} + \mathcal{O}(\mathrm{e}^{-c N^{\varepsilon}}).$$

Summing all terms $j \leqslant n_E$ in (5.4) and estimating the remaining series using $\|B_N\| \leqslant 1$, we obtain

$$\|(B_N - \lambda)\, \Psi^{s}_{\lambda,x_0}\| = \mathcal{O}(|\lambda|^{n_E}).$$

From the definition of n_E, we have $|\lambda|^{n_E} = N^{-\alpha}$.

The asymptotic normalization of Ψ^{s}_{λ,x_0} is proven by estimating the overlaps between coherent states $\psi_{x_{-j},s_{-j}}$, $\psi_{x_{-j'},s_{-j'}}$ for $j, j' \leqslant n_E$. Because the sets M_{j+1} and $M_{j'+1}$ are disjoint for $j \neq j'$, the above mentioned localization properties imply that

$$\forall j, \quad j' \leqslant n_E, \qquad \langle \psi_{x_{-j},s_{-j}}, \psi_{x_{-j'},s_{-j'}}\rangle = \delta_{j',j} + \mathcal{O}(\mathrm{e}^{-c N^{\varepsilon}}),$$

and the normalization estimate follows. This achieves the proof of (i).

To prove (ii), let us consider an arbitrary $x_0 = (q_0, p_0) \in \Gamma_+^{(1)}$. If $q_0 \notin \{1/3, 2/3\}$, we take δ small enough such that $q_0 \in (1/3 + \delta, 2/3 - \delta)$ and set $q_0(N) = q_0$. Otherwise, we may let δ slowly decrease with N and find a sequence $(q_0(N))$ such that $q_0(N) \in (1/3 + \delta(N), 2/3 - \delta(N))$ and $q_0(N) \xrightarrow{N\to\infty} q_0$. On the other hand, $p_0 \in \mathcal{C}_{\boldsymbol{r}_{\mathrm{sym}}} = p(\boldsymbol{\epsilon})$ for a certain sequence $\boldsymbol{\epsilon}$, with all $\epsilon_{-j} \in \{0, 2\}$. For each N, we inspect the word $\epsilon_{-n-1}\epsilon_{-n-2}$ of that sequence, where $n = [n_E]$ (see (5.5)). If this word is in the set $\{02, 20\}$ we keep $p_0(N) = p_0$, otherwise we replace this word by 02 (keeping the other symbols unchanged) to define $p_0(N)$. The point $x_0(N) = (q_0(N), p_0(N))$ is then in $\mathcal{D}_{n,\delta,\gamma}$, and $x_0(N) \xrightarrow{N\to\infty} x_0$.

The convergence to the measure $\mu_{x_0,\Lambda}$ is due to the localization of the coherent states $\psi_{x_{-j},s_{-j}}$ for $j \leqslant n_E$. □

6. A solvable toy model: Walsh-quantization of the open baker

In this section we study an alternative quantization of the open symmetric baker $B_{\boldsymbol{r}_{\mathrm{sym}}}$, introduced in [30, 31]. A similar quantization of the closed baker $\hat{B}_{\boldsymbol{r}_{\mathrm{sym}}}$ was proposed and studied in [11, 36, 45], mainly motivated by research in quantum computation. From now on, we will drop the index $\boldsymbol{r}_{\mathrm{sym}}$ from our notations and call $B = B_{\boldsymbol{r}_{\mathrm{sym}}}$.

A 'simplified' quantization of B was introduced in [30, 31], as a $N \times N$ matrix $\tilde{B}_N$, obtained by only keeping the 'skeleton' of B_N (see (3.12)). Although $\tilde{B}_N$ can be defined for any N, its spectrum can be explicitly computed only if $N = 3^k$ for some $k \in \mathbb{N}$. As explained in those references, $\tilde{B}_N$ can be interpreted as the 'Weyl' quantization of a multivalued map $\tilde{B}$ built upon B. In the present work we will stick to a different interpretation of $\tilde{B}_N$, valid in the case $N = 3^k$: one can then introduce a (Walsh) quantization for observables on $\mathbb{T}^2$, which is not equivalent to the anti-Wick quantization of section 3.2.1. We will check below that the matrices $\tilde{B}_N$ satisfy property (3.4) with respect to that quantization and the open baker B. Finally, this Walsh quantization is also suited to quantizing observables on the symbol space Σ: the matrices $\tilde{B}_N$ then quantize the open shift σ. We will see that this interpretation is more 'convenient', because it avoids problems due to discontinuities.

We first recall the definition of the Walsh-quantization of observables on $\mathbb{T}^2$ (or Σ) and the associated Walsh–Husimi measures.

6.1. Walsh transform and coherent states

6.1.1. Walsh coherent states. The Walsh-quantization of observables on $\mathbb{T}^2$ uses the decomposition of the quantum Hilbert space $\mathcal{H}_N$ into a tensor product of 'qubits', namely, $\mathcal{H}_N = (\mathbb{C}^3)^{\otimes k}$ (clearly, this makes sense only for $N = 3^k$). Each discrete position $q_j = j/N$ can be represented by its ternary sequence $q_j = 0 \cdot \epsilon_0\epsilon_2 \cdots \epsilon_{k-1}$, with symbols $\epsilon_i \in \{0, 1, 2\}$. Accordingly, each position eigenstate $\boldsymbol{q}_j$ can be represented as a tensor product:

$$\boldsymbol{q}_j = e_{\epsilon_0} \otimes e_{\epsilon_2} \otimes \cdots \otimes e_{\epsilon_{k-1}} = |[\epsilon_0\epsilon_2 \ldots \epsilon_{k-1}]_k\rangle \,,$$

where $\{e_0, e_1, e_2\}$ is the canonical (orthonormal) basis of $\mathbb{C}^3$. The notation on the right-hand side emphasizes the fact that this state is associated with the cylinder $[\boldsymbol{\epsilon}] = [\boldsymbol{\epsilon}]_k$ with k symbols on the right of the comma and no symbols on the left. Its image on $\mathbb{T}^2$ is a rectangle $[\boldsymbol{\epsilon}]_k$ of height unity and width $3^{-k} = 1/N$.

The Walsh-quantization consists of replacing the discrete Fourier transform (3.11) on $\mathcal{H}_N$ by the Walsh(–Fourier) transform W_N, which is a unitary operator preserving the tensor product structure of $\mathcal{H}_N$. We define it through its inverse W_N^*, which maps the position basis to the orthonormal basis of 'Walsh momentum states': for any $j \equiv \epsilon_0 \ldots \epsilon_{k-1}$,

$$\boldsymbol{p}_j = W_N^* \,\boldsymbol{q}_j \stackrel{\text{def}}{=} F_3^* e_{\epsilon_{k-1}} \otimes F_3^* e_{\epsilon_{k-2}} \otimes \cdots \otimes F_3^* e_{\epsilon_1} \otimes F_3^* e_{\epsilon_0}$$

(here F_3^* is the inverse Fourier transform on $\mathbb{C}^3$). To agree with our notations of section 2.2, we will index the symbols relative to the momentum coordinate by *negative* integers, so the Walsh momentum states will be denoted by

$$\boldsymbol{p}_j = |[\epsilon_{-k} \ldots \epsilon_{-1}]_0\rangle \,, \qquad \text{where} \quad p_j = j/N = 0 \cdot \epsilon_{-1} \ldots \epsilon_{-k} \,.$$

This momentum state is associated with a rectangle of height 3^{-k} and width unity.

More generally, for any $\ell \in \{0, \ldots, k\}$ and any sequence $\boldsymbol{\epsilon} = \epsilon_{-k+\ell} \ldots \epsilon_{-1} \cdot \epsilon_0 \ldots \epsilon_{\ell-1}$, one can construct a (Walsh-)coherent state

$$|[\boldsymbol{\epsilon}]_\ell\rangle \stackrel{\text{def}}{=} e_{\epsilon_0} \otimes \cdots e_{\epsilon_{\ell-1}} \otimes F_3^* e_{\epsilon_{-k+\ell}} \otimes \cdots F_3^* e_{\epsilon_{-1}} \,. \tag{6.1}$$

This state is localized in the rectangle $[\boldsymbol{\epsilon}]_\ell$ with height $3^{-k+\ell}$ and width $3^{-\ell}$, so it still has an area $1/N$ (all such rectangles are minimal-uncertainty or 'quantum' rectangles).

Like the squeezing parameter s of Gaussian wavepackets (see section 3.2.1), the index ℓ describes the aspect ratio of the coherent state. When going to the semiclassical limit $k \to \infty$, we will always select $\ell(k) \sim k/2$, which corresponds to an 'isotropic' squeezing. One important difference between the Gaussian and the Walsh coherent states lies in the fact that the latter are *strictly* localized both in momentum and position. Another difference is that, for each ℓ, the ℓ-coherent states make up a finite orthonormal basis of $\mathcal{H}_N$, instead of a continuous overcomplete family.

6.1.2. Walsh-quantization for observables on Σ. As explained in [1], it is more natural to Walsh-quantize observables on the symbol space Σ than on $\mathbb{T}^2$. Indeed, if one equips Σ with the metric structure

$$d_\Sigma(\epsilon, \epsilon') = \max(3^{-n_+}, 3^{-n_-}), \qquad n_+ = \min\left\{n \geqslant 0,\ \epsilon_n \neq \epsilon'_n\right\},$$
$$n_- = \min\left\{n \geqslant 0,\ \epsilon_{-n-1} \neq \epsilon'_{-n-1}\right\},$$

the closed shift $\hat{\sigma}$ then acts as a *Lipschitz* map on Σ, and the hole $\{\epsilon_0 = 1\}$ is at a finite distance from its complement in Σ. Indeed, the 'lift' from $\mathbb{T}^2$ to Σ has the effect to 'blow up' the lines of discontinuity of B.

The conjugacy $J : \Sigma \to \mathbb{T}^2$ is also Lipschitz, so any Lipschitz function $F \in \mathrm{Lip}(\mathbb{T}^2)$ is pushed to a function $f = F \circ J \in \mathrm{Lip}(\Sigma)$. However, the converse is not true: if we use the inverse map $(J_{|\Sigma'})^{-1} : \mathbb{T}^2 \to \Sigma'$ and take an arbitrary $f \in \mathrm{Lip}(\Sigma)$, the function $F = f \circ (J_{|\Sigma'})^{-1}$ is generally discontinuous on $\mathbb{T}^2$.

Let us select some $\ell \sim k/2$. The Walsh-quantization of a function $f \in \mathrm{Lip}(\Sigma)$ is defined as the following operator on $\mathcal{H}_N$:

$$\mathrm{Op}_N^\ell(f) \stackrel{\mathrm{def}}{=} 3^k \sum_{[\epsilon]_\ell} |[\epsilon]_\ell\rangle\langle[\epsilon]_\ell| \int_{[\epsilon]_\ell} f\, \mathrm{d}\mu_L. \tag{6.2}$$

Here the sum goes over all 'quantum' cylinders $[\epsilon]_\ell$, that is, over all 3^k sequences $\epsilon = \epsilon_{-k+\ell} \ldots \epsilon_{-1} \cdot \epsilon_0 \ldots \epsilon_{\ell-1}$. The integral over f is performed using the uniform Bernoulli measure μ_L, which is equivalent to the Liouville measure on $\mathbb{T}^2$ (see the end of section 2.2). Note the formal similarity of this quantization with the anti-Wick quantization (3.7).

It is shown in [1, proposition 3.1] that this quantization satisfies axioms 1 (with Lipschitz observables) and that two quantizations $\mathrm{Op}_N^{\ell_1}$, $\mathrm{Op}_N^{\ell_2}$ are semiclassically equivalent if both $\ell_1,\ \ell_2 \sim k/2$.

By duality, we define the Walsh–Husimi measure of a quantum state ψ_N. The corresponding density is constant in each ℓ-cylinder $[\epsilon]_\ell$:

$$\forall \alpha \in [\epsilon]_\ell, \qquad WH^\ell_{\psi_N}(\alpha) = 3^k\, |\langle[\epsilon]_\ell, \psi_N\rangle|^2 . \tag{6.3}$$

This density is originally defined on Σ but can be pushed forward to $\mathbb{T}^2$. Semiclassical measures are defined as the weak-$*$ limits of sequences $(WH^\ell_{\psi_N})$, where $N = 3^k \to \infty$ and $\ell = \ell(k) \sim k/2$. One first obtains a measure μ^Σ on Σ, which can be pushed on $\mathbb{T}^2$ into $\mu = J^*\mu^\Sigma$ (equivalently, μ is the weak-$*$ limit of the Walsh–Husimi measures on $\mathbb{T}^2$).

6.2. Walsh-quantization of the open baker

We now recall the Walsh-quantization of the open baker $B = B_{r_{\mathrm{sym}}}$, as defined in [30, 31]. Mimicking the standard quantization (3.12), we replace the Fourier transforms F_N^*, $F_{N/3}$ by their Walsh analogues W_N, $W_{N/3}$ (with $N = 3^k$, $k \geqslant 0$), so that the Walsh-quantized open baker is given by the following matrix on the position basis:

$$\tilde{B}_N \stackrel{\mathrm{def}}{=} W_N^* \begin{pmatrix} W_{N/3} & & \\ & 0 & \\ & & W_{N/3} \end{pmatrix}. \tag{6.4}$$

For any set of vectors $v_0, \ldots, v_{k-1} \in \mathbb{C}^3$, this operator acts as follows on any tensor product state $v_0 \otimes \cdots \otimes v_{k-1}$:

$$\tilde{B}_N(v_0 \otimes v_1 \ldots \otimes v_{k-1}) = v_1 \otimes \ldots \otimes v_{k-1} \otimes \tilde{F}_3^* v_0 . \tag{6.5}$$

Here $\tilde{F}_3^* = F_3^*\, \pi_{02}$, where π_{02} is the orthogonal projector on $\mathbb{C}e_0 \oplus \mathbb{C}e_2$ in $\mathbb{C}^3$.

One can generalize the quantum-correspondence of [1, proposition 3.2] to the open shift σ and prove the Walsh version of axioms 2.

Proposition 7.

(*i*) *Take any* $f \in \mathrm{Lip}(\Sigma)$. *Then, in the limit* $N = 3^k \to \infty$, $\ell \sim k/2$,

$$\tilde{B}_N^\dagger \, \mathrm{Op}_N^\ell(f) \, \tilde{B}_N \sim \mathrm{Op}_N^\ell \mathbb{1}_{\{\epsilon_0 \neq 1\}} \times f \circ \sigma) .$$

Note that the function $\mathbb{1}_{\{\epsilon_0 \neq 1\}} \times f \circ \sigma \in \mathrm{Lip}(\Sigma)$.

(*ii*) *If we take* $F \in \mathrm{Lip}_c(C(B^{-1}) \sqcup \mathring{H}^{-1})$ *and* $f = F \circ J$, *we have*

$$\mathbb{1}_{\{\epsilon_0 \neq 1\}} \times f \circ \sigma = \mathbb{1}_M \times F \circ B) \circ J .$$

The points (i) and (ii) show that the family $(\tilde{B}_N)$ satisfies axioms 2 with respect to the map B on $\mathbb{T}^2$ and quantization (6.2) of observables in $\mathrm{Lip}(\mathbb{T}^2)$. Point (i) alone shows that $(\tilde{B}_N)$ is a quantization of the open shift σ on Σ. As noted above, the latter interpretation allows us to get rid of problems of discontinuities.

Proof. Applying (6.5) to a coherent state $|[\epsilon]_\ell\rangle$, we get the exact evolution

$$\tilde{B}_N^\dagger \, |[\epsilon]_\ell\rangle = (1 - \delta_{\epsilon_{-1},1}) \, |[\sigma^{-1}(\epsilon)]_{\ell+1}\rangle .$$

That is, the coherent state $|[\epsilon]_\ell\rangle$ is either killed if $\sigma^{-1}([\epsilon]_\ell) = \infty$ or transformed into a coherent state associated with the cylinder $[\sigma^{-1}(\epsilon)]_{\ell+1} = \sigma^{-1}([\epsilon]_\ell)$. This exact expression, which is the quantum counterpart of the classical shift (2.9), should be compared with the approximate expression (3.13). From there, a straighforward computation shows that, for any $f \in Lip(\Sigma)$:

$$\tilde{B}_N^\dagger \, \mathrm{Op}_N^\ell(f) \, \tilde{B}_N = \mathrm{Op}_N^{\ell+1}((1 - \delta_{\epsilon_0,1}) \times f \circ \sigma) .$$

The semiclassical equivalence $\mathrm{Op}_N^{\ell+1} \sim \mathrm{Op}_N^\ell$ finishes the proof of (i).

To prove (ii), we note that, if $F \in \mathrm{Lip}_c(C(B^{-1}) \sqcup \mathring{H}^{-1})$ and $f = F \circ J$, the function $F \circ B$ is supported away from $D(B)$, and both functions $\mathbb{1}_{\{\epsilon_0 \neq 1\}} \times f \circ \sigma$ and $\mathbb{1}_{\{\epsilon \neq 1\}} \times F \circ B \circ J$ are supported inside Σ''. Semiconjugacy (2.11) shows that these two functions are equal. Finally, we note that, for $\epsilon \in \Sigma''$, one has $\mathbb{1}_{\{\epsilon_0 \neq 1\}}(\epsilon) = \mathbb{1}_M \circ J(\epsilon)$. □

6.2.1. Spectrum of the Walsh open baker. The simple expression (6.5) allows us to explicitly compute the spectrum of $\tilde{B}_N$ (see [31, proposition 5.5]). This spectrum is determined by the two nontrivial eigenvalues λ_- and λ_+ of the matrix $\tilde{F}_3^*$. These eigenvalues have moduli $|\lambda_+| \approx 0.8443$, $|\lambda_-| \approx 0.6838$. The spectrum of $\tilde{B}_N$ has a gap: the long-living eigenvalues are contained in the annulus $\{|\lambda_-| \leqslant |\lambda| \leqslant |\lambda_+|\}$, while the rest of the spectrum lies at the origin. Most of the eigenvalues are degenerate. If we count multiplicities, the long-living ($\equiv$ nontrivial) spectrum satisfies the following asymptotics when $k \to \infty$:

$$\forall r > 0, \qquad \#\{\lambda_j \in \mathrm{Spec}(\tilde{B}_N) \,,\ |\lambda_j| \geqslant r\} = C_r \, 2^k + o(2^k) ,$$
$$C_r = \begin{cases} 1 , & r < r_0, \\ 0 , & r > r_0, \end{cases} \quad , \quad r_0 \overset{\text{def}}{=} |\lambda_- \lambda_+|^{1/2} = 3^{-1/4} . \tag{6.6}$$

The nontrivial spectrum is spanned by a subspace $\mathcal{H}_{N,\mathrm{long}}$ of dimension 2^k. Since the trapped set for $B = B_{r_{\mathrm{sym}}}$ has dimension $2d = 2\,(\log 2/\log 3)$, the above asymptotics agrees with the fractal Weyl law (4.1). Although the density of resonances (counted with multiplicities) is peaked near the circle $\{|\lambda| = r_0\}$, the spectrum (as a set) densely fills the annulus $\{|\lambda_-| \leqslant |\lambda| \leqslant |\lambda_+|\}$ when $k \to \infty$ (see figure 8). In the next section we construct some long-living eigenstates of $\tilde{B}_N$ and analyse their Walsh–Husimi measures. Due to the spectral degeneracies, there

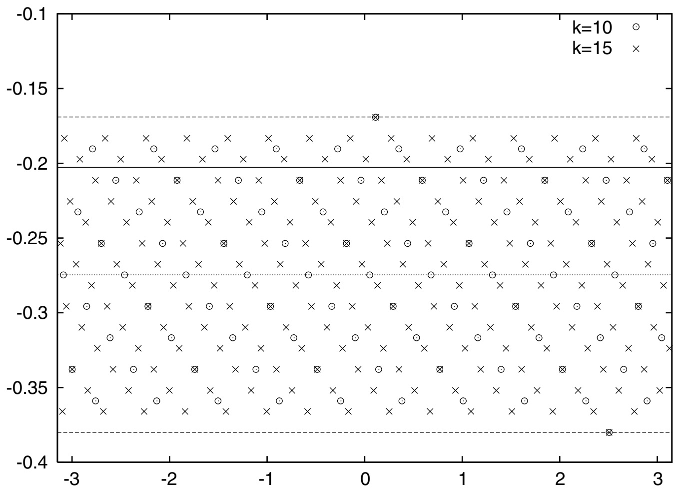

Figure 8. Nontrivial spectrum of the Walsh open baker $\tilde{B}_N$ for $N = 3^{10}$ (circles) and 3^{15} (crosses), using a logarithmic representation (horizontal $= \arg\lambda_j$, vertical $= \log|\lambda_j|$). We plot horizontal lines at the extremal radii $|\lambda_\pm|$ of the spectrum (dashed), at the radius $r_0 = |\lambda_-\lambda_+|^{1/2}$ of highest degeneracies (dotted) and at the radius corresponding to the natural measure (full).

is generally large freedom for selecting eigenstates (ψ_N) associated with a sequence of eigenvalues (λ_N). Intuitively, this freedom should provide more possibilities for semiclassical measures.

We mention that Keating and coworkers have recently studied the eigenstates of a slightly different version of the Walsh–baker, namely, a matrix $\tilde{B}'_N$ obtained by replacing F_3 by the 'half-integer Fourier transform' $(G_3)_{jj'} = 3^{-1/2}\mathrm{e}^{-2\mathrm{i}\pi(j+1/2)(j'+1/2)/3}$ (see [19]).

6.3. Long-living eigenstates of the Walsh open baker

We first provide the analogue of theorem 1 for the Walsh–baker. We recall that a semiclassical measure μ^Σ (or its push-forward μ) is now a weak-$*$ limit of some sequence of Walsh–Husimi measures $(WH^\ell_{\psi_N})_{N=3^k\to\infty}$, where ψ_N are eigenstates of $\tilde{B}_N$ and the index $\ell \approx k/2$.

From the quantum-classical correspondence of proposition 7 and using proposition 4, we deduce the following corollary.

Corollary 1. *Let μ^Σ be a semiclassical measure for a sequence of long-living eigenstates (ψ_N, λ_N) of the Walsh–baker $\tilde{B}_N$. Then*

- (i) *μ^Σ is an eigenmeasure for the open shift σ on Σ and the corresponding decay rate Λ satisfies $\Lambda = \lim_{N\to\infty}|\lambda_N|^2$,*
- (ii) *if $\mu^\Sigma(\Sigma\setminus\Sigma'') = 0$ (where Σ'' is defined in (2.10)), then $\mu = J^*\mu^\Sigma$ is an eigenmeasure of the open baker B, with the decay rate Λ.*

From the structure of $\mathrm{Spec}(\tilde{B}_N)$ explained above, there exist sequences of eigenvalues $(\lambda_N)_{N\to\infty}$ converging to any circle of radius $\lambda \in [|\lambda_-|, |\lambda_+|]$. We also know that any semiclassical measure is an eigenmeasure of σ, so it is meaningful to ask question 2 in the present framework (setting $\lambda_{\max/\min} = |\lambda_\pm|$). We add the following question: are there semiclassical measures μ^Σ such that $\mu^\Sigma(\Sigma'') < 1$? In the next section we give partial answers to these questions.

Concerning the last point in question 2, we note that the 'physical' decay rate for $B = B_{\boldsymbol{r}_{\rm sym}}$ is $\Lambda_{\rm nat} = 2/3$. The circle $\left\{|\lambda| = \sqrt{\Lambda_{\rm nat}}\right\}$ is contained inside the annulus $\{|\lambda_-| \leqslant |\lambda| \leqslant |\lambda_+|\}$ where the nontrivial spectrum of $\tilde{B}_N$ is semiclassically dense, although it differs from the circle $\{|\lambda| = r_0\}$ where the spectral density is peaked, see figure 8. Still, there exist semiclassical measures of $\tilde{B}_N$ with eigenvalue $\Lambda_{\rm nat}$, and it is relevant to ask whether $\mu_{\rm nat}$ can be one of these. At present we are not able to answer that question. The next section shows that there are plenty of semiclassical measures with eigenvalue $\Lambda_{\rm nat}$, so even if $\mu_{\rm nat}$ is a semiclassical measure with $\Lambda = \Lambda_{\rm nat}$, it is certainly not the only one.

6.4. Constructing the eigenstates of $\tilde{B}_N$

In this section we construct one particular (right) eigenbasis of $\tilde{B}_N$ restricted to the subspace $\mathcal{H}_{N,\rm long}$ of long-living eigenstates. The construction starts from the (right) eigenvectors $v_\pm \in \mathbb{C}^3$ of $\tilde{F}_3^*$ associated with $\lambda_\pm$. Note that these two vectors (which we take normalized) are not orthogonal to each other. For any sequence $\boldsymbol{\eta} = \eta_0 \ldots \eta_{k-1}$, $\eta_i \in \{\pm\}$, we form the tensor product state

$$|\boldsymbol{\eta}\rangle \stackrel{\rm def}{=} v_{\eta_0} \otimes v_{\eta_1} \ldots \otimes v_{\eta_{k-1}} .$$

Action (6.5) of $\tilde{B}_N$ implies that

$$\tilde{B}_N |\boldsymbol{\eta}\rangle = \lambda_{\eta_0} \, |\tau(\boldsymbol{\eta})\rangle ,$$

where τ acts as a cyclic shift on the sequence: $\tau(\eta_0 \ldots \eta_{k-1}) = \eta_1 \ldots \eta_{k-1}\eta_0$. The orbit $\{\tau^j(\boldsymbol{\eta}),\ j \in \mathbb{Z}\}$ contains ℓ_η elements, where the *period* ℓ_η of the sequence $\boldsymbol{\eta}$ necessarily divides k. The states $\{|\tau^j(\boldsymbol{\eta})\rangle,\ j = 0, \ldots, \ell_\eta - 1\}$ are not orthogonal to each other but form a linearly independent family, which generates the $\tilde{B}_N$-invariant subspace $\mathcal{H}_{\boldsymbol{\eta}} \subset \mathcal{H}_{N,\rm long}$. The eigenvalues of $\tilde{B}_N$ restricted to $\mathcal{H}_{\boldsymbol{\eta}}$ are of the form $\lambda_{\boldsymbol{\eta},r} = {\rm e}^{2{\rm i}\pi r/\ell_\eta} \left(\prod_{j=0}^{\ell_\eta-1} \lambda_{\eta_j}\right)^{1/\ell_\eta}$ (with indices $r = 1, \ldots, \ell_\eta$), and the corresponding eigenstates read as

$$|\psi_{\boldsymbol{\eta},r}\rangle = \frac{1}{\sqrt{\mathcal{N}_{\boldsymbol{\eta},r}}} \sum_{j=0}^{\ell_\eta-1} c_{\boldsymbol{\eta},r,j} \, |\tau^j(\boldsymbol{\eta})\rangle , \qquad c_{\boldsymbol{\eta},r,j} = \prod_{m=0}^{j-1} \frac{\lambda_{\eta_m}}{\lambda_{\boldsymbol{\eta},r}} , \tag{6.7}$$

where $\mathcal{N}_{\boldsymbol{\eta},r} > 0$ is the factor which normalizes $|\psi_{\boldsymbol{\eta},r}\rangle$. Up to a phase, this state is unchanged if $\boldsymbol{\eta}$ is replaced by $\tau(\boldsymbol{\eta})$. In the following subsections we explicitly compute the Walsh–Husimi measures of some of these eigenstates.

6.4.1. Extremal eigenstates. The simplest case is provided by the sequence $\boldsymbol{\eta} = + + \cdots +$, which has period 1, so that $|\psi_{+,N}\rangle = |\boldsymbol{\eta}\rangle = v_+^{\otimes k}$ is the (unique) eigenstate associated with the largest eigenvalue λ_+ (this is the longest-living eigenstate). For any choice of index $0 \leqslant \ell \leqslant k$, the Walsh–Husimi measure of $|\psi_{+,N}\rangle$ factorizes:

$$\text{for all } \ell\text{-rectangle } [\boldsymbol{\epsilon}]_\ell, \qquad WH^\ell_{\psi_{+,N}}([\boldsymbol{\epsilon}]_\ell) = \prod_{j=0}^{\ell-1} |\langle v_+,\, e_{\epsilon_j}\rangle|^2 \prod_{j=-1}^{-k+\ell} |\langle v_+,\, F_3^*\, e_{\epsilon_j}\rangle|^2 .$$

The second product involves the vector $w_+ \stackrel{\rm def}{=} F_3 v_+$, with components $w_{+,\epsilon} = (1-\delta_{\epsilon,1}) v_{+,\epsilon}/\lambda_+$. Following the notations of section 2.3.5, let $\mu^\Sigma_{\mathbf{P}_+}$ be the Bernoulli eigenmeasure of σ with weights $P_{+,\epsilon} = |v_{+,\epsilon}|^2$, $P^*_{+,\epsilon} = |w_{+,\epsilon}|^2$. The above expression shows that the Husimi measure $WH^\ell_{\psi_{+,N}}$ is equal to the measure $\mu^\Sigma_{\mathbf{P}_+}$, *conditioned on the grid formed by the ℓ-cylinders*. Since the diameters of the cylinders decrease to zero as $k \to \infty$, $\ell(k) \sim k/2$, the Husimi measures $(H^\ell_{\psi_{+,N}})$ converge to $\mu^\Sigma_{\mathbf{P}_+}$.

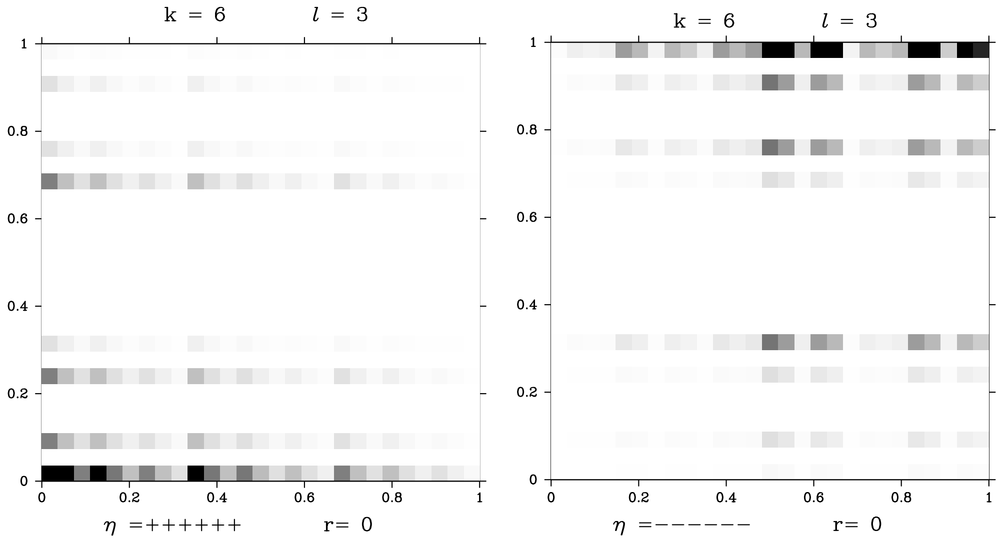

Figure 9. Walsh–Husimi densities for the extremal eigenstates $\psi_{+,N}$, $\psi_{-,N}$ of $\tilde{B}_N$, with $N = 3^6$, $\ell = 3$. These are coarse-grained versions of the Bernoulli measures $\mu_{\boldsymbol{P}_+}$ and $\mu_{\boldsymbol{P}_-}$.

One can similarly show that the Husimi functions of the eigenstates $\psi_- = v_-^{\otimes k}$, associated with the smallest nontrivial eigenvalue λ_-, converge to the Bernoulli eigenmeasure $\mu^{\Sigma}_{\boldsymbol{P}_-}$, with weights $P_{-,\epsilon} = |v_{-,\epsilon}|^2$, $P^*_{-,\epsilon} = |w_{-,\epsilon}|^2$, where $w_- = F_3 v_-$.

In figure 9 we plot the Walsh–Husimi densities (pushed forward on $\mathbb{T}^2$) for $\psi_{+,N}$ and $\psi_{-,N}$, using the 'isotropic" $\ell = k/2$. These give a clear idea of the self-similar structure of the respective semiclassical measures $\mu_{\boldsymbol{P}_+}$ and $\mu_{\boldsymbol{P}_-}$. The weights have the approximate values $\boldsymbol{P}_+ \approx (0.579, 0.287, 0.134)$, $\boldsymbol{P}_- \approx (0.088, 0.532, 0.380)$.

Considering the fact that the eigenvalues λ_N close to the circles of radii $|\lambda_+|$ and $|\lambda_-|$ have small degeneracies, we propose the following conjecture.

Conjecture 3. *Any sequence of eigenstates $(\psi_N)_{N\to\infty}$ with eigenvalues $|\lambda_N| \to |\lambda_+|$ (respectively, $|\lambda_N| \to |\lambda_-|$) converges to the semiclassical measure $\mu^{\Sigma}_{\boldsymbol{P}_+}$ (respectively, $\mu^{\Sigma}_{\boldsymbol{P}_-}$).*

This conjecture can be proven for the version of the Walsh baker $\tilde{B}'_N$ studied in [19]: in this case the two eigenvectors of $\tilde{G}^*_3$ replacing $v_\pm$ are orthogonal to each other, which greatly simplifies the analysis. The limit measure $\mu^{\Sigma}_{\boldsymbol{P}'_+}$ is then the 'uniform' measure on the trapped set $\{\epsilon_n \neq 1,\ n \in \mathbb{Z}\}$, with $\boldsymbol{P}'_+ = \boldsymbol{P}'^{*}_+ = (1/2, 0, 1/2)$.

6.4.2. Semiclassical measures in the 'bulk'. In this section we investigate some eigenstates of the form (6.7), with eigenvalues λ_N situated in the 'bulk' of the nontrivial spectrum, that is, $|\lambda_N| \in (|\lambda_-|, |\lambda_+|)$.

In the next proposition we show that there can be two different semiclassical measures with the same decay rate Λ. This answers in the *negative* the first point in question 2.

Proposition 8. *Choose a rational number $t = m/n$, with $n \geqslant 1$, $1 \leqslant m \leqslant n-1$.*

(*i*) *Select a sequence $\boldsymbol{\eta}_n$ with m $(+)$ and $n-m$ $(-)$. For all $k' \geqslant 1$, form the repeated sequence $(\boldsymbol{\eta}_n)^{k'}$ and choose $r = r(k') \in \mathbb{Z}$ arbitrarily. Then the sequence of eigenstates $(\psi_{(\boldsymbol{\eta}_n)^{k'},r(k')})_{k=nk'\to\infty}$ converges to the semiclassical measure $\mu^{\Sigma}_{\boldsymbol{\eta}_n}$, which is a linear*

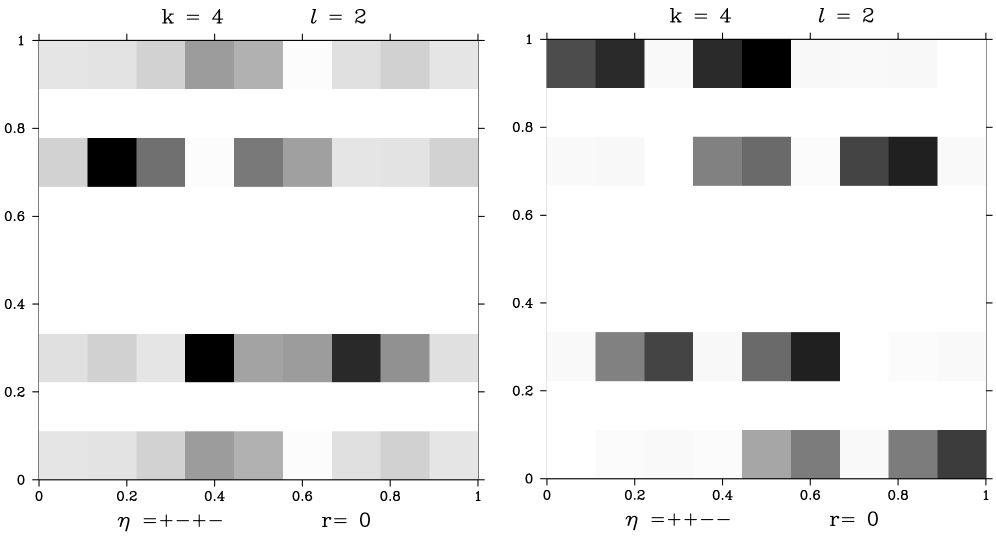

Figure 10. Walsh–Husimi densities of two eigenstates $\psi_{\eta,0}$ constructed from the sequences $\eta_4 = +-+-$ (left) and $\eta_4' = ++--$ (right).

combination of Bernoulli measures for the iterated shift σ^n (see (6.12)). This measure is independent of the choice of $(r(k'))$ and has a decay rate $\Lambda_t \stackrel{\text{def}}{=} |\lambda_-^{1-t}\,\lambda_+^t|^2$. Its push-forward is an eigenmeasure of B.

(ii) If $\boldsymbol{\eta}_n$ and $\boldsymbol{\eta}_n'$ are two sequences with m $(+)$ and $n-m$ $(-)$, which are not related by a cyclic permutation, then the semiclassical measures $\mu^{\Sigma}_{\eta_n}$, $\mu^{\Sigma}_{\eta_n'}$ are mutually singular and so are their push-forwards on $\mathbb{T}^2$.

Note that the radius $\sqrt{\Lambda_t}$ lies in the 'bulk' of the nontrivial spectrum, but can be different from the radius r_0 where the spectral density is peaked. In figure 10 we plot the Husimi functions of two states $\psi_{\eta_4,0}$, $\psi_{\eta_4',0}$ constructed from two 4-sequences $\boldsymbol{\eta}_4$, $\boldsymbol{\eta}_4'$ not cyclically related. The two functions, which give a rough idea of the limit measures μ_{η_4}, $\mu_{\eta_4'}$, seemingly concentrate on different parts of the phase space.

Proof. In short we call $\boldsymbol{\eta} = (\boldsymbol{\eta}_n)^{k'}$ which has a period $\ell_\eta = \ell_{\eta_n}$ and $r = r(k')$. From (6.7), each state $|\psi_{\eta,r}\rangle$ is a combination of ℓ_η states $|\tau^j(\boldsymbol{\eta})\rangle$, and its eigenvalue has an exact modulus $\sqrt{\Lambda_t}$. When $k' \to \infty$, these states are asymptotically orthogonal to each other. Indeed, their overlaps can be decomposed as

$$\langle \boldsymbol{\eta}|\tau^j(\boldsymbol{\eta})\rangle = (\langle \boldsymbol{\eta}_n|\tau^j(\boldsymbol{\eta}_n)\rangle)^{k'} \qquad j = 0,\ldots,\ell_\eta - 1\,,$$

and for any $j \neq 0 \bmod \ell_\eta$ we have $\boldsymbol{\eta}_n \neq \tau^j(\boldsymbol{\eta}_n)$, which implies $|\langle \boldsymbol{\eta}_n|\tau^j(\boldsymbol{\eta}_n)\rangle| \leqslant c$, with $c \stackrel{\text{def}}{=} |\langle v_+, v_-\rangle|^2 < 1$. As a result, the normalization factor of $\psi_{\eta,r}$ satisfies

$$\mathcal{N}_{\eta,r} = \sum_{j=0}^{\ell_\eta - 1} |c_{\eta,r,j}|^2 + \mathcal{O}(c^{k'})\,, \qquad k' \to \infty\,.$$

To study the semiclassical measures of the sequence $(\psi_{\eta,r})_{k'\to\infty}$, we fix some cylinder $[\alpha] = [\alpha_{-l'}\ldots\alpha_{-1}\cdot\alpha_0\ldots\alpha_{l-1}]$ and compute the weight of the measures $H^{\ell}_{\psi_{\eta,r}}$. If k is large enough and $\ell \sim k/2$, the conditions $\ell > l$ and $k - \ell > l'$ are fulfilled, and this weight can be written as

$$H^{\ell}_{\psi_{\eta,r}}([\boldsymbol{\alpha}]) = \langle \psi_{\eta,r} | \Pi_{[\alpha]} | \psi_{\eta,r} \rangle = \sum_{j,j'=0}^{\ell_\eta - 1} c_{\eta,r,j}\, \bar{c}_{\eta,r,j'}\, \langle \tau^{j'}(\boldsymbol{\eta}) | \Pi_{[\alpha]} | \tau^{j}(\boldsymbol{\eta}) \rangle\,, \tag{6.8}$$

where the projector on $[\boldsymbol{\alpha}]$ is a tensor product operator:

$$\Pi_{[\alpha]} = \pi_{\alpha_0} \otimes \pi_{\alpha_1} \ldots \pi_{\alpha_{l-1}} \otimes (I)^{\otimes k-l-l'} \otimes F_3^* \pi_{\alpha_{-l'}} F_3 \otimes \cdots \otimes F_3^* \pi_{\alpha_{-1}} F_3\,.$$

The tensor factor $(I)^{k-l-l'-1}$ implies that each matrix element $\langle \tau^{j'}(\boldsymbol{\eta}) | \Pi_{[\alpha]} | \tau^{j}(\boldsymbol{\eta}) \rangle$ contains a factor $(\langle \tau^{j'}(\boldsymbol{\eta}_n) | \tau^{j}(\boldsymbol{\eta}_n) \rangle)^{k'-\mathcal{O}(1)}$; for the same reasons as above, this element is $\mathcal{O}(c^{k'})$ if $j \neq j'$. We are then led to consider only the diagonal elements $j = j'$:

$$\forall j = 0, \ldots, \ell_\eta - 1, \qquad \langle \tau^{j}(\boldsymbol{\eta}) | \Pi_{[\alpha]} | \tau^{j}(\boldsymbol{\eta}) \rangle = \prod_{i=0}^{l-1} P_{\eta_{j+i},\alpha_i} \prod_{i'=1}^{l'} P^*_{\eta_{j-i'},\alpha_{-i'}}\,. \tag{6.9}$$

As above the weights $P_{\pm,\epsilon} = |v_{\pm,\epsilon}|^2$, $P^*_{\pm,\epsilon} = |w_{\pm,\epsilon}|^2$, and the definition of η_i was extended to $i \in \mathbb{Z}$ by periodicity. We claim that the right-hand side exactly corresponds to $\tilde{\mu}^{\Sigma}_{\tau^j(\eta_n)}([\boldsymbol{\alpha}])$, where $\tilde{\mu}^{\Sigma}_{\tau^j(\eta_n)}$ is a certain Bernoulli eigenmeasure for the iterated shift σ^n. The latter can be seen as a simple shift on the symbol space $\tilde{\Sigma}$ constructed from 3^n symbols $\tilde{\epsilon} \in \{0, \ldots, 3^n - 1\}$: each $\tilde{\epsilon}$ is in one-to-one correspondence with a certain n-sequence $\epsilon_0 \ldots \epsilon_{n-1}$. Adapting the formalism of section 2.3.5 to this new symbol space, the Bernoulli measure $\tilde{\mu}^{\Sigma}_{\tau^j(\eta_n)}$ corresponds to the following weight distributions $\tilde{\mathbf{P}}$, $\tilde{\mathbf{P}}^*$:

$$\text{for } \tilde{\epsilon} \equiv \epsilon_0 \ldots \epsilon_{n-1}, \qquad \tilde{P}_{\tilde{\epsilon}} = \prod_{i=0}^{n-1} P_{\eta_{n,j+i},\epsilon_i} \quad \text{and} \quad \tilde{P}^*_{\tilde{\epsilon}} = \prod_{i=1}^{n} P^*_{\eta_{n,j-i},\epsilon_{n-i}}\,. \tag{6.10}$$

The measures $\tilde{\mu}^{\Sigma}_{\tau^j(\eta_n)}$, $j = 0, \ldots, \ell_\eta - 1$, are related to one another through σ:

$$\sigma^*\, \tilde{\mu}^{\Sigma}_{\tau^j(\eta_n)} = |\lambda_{\eta_{n,j}}|^2\, \tilde{\mu}^{\Sigma}_{\tau^{j+1}(\eta_n)}\,. \tag{6.11}$$

Finally, the semiclassical measure associated with the sequence $(\psi_{\eta,r})_{k=nk'\to\infty}$ is

$$\mu^{\Sigma}_{\eta_n} = \frac{1}{\mathcal{N}_{\eta_n}} \sum_{j=0}^{\ell_\eta - 1} C_{\eta_n,j}\, \tilde{\mu}^{\Sigma}_{\tau^j(\eta_n)}\,, \qquad \text{where} \quad C_{\eta_n,j} = \prod_{m=0}^{j-1} \frac{|\lambda_{\eta_{n,m}}|^2}{\Lambda_t}\,,$$

$$\mathcal{N}_{\eta_n} = \sum_{j=0}^{n-1} C_{\eta_n,j}\,. \tag{6.12}$$

This is a probability eigenmeasure of σ, with the decay rate Λ_t. It only depends on the orbit $\{\tau^j \boldsymbol{\eta}_n,\ j = 0, \ldots, \ell_{\eta_n} - 1\}$ and not on the choice of $(r(k'))$. This measure does not charge $\Sigma \setminus \Sigma'$, so its push-forward is an eigenmeasure of B.

The proof of statement (ii) goes as follows: since $\boldsymbol{\eta}_n$ and $\boldsymbol{\eta}'_n$ are not cyclically related, for any $j, j' \in \mathbb{Z}$, the weight distributions $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{P}}'$ defining, respectively, the Bernoulli measures $\tilde{\mu}^{\Sigma}_{\tau^j(\eta_n)}$ and $\tilde{\mu}^{\Sigma}_{\tau^{j'}(\eta'_n)}$ (see (6.10)) are different. As a result, these two measures are mutually singular (see the end of section 2.3.5), and so are the two linear combinations $\mu^{\Sigma}_{\eta_n}, \mu^{\Sigma}_{\eta'_n}$. Because the weights $\tilde{\boldsymbol{P}}, \tilde{\boldsymbol{P}}'$ are different from $(1, 0, \ldots, 0)$ and $(0, \ldots, 1)$, the push-forwards $\mu_{\eta_n}, \mu_{\eta'_n}$ are also mutually singular. □

By a standard density argument, we can exhibit semiclassical measures for arbitrary decay rates in $[|\lambda_-|^2, |\lambda_+|^2]$.

Corollary 2. *Consider a real number $t \in [0, 1]$ and a sequence of rationals $(t_p = (m_p/n_p) \in [0, 1])_{p\to\infty}$ converging to t when $p \to \infty$. For each p, let $\boldsymbol{\eta}_p$ be a sequence with m_p $(+)$ and $n_p - m_p$ $(-)$ and $\mu^{\Sigma}_{\boldsymbol{\eta}_p}$ the σ-eigenmeasure constructed above. Then any weak-$*$ limit of the sequence $(\mu^{\Sigma}_{\boldsymbol{\eta}_p})_{p\to\infty}$ is a semiclassical measure of $(\tilde{B}_N)$; it is a σ-eigenmeasure of the decay rate $\Lambda_t = |\lambda_-^{1-t}\,\lambda_+^t|^2$, and its push-forward is an eigenmeasure of B.*

Let us call $\mathfrak{M}^{\Sigma}(\Lambda_t)$ the family of semiclassical measures obtained this way, and $\mathfrak{M}^{\Sigma} = \cup_{t\in[0,1]}\mathfrak{M}^{\Sigma}(\Lambda_t)$. We do not know whether the family $\mathfrak{M}^{\Sigma}$ exhausts the full set of semiclassical measures for $(\tilde{B}_N)$. Still, we can address the third point in question 2 with respect to this family.

Proposition 9. *The family of eigenmeasures $\mathfrak{M}^{\Sigma}$ does not contain the natural measure $\mu^{\Sigma}_{\rm nat} = \mu^{\Sigma}_{\boldsymbol{r}_{\rm sym}}$. The same statement holds after push-forward on $\mathbb{T}^2$.*

Proof. For any $\epsilon_0 \in \{0, 1, 2\}$, let us compute the weight of a measure $\mu^{\Sigma}_{\boldsymbol{\eta}_p} \in \mathfrak{M}^{\Sigma}$ on the cylinder $[\epsilon_0]$ (corresponding to the vertical rectangle $\overline{R_{\epsilon_0}}$). For the natural measure we have $\mu^{\Sigma}_{\rm nat}([\epsilon_0]) = 1/3$ for $\epsilon_0 = 0, 1, 2$.

From (6.10), for any $j \in \mathbb{Z}$ one has $\tilde{\mu}^{\Sigma}_{\tau^j(\boldsymbol{\eta}_p)}([\epsilon_0]) = P_{\eta_{p,j},\epsilon_0}$. Combining this with (6.12), we get

$$\mu^{\Sigma}_{\boldsymbol{\eta}_p}([\epsilon_0]) = C_+\,P_{+,\epsilon_0} + (1 - C_+)\,P_{-,\epsilon_0}, \qquad \text{where} \quad C_+ = \frac{1}{\mathcal{N}_{\eta_p}} \sum_{j\,:\,\eta_{p,j}=(+)} C_{\eta_p,j}\,.$$

For any $\mu^{\Sigma} \in \mathfrak{M}$, the weights $\mu^{\Sigma}([\epsilon_0])$ will take the same form, for some $C_+ \in [0, 1]$. Using the approximate expressions for the weights given in section 6.4.1, one finds that the condition $\mu^{\Sigma}([\epsilon_0]) = 1/3$ cannot be satisfied simultaneously for $\epsilon_0 = 0, 1, 2$. □

7. Concluding remarks

The main result of this paper is the semiclassical connection between, on the one hand, eigenfunctions of a quantum open map (which mimic 'resonance eigenfunctions') and, on the other hand, eigenmeasures of the classical open map.

We proved that, modulo some problems at the discontinuities of the classical map, semiclassical measures associated with long-living resonant are eigenmeasures of the classical dynamics, and their decay rate is directly related with those of the corresponding resonant eigenstates (see theorem 1). This result, which basically derives from Egorov's theorem, has been expressed in a quite general framework and applied to the specific example of the open baker's map. An analogue has been proven for the more realistic setting of Hamiltonian scattering [32, theorem 3].

Although the construction and classification of eigenmeasures with decay rates $\Lambda < 1$ is quite easy, the classification of semiclassical measures among all possible eigenmeasures remains largely open (see question 2). The solvable model provided by the Walsh-quantized baker provides some hints to these classification in the form of an explicit family of semiclassical measures, but we have no idea whether these results apply to more general systems, not even the 'standard' quantum open baker. Indeed, the high degeneracies of the Walsh–baker may be responsible for a nongeneric profusion of semiclassical measures for this model. Interestingly, the natural eigenmeasure does not seem to play a particular role at the quantum level.

A tempting way of constraining the set of semiclassical measures would be to adapt the 'entropic' methods of [1] to open chaotic maps. A desirable output of these methods would be, for instance, to forbid semiclassical measures from being of the pure point type described in section 2.3.3.

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