Université Paris-Sud • M2 CS & EDP
Introduction to the spectral theory (2014–2015)
Webpage: `http://www.math.u-psud.fr/~pankrash/2014spec/`
Final Examination

The solutions must be written either in English or in French. The result of any question (even if you have no solution) can be used to study the subsequent questions and exercises.

Exercise 1. Let $\Omega \subset \mathbb{R}^d$ be a non-empty open set and $V \in L^1_{\text{loc}}(\Omega)$, $V \geq 0$. Consider the following form t in the Hilbert space $L^2(\Omega)$:

$$t(u,u) = \int_\Omega \big(|\nabla u|^2 + V|u|^2\big)dx, \quad D(t) = \Big\{u \in H^1_0(\Omega) : \int_\Omega V|u|^2 dx < \infty\Big\},$$

and let T be the associated self-adjoint operator. For $R > 0$ set $\Omega_R := \big\{x \in \Omega : |x| < R\big\}$,

$$\Sigma(R) := \inf_{\substack{u \in D(t)\\ u=0 \text{ in } \Omega_R\\ u \neq 0}} \frac{t(u,u)}{\|u\|^2}, \quad \text{and} \quad \Sigma := \lim_{R\to+\infty} \Sigma(R) \in [0, +\infty].$$

The aim of the present exercise is to show the equality $\Sigma = \inf \operatorname{spec}_{\text{ess}} T$.

(1) Let $\lambda \in \operatorname{spec}_{\text{ess}} T$ and (u_n) be a singular Weyl sequence for T and λ with $\|u_n\| \equiv 1$. Pick an arbitrary $R > 0$. We would like to show that

$$\|u_n\|_{L^2(\Omega_R)} \to 0 \text{ as } n \to \infty. \tag{A}$$

- (a) Show that $\lim_{n\to\infty}\langle u_n, Tu_n\rangle = \lambda$ and that the sequence (u_n) is bounded in $H^1(\Omega)$.
- (b) Let $\varepsilon > 0$. Assume that there exists a subsequence (u_{n_k}) with $\|u_{n_k}\|_{L^2(\Omega_R)} \geq \varepsilon$. Show that there is a subsequence $(u_{n_{k_m}})$ which is norm-convergent in $L^2(\Omega_R)$.
- (c) Obtain a contradiction, then prove the claim (A).

(2) Let λ and u_n be as in question (1). Pick $R > 0$ and consider a C^∞-function $\chi : \mathbb{R}^d \to [0,1]$ such that:

$$\chi(x) = 0 \text{ for } |x| \leq R, \quad \chi(x) = 1 \text{ for } |x| \geq R+1.$$

- (a) Let $\varepsilon > 0$. Show that that for large n we have $\displaystyle\int_\Omega |\chi|^2\big(|\nabla u_n|^2 + V|u_n|^2\big)dx \leq \lambda + \varepsilon$.
- (b) Let $\varepsilon > 0$. Show that that for large n there holds $\displaystyle\int_\Omega \Big(\big|\nabla(\chi u_n)\big|^2 + V|\chi u_n|^2\Big)dx \leq \lambda + \varepsilon$.
- (c) Show that $\|\chi u_n\|_{L^2(\Omega)} \to 1$ as $n \to \infty$.
- (d) Let $\varepsilon > 0$. Show that for large n one has $\dfrac{t(\chi u_n, \chi u_n)}{\langle \chi u_n, \chi u_n\rangle} \leq \lambda + \varepsilon$.
- (e) Show the inequality $\Sigma \leq \inf \operatorname{spec}_{\text{ess}} T$.

(3) Let $\mu < \inf \operatorname{spec}_{\text{ess}} T$. Consider the spectral projector $\Pi := E_T\big((-\infty, \mu]\big)$ and an orthonormal basis (χ_j), $j = 1, \dots, N$, in $\operatorname{ran}\Pi$.

- (a) Show that $t(u,u) \geq \mu\big\|(1-\Pi)u\big\|^2$ for any $u \in D(t)$.
- (b) Let $u \in L^2(\Omega)$ such that $u = 0$ in Ω_R. Show the estimate

$$\big\|\Pi u\big\|^2 \leq \Big(\sum_{j=1}^N \int_{\Omega\setminus\Omega_R} \big|\chi_j(x)\big|^2 dx\Big)\|u\|^2.$$

- (c) Let $\varepsilon > 0$. Show that one can find $R > 0$ such that for any $u \in L^2(\Omega)$ with $u = 0$ in Ω_R there holds $\|(1-\Pi)u\|^2 \geq (1-\varepsilon)\|u\|^2$.
- (d) Let $\varepsilon > 0$. Show that for large R one has $\Sigma(R) \geq \mu(1-\varepsilon)$, then deduce the inequality $\Sigma \geq \inf \operatorname{spec}_{\text{ess}} T$.

Exercise 2. We work in the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^d)$. Let $V \in C^0(\mathbb{R}^d)$, $V \geq 0$. Consider the naturally defined self-adjoint operator $T = -\Delta + V$ in $\mathcal{H}$.

(1) Show that T is exactly the self-adjoint operator generated by the sesqulinear form

$$t(u,u) = \int_{\mathbb{R}^d} \big(|\nabla u|^2 + V|u|^2\big)dx, \quad D(t) = \Big\{u \in H^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} V|u|^2 dx < \infty\Big\}.$$

(2) Show the inequality $\inf \operatorname{spec}_{\mathrm{ess}} T \geq \lim_{R\to+\infty} \inf_{|x|\geq R} V(x)$.

(3) Assume that $\lim_{|x|\to+\infty} V(x) = a$. Show that $\operatorname{spec}_{\mathrm{ess}} T = [a, +\infty)$.

Exercise 3.

(1) Describe (without proof) the spectrum of the following operator T_α in $L^2(\mathbb{R}_+)$:

$$T_\alpha u = -u'', \quad D(T) = \big\{u \in H^2(\mathbb{R}_+) : u'(0) + \alpha u(0) = 0\big\},$$

where $\alpha \in \mathbb{R}$.

(2) Let $\alpha > 0$. Consider the sesquilinear form

$$q(u,u) = \iint_{\mathbb{R}_+\times\mathbb{R}_+} |\nabla u|^2 dx - \alpha \int_{\mathbb{R}_+} \big|u(x_1, 0)\big|^2 dx_1, \quad D(q) = H^1(\mathbb{R}_+ \times \mathbb{R}_+).$$

Let Q be the self-adjoint operator generated by q, acting in $L^2(\mathbb{R}_+ \times \mathbb{R}_+)$.

(a) Show the equality $Q = T_0 \otimes 1 + 1 \otimes T_\alpha$.

(b) Describe the spectrum of Q.

(3) Take any $b > 0$ and consider the domain $\Omega := \{x_2 \geq bx_1\} \cup \{x_2 \geq 0\} \subset \mathbb{R}^2$. Now take $\alpha > 0$, consider the sesqulinear form

$$t(u,u) = \iint_\Omega |\nabla u|^2 dx - \alpha \int_{\partial\Omega} |u|^2 ds, \quad D(t) = H^1(\Omega),$$

where ds is the one-dimensional Hausdorff measure on $\partial\Omega$, and denote by T the associated self-adjoint operator in $L^2(\Omega)$.

(a) Using the results of item 2 and a suitable decomposition of Ω show that $T \geq -\alpha^2$.

(b) Using suitable Weyl sequences show that $[-\alpha^2, +\infty) = \operatorname{spec} T$.