Lecture Notes[1] for the course

Introduction to Spectral Theory

Master Mathématiques et Applications

parcours *Analyse, Modélisation, Simulation* et

Analyse, Arithmétique, Géométrie

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1. following an earlier version by Konstantin Pankrashkin

Contents

Some notations

We list some notations used throughout the text.

The symbol $\mathbb{N}$ denotes the set of the natural numbers starting from 0.

If $(M, \mathcal{T}, \mu)$ is a measure space and $f : M \to \mathbb{C}$ is a measurable function, then we denote the *essential range* and the *essential supremum* of f w.r.t. the measure μ:

$$\text{ess}_\mu \operatorname{Ran} f \stackrel{\text{def}}{=} \Big\{ z \in \mathbb{C} : \mu\big\{ x \in M : \big|z - f(x)\big| < \epsilon \big\} > 0 \text{ for all } \epsilon > 0 \Big\},$$
$$\text{ess}_\mu \sup |f| \stackrel{\text{def}}{=} \inf \Big\{ a \in \mathbb{R} : \mu\big\{ x \in M : \big|f(x)\big| > a \big\} = 0 \Big\}.$$

If the measure μ is obvious in the context, we will omit to indicate it in the notations.

In the following, we will consider linear operators acting on a comple Banach space, which we will usually denote by the letter $\mathcal{B}$, but a large part of the notes will be focussed on the case of Hilbert spaces. What we call a *Hilbert space* will mean a *separable complex Hilbert space*, which we will generally denote by $\mathcal{H}$.

Because we'll have in mind mostly Hilbert spaces made of functions on $\mathbb{R}^d$ or some domain $\Omega \subset \mathbb{R}^d$, we will denote the "vectors" of $\mathcal{H}$ by $u, v, w \ldots$. For two vectors $u, v \in \mathcal{H}$, $\langle u, v \rangle$ will denote the sesquilinear scalar product of u and v. If several Hilbert spaces are considered in the problem, we will specify the scalar product with the notation $\langle u, v \rangle_{\mathcal{H}}$. To respect the convention in quantum mechanics, our scalar products will always be linear with respect to the *second* argument, and as antilinear with respect to the first one:

$$\forall \alpha \in \mathbb{C} \quad \langle u, \alpha v \rangle = \langle \bar{\alpha} u, v \rangle = \alpha \langle u, v \rangle.$$

For example, the scalar product in the Lebesgue space $L^2(\mathbb{R})$ is defined by

$$\langle f, g \rangle_{L^2} = \int_{\mathbb{R}} \overline{f(x)} g(x)\, dx.$$

If A is a finite or countable set, $\ell^2(A)$ denotes the vector space of square-summable functions $u : A \to \mathbb{C}$:

$$\sum_{a \in A} \big|u(a)\big|^2 < \infty.$$

This forms a Hilbert space, equipped with the scalar product

$$\langle u, v \rangle = \sum_{a \in A} \overline{u(a)} v(a).$$

Note that when $A = \mathbb{N}$ or $A = \mathbb{Z}$ the functions u are sometimes written as sequences: $u(a) = u_a$.

$\mathcal{L}(\mathcal{B})$ and $\mathcal{K}(\mathcal{B})$ denote the spaces of bounded linear operators, respectively of compact operators from $\mathcal{B}$ to $\mathcal{B}$. A similar notation applies also to bounded, resp. compact opertors on a Hilbert space $\mathcal{H}$.

Some functional spaces

If $\Omega \subset \mathbb{R}^d$ is an domain (= convex open set) and $k \in \mathbb{N}$, then $H^k(\Omega)$ denotes the kth Sobolev space on Ω, i.e. the space of functions in L^2, whose partial derivatives up to order k are also in $L^2(\Omega)$. The Sobolev space H^k is equipped with the scalar product:

(0.0.1)
$$\langle u, v\rangle_{H^k} = \sum_{|\alpha|\leq k} \langle \partial^\alpha u, \partial^\alpha v\rangle_{L^2},$$

where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ is a multiindex, and $\partial^\alpha = \partial_{x_1}^{\alpha^1} \cdots \partial_{x_d}^{\alpha^d}$ is the multi-derivative operators. It is complete w.r.t. the norm associated with this scalar product.

We will use a notation frequent in the theory of partial differential equations: the symmetric derivative operator $D_x = \frac{1}{i}\partial_x$, as well as its multi-derivative version

$$D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_d}^{\alpha_d} = (-i)^{|\alpha|}\partial^\alpha, \quad \alpha \in \mathbb{N}^d.$$

By $H_0^k(\Omega)$ we denote the completion in $H^k(\Omega)$ of the subspace $C_c^\infty(\Omega)$ (with respect to the norm of $H^k(\Omega)$). The symbol $C^k(\Omega)$ denotes the space of functions on Ω whose partial derivatives up to order k are continuous; in particular, the set of the continuous functions is denoted as $C^0(\Omega)$. This should not be confused with the notation $C_0(\mathbb{R}^d)$ for the space of continuous functions $f : \mathbb{R}^d \to \mathbb{C}$ vanishing at infinity: $\lim_{|x|\to\infty} f(x) = 0$. The subscript $_{\text{comp}}$ on a functional space indicates that its elements have compact supports: for instance $H^1_{\text{comp}}(\mathbb{R}^d)$ is the space of functions in $H^1(\mathbb{R}^d)$ having compact supports.

We denote by $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ the Fourier transform, defined for $f \in \mathcal{S}(\mathbb{R}^d)$ by:

$$\mathcal{F}f(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x)\, e^{-i\xi\cdot x}\, dx.$$

The normalization makes this transform unitary on $L^2(\mathbb{R}^d, dx)$. The Fourier transformed function $\mathcal{F}f$ will sometimes be denoted by $\hat{f}$.

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Recommended books

During the preparation of the notes, we used part of the lecture notes by Bernard Helffer [6]. Other recommended books are the one E. B. Davies [4] and the book of G. Teschl [13] (available online). A detailed account is given in the series by Reed&Simon, in particular [7, 8, 9].

Additional references for particular topics will be given throughout the text.

Chapter 1

What is a Spectrum ?

1.1 The spectrum in physics

The term "spectrum" first appeared in different domains of physics; originally it described the decomposition of the light observed from the spatial objects (like the sun, or other stars), when observed through a device able to separate the different colors (that is, the different frequencies of the received light). Quite often, one could observe peaks of luminosity at certain frequencies, above a more or less uniform "background". Chemists observed that the light emitted by some gases always produced peaks at the same frequencies: the emitted spectra were thus characteristic of chemical elements, and allowed to analyse chemical reactions inside stars.

In the study of electric circuits and electronics, one often observes a time signal (e.g. of the voltage along some part of the circuit). This time signal $S(t)$ can be analyzed through the Fourier transform, or the Laplace transform

$$\hat{S}(\omega) = \int_0^\infty e^{-i\omega t} S(t)\, dt$$

(we assume that the signal vanishes for negative times). Often one cannot detect the phase of $\hat{S}(\omega)$, but only observes $|\hat{S}(\omega)|^2$, which is called the *power spectrum* of the signal $S(t)$. For instance, the RLC circuit leads to a power spectrum which is peaked at the characteristic frequency $\omega_0 = \frac{1}{\sqrt{LC}}$, the width of the peak depending on the size of the resistance R.

In both examples, the spectrum corresponds to a decomposition in frequency. The hope is to analyze a (possibly complicated) time signal, through a (hopefully small) set of characteristic frequencies, which would contain most of the "interesting" information of the signal.

1.2 An example: Schrödinger evolution in quantum mechanics

This analysis is most relevant when the dynamics under study can be modeled by a semigroup generated by a *linear operator*. We will take for example the case of Quantum Mechanics, where the notion of spectrum acquired a central place, which acted as a strong incentive to the fast development of spectral theory in mathematics.

The state of a quantum particle evolving in some domain ("box") $\Omega \subset \mathbb{R}^d$ is represented by a time-dependent wavefunction

$$\psi : \mathbb{R} \ni t \mapsto \psi(t) \in L^2(\Omega)\,.$$

The state of the particle at time $t \in \mathbb{R}$ is represented by the function $\psi(t) \in L^2(\Omega)$. Quantum mechanics is a probabilistic theory: if one uses a device to measure the position of the particle at time t, then $|\psi(t,x)|^2$ represents the probability density to detect the particle at the point x. With this probabilistic interpretation in mind, one needs to enforce the normalization:

$$\forall t \in \mathbb{R}, \quad \|\psi(t)\|_{L^2} = 1.$$

Quantum mechanics prescribes the law of evolution of $\psi(t)$: it is given by the (time dependent) Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi(t,x) = -\frac{\hbar^2}{2m}\Delta\psi(t,x) + V(x)\psi(t,x),$$

where $\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ is the Laplacian, and the real valued function $V : \Omega \to \mathbb{R}$ represents the potential energy of the particle (e.g. the electric potential, if the particle carries an electric charge).

By rescaling the units of time and space, we can remove the physical constants, to obtain[1]:

$$i\frac{\partial}{\partial t}\psi(t,x) = -\Delta\psi(t,x) + V(x)\psi(t,x) = [P\,\psi](t) \tag{1.2.1}$$

where $P = -\Delta + V$ appears as a linear operator acting on the Hilbert space $\mathcal{H} = L^2(\Omega)$; it is called a Schrödinger operator, or also the *Hamiltonian* of this quantum system. This equation therefore takes the form of a linear evolution equation, where the operator P acts as the generator of a semigroup on $\mathcal{H}$.

Several mathematical questions pop up. A generic function $\psi \in L^2$ does not admit derivatives in L^2, so $\Delta\psi$ is not well-defined on L^2. This means that the operator Δ is not defined on the whole of L^2, but only on a linear subspace of that space, namely the Sobolev space $H^2(\Omega)$. If the potential V is bounded on Ω, then H is still well-defined on $H^2(\Omega)$. We call $H^2(\Omega)$ the *domain* of the operator P, denoted by $D(P)$. In this course we will pay a special attention to the domains of operators.

Another question (both physical and mathematical) concerns the boundary behaviour of the functions $\psi(t)$: from physical ground, we may want to assume that the wavefunctions $\psi(t,x)$ vanish when x approaches the boundary of the box $\partial\Omega$. One may want to take into account such a physical constraint, when defining the domain of P.

1.2.1 The Schrödinger group

Semigroup theory, in particular the Hille-Yosida theorem, teaches us that, under favorable conditions on the operator $P : D(P) \to \mathcal{H}$, this operator will generate a semigroup of evolution, meaning that for any initial data $\psi(0) \in D(P)$, the equation (1.2.1) admits a unique solution $\psi \in C^1(\mathbb{R}_+, \mathcal{H})$, defined

1. Implicitly, the functions ψ and V have been modified by the rescaling, but we keep the same notations.

through a semigroup of bounded operators $S(t) : \mathcal{H} \to \mathcal{H}$: $\psi(t) = S(t)\psi(0)$. What is remarkable is that this semigroup extends to the full Hilbert space $\mathcal{H}$, namely the evolution is actually defined even for initial data $\psi(0) \notin D(P)$.

The "favorable conditions" on the operator H can be expressed in terms of the *resolvent* of the operator, which will play a crucial role in these notes. We will give a more formal definition of the resolvent, but roughly speaking it is a family of bounded operators $R(z) = (P - z)^{-1} : \mathcal{H} \to \mathcal{H}$, depending on complex parameter z defined on some open subset of $\mathbb{C}$.

In the case of the Schrödinger operator P acting on $L^2(\Omega)$, which is symmetric, these conditions can be replaced by a positivity argument, provided the potential V is bounded from below. We will see that, if one makes "good" choices of domain $D(P)$, the operator P is not only symmetric, but actually *selfadjoint*. In this case, the semigroup generated by P extends to a *unitary* group $(U^t)_{t\in\mathbb{R}}$ on $L^2(\Omega)$, which describes the quantum evolution:

$$\forall t \in \mathbb{R}, \quad \psi(t) = U^t\psi(0).$$

Formally, we may write $U^t = e^{-itP}$, eventhough the exponentiation of P cannot be defined by a series.

1.2.2 Spectral expansion

In order to describe more quantitatively the behaviour of $\psi(t) = U^t\psi(0)$, one is lead to study the *spectrum* of the operator P. Let us restrict ourelves to the case where

i) the "box" Ω is bounded,

ii) one imposes Dirichlet boundary conditons on Ω,

iii) and the potential $V \in L^\infty(\Omega)$.

In that case, we will show that the spectrum of P is purely discrete: it is composed of a countable set of real eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ of finite multiplicities, associated with a family of eigenfunctions $(\varphi_j)_{j\in\mathbb{N}}$ which form an orthonormal Hilbert basis of $L^2(\Omega)$. This spectral information allows to expand the evolved state $\psi(t)$, taking into account the decomposition of $\psi(0)$ in this eigenbasis:

$$\psi(0) = \sum_{j\in\mathbb{N}} \langle\varphi_j, \psi(0)\rangle\, \varphi_j \qquad \forall t \in \mathbb{R}, \quad \psi(t) = \sum_{j\in\mathbb{N}} e^{-it\lambda_j}\, \langle\varphi_j, \psi(0)\rangle\, \varphi_j. \tag{1.2.2}$$

We note that the spectrum of the differential operator P generally depends on the choice of its domain $D(P)$, and so does the expansion (1.2.2). For instance, requiring Dirichlet, vs. Neumann boundary conditions, leads to two different discrete spectra for P. This shows that the question of domain is not just a mathematical subtlety, but it directly impacts the evolution of the quantum state.

1.2.3 Stationary states

The above expansion shows that, if the initial state is an eigenstate of P, namely $\psi(0) = \varphi_j$ for some j, then the evolution of ψ is very simple:

$$\psi(t) = e^{-it\lambda_j}\,\psi(0) = e^{-it\lambda_j}\,\varphi_j\,.$$

The global phase factor $e^{-it\lambda_j}$ is not detectable physically, which explains why such a particle is said to occupy a *stationary* state. A large part of atomic and molecular physics consists in computing the eigenvalues (λ_j) and eigenstates (φ_j) of the corresponding Hamiltonian operator. Once this *spectrum* is known, the evolution of the atom (or molecule) is often described in physics textbooks as a sequence of "jumps" between different stationary states, induced by the interaction with an external electromagnetic field (one speaks of light emission or absorption, depending on whether the eigenvalue goes down or up). Such an evolution through "jumps" cannot simply result from the Schrödinger group described above, and we will not try to do it here. Yet, it shows the importance of identifying the spectra of Schrödinger operators in quantum physics.

1.3 Example of the heat equation

Let us briefly describe another equation making use of the spectral decomposition of the Laplacian on a bouded open set $\Omega \subset \mathbb{R}^d$. The heat equation

$$\partial_t \theta(t,x) = \Delta\theta(t,x)$$

describes the evolution of the temperature $\theta(t,x)$ in a body Ω, when this body is inserted in a thermostat of given temperature $\theta_{th} \in \mathbb{R}$, starting from a given temperature distribution $\theta(0,x)$. The function $u(t) = \theta(t) - \theta_{th}$ describes the relative temperature. The physical condition of thermal contact at the boundary of Ω imposes the constraint $\theta(t,x) = \theta_{th}$ for all $t > 0$, $x \in \partial\Omega$. It is easier to consider the relative temperature $u(t,x) \stackrel{\text{def}}{=} \theta(t,x) - \theta_{th}$, which satisfies the Dirichlet boundary conditions, and satisfies the same heat equation as θ. The discrete spectrum of $P = -\Delta$ implies the following spectral expansion for the function u:

$$u(t) = \sum_{j\in\mathbb{N}} e^{-t\lambda_j}\,\langle\varphi_j, u_0\rangle\,\varphi_j. \tag{1.3.3}$$

As opposed to the expansion (1.2.2), we see that the above expansion is dominated by its first few terms when $t \to \infty$. To understand the long time behaviour of the heat equation, it is not necessary to identify the full spectrum, but only the "bottom" of the spectrum of P.

This example shows that, quite often, a partial description of the spectrum (like the identification of the bottom of the spectrum, or the presence of a *spectral gap* at the bottom), already provides relevant physical information for equations like the heat equation.

Focussing on selfadjoint operators on Hilbert spaces

In situations where the spectrum of P is not purely discrete, a similar (yet more complicated) decomposition can be written. Such a decomposition uses the *spectral theorem for selfadjoint operators*.

The power of this theorem, and its relevance for quantum mechanics, induce us to devote a large part of the present notes to the specific case of selfadjoint operators defined on a Hilbert space. We will already see that the precise identification of such operators (including their domains) requires some care. A nice way to construct such selfadjoint operators is through the use of *quadratic forms*.

Yet, spectral expansions (possibly with some remainder term) can also be helpful in nonselfadjoint situations, for instance when the natural functional space is not a Hilbert space, but only a Banach space, for instance a space of Lebesgue type $L^p(\Omega)$, a Sobolev space based on such an L^p. Alternatively, the space of continuous functions $C^0(\Omega)$, or of finitely differentiable functions $C^k(\Omega)$ is often useful when describing operators issued from the theory of dynamical systems, which is another application of spectral theory.

Chapter 2

Bounded vs. Unbounded operators

In this section, after recalling the definition of a bounded operator on a Hilbert (or Banach) space, we start to describe a more general class of linear operators, namely (densely defined) unbounded operators, which will constitute the main focus of these notes. The Schrödinger operator $P = -\Delta + V$ on $L^2(\Omega)$ mentioned in the introduction is and example of such unbounded operators; actually, all differential operators belong to that class, which explains the importance of the study of these operators towards understanding linear (and actually, also nonlinear) Partial Differential Equations.

2.1 Some definitions

A *linear operator* T on a Banach space $\mathcal{B}$ is a linear map from a subspace $D(T) \subset \mathcal{B}$ (called the *domain* of T) to $\mathcal{B}$. The domain is an important component of the definition of the operator, so one should actually denote the operator by the pair $(T, D(T))$. Yet, we will often omit to mention the domain, keeping the shorter notation T.

Across these notes, we will implicitly assume that the domain $D(T)$ is a *dense* subspace of $\mathcal{B}$ (w.r.to the natural topology of $\mathcal{B}$), unless the opposite is explicitly stated.

The *range* of $(T, D(T))$ is the set $\mathrm{Ran}\, T \stackrel{\text{def}}{=} \{Tu : u \in D(T)\}$; this is obviously a linear subspace of $\mathcal{B}$. We say that a linear operator T is *bounded* if the quantity

$$\mu(T) \stackrel{\text{def}}{=} \sup_{\substack{u \in D(T) \\ u \neq 0}} \frac{\|Tu\|}{\|u\|}$$

is finite. On the opposite, an operator $(T, D(T))$ will be said to be *unbounded* if $\mu(T) = \infty$.

If $D(T) = \mathcal{B}$ and T is bounded, then the operator $T : \mathcal{B} \to \mathcal{B}$ is continuous. The set of continuous operators on $\mathcal{B}$ forms a vector space, denoted by $\mathcal{L}(\mathcal{B})$. Equipped with the norm $\|T\| \stackrel{\text{def}}{=} \mu(T)$, this space has the structure of a *Banach algebra*: it is a Banach space, and also hosts an interal product $S, T \in \mathcal{L}(\mathcal{B}) \mapsto ST = S \circ T \in \mathcal{L}(\mathcal{B})$, with the inequality $\|ST\| \leq \|S\| \, \|T\|$.

Proposition 2.1.1 Assume $(T, D(T))$ is a *bounded* linear operator on $\mathcal{B}$ with a dense domain $D(T)$. Then T can be uniquely extended to a continuous linear operator defined on all of $\mathcal{B}$. This extension is called the closure the T, and is usually denoted by $\overline{T}$.

Proof.— Let us consider an element $u \in \mathcal{B} \setminus D(T)$. By the density of $D(T)$ in $\mathcal{B}$, we may consider a sequence $(u_n \in D(T))_{n\in\mathbb{N}}$ converging to u in $\mathcal{B}$. The sequence $(Tu_n)_{n\in\mathbb{N}}$ satisfies $\|Tu_n - Tu_m\| \leq \|T\|\|u_n - u_m\|$, hence it is a Cauchy sequence in $\mathcal{B}$, and admits a limit $w \in \mathcal{B}$. Let us decide that w is the image of u through an extended operator $\overline{T}$; we need to check that this image does not depend on the choice of sequence converging to u. Indeed, if $(\tilde{u}_n)$ is another sequence converging to u, with $T\tilde{u}_n$ converging to some $\tilde{w} \in \mathcal{B}$, then considering the alternating sequence $(u_0, \tilde{u}_0, u_1, \tilde{u}_1, \ldots)$ shows that $w = \tilde{w}$, therefore the image of u is unique. It is easy to check that the resulting operator $\overline{T}$ is linear, and bounded, with the same norm $\|\overline{T}\| = \|T\|$. $\square$

2.1.1 Closed unbounded operators

If $(T, D(T))$ is unbounded, it is not possible to extend it to all of $\mathcal{B}$ in a natural way. Yet, we can aim at an alternative property, *closedness*, which refers to a topological property of the *graph* of T.

Definition 2.1.2 (Graph of a linear operator) The *graph* of a linear operator $(T, D(T))$ is the set

$$\mathrm{gr}\, T \overset{\text{def}}{=} \big\{(u, Tu) :\, u \in D(T)\big\} \subset \mathcal{B} \times \mathcal{B}.$$

This is obviously a linear subspace of $\mathcal{B} \times \mathcal{B}$.

For two linear operators T_1 and T_2 in $\mathcal{B}$, we write $T_1 \subset T_2$ if $\mathrm{gr}\, T_1 \subset \mathrm{gr}\, T_2$. Namely, $T_1 \subset T_2$ means that $D(T_1) \subset D(T_2)$ and that $T_2 u = T_1 u$ for all $u \in D(T_1)$; the operator T_2 is then called an *extension* of T_1, while T_1 is called a *restriction* of T_2.

Definition 2.1.3 (Closed operator, closable operator)

- An operator $(T, D(T))$ on $\mathcal{B}$ is called *closed* if its graph is a closed subspace in $\mathcal{B} \times \mathcal{B}$.
- An operator $(T, D(T))$ on $\mathcal{B}$ is called *closable*, if the closure $\overline{\mathrm{gr}\, T}$ of the graph of T in $\mathcal{B} \times \mathcal{B}$ is still the graph of a certain operator, which we call $\overline{T}$. The latter operator $\overline{T}$ is called the *closure* of T.

An easy exercise shows that any bounded operator $T \in \mathcal{L}(\mathcal{B})$ is closed. Similarly, if we start from a bounded operator $(T, D(T))$ defined on a dense domain, the extension $\overline{T}$ constructed in Proposition 2.1.1 is the closure of T.

The closedness property can be characterized in terms of sequences.

Proposition 2.1.4 A linear operator T in $\mathcal{B}$ is closed if and only if, for any sequence $(u_n)_{n\in\mathbb{N}}$ satisfying the following two conditions:

i) the sequence $(u_n)_{n\in\mathbb{N}}$ converges to some element $u \in \mathcal{B}$,

ii) the sequence $(Tu_n)_{n\in\mathbb{N}}$ converges to $v \in \mathcal{B}$,

then one has $u \in D(T)$ and $v = Tu$.

Another characterization of the closedness can be obtained by introducing an auxiliary norm on $D(T)$, called the *graph norm*.

Definition 2.1.5 (Graph norm) Let $(T, D(T))$ be a linear operator on $\mathcal{B}$. We define on $D(T)$ the function:

$$u \mapsto \|u\|_T \stackrel{\text{def}}{=} \|u\|_{\mathcal{B}} + \|Tu\|_{\mathcal{B}}.$$

One easily checks that it makes up a norm on $D(T)$. We call it the *graph norm for* T.

If $\mathcal{B} = \mathcal{H}$ is a Hilbert space, the graph norm is usually defined alternatively as

$$\|u\|'_T \stackrel{\text{def}}{=} \sqrt{\|u\|^2_{\mathcal{H}} + \|Tu\|_{\mathcal{H}}}$$

This definition has the advantage to be a Hilbert norm, associated with the scalar product $\langle u, v\rangle_T = \langle u, v\rangle + \langle Tu, Tv\rangle$. This norm is equivalent with $\|\cdot\|_T$.

If T is bounded, the graph norm is equivalent with the standard norm. But this is not the case for an unbounded operator.

The closedness property can then be characterized as follows.

Proposition 2.1.6 Let $(T, D(T))$ be a linear operator on $\mathcal{B}$.

i) $(T, D(T))$ is closed iff the domain $D(T)$, equipped with the graph norm, is a complete Banach space.

ii) if $(T, D(T))$ is closable, then the domain $D(T)$, equipped with the graph norm, can be completed inside $\mathcal{B}$, namely its completion $\overline{D(T)}^{\|\cdot\|_T}$ can be identified with a certain subspace of $\mathcal{B}$, thereby extending the norm $\|\cdot\|_T$ to that subspace. This subspace is then the domain $D(\overline{T})$ of the operator $\overline{T}$.

The second point is a bit subtle: a normed space like $(D(T), \|\cdot\|_T)$ will always admit a formal completion, that is a Banach space $\tilde{\mathcal{B}}$ such that $D(T)$ embeds into $\tilde{\mathcal{B}}$ isometrically, and in a dense way. However, in general it is not clear whether $\tilde{\mathcal{B}}$ can be identified with a subspace of the initial Banach space $\mathcal{B}$. See Example 2.1.11 for a counter-example to this property.

Proposition 2.1.7 (Closed graph theorem) A linear operator T on $\mathcal{B}$ with $D(T) = \mathcal{B}$ is closed if and only if it is bounded.

Proof.— The implication bounded $\Longrightarrow$ closed is obvious. Conversely, let us assume that T is closed. Its graph gr$\,T$ is thus a closed linear subspace of the Banach space $\mathcal{B} \times \mathcal{B}$, hence gr$\,T$ can be viewed itself as a Banach space. Consider the two natural projections $p_1, p_2 : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$; they are obviously continuous linear maps. Their restrictions on gr$\,T \to \mathcal{B}$ are still continuous. In particular, the first projection p_1 : gr$\,T \to \mathcal{B}$ is a continuous bijection. The isomorphism theorem states that the inverse map $q : \mathcal{B} \to$ gr$\,T$ is also a continuous bijection. Finally, the composition $p_2 \circ q : \mathcal{B} \to \mathcal{B}$ is continuous. But note that $p_2 \circ q$ is nothing but T itself.

$$
\begin{array}{ccc}
(u, Tu) & \underset{q}{\overset{p_2}{\leftrightarrows}} & Tu \\
\downarrow p_1 & \swarrow T & \\
u & &
\end{array}
$$

□

We now give some examples of closed unbounded operators.

Example 2.1.8 (Multiplication operator) Take $\mathcal{H} = L^2(\mathbb{R}^d)$ and pick $f \in L^\infty_{\text{loc}}(\mathbb{R}^d)$. Define a linear operator M_f in $\mathcal{H}$ as follows:

$$D(M_f) = \{u \in L^2(\mathbb{R}^d) : fu \in L^2(\mathbb{R}^d)\} \text{ and } \quad M_f u = fu \text{ for } u \in D(M_f).$$

It can be easily seen that $D(M_f)$, equipped with the graph norm $\|\cdot\|'_{M_f}$, coincides with the weighted space $L^2\big(\mathbb{R}^d, (1+|f|^2)dx\big)$, which is a Hilbert space, hence complete. This shows that M_f is closed.

Exercise 2.1.9 For any $p \in [1, \infty[$, one may define a closed multiplication operator M_f on $\mathcal{B} = L^p(\mathbb{R}^d)$ in a similar way.

Using the Fourier transform, we are able to transform multiplication operators on $\mathcal{H} = L^2$ into differential operators. Let us start with the most famous one, the Laplacian on $\mathbb{R}^d$, which will appear many times in those notes.

Example 2.1.10 (Laplacians in $\mathbb{R}^d$) Take $\mathcal{H} = L^2(\mathbb{R}^d)$ and consider two operators in $\mathcal{H}$:

$$\begin{aligned} T_0 u &= -\Delta u, && D(T_0) = C_c^\infty(\mathbb{R}^d), \\ T_1 u &= -\Delta u, && D(T_1) = H^2(\mathbb{R}^d) \quad \text{(second Sobolev space)}. \end{aligned}$$

We are going to show that $\overline{T_0} = T_1$ (this implies that T_1 is closed, while T_0 is not).

For this aim, we will use the Fourier transform to transform the differential operator Δ into a multiplication operator.

When acting on a function $f \in \mathcal{S}(\mathbb{R}^d)$, we have the identity

$$\mathcal{F}\Delta f(\xi) = -|\xi|^2 \mathcal{F} f(\xi), \quad \xi \in \mathbb{R}^d,$$

showing that $-\Delta$ is conjugate to the multiplication operator by $|\xi|^2$.

By duality, the above identity holds as well for distributions $f \in \mathcal{S}'(\mathbb{R}^d)$. But we would like to restrict $-\Delta$ to the Sobolev space $H^2(\mathbb{R}^d)$. How does this space translate on the Fourier side?

$$f \in H^2(\mathbb{R}^d) \Longleftrightarrow \hat{f}, \xi_j \hat{f}, \xi_j \xi_k \hat{f} \in L^2(\mathbb{R}^d), \quad \text{for any indices } j, k.$$

The conditions on the right-hand side can be simplified. Indeed, the bounds:

$$\forall \xi \in \mathbb{R}^d,\ \forall j, k = 1, \ldots, d, \qquad |\xi_j \xi_k| \leq \frac{\xi_j^2 + \xi_k^2}{2} \leq |\xi|^2, \qquad |\xi_j| \leq \frac{1 + \xi_j^2}{2} \leq (1 + |\xi|^2), \tag{2.1.1}$$

imply that

$$\begin{aligned} f \in H^2(\mathbb{R}^d) &\Longleftrightarrow (1 + |\xi|^2)\hat{f} \in L^2(\mathbb{R}^d) \\ &\Longleftrightarrow (1 - \Delta) f \in L^2(\mathbb{R}^d) \\ &\Longleftrightarrow f,\ \Delta f \in L^2(\mathbb{R}^d)\,. \end{aligned} \tag{2.1.2}$$

The first line shows that the operator T_1, with domain $H^2(\mathbb{R}^d)$, is unitarily conjugate through the Fourier transform to the operator $\hat{T}$ defined by

$$D(\hat{T}) = \{g \in L^2 :\, |\xi|^2 g \in L^2\}, \quad \hat{T} g(\xi) = |\xi|^2 g(\xi).$$

In other words, we have the exact conjugacy

$$T_1 = \mathcal{F}^{-1}\, \hat{T}\, \mathcal{F}, \quad D(T_1) = \mathcal{F}^{-1} D(\hat{T}).$$

This conjugacy shows the following relation between the graphs of the two operators:

$$\mathsf{gr}\, T_1 = \{(\mathcal{F}^{-1} u, \mathcal{F}^{-1} \hat{T} u) :\, u \in D(\hat{T})\} = K(\mathsf{gr}\, \hat{T}),$$

where K is the linear operator on $L^2 \times L^2$ defined by $K(u, v) = (\mathcal{F}^{-1} u, \mathcal{F}^{-1} v)$. The unitarity of $\mathcal{F}$ implies that K acts unitarily on $L^2 \times L^2$, in particular it maps closed sets to closed sets.

Now, the example 2.1.8 shows that the multiplication operator $\hat{T}$ is closed on $L^2(\mathbb{R}^d)$, which means that $\mathsf{gr}\, \hat{T}$ is closed in $L^2 \times L^2$. Finally, $\mathsf{gr}\, T_1 = K(\mathsf{gr}\, \hat{T})$ is a closed set too, hence T_1 is closed.

Since T_0 is a restriction of the closed operator T_1, namely $D(T_0) \subset D(T_1)$, it follows that the closure of $D(T_0)$ is contained in the closed subspace $D(T_1)$, which implies that $\overline{D(T_0)}$ is a graph. Hence T_0 is closable, and the domain of its closure $D(\overline{T}_0)$ is the closure of $D(T_0)$ in the graph norm of T_0 (Proposition 2.1.6).

What is this graph norm? The inequalities (2.1.1) show that the standard norm on H^2, expressed through the Fourier conjugacy, reads:

$$\|f\|_{H^2}^2 = \|\hat{f}\|_{L^2}^2 + \sum_j \|\xi_j \hat{f}\|_{L^2}^2 + \sum_{j,k} \|\xi_j \xi_k \hat{f}\|_{L^2}^2,$$

This norm is equivalent with the modified norm

$$\|f\|_{modif}^2 \stackrel{\text{def}}{=} \|\hat{f}\|_{L^2}^2 + \||\xi|^2 \hat{f}\|^2 = \|f\|_{L^2}^2 + \|\Delta f\|_{L^2}^2,$$

namely the graph norm of T_0, so the two norms generate the same topology. The space $D(T_1) = H^2$ is hence complete w.r.to the norm $\|\cdot\|_{modif} = \|\cdot\|_{T_1} \sim \|\cdot\|_{H^2}$ (using the first item of Proposition 2.1.6, this is a second way to prove that T_1 is closed).

Finally, we know that $D(T_0) = C_c^\infty$ is a dense subspace in H^2 (w.r.to the corresponding Sobolev norm), hence its closure in H^2 is the full space $H^2 = D(T_1)$. In conclusion, $D(\overline{T_0}) = H^2 = D(T_1)$, or equivalently $\overline{T_0} = T_1$.

Let us now exhibit a rather simple operator which does NOT admit a closure.

Example 2.1.11 (Non-closable operator) Take $\mathcal{B} = L^p(\mathbb{R})$ for some $p \in [1, \infty[$, and pick a nontrivial function $g \in \mathcal{B}$. Consider the rank-1 operator L defined on $D(L) = C^0(\mathbb{R}) \cap L^p(\mathbb{R})$ by $Lf = f(0)g$. Let us show that this operator is *not* closable.

Choose some nontrivial function $f \in D(L)$. It is easy to construct two sequences $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ in $D(L)$ such that both converge in L^p to f, but with $f_n(0) = 0$ and $g_n(0) = 1$ for all n. Then for all n we have $Lf_n = 0$, while $Lg_n = g$: both sequences Lf_n and Lg_n converge to different limits. This shows that the closure of $\mathrm{gr}\, L$ in $\mathcal{B} \times \mathcal{B}$ is not a graph, since it contains both elements $(f, 0)$ and (f, g). Hence L is not closable.

If we try to complete $D(L)$ w.r.to the graph norm $\|\cdot\|_L$, we will obtain a space $\tilde{\mathcal{B}}$ isometric to $L^p(\mathbb{R}) \times \mathbb{R}$, which takes into account both the limiting function $\lim_n f \in L^p(\mathbb{R})$, and the limiting values $\lim_n f_n(0)$. The space $\tilde{\mathcal{B}}$ is "larger" than $L^p(\mathbb{R})$, since it records the extra information of the value taken by the function at zero.

The next example generalizes the case of the Laplacian, and shows that considering differential operators acting on a domain $\Omega \subsetneq \mathbb{R}^d$ with boundaries makes the analysis more tricky.

2.1.2 Partial differential operators

Let Ω be an open subset of $\mathbb{R}^d$ and $P(x, D_x)$ be a partial differential expression with C^∞ coefficients:

$$P(x, D_x) = \sum_{|\alpha| \le m} a_\alpha(x)\, D^\alpha, \qquad a_\alpha \in C^\infty(\Omega),$$

where we use the notation $D_x = \frac{1}{i}\partial_x$, and $D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_d}^{\alpha_d}$ for multiple derivatives. Choosing as reference space $\mathcal{H} = L^2(\Omega)$, this differential expression defines a linear operator P on the domain $D(P) = C_c^\infty(\Omega)$, $Pu(x) = P(x, D_x)u(x)$. Like in the example of the Laplacian, we try to extend P to some larger subspace of L^2.

The theory of distributions teaches us that, for any $u \in L^2$, the expression $P(x, D_x)u$ makes sense as a well-defined distribution in $\mathcal{D}'(\Omega)$, yet generally this distribution is not in L^2. However, if a sequence $(u_n \in D(P))$ converges to u in L^2, and satisfies $Pu_n \to v$ in L^2, then the two limits hold as well in $\mathcal{D}'$. Because P acts continuously $\mathcal{D}' \to \mathcal{D}'$, the limit v must be equal to the (unique) distribution defined by Pu. Hence, the limit v is independent of the sequence (u_n) converging towards u. This shows that the closure of gr P in $L^2 \times L^2$ is a graph, hence that P is closable. Its closure $\overline{P} = P_{\text{min}}$ is called the minimal closed extension, or *minimal operator*. The above reasoning also shows that gr $\overline{P} = \overline{\text{gr}\, P}$ must be included in the set

$$\big\{(u, f) \in \mathcal{H} \times \mathcal{H} \,:\, P(x, D_x)u = f \text{ in } \mathcal{D}'(\Omega)\big\}. \tag{2.1.3}$$

The above set defines a closed graph in $L^2 \times L^2$, the corresponding operator is called the maximal extension, or the *maximal operator*, and is denoted by P_{max}. Its domain is

$$D(P_{\text{max}}) = \big\{u \in \mathcal{H} :\, P(x, D_x)u \in \mathcal{H}\big\},$$

where, as above, $P(x, D_x)u$ is understood in the sense of distributions.

We have already shown the inclusion $P_{\text{min}} \subset P_{\text{max}}$, and we saw in the Example 2.1.10 of the Laplacian on $\mathbb{R}^d$, that one can have $P_{\text{min}} = P_{\text{max}}$. But one may easily find examples where this equality does not hold.

Example 2.1.12 If we take $P(x, D_x) = d/dx$ and $\Omega = \mathbb{R}_+^*$, with domain $D(P) = C_c^\infty(\mathbb{R}_+^*)$, we find for the minimal closed extension

$$D(P_{\text{min}}) = \overline{C_c^\infty(\mathbb{R}_+^*)}^{H_1} = H_0^1(\mathbb{R}_+^*),$$

(the space of H^1 functions vanishing at $x = 0$), since the graph norm $\|\cdot\|_P$ is equivalent with the H^1 norm. On the other hand,

$$D(P_{\text{max}}) = \{u \in L^2(\mathbb{R}_+^*),\ u' \in L^2(\mathbb{R}_+^*)\} = H^1(\mathbb{R}_+^*)$$

(with no condition at $x = 0$).

In general, one may expect that $P_{\text{min}} \subsetneqq P_{\text{max}}$ if Ω has a boundary.

Such questions become more involved if one studies partial differential operators with more singular coefficients (e.g. with coefficients which are not smooth but just belong to some L^p), since one cannot easily define their action on distributions. During the course, we will nevertheless deal with certain classes of such operators (one easy case is the multiplication operator by an L^∞_{loc} function of Example 2.1.8).

In the next section, we restrict ourselves to operators P defined on a *Hilbert space*. In this framework, we will define and study the *adjoint operator* of P; we will see that the very definition of the adjoint is not obvious, in cases where P is unbounded on $\mathcal{H}$.

Note that adjoints can also be defined on Banach space $\mathcal{B}$, yet the adjoint operator then acts on the dual space $\mathcal{B}'$, which is generally different from $\mathcal{B}$. We will not describe this situation in those notes.

2.2 Adjoint of an operator on a Hilbert space

In this section all operators are defined on a Hilbert space $\mathcal{H}$.

2.2.1 Adjoint of a continuous operator

For a continuous operator $T \in \mathcal{L}(\mathcal{H})$, its adjoint T^* is defined by the identity

(2.2.4) $$\langle u, Tv\rangle = \langle T^* u, v\rangle \text{ for all } u, v \in \mathcal{H}.$$

The fact that these identities uniquely define the operator T^* comes from the Riesz representation theorem: for each $u \in \mathcal{H}$ the map $\mathcal{H} \ni v \mapsto \langle u, Tv\rangle \in \mathbb{C}$ is a *continuous* linear functional; the Riesz theorem states that there exists a unique vector $w \in \mathcal{H}$ such that $\langle u, Tv\rangle = \langle w, v\rangle$ for all $v \in \mathcal{H}$. One can then easily check that the map $u \mapsto w$ is linear, and by estimating the above scalar product with $v = w$, one finds that this map is bounded:

$$\langle w, w\rangle = \langle u, Tw\rangle \Longrightarrow \|w\|^2 \leq \|u\|\|T\|\|w\| \Longrightarrow \|w\| \leq \|T\|\|u\|$$

We may hence denote this map by: $w = T^* u$, thus defining the operator T^*. The above bound shows that $\|T^*\| \leq \|T\|$. Actually, the symmetry of (2.2.4) shows that $(T^*)^* = T$, hence we actually have $\|T^*\| = \|T\|$.

2.2.2 Adjoint of an unbounded operator

Let us try to generalize this construction for an unbounded operator T. As we will see, the main difficulty consists in properly defining the domain of T^*.

Definition 2.2.1 (Adjoint operator) Let $(T, D(T))$ be a linear operator in $\mathcal{H}$, with $D(T)$ dense in $\mathcal{H}$. We then define its *adjoint operator* $(T^*, D(T^*))$ as follows.

The domain $D(T^*)$ consists of the vectors $u \in \mathcal{H}$ for which the map $D(T) \ni v \mapsto \langle u, Tv\rangle \in \mathbb{C}$ is a bounded linear form on $\mathcal{H}$. For such u there exists, by the Riesz theorem, a unique vector (which we denote by $T^* u$) such that $\langle u, Tv\rangle = \langle T^* u, v\rangle$ for all $v \in D(T)$.

We notice that our assumption of a *dense* domain $\overline{D(T)} = \mathcal{H}$ is crucial here: if it is not satisfied, then there are several ways to extend the linear form defined on $D(T)$, into a bounded linear form on $\mathcal{H}$. Equivalently, the vector $T^* u$ is not uniquely determined, since one can add to $T^* u$ an arbitrary vector in $D(T)^\perp$. Hence, when we mention the adjoint of an operator T, we always implicitly assume that $D(T)$ is dense.

Let us give a geometric interpretation of the adjoint operator. Consider the linear "$-\pi/2$ rotation" operator

$$J : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}, \quad J(u, v) = (v, -u)\,.$$

We notice that J commutes with taking the orthogonal complement in $\mathcal{H} \times \mathcal{H}$: for any subset $V \subset \mathcal{H} \times \mathcal{H}$, $J(V)^\perp = J(V^\perp)$.

Proposition 2.2.2 (Geometric interpretation of the adjoint) Let T be a linear operator in $\mathcal{H}$, with dense domain $D(T)$. Then the graph of the adjoint operator T^* is given by:

$$\operatorname{gr} T^* = J(\operatorname{gr} T)^\perp = J\big((\operatorname{gr} T)^\perp\big). \tag{2.2.5}$$

Proof.— By definition, $u \in D(T^*)$ iff there exists a vector T^*u such that, for any $v \in D(T)$,

$$\begin{aligned} 0 &= \langle u, Tv\rangle_{\mathcal{H}} - \langle T^*u, v\rangle_{\mathcal{H}} \\ &= \big\langle (u, T^*u), (Tv, -v)\big\rangle_{\mathcal{H}\times\mathcal{H}} \\ &= \big\langle (u, T^*u), J(v, Tv)\big\rangle_{\mathcal{H}\times\mathcal{H}}\,. \end{aligned} \tag{2.2.6}$$

Equivalently, $u \in D(T^*)$ iff there exists $T^*u \in \mathcal{H}$ such that (u, T^*u) is orthogonal to the subspace $J(\operatorname{gr} T)$. Hence, the set of admissible pairs (u, T^*u) is given by the orthogonal complement to $J(\operatorname{gr} T)$. We know that these pairs form a graph (to each admissible u corresponds a unique T^*u). We finally get the required identify $\operatorname{gr} T^* = J(\operatorname{gr} T)^\perp$. □

A byproduct of the equalities (2.2.6) is the identity

$$\operatorname{Ker} T^* = \big(\operatorname{Ran} T\big)^\perp\,.$$

As a simple application we obtain

Proposition 2.2.3 $i)$ The adjoint T^* is a closed operator.

$ii)$ If T is closable, then $T^* = (\overline{T})^*$.

Proof.— In (2.2.5), we remember that the orthogonal complement of a subspace is always a *closed* subspace, so $\operatorname{gr} T^*$ is closed, meaning that T^* is a closed operator.

Besides, the map J is continuous, and the complement of a subspace is equal to the complement of its closure, so

$$J(\operatorname{gr} T)^\perp = \overline{J(\operatorname{gr} T)}^\perp = J(\overline{\operatorname{gr} T})^\perp = J(\operatorname{gr} \overline{T})^\perp,$$

which proves the second item. □

So far, we do not know if the domain of the adjoint operator could be nontrivial. This is discussed in the following proposition.

Proposition 2.2.4 (Domain of the adjoint) Let $(T, D(T))$ be a closable operator on $\mathcal{H}$, with dense domain. Then
$i)$ $D(T^*)$ is a dense subspace of $\mathcal{H}$;
$ii)$ $T^{**} \stackrel{\text{def}}{=} (T^*)^* = \overline{T}$.

Proof.— The item $ii)$ easily follows from $i)$ and Eq. (2.2.5): one remarks that $J^2 = -1$, and that taking twice the orthogonal complement results in taking the closure of the graph, hence gr$\,\overline{T}$.

Now let us prove $i)$. Assume the opposite conclusion, namely that some nonzero vector $w \in \mathcal{H}$ is orthogonal to $D(T^*)$: $\langle u, w\rangle = 0$ for all $u \in D(T^*)$. Then for all $u \in D(T^*)$ one has

$$\langle J(u, T^*u), (0, w)\rangle_{\mathcal{H}\times\mathcal{H}} = \langle u, w\rangle + \langle T^*u, 0\rangle = 0,$$

which means that $(0, w) \in J(\text{gr}\, T^*)^\perp = \overline{\text{gr}\, T}$. Since the operator T is closable, the closure $\overline{\text{gr}\, T}$ must be a graph, which imposes $w = 0$, so we have a contradiction. □

Remark 2.2.5 In the above Proposition, the closability of T is a necessary assumption. Indeed, let us come back to the Example 2.1.11 of the nonclosable operator L. The adjoint of this operator has for domain $D(L^*) = \{g\}^\perp$, a closed subspace of codimension 1, hence not dense on L^2. Note that the operator L^* vanishes on this domain.

Let us consider some examples of adjoints of closable operators.

Example 2.2.6 (Adjoint of a bounded operator) The general definition (2.2.1) for the adjoint operator is compatible with the definition of the adjoint of a continuous linear operators given in section 2.2.1: in case T is bounded and $D(T) = \mathcal{H}$, the domain of the adjoint is $D(T^*) = \mathcal{H}$, and the relation $\langle u, Tv\rangle = \langle T^*u, v\rangle$ for all $u, v \in \mathcal{H}$ fully defines T^*.

Example 2.2.7 (Laplacian on $\mathbb{R}^d$) Let us consider again the operators T_0 and T_1 from Example 2.1.10, and show that $T_0^* = T_1$.

By definition, the domain $D(T_0^*)$ consists of the functions $u \in L^2(\mathbb{R}^d)$ for which there exists a vector $f \in L^2(\mathbb{R}^d)$ such that

$$\forall v \in D(T_0) = C_c^\infty(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} \overline{u(x)}(-\Delta v)(x)dx = \int_{\mathbb{R}^d} \overline{f(x)} v(x)dx\,.$$

This equation exactly means that $f = -\Delta u$ as a distribution in $\mathcal{D}'(\mathbb{R}^d)$. Therefore, $D(T_0^*)$ consists of the functions $u \in L^2$ such that the distribution $-\Delta u$ is actually in L^2. The identities (2.1.2) showed that this is exactly the space $H^2(\mathbb{R}^d) = D(T_1)$. So $D(T_0^*) = D(T_1)$, and the two operators both act by $u \mapsto -\Delta u$, they are thus identical.

Let us come back to the simplest differential operator, which appeared in Example 2.1.12.

Example 2.2.8 Consider the operator A_0 acting through $A_0 u = D_x u \stackrel{\text{def}}{=} -i\partial_x u$ for $u \in D(A_0) = C_c^\infty(]0,1[)$. In Example 2.1.12 we showed that the "minimal operator" $\overline{A_0}$ admits the domain $D(\overline{A_0}) = H_0^1(]0,1[)$.

Let us show that the adjoint A_0^* admits the larger domain $D(A_0^*) = H^1(]0,1[)$. Indeed, if $v \in C_c^\infty(]0,1[)$, the equation

$$\langle u, D_x v\rangle = \langle D_x u, v\rangle$$

holds for any $u \in C^\infty(]0,1[)$ thanks to an integration by parts, and the resulting linear form in v can be continuously extended to all of $v \in L^2$, as long as $D_x u \in L^2$, hence as long as $u \in H^1(]0,1[)$.

To anticipate the Definition 2.2.10 below, the operator A_0 is symmetric, but not essentially selfadjoint, since $\overline{A_0} \subsetneqq A_0^*$. Equivalently, the non-inclusion $A_0^* \not\subset A_0^{**} = \overline{A_0}$ shows that the operator A_0^* is *not* symmetric.

Exercise 2.2.9 Remember the multiplication operator M_f from Example 2.1.8, for a complex valued function f. Show that $(M_f)^* = M_{\bar{f}}$.

2.2.3 Symmetric and Selfajdoint operators

The following definition introduces classes of linear operators defined on a Hilbert space, which will be studied intensively in this course.

Definition 2.2.10 (Symmetric, self-adjoint, essentially self-adjoint ops) An operator $(T, D(T))$ on a Hilbert space is said to be *symmetric* (or *Hermitian*) if

$$\langle u, Tv\rangle = \langle Tu, v\rangle \quad \text{for all } u, v \in D(T).$$

Equivalently, T is symmetric iff $T \subset T^*$ (that is, T^* is an extension of T).

- T is called *selfadjoint* if $T = T^*$ (in particular, $D(T) = D(T^*)$)
- T is called *essentially selfadjoint* if T is closable and $\overline{T}$ is self-adjoint: $\overline{T} = (\overline{T})^* = T^*$.

An important feature of symmetric operators is their closability:

Proposition 2.2.11 A symmetric operator $(T, D(T))$ is necessarily closable.

Proof.— Indeed, for a symmetric operator T we have $\operatorname{gr} T \subset \operatorname{gr} T^*$ and, due to the closedness of T^*, $\overline{\operatorname{gr} T} \subset \operatorname{gr} T^*$ is a graph, the graph of the closure $\overline{T}$. $\square$

Example 2.2.12 (Free Laplacian on $\mathbb{R}^d$) The Laplacian T_1 from Example 2.1.10 is selfadjoint. Indeed, we have shown in Ex. 2.2.7 that $T_0^* = T_1$, hence $T_1^* = T_0^{**} = \overline{T_0} = T_1$, where the last equality uses Ex. 2.1.10. This shows that T_1 is selfadjoint, while its restriction T_0 is essentially selfadjoint.

The operator T_1 is called the *free Laplacian* on $\mathbb{R}^d$.

Example 2.2.13 (Continuous symmetric operators are self-adjoint) For $T \in \mathcal{L}(\mathcal{H})$, being symmetric is equivalent to being selfadjoint, since the domains of T and T^* are both the full space $\mathcal{H}$.

For unbounded operators, proving the symmetry is often easy, but proving selfadjointness requires a precise identification of the domains, which may be quite difficult in general. This is a reason why, in the next section, we will appeal to quadratic forms to construct selfadjoints operators.

Example 2.2.14 (Self-adjoint multiplication operators) As follows from example 2.2.9, the multiplication operator M_f on $L^2(\mathbb{R}^d)$ from example 2.1.8 is self-adjoint iff $f(x) \in \mathbb{R}$ for a.e. $x \in \mathbb{R}^d$.

The following proposition will allow to construct a large class of self-adjoint operators.

Proposition 2.2.15 Let T be an *injective* selfadjoint operator, then its inverse T^{-1} is also selfadjoint (notice that the inverse may be unbounded).

Proof.— We show first that $D(T^{-1}) \stackrel{\text{def}}{=} \operatorname{Ran} T$ is dense in $\mathcal{H}$. Let $u \perp \operatorname{Ran} T$, then $\langle u, Tv\rangle = 0$ for all $v \in D(T)$. This can be rewritten as $\langle u, Tv\rangle = \langle 0, v\rangle$ for all $v \in D(T)$, which shows that $u \in D(T^*)$, with image $T^*u = 0$. Since by assumption $T^* = T$, we have $u \in D(T)$ and $Tu = 0$. Since T in injective, the vector u must be trivial. Hence $\operatorname{Ran} T$ is dense.

Now consider the "switch operator" $S : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ given by $S(u, v) = (v, u)$. One has then $\operatorname{gr} T^{-1} = S(\operatorname{gr} T)$. We conclude the proof by noting that S commutes with the operation of the orthogonal complement in $\mathcal{H} \times \mathcal{H}$ and anticommutes with J. From the assumption $\operatorname{gr} T = \operatorname{gr} T^* = J(\operatorname{gr} T)^{\perp}$, we draw:

$$
\begin{aligned}
\operatorname{gr} T^{-1} &\stackrel{\text{def}}{=} S(\operatorname{gr} T) \stackrel{ass.}{=} S(\operatorname{gr} T^*) = S\big(J(\operatorname{gr} T)^{\perp}\big) \\
&= -JS\big((\operatorname{gr} T)^{\perp}\big) = J\big(S \operatorname{gr} T\big)^{\perp} J(\operatorname{gr} T^{-1})^{\perp} = \operatorname{gr}(T^{-1})^*.
\end{aligned}
$$

□

It is in general rather easy to check that an operator is symmetric (for differential operators, this fact often involves some form of integration by parts). There exist explicit criteria to check (essential) selfadjointness.

Proposition 2.2.16 Assume the operator $(T, D(T))$ is symmetric on the Hilbert space $\mathcal{H}$. Then the following properties are equivalent:

$i)$ $(T, D(T))$ is essentially selfadjoint (selfadjoint);

$ii)$ $\mathrm{Ker}(T^* + i) = \mathrm{Ker}(T^* - i) = \{0\}$ (and furthermore $(T, D(T))$ is closed);

$iii)$ $\mathrm{Ran}(T + i) = \mathrm{Ran}(T - i)$ is dense in $\mathcal{H}$ (is equal to $\mathcal{H}$).

Proof.— We will give the proofs for the selfadjoint case only, the small adaptations necessary for the essentially selfadjoint case being left to the reader.

$i) \implies ii)$: easy.

$ii) \implies iii)$: we have $0 = \mathrm{Ker}(T^* \pm i) = \mathrm{Ran}(T \mp i)^\perp$, which shows that $\mathrm{Ran}(T \pm i)$ is dense. Assuming the closedness of T, we want to show the closedness of $\mathrm{Ran}(T \pm i)$. For this, we use "Pythagore's theorem":

$$\|(T + i)u\|^2 = \langle (T + i)u, (T + i)u\rangle = \langle Tu, Tu\rangle + \langle u, u\rangle$$

Assume a sequence $(u_n \in D(T))$ is such that $(T + i)u_n$ is Cauchy. The sequence $((T + i)u_n)$ is Cauchy, hence the above equality shows that so are (u_n) and (Tu_n). The closedness of A then implies that $u_n \to u$ and $Tu_n \to Tu$, hence $(T + i)u_n \to (T + i)u \in \mathrm{Ran}(T + i)$. As a result, $\mathrm{Ran}(T + i)$ is closed, and is equal to $\mathcal{H}$. The proof for $\mathrm{Ran}(T - i)$ is identical.

$iii) \implies i)$ The symmetry means that $T \subset T^*$, and we want to show the inverse inclusion $T^* \subset T$.

Take $v \in D(T^*)$, so that $(T^* + i)v \in \mathcal{H}$. From the assumption that $\mathrm{Ran}(T + i) = \mathcal{H}$, there exists $u \in D(T)$ such that $(T^*+i)v = (T+i)u$; since $T \subset T^*$, this identity also reads $(T^*+i)v = (T^*+i)u$, hence $v - u \in \mathrm{Ker}(T^* + i) = \mathrm{Ran}(T - i)^\perp$. The assumption $\mathrm{Ran}(T - i) = \mathcal{H}$ shows that $u = v$, so that $D(T^*) \subset D(T)$. □

Remark 2.2.17 (Why focus on selfadjoint operators?) As mentioned in the introduction, selfadjoint operators lie at the heart of quantum mechanics, not just in as Hamiltonians generating the quantum evolution, but also as *quantum observables*, selfadjoint operators reprensenting the quantities which can (in theory) be measured in an experiment.

Mathematically, selfadjoint operators enjoy a very special spectral structure: we will establish the *spectral theorem* for selfadjoint operators, which provides a general description of these operators, in terms of their spectral measure. From this theorem we will also construct a *functional calculus* for selfadjoint operators, that is define operators of the form $f(T)$, where T is selfadjoint, and $f : \mathbb{R} \to \mathbb{C}$ is an arbitrary function.

2.3 Exercises

Exercise 2.3.1 (a) Let H_1 and H_2 be Hilbert spaces. Let A be a linear operator in H_1, B be a linear operator in H_2. Assume that there exists a unitary operator $U : H_2 \to H_1$ such that $D(A) = UD(B)$ and that $U^*AUf = Bf$ for all $f \in D(B)$; such A and B are called *unitary equivalent*.

Let two operators A and B be unitarily equivalent. Show that A is closed/symmetric/self-adjoint iff B has the same property.

(b) Let (λ_n) be an arbitrary sequence of complex numbers, $n \in \mathbb{N}$. In the Hilbert space $\ell^2(\mathbb{N})$ consider the operator S:

$$D(S) = \big\{(x_n) : \text{there exists } N \text{ such that } x_n = 0 \text{ for } n > N\big\}, \quad S(x_n) = (\lambda_n x_n).$$

Describe the closure of S.

(c) Now let H be a separable Hilbert space and T be a linear operator in H with the following property there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H with $e_n \in D(T)$ and $Te_n = \lambda_n e_n$ for all $n \in \mathbb{N}$, where λ_n are some complex numbers.

i) Describe the closure $\overline{T}$ of T. Hint: one may use (a) and (b).

ii) Describe the adjoint T^* of T.

iii) Let all λ_n be real. Show that the operator $\overline{T}$ is self-adjoint.

Exercise 2.3.2 Let A and B be self-adjoint operators in a Hilbert space H such that $D(A) \subset D(B)$ and $Au = Bu$ for all $u \in D(A)$. Show that $D(A) = D(B)$. (This property is called the *maximality* of self-adjoint operators.)

Exercise 2.3.3 We consider a linear operator A on a Hilbert space $\mathcal{H}$, and a *continuous* operator B on the same space; we define their sum $A + B$ as the operator S with domain $D(S) = D(A)$, such that $Su \stackrel{\text{def}}{=} Au + Bu$ for each $u \in D(S)$. (We note that defining the sum of two unbounded operators is a nontrivial task in general, due to questions of domains.)

(a) Assumed A is a closed operator and B is continuous. Show that $A + B$ is closed.

(b) Assume, in addition, that A is densely defined. Show that $(A + B)^* = A^* + B^*$ (here the sum $A^* + B^*$ is defined similarly as $A + B$).

Exercise 2.3.4 Let $\mathcal{H} = L^2(]0, 1[)$. For $\alpha \in \mathbb{C}$, consider the operator T_α acting as $T_\alpha f = if'$ on the domain

$$D(T_\alpha) = \Big\{f \in C^\infty\big([0, 1]\big) : f(1) = \alpha f(0)\Big\}.$$

(a) Describe the adjoint of T_α.

(b) Describe the closure $S_\alpha \stackrel{\text{def}}{=} \overline{T_\alpha}$.

(c) Find all α for which S_α is selfadjoint.