THE TOPOLOGICAL STRUCTURE OF SCALING LIMITS OF LARGE PLANAR MAPS

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Abstract

We discuss scaling limits of large bipartite planar maps. If $p \geq 2$ is a fixed integer, we consider, for every integer $n \geq 2$, a random planar map M_n which is uniformly distributed over the set of all rooted $2p$-angulations with n faces. Then, at least along a suitable subsequence, the metric space consisting of the set of vertices of M_n, equipped with the graph distance rescaled by the factor $n^{-1/4}$, converges in distribution as $n \to \infty$ towards a limiting random compact metric space, in the sense of the Gromov-Hausdorff distance. We prove that the topology of the limiting space is uniquely determined independently of p and of the subsequence, and that this space can be obtained as the quotient of the Continuum Random Tree for an equivalence relation which is defined from Brownian labels attached to the vertices. We also verify that the Hausdorff dimension of the limit is almost surely equal to 4.

1 Introduction

The main purpose of the present work is to investigate continuous limits of rescaled planar maps. We concentrate on bipartite planar maps, which are known to be in one-to-one correspondence with certain labeled trees called mobiles (Bouttier, Di Francesco, Guitter [8]). In view of the correspondence between maps and mobiles, it seems plausible that scaling limits of large bipartite planar maps can be described in terms of continuous random trees. This idea already appeared in the pioneering work of Chassaing and Schaeffer [12], and was then developed by Marckert and Mokkadem [27], who defined and studied the so-called Brownian map. It was argued in [27] that the Brownian map is in some weak sense the limit of rescaled uniformly distributed random quadrangulations of the plane (see also Marckert and Miermont [26] for recent work along the same lines). The point of view of the present paper is however different from the one in [27] or in [26]. For every given planar map M, we equip the set $\mathbf{m}$ of its

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vertices with the graph distance, and our aim is to study the resulting compact metric space when the number of faces of the map tends to infinity. Assuming that the map M is chosen uniformly over the set of all rooted $2p$-angulations with n faces, we discuss the convergence in distribution when n tends to infinity of the associated random metric spaces, rescaled with the factor $n^{-1/4}$, in the sense of the Gromov-Hausdorff distance between compact metric spaces (see e.g. Chapter 7 of [10], or subsection 2.3 below, for the definition of the Gromov-Hausdorff distance). This is in contrast with [27], where the continuous limit of rescaled discrete maps is studied in the sense of the convergence of the coding functions rather than of the maps themselves (cf the definition of the distance between two "abstract maps" on page 2171 of [27]). We believe that the Gromov-Hausdorff convergence is more natural and better suited to applications to asymptotic properties of finite maps (see Corollary 1.2 of [24] for such an application).

Before we describe our main results in a more precise way, we need to set some definitions. Recall that a planar map is a proper embedding, without edge crossings, of a connected graph in the two-dimensional sphere. Loops and multiple edges are a priori allowed. The faces of the map are the connected components of the complement of the union of edges. A planar map is rooted if it has a distinguished oriented edge called the root edge, whose origin is called the root vertex. The set of vertices will always be equipped with the graph distance: If a and a' are two vertices, $d_{gr}(a, a')$ is the minimal number of edges on a path from a to a'. Two rooted planar maps are said to be equivalent if the second one is the image of the first one under an orientation-preserving homeomorphism of the sphere, which also preserves the root edges. From now on we deal only with equivalence classes of rooted planar maps. Given an integer $p \geq 2$, a $2p$-angulation is a planar map where each face has degree $2p$, that is $2p$ adjacent edges (one should count edge sides, so that if an edge lies entirely inside a face it is counted twice). We denote by $\mathcal{M}_n^p$ the set of all rooted $2p$-angulations with n faces.

Let us now discuss the continuous trees that will arise in scaling limits of planar maps. We write $\mathcal{T}_{\mathbf{e}}$ for the continuum random tree or CRT, which was introduced and studied by Aldous [2], [3]. The CRT can be viewed as a random variable taking values in the space of all rooted compact real trees (see e.g. [22], or subsection 2.3 below). It turns out that the CRT is the limit in distribution of several (suitably rescaled) classes of discrete trees when the number of edges tends to infinity. For instance, it is relatively easy to show that if τ_n is distributed uniformly over the set of all plane trees with n edges, then the vertex set of τ_n, viewed as a metric space for the graph distance rescaled by the factor $(2n)^{-1/2}$, will converge in distribution to the CRT as $n \to \infty$, in the sense of the Gromov-Hausdorff distance. Our notation $\mathcal{T}_{\mathbf{e}}$ reflects the fact that the CRT can be defined as the real tree coded by a normalized Brownian excursion $\mathbf{e} = (\mathbf{e}_t)_{0 \leq t \leq 1}$. This coding, which plays a major role in the present work, is recalled in subsection 2.3 below. In addition to the usual genealogical order of the tree, the CRT $\mathcal{T}_{\mathbf{e}}$ inherits a lexicographical order from the coding, in a way analogous to the ordering of (discrete) plane trees from the left to the right. We write $d_{\mathbf{e}}$ for the distance on the tree $\mathcal{T}_{\mathbf{e}}$ and ρ for the root of $\mathcal{T}_{\mathbf{e}}$.

We can assign Brownian labels to the vertices of the CRT. This means that given $\mathcal{T}_{\mathbf{e}}$, we consider a centered Gaussian process $(Z_a)_{a \in \mathcal{T}_{\mathbf{e}}}$, such that $Z_\rho = 0$ and the variance of $Z_a - Z_b$ is equal to $d_{\mathbf{e}}(a, b)$ for every $a, b \in \mathcal{T}_{\mathbf{e}}$. The pair $(\mathcal{T}_{\mathbf{e}}, (Z_a)_{a \in \mathcal{T}_{\mathbf{e}}})$ is the probabilistic object that allows us to describe the continuous limit of random planar maps. We use the Brownian labels to define a mapping D° from $\mathcal{T}_{\mathbf{e}} \times \mathcal{T}_{\mathbf{e}}$ into $\mathbb{R}_+$, via the formula

$$D^\circ(a,b) = Z_a + Z_b - 2 \inf_{c\in[a,b]} Z_c$$

where $[a,b]$ denotes the "lexicographical" interval between a and b. The preceding definition is a little informal, since there are two lexicographical intervals between a and b, corresponding to the two possible ways of going from a to b around the tree. It should be understood that we take the lexicographical interval that minimizes the value of $D^\circ(a,b)$ as defined above (see Section 3 below for a more rigorous presentation). The intuition behind the definition of D° comes from the discrete picture where each (bipartite) planar map is coded by a labeled tree, in such a way that vertices of the map other than the root are in one-to-one correspondence with vertices of the tree ([8], see subsection 2.1 below). From the properties of this coding, and more precisely from the way edges of the map are reconstructed from the labels in the tree, one sees that any two vertices a and b that satisfy a discrete version of the relation $D^\circ(a,b) = 0$ will be connected by an edge of the map. See subsection 2.1 for more details.

The function D° does not satisfy the triangle inequality, but we may set

$$D^*(a,b) = \inf \left\{ \sum_{i=1}^{q} D^\circ(a_{i-1}, a_i) \right\}$$

where the infimum is over all choices of the integer $q \geq 1$ and of the finite sequence $a_0, a_1, \ldots, a_q$ in $\mathcal{T}_{\mathbf{e}}$ such that $a_0 = a$ and $a_q = b$. We then define an equivalence relation on $\mathcal{T}_{\mathbf{e}}$ by setting $a \approx b$ if and only if $D^*(a,b) = 0$. Although this is not obvious, it turns out that the latter condition is equivalent to $D^\circ(a,b) = 0$, outside a set of probability zero. Moreover one can check that equivalence classes for $\approx$ contain 1, 2 or at most 3 points, almost surely. The quotient space $\mathcal{T}_{\mathbf{e}} / \approx$ equipped with the metric D^* is a compact metric space, and it is easily seen that its topology coincides with the quotient topology on $\mathcal{T}_{\mathbf{e}} / \approx$.

Let us now come to our main results. For every integer $n \geq 2$, let M_n be a random rooted $2p$-angulation uniformly distributed over $\mathcal{M}_n^p$. Denote by $\mathbf{m}_n$ the set of vertices of M_n and by d_n the graph distance on $\mathbf{m}_n$. We view $(\mathbf{m}_n, d_n)$ as a random variable taking values in the space of isometry classes of compact metric spaces. Recall that the latter space equipped with the Gromov-Hausdorff distance is a Polish space, as a simple consequence of Gromov's compactness theorem ([10], Theorem 7.4.15). It can be checked that the sequence of the laws of $(\mathbf{m}_n, n^{-1/4} d_n)$ is tight, and so, at least along a subsequence, we may assume that $(\mathbf{m}_n, n^{-1/4} d_n)$ converges in distribution towards a certain random compact metric space. The principal contribution of the present work is to show that this limiting (random) compact metric space is homeomorphic to the quotient $\mathcal{T}_{\mathbf{e}}/\approx$ equipped with the metric D^*.

Precisely, our main result (Theorem 3.4) can be stated as follows. From any sequence of integers converging to $+\infty$, we can extract a subsequence and for every n belonging to this subsequence we can construct a random $2p$-angulation M_n that is uniformly distributed over $\mathcal{M}_n^p$, in such a way that we have the almost sure convergence

$$\left(\mathbf{m}_n, \Big(\frac{9}{4p(p-1)}\Big)^{1/4} n^{-1/4} d_n\right) \underset{n\to\infty}{\longrightarrow} (\mathcal{T}_{\mathbf{e}} / \approx, D) \tag{1}$$

in the sense of the Gromov-Hausdorff distance. Here D is a (random) metric on the quotient space $\mathcal{T}_{\mathbf{e}} / \approx$, such that $D(a,b) \leq D^*(a,b)$ for every a, b. The random metric D may a priori depend on the choice of the subsequence and on the value of p. However, since $\mathcal{T}_{\mathbf{e}} / \approx$ equipped with the metric D^* is compact and $D \leq D^*$, a standard argument shows that the metric spaces $(\mathcal{T}_{\mathbf{e}} / \approx, D)$ and $(\mathcal{T}_{\mathbf{e}} / \approx, D^*)$ are homeomorphic, so that the topological structure of the limit in (1) is uniquely determined as the quotient topology on $\mathcal{T}_{\mathbf{e}} / \approx$. In the companion paper [24], we prove that $(\mathcal{T}_{\mathbf{e}} / \approx, D^*)$, is a.s. homeomorphic to the sphere S^2. Combining this result with the present work, we obtain that any weak limit (in the sense of the Gromov-Hausdorff distance) of rescaled uniform $2p$-angulations with n faces is a.s. homeomorphic to S^2.

We conjecture that $D = D^*$, and then the convergence (1) would not require the use of a subsequence, and the limit would not depend on p (the constant $(9/(4p(p-1))^{1/4}$ in (1) is relevant mainly because we expect the limit to be independent of p). Although we are not able to prove this, we can derive enough information about the limiting metric space in (1) to prove that its Hausdorff dimension is equal to 4 almost surely (Theorem 6.1).

Let us briefly comment on the proof of our main result. The compactness argument that we use to get the existence of a limit in (1) along a suitable subsequence also shows that this limit can be written as a quotient of the CRT $\mathcal{T}_{\mathbf{e}}$ corresponding to a certain random pseudo-metric D. The point is then to check that $a \approx b$ holds if and only if $D(a, b) = 0$. In other words, the points of the CRT that we need to identify in order to get the limit in (1) are given by the equivalence relation $\approx$, which is defined in terms of D^* or of D°. Once we know that $D \leq D^*$, it is obvious that $a \approx b$ implies $D(a, b) = 0$. The hard core of the proof is thus to check the reverse implication. The above-mentioned interpretation of the condition $D^\circ(a, b) = 0$ in the discrete setting makes it clear that any two points satisfying this condition must be identified. However, other pairs of points could conceivably have been identified. Roughly speaking, the proof that this is not the case proceeds as follows. Given a and b in $\mathcal{T}_{\mathbf{e}}$, we can construct corresponding vertices a_n and b_n in M_n such that the sequence (a_n) converges to a and the sequence (b_n) converges to b, in some suitable sense. The condition $D(a, b) = 0$ entails that $d_n(a_n, b_n) = o(n^{1/4})$ as $n \to \infty$. We can then use this estimate together with some combinatorial considerations and certain delicate properties of the "Brownian tree" $(\mathcal{T}_{\mathbf{e}}, (Z_a)_{a \in \mathcal{T}_{\mathbf{e}}})$, in order to conclude that we must have $D^\circ(a, b) = 0$.

Let us discuss previous work related to the subject of the present article. Planar maps were first studied by Tutte [30] in connection with his work on the four colors theorem. Because of their relations with Feynman diagrams, planar maps soon attracted the attention of specialists of theoretical physics. The pioneering papers [18] and [9] related enumeration problems for planar maps with asymptotics of matrix integrals. The interest for random planar maps in theoretical physics grew significantly when these combinatorial objects were interpreted as models of random surfaces, especially in the setting of the theory of quantum gravity (see in particular [14] and the book [4]). On the other hand, the idea of coding planar maps with simpler combinatorial objects such as labeled trees appeared in Cori and Vauquelin [13] and was much developed in Schaeffer's thesis [29]. In the present work, we use a version of the bijections between maps and trees that was obtained in the recent paper of Bouttier, Di Francesco and Guitter [8]. See Bouttier's thesis [7] and the references therein for applications of these bijections to the statistical physics of random surfaces. Other applications in the spirit of the present work can be found in the recent papers [12], [26] and [27] that were mentioned earlier. Note in particular that the random metric space $(\mathcal{T}_{\mathbf{e}} / \approx, D^*)$ that is discussed above is essentially equivalent to the Brownian map of [27], although the presentation there is different.

See also [5], [6], [11] and [20] for various results about random infinite planar triangulations and quadrangulations and their asymptotic properties.

The paper is organized as follows. Section 2 gives a number of preliminaries concerning bijections between maps and trees, the coding of real trees and the construction of the Brownian tree $(\mathcal{T}_{\mathbf{e}}, (Z_a)_{a\in\mathcal{T}_{\mathbf{e}}})$. We also state three important lemmas about the Brownian tree. Section 3 contains our main results. The presentation is slightly different (although equivalent) from the one that is given above, because we prefer to argue with the tree $\mathcal{T}_{\mathbf{e}}$ re-rooted at the vertex with the minimal label, and the labels Z_a shifted accordingly so that the label of the root is still zero. Indeed, it is the genealogical structure of this re-rooted tree that plays a major role in our approach. Section 4 is devoted to the main step of our arguments, that is the proof that $D(a,b)=0$ implies $D^\circ(a,b)=0$. Section 5 gives the proof of three technical lemmas that were stated in Section 2. The proofs of these lemmas depend on some rather intricate properties of Brownian trees, which we found convenient to derive using the path-valued process called the Brownian snake [21]. In order to make most of the paper accessible to the reader who is unfamiliar with the Brownian snake, we have preferred to postpone these proofs to Section 5. At last, Section 6 contains the calculation of the Hausdorff dimension of the limiting metric space.

As a final remark, it is very plausible that our results can be extended to the more general setting of Boltzmann distributions on bipartite maps, which is considered in [26] and in [31]. We have chosen to concentrate on the particular case of uniform $2p$-angulations for the sake of simplicity and to keep the present work to a reasonable size.

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2 Preliminaries

2.1 Planar maps and the Bouttier-Di Francesco-Guitter bijection

Recall that we have fixed an integer $p \geq 2$ and that $\mathcal{M}_n^p$ denotes the set of all rooted $2p$-angulations with n faces. We start this section with a precise description of the Bouttier-Di Francesco-Guitter bijection between $\mathcal{M}_n^p$ and the set of all p-mobiles with n black vertices.

We use the standard formalism for plane trees as found in [28] for instance. Let

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

where $\mathbb{N} = \{1,2,\ldots\}$ and by convention $\mathbb{N}^0 = \{\varnothing\}$. The generation of $u=(u_1,\ldots,u_n)\in\mathbb{N}^n$ is $|u|=n$. If $u=(u_1,\ldots u_m)$ and $v=(v_1,\ldots,v_n)$ belong to $\mathcal{U}$, $uv=(u_1,\ldots u_m,v_1,\ldots,v_n)$ denotes the concatenation of u and v. In particular $u\varnothing=\varnothing u=u$. If v is of the form $v=uj$ for $u\in\mathcal{U}$ and $j\in\mathbb{N}$, we say that u is the *parent* of v, or that v is a *child* of u. More generally, if v is of the form $v=uw$ for $u,w\in\mathcal{U}$, we say that u is an *ancestor* of v, or that v is a *descendant* of u.

A plane tree τ is a finite subset of $\mathcal{U}$ such that:

(i) $\varnothing \in \tau$.

(ii) If $v \in \tau$ and $v \neq \varnothing$, the parent of u belongs to τ.

(iii) For every $u \in \tau$, there exists an integer $k_u(\tau) \geq 0$ such that $uj \in \tau$ if and only if $1 \leq j \leq k_u(\tau)$.

A p-tree is a plane tree τ that satisfies the following additional property:

(iv) For every $u \in \tau$ such that $|u|$ is odd, $k_u(\tau) = p - 1$.

If τ is a p-tree, vertices u of τ such that $|u|$ is even are called white vertices, and vertices of u such that $|u|$ is odd are called black vertices. We denote by τ° the set of all white vertices of τ and by $\tau^\bullet$ the set of all black vertices. See the left side of Fig.1 for an example of a 3-tree.

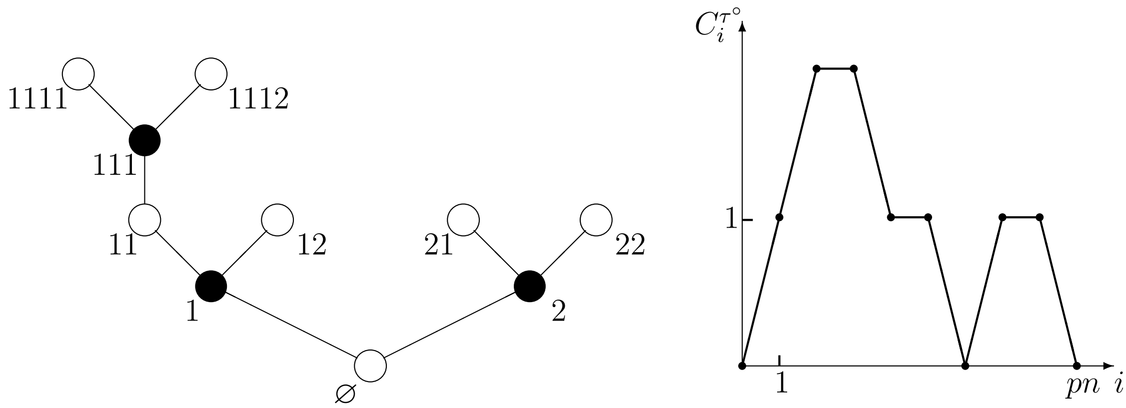

Figure 1. A 3-tree τ and the associated contour function C^{τ° of τ°.

A (rooted) p-mobile is a pair $\theta = (\tau, (\ell_u)_{u \in \tau^\circ})$ that consists of a p-tree τ and a collection of integer labels attached to the white vertices of τ, such that the following properties hold:

(a) $\ell_\varnothing = 1$ and $\ell_u \geq 1$ for each $u \in \tau^\circ$.

(b) Let $u \in \tau^\bullet$, let $u_{(0)}$ be the parent of u and let $u_{(j)} = uj$ for every $1 \leq j \leq p-1$. Then for every $j \in \{0, 1, \ldots, p-1\}$, $\ell_{u_{(j+1)}} \geq \ell_{u_{(j)}} - 1$, where by convention $u_{(p)} = u_{(0)}$.

The left side of Fig.2 gives an example of a p-mobile with $p = 3$. The numbers appearing inside the circles representing white vertices are the labels assigned to these vertices. Condition (b) above means that if one lists the white vertices adjacent to a given black vertex in clockwise order, the labels of these vertices can decrease by at most one at each step.

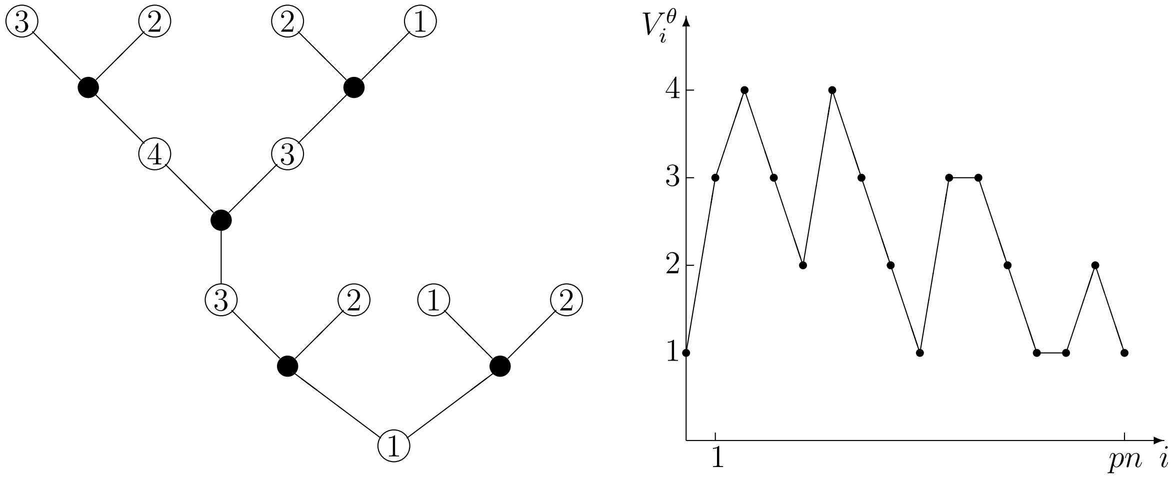

Figure 2. A 3-mobile θ with 5 black vertices and the associated spatial contour function.

We will now describe the Bouttier-Di Francesco-Guitter bijection between $\mathcal{M}_n^p$ and the set of all p-mobiles with n black vertices. This bijection can be found in Section 2 of [8] in the more general setting of bipartite planar maps. Also [8] deals with pointed planar maps rather than with rooted planar maps. It is however easy to verify that the results described below are simple consequences of [8].

Let τ be a p-tree with n black vertices and let $k = \#\tau - 1 = pn$. The depth-first search sequence of τ is the sequence $u_0, u_1, \dots, u_{2k}$ of vertices of τ which is obtained by induction as follows. First $u_0 = \varnothing$, and then for every $i \in \{0, \dots, 2k-1\}$, u_{i+1} is either the first child of u_i that has not yet appeared in the sequence $u_0, \dots, u_i$, or the parent of u_i if all children of u_i already appear in the sequence $u_0, \dots, u_i$. It is easy to verify that $u_{2k} = \varnothing$ and that all vertices of τ appear in the sequence $u_0, u_1, \dots, u_{2k}$ (of course some of them appear more than once).

It is immediate to see that vertices u_i are white when i is even and black when i is odd. The depth-first search sequence of τ° is by definition the sequence $v_0, \dots, v_k$ defined by $v_i = u_{2i}$ for every $i \in \{0, 1, \dots, k\}$.

Now let $(\tau, (\ell_u)_{u \in \tau^\circ})$ be a p-mobile with n black vertices. Denote by $v_0, v_1, \dots, v_{pn}$ the depth-first search sequence of τ°. Suppose that the tree τ_n is drawn in the plane as pictured on Fig.3 and add an extra vertex ∂. We associate with $(\tau, (\ell_u)_{u \in \tau^\circ})$ a rooted $2p$-angulation M with n faces, whose set of vertices is

$$\tau^\circ \cup \{\partial\}$$

and whose edges are obtained by the following device: For every $i \in \{0, 1, \dots, pn-1\}$,

- if $\ell_{v_i} = 1$, draw an edge between v_i and ∂ ;
- if $\ell_{v_i} \geq 2$, draw an edge between v_i and the first vertex in the sequence $v_{i+1}, \dots, v_{pn}$ whose label is $\ell_{v_i} - 1$ (this vertex will be called a *successor* of v_i – note that a given vertex v can appear several times in the depth-first search sequence and so may have several different successors).

Notice that $\ell_{v_{pn}} = \ell_\varnothing = 1$ and that condition (b) in the definition of a p-tree entails that $\ell_{v_{i+1}} \geq \ell_{v_i} - 1$ for every $i \in \{0, 1, \dots, pn-1\}$. This ensures that whenever $\ell_{v_i} \geq 2$ there is at least one vertex among $v_{i+1}, v_{i+2}, \ldots, v_{pn}$ with label $\ell_{v_i} - 1$. The construction can be made in such a way that edges do not intersect: For every vertex v, each index i such that $v_i = v$ corresponds to a "corner" of v, and the associated edge starts from this corner. We refer to Section 2 of [8] for a more detailed description (here we will only need the fact that edges are generated in the way described above). The resulting planar graph M is a $2p$-angulation, which is rooted at the oriented edge between ∂ and $v_0 = \varnothing$, corresponding to $i = 0$ in the previous construction. Each black vertex of τ is associated with a face of the map M. Furthermore the graph distance in M between the root vertex ∂ and another vertex $u \in \tau^\circ$ is equal to ℓ_u. See Fig.3 for the 6-angulation associated with the 3-mobile of Fig.2.

It follows from [8] that the preceding construction yields a bijection between the set $\mathbb{T}_n^p$ of all p-mobiles with n black vertices and the set $\mathcal{M}_n^p$.

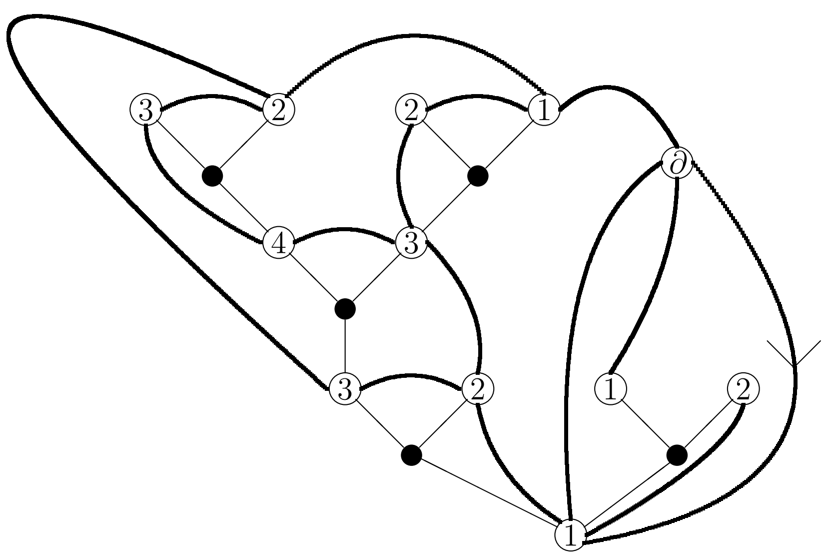

Figure 3. The Bouttier-Di Francesco-Guitter bijection: A rooted 3-mobile with 5 black vertices and the associated rooted 6-angulation with 5 faces. The root of the map is the edge between the vertex ∂ and the root of the tree at the right end of the figure.

2.2 Genealogical structure of maps

Let $\theta = (\tau, (\ell_u)_{u \in \tau^\circ})$ be a p-mobile with n black vertices. The set τ° of white vertices can also be viewed as a graph, by declaring that there is an edge between u and v if and only if u is the grandparent of v (that is, there exist j and $k \in \mathbb{N}$ such that $v = ujk$) or conversely v is the grandparent of u. Obviously τ° is a tree in the graph-theoretic sense. If $u, v \in \tau^\circ$, we then denote by $[\![u, v]\!]$ the set of points of τ° that lie on the unique shortest path from u to v in τ°. As usual, $]\!]u, v[\![= [\![u, v]\!] \setminus \{u, v\}$. We also denote by $u \wedge v$ the "most recent common ancestor" of u and v in τ°, which may be defined by $[\![\varnothing, u \wedge v]\!] = [\![\varnothing, u]\!] \cap [\![\varnothing, v]\!]$. Notice that $u \wedge v$ is not necessarily the most recent ancestor of u and v in the tree τ.

We denote by $\prec$ the genealogical relation on τ°: $u \prec v$ if and only if u is an ancestor of v (in the tree τ). We use $u \leq v$ for the lexicographical order on τ°. As usual $u < v$ if and only if $u \leq v$ and $u \neq v$. It will also be convenient to introduce a "reverse" lexicographical order denoted by $\ll$. This is the total order on τ° defined as follows. If neither of the relations $u \prec v$ and $v \prec u$ holds, then $u \ll v$ if and only if $u \leq v$. On the other hand, if $u \prec v$, then $v \ll u$ (although $u \leq v$).

Let $v_0, v_1, \ldots, v_{pn}$ be the depth-first search sequence of τ°, as defined in the preceding subsection. If $x, y \in \tau^\circ$, the condition $x \leq y$ is equivalent to the fact that the first occurence of x in the sequence $v_0, \ldots, v_{pn}$ occurs before the first occurence of y. Similarly, the condition $x \ll y$ is equivalent to the fact that the last occurence of x occurs before the last occurence of y. The *contour function* of τ° is the discrete sequence $C_0^{\tau^\circ}, C_1^{\tau^\circ}, \ldots, C_{pn}^{\tau^\circ}$ defined by

$$C_i^{\tau^\circ} = \frac{1}{2}\,|v_i| \ , \text{ for every } 0 \leq i \leq pn.$$

See Fig.1 for an example with $p = n = 3$. It is easy to verify that the contour function determines τ°, which in turn determines the p-tree τ uniquely. We will also use the *spatial contour function* of $\theta = (\tau, (\ell_u)_{u \in \tau^\circ})$, which is the discrete sequence $(V_0^\theta, V_1^\theta, \ldots, V_{pn}^\theta)$ defined by

$$V_i^\theta = \ell_{v_i} \ , \text{ for every } 0 \leq i \leq pn.$$

From property (b) of the labels and the definition of the depth-first search sequence, it is clear that $V_{i+1}^\theta \geq V_i^\theta - 1$ for every $0 \leq i \leq pn-1$ (cf Fig.2). This fact will be used many times below.

The pair $(C^{\tau^\circ}, V^\theta)$ determines θ uniquely. For our purposes it will sometimes be convenient to view C^{τ° or V^θ as functions of the continuous parameter $t \in [0, pn]$, simply by interpolating linearly on the intervals $[i-1, i]$, $1 \leq i \leq n$ (as it is suggested by Figs 1 and 2).

Let $[pn]$ stand for the set $\{0, 1, \ldots, pn\}$. Define an equivalence relation $\sim$ on $[pn]$ by setting $i \sim j$ if and only if $v_i = v_j$. The quotient space $[pn]/\sim$ is then obviously identified with τ°. This identification plays an important role throughout this work. If $i \leq j$, the relation $i \sim j$ implies

$$\inf_{i \leq k \leq j} C_k^{\tau^\circ} = C_i^{\tau^\circ} = C_j^{\tau^\circ}.$$

The converse is not true (except if $p = 2$) but the conditions $j > i+1$, $C_i^{\tau^\circ} = C_j^{\tau^\circ}$ and

$$C_k^{\tau^\circ} > C_i^{\tau^\circ} \ , \text{ for every } k \in]i, j[\cap \mathbb{Z}$$

imply that $i \sim j$. Similarly, if $i < j$, the condition $v_i \prec v_j$ implies

$$\inf_{i \leq k \leq j} C_k^{\tau^\circ} = C_i^{\tau^\circ}.$$

The converse is not true, but the condition

$$\inf_{i < k \leq j} C_k^{\tau^\circ} > C_i^{\tau^\circ}$$

forces $v_i \prec v_j$.

Let $u, v \in \tau^\circ$ with $u \prec v$, and let $w \in]\!]u, v[\![$. The set

$$\tau^\circ_{(v,w)} := \{x \in \tau^\circ : x \wedge v = w \text{ and } x \leq v\}$$

is called the subtree from the left side of $[\![u,v]\!]$ with root w. Similarly, the set

$$\widetilde{\tau}^{\circ}_{(v,w)} := \{x \in \tau^{\circ} : x \wedge v = w \text{ and } v \ll x\}$$

is called the subtree from the right side of $[\![u,v]\!]$ with root w. Let $j \in [pn]$ be such that $v_j = v$, and set

$$k = \inf\{i \in \{0,1,\dots,j\} : v_i = w\},$$
$$k' = \sup\{i \in \{0,1,\dots,j\} : v_i = w\}.$$

Then $\tau^{\circ}_{(v,w)}$ exactly consists of the vertices v_i for $k \le i \le k'$: We will say that $[k,k'] \cap \mathbb{Z}$ is the interval coding $\tau^{\circ}_{(v,w)}$. Similar remarks apply to $\widetilde{\tau}^{\circ}_{(v,w)}$.

Recall from the preceding subsection that the p-mobile $(\tau,(\ell_u)_{u\in\tau^{\circ}})$ corresponds to a $2p$-angulation M via the Bouttier-Di Francesco-Guitter bijection. Through this correspondence, vertices of M (with the exception of the root vertex ∂) are identified with elements of τ°. From now on, we systematically do this identification. Let d_M stand for the graph distance on the set of vertices of M. A geodesic path in M is a discrete path $\gamma = (\gamma(i), 0 \le i \le k)$ in M such that $d_M(\gamma(i),\gamma(j)) = |i-j|$ for every $i,j \in \{0,\dots,k\}$.

The following lemma plays an important role in our proofs.

Lemma 2.1 *Let $\gamma = (\gamma(i), 0 \le i \le k)$ be a geodesic path in M which does not visit the root vertex ∂. Let $u = \gamma(0)$ be the starting point of the path γ and let $y = \gamma(k)$ be its final point. Let τ_1 be a subtree from the left side of $[\![\varnothing,y]\!]$ (respectively from the right side of $[\![\varnothing,y]\!]$) with root $w \in]\!]\varnothing,y[\![$. Let $v = \gamma(1)$ be the point following u on the path γ, and assume that:*

(i) *$v \in \tau_1$ and $v \neq w$.*

(ii) *$u \le w$ (resp. $w \ll u$).*

(iii) *For every $i \in \{1,\dots,k\}$, one has $w \le \gamma(i)$ (resp. $\gamma(i) \ll w$).*

Then, for any point x of $\tau_1 \backslash \{w\}$ such that

$$\ell_z > \sup_{0\le i\le k} \ell_{\gamma(i)}, \text{ for every } z \in [\![w,x]\!] \tag{2}$$

one has

$$d_M(x,y) \le d_M(u,y) + \ell_x - \inf_{0\le i\le k} \ell_{\gamma(i)}.$$

Proof: We only treat the case when τ_1 is a subtree from the left side of $[\![\varnothing,y]\!]$. We fix a point $x \in \tau_1\backslash\{w\}$ such that (2) holds. Denote by b the first point on the geodesic γ such that $x \le b$. This makes sense because $x \le y$ by the definition of subtrees. Also $b \neq u$ because $u \le w$ and $w < x$. So we can also introduce the point a preceding b on the geodesic γ.

Let us first assume that $b \neq v$, or equivalently $a \neq u$. Then $w \le a$ by (iii), and $a \le x$, which forces $a \in \tau_1$. On the other hand, assumption (2) guarantees that $a \notin [\![w,x]\!]$ and since $a \in \tau_1$ we get that a is not an ancestor of x. Since $a \le x \le b$, it follows that a cannot be an ancestor of b. Any occurence of a in the depth-first search sequence of τ° thus happens before the first occurence of b in this sequence. Now notice that a and b are connected by an edge of the map M, and recall the construction of these edges at the end of the preceding subsection. It follows that $\ell_z \geq \ell_a$ for every vertex z such that $a \ll z < b$, whereas $\ell_b = \ell_a - 1$. Note that any vertex z such that $x \leq z < b$ satisfies $a \ll z < b$: Indeed, the property $a \leq x \leq z$ and the fact that a is not ancestor of x imply that $a \ll z$. Set $q = \ell_x - \ell_b \geq 1$. We let i_0 be the first index such that $v_{i_0} = x$, and we define $i_1, \ldots, i_q$ by setting

$$i_j = \inf\{i \geq i_0 : \ell_{v_i} = \ell_x - j\} \text{ for every } 1 \leq j \leq q.$$

By the preceding considerations, we have $\ell_z \geq \ell_a = \ell_x - q + 1$ for every z such that $x \leq z < b$. It follows that $v_{i_q} = b$. On the other hand, $v_{i_0} = x$ and $d_M(v_{i_j}, v_{i_{j+1}}) = 1$ for every $0 \leq j \leq q-1$, by the construction of edges in M. We thus get

$$d_M(x, b) \leq q = \ell_x - \ell_b.$$

Finally,

$$d_M(x, y) \leq d_M(b, y) + d_M(x, b) \leq d_M(u, y) + \ell_x - \ell_b,$$

which gives the desired bound.

In the case when $a = u$ and $b = v$, the argument is almost the same. Note that $u < w$ ($u = w$ is excluded by (2)) and $x < b = v$ as previously. The existence of an edge between u and v warrants that $\ell_v = \ell_u - 1$ and that $\ell_z \geq \ell_u$ for every vertex z of τ_1 such that $z < v$. In the same way as before, we get $d_M(x, v) \leq \ell_x - \ell_v$ which leads to the desired bound. □

2.3 Real trees

We will now discuss the continuous trees that are scaling limits of our discrete plane trees. We start with a basic definition.

Definition 2.1 *A metric space $(\mathcal{T}, d)$ is a real tree if the following two properties hold for every $a, b \in \mathcal{T}$.*

(i) *There is a unique isometric map $f_{a,b}$ from $[0, d(a,b)]$ into $\mathcal{T}$ such that $f_{a,b}(0) = a$ and $f_{a,b}(d(a,b)) = b$.*

(ii) *If q is a continuous injective map from $[0,1]$ into $\mathcal{T}$, such that $q(0) = a$ and $q(1) = b$, we have*

$$q([0,1]) = f_{a,b}([0, d(a,b)]).$$

A rooted real tree is a real tree $(\mathcal{T}, d)$ with a distinguished vertex $\rho = \rho(\mathcal{T})$ called the root.

In what follows, real trees will always be rooted and compact, even if this is not mentioned explicitly.

Let us consider a rooted real tree $(\mathcal{T}, d)$. The range of the mapping $f_{a,b}$ in (i) is denoted by $[\![a, b]\!]$ (this is the line segment between a and b in the tree), and we also use the obvious notation $]\!]a,b[\![$. In particular, for every $a \in \mathcal{T}$, $[\![\rho,a]\!]$ is the path going from the root to a, which we will interpret as the ancestral line of vertex a. More precisely we can define a partial order on the tree, called the genealogical order, by setting $a \prec b$ if and only if $a \in [\![\rho,b]\!]$. If $a,b \in \mathcal{T}$, there is a unique $c \in \mathcal{T}$ such that $[\![\rho,a]\!] \cap [\![\rho,b]\!] = [\![\rho,c]\!]$. We write $c = a \wedge b$ and call c the most recent common ancestor to a and b. The multiplicity of a vertex $a \in \mathcal{T}$ is the number of connected components of $\mathcal{T}\backslash\{a\}$. In particular, a is called a leaf if it has multiplicity one.

In a way similar to the discrete case, real trees can be coded by "contour functions". If E and F are two topological spaces, we write $C(E,F)$ for the space of all continuous functions from E into F. Let $\sigma > 0$ and let $g \in C([0,\sigma],[0,\infty[)$ be such that $g(0) = g(\sigma) = 0$. To avoid trivialities, we will also assume that g is not identically zero. For every $s,t \in [0,\sigma]$, we set

$$m_g(s,t) = \inf_{r \in [s\wedge t, s\vee t]} g(r),$$

and

$$d_g(s,t) = g(s) + g(t) - 2m_g(s,t).$$

It is easy to verify that d_g is a pseudo-metric on $[0,\sigma]$. As usual, we introduce the equivalence relation $s \simeq_g t$ if and only if $d_g(s,t) = 0$ (or equivalently if and only if $g(s) = g(t) = m_g(s,t)$). The function d_g induces a distance on the quotient space $\mathcal{T}_g := [0,\sigma] / \simeq_g$, and we keep the notation d_g for this distance. We denote by $p_g : [0,\sigma] \longrightarrow \mathcal{T}_g$ the canonical projection. Clearly p_g is continuous (when $[0,\sigma]$ is equipped with the Euclidean metric and $\mathcal{T}_g$ with the metric d_g), and therefore $\mathcal{T}_g = p_g([0,\sigma])$ is a compact metric space.

By Theorem 2.1 of [15], the metric space $(\mathcal{T}_g, d_g)$ is a real tree. We will always view $(\mathcal{T}_g, d_g)$ as a rooted real tree with root $\rho = p_g(0) = p_g(\sigma)$. Then, if $s,t \in [0,\sigma]$, the property $p_g(s) \prec p_g(t)$ holds if and only if $g(s) = m_g(s,t)$.

Let us recall the definition of the Gromov-Hausdorff distance. Let (E_1,d_1) and (E_2,d_2) be two compact metric spaces. The Gromov-Hausdorff distance between (E_1,d_1) and (E_2,d_2) is

$$d_{GH}(E_1,E_2) = \inf\Big(d_{Haus}(\varphi_1(E_1),\varphi_2(E_2))\Big),$$

where the infimum is over all isometric embeddings $\varphi_1 : E_1 \longrightarrow E$ and $\varphi_2 : E_2 \longrightarrow E$ of E_1 and E_2 into the same metric space (E,d), and d_{Haus} stands for the usual Hausdorff distance between compact subsets of E. Then Lemma 2.3 of [15] shows that $\mathcal{T}_g$ depends continuously on g, in the sense that

$$d_{GH}(\mathcal{T}_g, \mathcal{T}'_g) \leq 2\|g - g'\|$$

where $\|g-g'\|$ is the supremum norm of $g - g'$.

In addition to the genealogical order $\prec$, the tree $\mathcal{T}_g$ inherits a lexicographical order from the coding through the function g. Precisely if $a,b \in \mathcal{T}_g$ we write $a \leq b$ if and only if $s \leq t$, where s, respectively t, is the smallest representative of a, resp. of b, in $[0,\sigma]$. We can also introduce a "reverse" lexicographical order $\ll$, by replacing smallest by greatest in the previous sentence. If neither of the relations $a \prec b$ or $b \prec a$ holds, we have $a \ll b$ if and only if $a \leq b$. On the other hand, if $a \prec b$, we have $b \ll a$.

Let $a,b \in \mathcal{T}_g$. If $a \leq b$, or if $a \ll b$, we define the lexicographical interval $[a,b]$ as the image under the projection p_g of the minimal interval $[s,t]$ such that $s \leq t$, $p_g(s) = a$ and $p_g(t) = b$.

If neither of the relations $a \leq b$ or $a \ll b$ holds, then there is no such interval and we take $[a, b] = \varnothing$. If $[a, b]$ is nonempty, then $[\![a, b]\!] \subset [a, b]$. Furthermore if $a \prec b$, then both $[a, b]$ and $[b, a]$ are nonempty, and $[a, b] \cap [b, a] = [\![a, b]\!]$.

Let $a, b \in \mathcal{T}_g$ with $a \prec b$, and let $c \in]\!]a, b[\![$. Suppose that the set

$$\mathcal{T}^1 = \{u \in \mathcal{T}_g : u \wedge b = c \text{ and } u \leq b\}$$

is not the singleton $\{c\}$. Then the set $\mathcal{T}^1$ is called a subtree from the left side of $[\![a, b]\!]$ with root c (it is straightforward to verify that $\mathcal{T}^1$ is itself a real tree). Moreover, if $s = \inf p_g^{-1}(a)$ and $t = \inf p_g^{-1}(b)$, there is a unique subinterval $[\alpha, \beta]$ of $]s, t[$ such that $\mathcal{T}^1 = p_g([\alpha, \beta])$, $p_g(\alpha) = p_g(\beta) = c$ and

$$\alpha = \sup\{r \in [s, t] : g(r) < g(\alpha)\}\ ,\ \beta = \sup\{r \in [s, t] : g(r) \leq g(\alpha)\}.$$

We say that $[\alpha, \beta]$ is the coding interval of $\mathcal{T}^1$. In a similar way we can define subtrees from the right side of $[\![a, b]\!]$: $\mathcal{T}^2$ is such a subtree if there exists $c' \in]\!]a, b[\![$ such that

$$\mathcal{T}^2 = \{u \in \mathcal{T}_g : u \wedge b = c' \text{ and } b \ll u\}$$

and $\mathcal{T}^2 \neq \{c'\}$.

2.4 Brownian trees and conditioned Brownian trees

We first explain how we can assign Brownian labels to the vertices of the real tree $(\mathcal{T}_g, d_g)$ defined in the previous subsection. To this end, we consider the centered real-valued Gaussian process $(\Gamma_t)_{t\in[0,\sigma]}$ with covariance function

$$\mathrm{cov}(\Gamma_s, \Gamma_t) = m_g(s, t) \tag{3}$$

for every $s, t \in [0, \sigma]$ (it is a simple exercise to check that $m_g(s, t)$ is a covariance function). Note that $\Gamma_0 = \Gamma_\sigma = 0$ and that the form of the covariance gives $E[(\Gamma_s - \Gamma_t)^2] = d_g(s, t)$. Suppose that g is Hölder continuous with some exponent $\delta > 0$, which will always hold in what follows. Then an application of the classical Kolmogorov lemma shows that the process $(\Gamma_t)_{t\in[0,\sigma]}$ has a continuous modification, and from now on we consider only this modification. We write $\mathbf{Q}_g$ for the distribution of $(\Gamma_t)_{t\in[0,\sigma]}$, which is a probability measure on the space $C([0, \sigma], \mathbb{R})$.

From the formula $E[(\Gamma_s - \Gamma_t)^2] = d_g(s, t)$ and a continuity argument, we immediately get that a.s. for every $s, t \in [0, \sigma]$ such that $s \simeq_g t$, we have $\Gamma_s = \Gamma_t$. Therefore we may also view Γ as a Gaussian process indexed by the tree $\mathcal{T}_g$. Indeed, it is natural to interpret $(\Gamma_a, a \in \mathcal{T}_g)$ as Brownian motion indexed by $\mathcal{T}_g$ and started from 0 at the root of $\mathcal{T}_g$. Note that formula (3) may be rewritten in the form

$$\mathrm{cov}(\Gamma_a, \Gamma_b) = d_g(\rho, a \wedge b)$$

for every $a, b \in \mathcal{T}_g$.

We now randomize the coding function g. Let $\mathbf{e} = (\mathbf{e}_t)_{t\in[0,1]}$ be the normalized Brownian excursion, and take $g = \mathbf{e}$ and $\sigma = 1$ in the previous discussion. The random real tree $(\mathcal{T}_\mathbf{e}, d_\mathbf{e})$ coded by $\mathbf{e}$ is the so-called CRT, or Continuum Random Tree. Using the fact that local minima of Brownian motion are distinct, one easily checks that points of $\mathcal{T}_{\mathbf{e}}$ can have multiplicity at most 3.

We then consider the real-valued process $(Z_t)_{t\in[0,1]}$ such that conditionally given $\mathbf{e}$, $(Z_t)_{t\in[0,1]}$ has distribution $\mathbf{Q_e}$. As explained above, we can also view $(Z_t)_{t\in[0,1]}$ as parametrized by the tree $\mathcal{T}_{\mathbf{e}}$, and then interpret $(Z_a)_{a\in\mathcal{T}_{\mathbf{e}}}$ as Brownian motion indexed by $\mathcal{T}_{\mathbf{e}}$. This interpretation creates some technical difficulties since $\mathcal{T}_{\mathbf{e}}$ is now a random index set – to circumvent these difficulties it is often more convenient to view Z as indexed by $[0,1]$, keeping in mind that Z_t only depends on the equivalence class of t in $\mathcal{T}_{\mathbf{e}}$.

In view of our applications it is important to consider the pair $(\mathbf{e}, Z)$ conditioned on the event

$$Z_t \geq 0 \quad \text{for every } t \in [0,1].$$

Here some justification is needed for the conditioning, since the latter event has probability zero. The paper [25] describes several limit procedures that allow one to make sense of the previous conditioning. These procedures all lead to the same limiting pair $(\overline{\mathbf{e}}, \overline{Z})$ which can be described as follows from the original pair $(\mathbf{e}, Z)$. Set

$$\underline{Z} = \inf_{t\in[0,1]} Z_t$$

and let s_* be the (almost surely) unique time in $[0,1]$ such that $Z_{s_*} = \underline{Z}$. The fact that $\underline{Z}$ is attained at a unique time ([25] Proposition 2.5) entails that the vertex $p_{\mathbf{e}}(s_*)$ is a leaf of the tree $\mathcal{T}_{\mathbf{e}}$. For every $s, t \in [0,1]$, set $s \oplus t = s + t$ if $s + t \leq 1$ and $s \oplus t = s + t - 1$ if $s + t > 1$. Then, for every $t \in [0,1]$,

- $\overline{\mathbf{e}}_t = \mathbf{e}_{s_*} + \mathbf{e}_{s_*\oplus t} - 2\, m_{\mathbf{e}}(s_*, s_* \oplus t)$;
- $\overline{Z}_t = Z_{s_*\oplus t} - Z_{s_*}$.

The formula for $\overline{Z}$ makes it obvious that $\overline{Z}_t \geq 0$ for every $t \geq 0$, in agreement with the above-mentioned conditioning. The function $\overline{\mathbf{e}}$ is continuous on $[0,1]$ and such that $\overline{\mathbf{e}}(0) = \overline{\mathbf{e}}(1) = 0$. Hence the tree $\mathcal{T}_{\overline{\mathbf{e}}}$ is well defined, and this tree is isometrically identified with the tree $\mathcal{T}_{\mathbf{e}}$ re-rooted at the (minimizing) vertex $p_{\mathbf{e}}(s_*)$: See Lemma 2.2 in [15]. Moreover we have $s \simeq_{\overline{\mathbf{e}}} t$ if and only if $s_* \oplus s \simeq_{\mathbf{e}} s_* \oplus t$ and so $\overline{Z}_t$ only depends on the equivalence class of t in the tree $\mathcal{T}_{\overline{\mathbf{e}}}$. Therefore we may and will sometimes view $\overline{Z}$ as indexed by vertices of the tree $\mathcal{T}_{\overline{\mathbf{e}}}$.

By a well-known property of the Brownian excursion, the law of the pair $(\mathbf{e}_t, Z_t)_{t\in[0,1]}$ is invariant under time reversal, meaning that $(\mathbf{e}_t, Z_t)_{t\in[0,1]}$ has the same distribution as $(\mathbf{e}_{1-t}, Z_{1-t})_{t\in[0,1]}$. A similar time-reversal invariance property then holds for the pair $(\overline{\mathbf{e}}_t, \overline{Z}_t)_{t\in[0,1]}$. In what follows we use the notation ρ for the root of $\mathcal{T}_{\mathbf{e}}$ and $\overline{\rho}$ for the root of $\mathcal{T}_{\overline{\mathbf{e}}}$.

We now state three important lemmas which are key ingredients of the proofs of our main results.

Lemma 2.2 *We say that $s \in [0,1[$ is an increase point of the pair $(\mathbf{e}, Z)$, respectively of the pair $(\overline{\mathbf{e}}, \overline{Z})$, if there exists $\varepsilon > 0$ such that $\mathbf{e}_t \geq \mathbf{e}_s$ and $Z_t \geq Z_s$, resp. $\overline{\mathbf{e}}_t \geq \overline{\mathbf{e}}_s$ and $\overline{Z}_t \geq \overline{Z}_s$, for every $t \in [s, (s+\varepsilon) \wedge 1]$. Then a.s. there is no increase point of $(\mathbf{e}, Z)$, and $s = 0$ is the only increase point of $(\overline{\mathbf{e}}, \overline{Z})$.*

Before stating the next lemma we need to introduce some additional notation. The uniform measure λ on $\mathcal{T}_{\mathbf{e}}$, resp. on $\mathcal{T}_{\overline{\mathbf{e}}}$, is the image of Lebesgue measure on $[0,1]$ under the canonical projection $p_{\mathbf{e}}$, resp. $p_{\overline{\mathbf{e}}}$. There is no ambiguity in using the same notation λ for both cases, since it really corresponds to the same measure when $\mathcal{T}_{\overline{\mathbf{e}}}$ is identified to $\mathcal{T}_{\mathbf{e}}$ up to re-rooting. We also let $\mathcal{I}$ and $\overline{\mathcal{I}}$ be the random measures on $\mathbb{R}$ defined by

$$\langle \mathcal{I}, f\rangle = \int_{\mathcal{T}_{\mathbf{e}}} \lambda(da)\, f(Z_a) = \int_0^1 dt\, f(Z_t)\ ,\ \langle \overline{\mathcal{I}}, f\rangle = \int_{\mathcal{T}_{\overline{\mathbf{e}}}} \lambda(da)\, f(\overline{Z}_a) = \int_0^1 dt\, f(\overline{Z}_t).$$

The random measure $\mathcal{I}$ is called (one-dimensional) ISE, for integrated super-Brownian excursion. Notice that $\overline{\mathcal{I}}$ is supported on $[0,\infty[$ and is just the image of $\mathcal{I}$ under the shift $x \longrightarrow x - \underline{Z}$.

Lemma 2.3 *For every $\alpha > 0$,*

$$\lim_{\varepsilon\to 0} \varepsilon^{-2} P(\overline{\mathcal{I}}([0,\varepsilon]) \geq \alpha\varepsilon^2) = 0.$$

Our last lemma is concerned with values of $\overline{Z}$ over subtrees of $\mathcal{T}_{\overline{\mathbf{e}}}$. Roughly speaking it asserts that, for a given $\beta > 0$ and a subtree $\mathcal{T}^1$ with root c, if both $\overline{Z}_c > \beta$ and the minimum of the values of $\overline{Z}$ over $\mathcal{T}^1$ is strictly less than β, then the mass (for the uniform measure λ) of those vertices x of $\mathcal{T}_{\overline{\mathbf{e}}}$ with label $\overline{Z}_x \in [\beta, \beta+\varepsilon]$, and such that the label of any ancestor of x in $\mathcal{T}^1$ is greater than β, will be of order at least ε^2. The precise statement is as follows.

Lemma 2.4 *Almost surely, for every $\mu > 0$, for every $a \in \mathcal{T}_{\overline{\mathbf{e}}}$ and every subtree $\mathcal{T}^1$ from $[\![\overline{\rho}, a]\!]$ with root $c \in]\!]\overline{\rho}, a[\![$, the condition*

$$\inf_{b\in\mathcal{T}^1} \overline{Z}_b < \overline{Z}_c - \mu$$

implies that

$$\liminf_{\varepsilon\to 0} \varepsilon^{-2}\, \lambda\Big(\Big\{x \in \mathcal{T}^1 : \overline{Z}_x \leq \overline{Z}_c - \mu + \varepsilon \text{ and } \overline{Z}_y \geq \overline{Z}_c - \mu + \frac{\varepsilon}{8} \text{ for every } y \in [\![c, x]\!]\Big\}\Big) > 0.$$

Although Lemma 2.4 is stated in terms of the pair $(\overline{\mathbf{e}}, \overline{Z})$, in view of our applications, the proof will show that this lemma reduces to a similar statement for the pair $(\mathbf{e}, Z)$.

The proof of the preceding three lemmas depends on some properties of the path-valued process called the Brownian snake, and recalling these properties at the present stage would take us too far from our main concern. For this reason, we prefer to postpone the proofs to Section 5.

2.5 Invariance principles

In this subsection, we recall the basic invariance principles that relate the discrete labeled trees of subsection 2.1 to the Brownian trees of subsection 2.4. Recall that the integer $p \geq 2$ is fixed.

Let $\theta_n = (\tau_n, (\ell^n_u)_{u\in\tau_n^\circ})$ be uniformly distributed over the set $\mathbb{T}^p_n$ of all p-mobiles with n black vertices. We denote by $C^n = (C^n_t)_{0\leq t\leq pn}$ the contour function of τ_n° and by $V^n = (V^n_t)_{0\leq t\leq pn}$ the spatial contour function of θ_n (it is convenient to view C^n and V^n as continuous functions of $t \in [0, pn]$, as explained in subsection 2.2). Recall that the pair (C^n, V^n) determines θ_n.

Theorem 2.5 *We have*

$$\left(\frac{1}{2}\sqrt{\frac{p}{p-1}}\, n^{-1/2}\, C^n_{pnt}, \left(\frac{9}{4p(p-1)}\right)^{1/4} n^{-1/4} V^n_{pnt}\right)_{0\le t\le 1} \xrightarrow[n\to\infty]{\text{(d)}} (\overline{\mathbf{e}}_t, \overline{Z}_t)_{0\le t\le 1}. \tag{4}$$

in the sense of weak convergence of the laws in the space of probability measures on $C([0,1],\mathbb{R}^2)$.

The case $p = 2$ of Theorem 2.5 is a special case of Theorem 2.1 in [23], which is itself a conditional version of invariance principles relating discrete snakes to the Brownian snake [19]. See the discussion in Section 8 of [23]. Similar results were obtained before by Chassaing and Schaeffer [12]. In the general case, Theorem 2.5 is a consequence of Theorem 3.3 in [31], and is also closely related to Theorem 11 in [26].

Although Theorem 2.5 will be our main tool, we will also need another asymptotic result, which does not easily follow from Theorem 2.5 but fortunately can be deduced from the results in [26]. Let M_n be the random element of $\mathcal{M}^p_n$ that corresponds to θ_n via the Bouttier-Di Francesco-Guitter bijection. Obviously M_n is uniformly distributed over $\mathcal{M}^p_n$. Conditionally on M_n, let us choose a vertex Y_n of M_n uniformly at random. The pair (M_n, Y_n) is then uniformly distributed over the set of all rooted and pointed $2p$-angulations with n faces. Theorem 3 (iii) of [26] gives precise information about the profile of distances to the point Y_n in the map M_n (to be precise, [26] imposes a special constraint on the orientation of the root edge depending on the distinguished point in the map, but since every rooted and pointed map with this constraint corresponds exactly to two unconstrained rooted and pointed maps, the results of [26] immediately carry over to our setting). In our special situation, we can restate this result as follows. We write d_n for the graph distance on the set $\mathbf{m}_n$ of vertices of M_n, and for every $R > 0$ and $x \in \mathbf{m}_n$ we denote by $B_n(x, R)$ the closed ball with radius R centered at x in the metric space $(\mathbf{m}_n, d_n)$.

Proposition 2.6 *For every* $\alpha, \beta > 0$,

$$P\Big[\frac{1}{(p-1)n}\#B_n(Y_n, \alpha n^{1/4}) \ge \beta\Big] \underset{n\to\infty}{\longrightarrow} P\Big[\overline{\mathcal{I}}\Big(\Big[0, \Big(\frac{9}{4p(p-1)}\Big)^{1/4}\alpha\Big]\Big) \ge \beta\Big].$$

Since Y_n is uniformly distributed over $\mathbf{m}_n$ and $\#(\mathbf{m}_n) = (p-1)n + 2$, the convergence of the proposition can be restated as follows. For every $\alpha, \beta > 0$,

$$E\Big[\frac{1}{(p-1)n}\#\{y \in \mathbf{m}_n : \#B_n(y, \alpha n^{1/4}) \ge \beta\,(p-1)n\}\Big] \underset{n\to\infty}{\longrightarrow} P\Big[\overline{\mathcal{I}}\Big(\Big[0, \Big(\frac{9}{4p(p-1)}\Big)^{1/4}\alpha\Big]\Big) \ge \beta\Big]. \tag{5}$$

3 Main results

Recall the notation introduced in the previous section. In particular, M_n is a random rooted $2p$-angulation which is uniformly distributed over the set $\mathcal{M}^p_n$, $\mathbf{m}_n$ denotes the set of vertices of M_n, and $\theta_n = (\tau_n, (\ell^n_u)_{u\in\tau^\circ_n})$ is the random mobile corresponding to M_n via the Bouttier-Di Francesco-Guitter bijection. We constantly use the identification

$$\mathbf{m}_n = \tau^\circ_n \cup \{\partial_n\}$$

where ∂_n is the root vertex of M_n. The graph distance on $\mathbf{m}_n$ is denoted by d_n. In particular, if $a, b \in \tau_n^\circ$, $d_n(a,b)$ denotes the graph distance between a and b viewed as vertices in the map M_n.

As in subsection 2.5, C^n and V^n are respectively the contour function of the tree τ_n° and the spatial contour function of θ_n.

Following subsection 2.2, the equivalence relation $\sim_n$ on $[pn] = \{0, 1, \ldots, pn\}$ is defined by declaring that $i \sim_n j$ if and only if the i-th vertex in the depth first-search sequence of τ_n° is the same as the j-th vertex in the same sequence. Recall that this implies

$$C_i^n = C_j^n = \inf_{i \wedge j \le k \le i \vee j} C_k^n.$$

The quotient set $[pn] / \sim_n$ is then canonically identified with τ_n° and thus with the set of vertices of M_n other than the root ∂_n. If $a \in \tau_n^\circ$ and $i \in [pn]$, we will abuse notation by writing $a \sim_n i$ if i is a representative of a viewed as an element of $[pn] / \sim_n$ (similar abuses of notation will occur for other equivalence relations). With this notation, if $a \sim_n i$, we have $d_n(\partial_n, a) = \ell_a^n = V_i^n$, by the properties of the Bouttier-Di Francesco-Guitter bijection. If $i, j \in [pn]$ and $a, b \in \tau_n^\circ$ are such that $a \sim_n i$ and $b \sim_n j$, we will also write $d_n(i,j) = d_n(a,b)$.

For every $i, j \in [pn]$, we put

$$d_n^\circ(i,j) = V_i^n + V_j^n - 2 \inf_{i \wedge j \le k \le i \vee j} V_k^n + 2.$$

Lemma 3.1 *For every $i, j \in [pn]$,*

$$d_n(i,j) \le d_n^\circ(i,j).$$

Proof: Fix $i \in [pn]$ and let $a \in \tau_n^\circ$ be such that $a \sim_n i$. Let $q = V_i^n = d_n(\partial_n, a)$. We set $i_q = i$ and for every $k \in \{1, \ldots, q-1\}$,

$$i_k = \inf\{\ell \ge i : V_\ell^n = k\}.$$

From the construction of edges in the Bouttier-Di Francesco-Guitter bijection, it is immediate to see that $d_n(i_k, i_{k-1}) = 1$ for every $2 \le k \le q$.

We also fix $j \in [pn]$ and let $b \in \tau_n^\circ$ be such that $b \sim_n j$, and we set $r = V_j^n = d_n(\partial_n, b)$. We define similarly the sequence $j_r = j, j_{r-1}, \ldots, j_1$. Then:

- Either $\inf_{i \wedge j \le k \le i \vee j} V_k^n = 1$ and the bound of the lemma is just the triangle inequality $d_n(i,j) = d_n(a,b) \le d_n(\partial_n, a) + d_n(\partial_n, b)$.
- Or $\inf_{i \wedge j \le k \le i \vee j} V_k^n = \ell \ge 2$, and we have $i_{\ell-1} = j_{\ell-1}$. The bound of the lemma follows by writing:

$$d_n(i,j) \le d_n(i_{\ell-1}, i_q) + d_n(j_{\ell-1}, j_r) \le q + r - 2\ell + 2.$$

□

We extend the definition of $d_n(i,j)$ and $d_n^\circ(i,j)$ to noninteger values of i and j by linear interpolation. If $s, t \in [0, pn]$, we set

$$d_n(s,t) \quad = \quad (s - \lfloor s \rfloor)(t - \lfloor t \rfloor) d_n(\lceil s \rceil, \lceil t \rceil) + (s - \lfloor s \rfloor)(\lceil t \rceil - t) d_n(\lceil s \rceil, \lfloor t \rfloor)$$

$$+ \quad (\lceil s\rceil - s)(t-\lfloor t\rfloor)d_n(\lfloor s\rfloor,\lceil t\rceil) + (\lceil s\rceil - s)(\lceil t\rceil - t)d_n(\lfloor s\rfloor,\lfloor t\rfloor),$$

with the notation $\lfloor s\rfloor = \sup\{k\in\mathbb{Z} : k\le s\}$ and $\lceil s\rceil = \inf\{k\in\mathbb{Z} : k > s\}$. We define $d_n^\circ(s,t)$ in a similar way. Obviously the bound $d_n(s,t)\le d_n^\circ(s,t)$ remains valid for reals $s,t\in[0,pn]$. Furthermore, the triangle inequality $d_n(s,u)\le d_n(s,t)+d_n(t,u)$ also holds for every $s,t,u\in[0,pn]$.

As a straightforward consequence of (4) and the definition of $d_n^\circ(s,t)$, we have

$$\left(\Big(\frac{9}{4p(p-1)}\Big)^{1/4} n^{-1/4} d_n^\circ(pns,pnt)\right)_{0\le s\le 1, 0\le t\le 1} \xrightarrow[n\to\infty]{\text{(d)}} (D^\circ(s,t))_{0\le s\le 1,0\le t\le 1} \tag{6}$$

where

$$D^\circ(s,t) = \overline{Z}_s + \overline{Z}_t - 2\inf_{s\wedge t\le r\le s\vee t} \overline{Z}_r$$

and the limit holds in the sense of weak convergence in the space of probability measures on $C([0,1]^2,\mathbb{R})$.

Proposition 3.2 *The sequence of the laws of the processes*

$$\big(n^{-1/4}\, d_n(pns,pnt)\big)_{0\le s\le 1,0\le t\le 1}$$

is tight in the space of probability measures on $C([0,1]^2,\mathbb{R})$. *Let* $\mathbf{C}$ *be the space of isometry classes of compact metric spaces, which is equipped with the Gromov-Hausdorff distance. The sequence of the laws of the metric spaces* $(\mathbf{m}_n, n^{-1/4}d_n)$ *is tight in the space of probability measures on* $\mathbf{C}$.

Proof: First observe that, for every $s,t,s',t'\in[0,1]$,

$$\begin{aligned}|n^{-1/4}\, d_n(pns,pnt) - n^{-1/4}\, d_n(pns',pnt')| &\le n^{-1/4}(d_n(pns,pns') + d_n(pnt,pnt')) \\ &\le n^{-1/4}(d_n^\circ(pns,pns') + d_n^\circ(pnt,pnt')).\end{aligned} \tag{7}$$

From the convergence (6), we have for every $\delta,\varepsilon>0$,

$$\limsup_{n\to\infty} P\left(\sup_{|s-s'|\le\delta} n^{-1/4}\, d_n^\circ(pns,pns')\ge\varepsilon\right) \le P\left(\sup_{|s-s'|\le\delta} D^\circ(s,s')\ge\Big(\frac{4p(p-1)}{9}\Big)^{1/4}\varepsilon\right). \tag{8}$$

Let $\eta>0$ and for every $k\ge 1$ set $\varepsilon_k = 2^{-k}$. We apply (8) with $\varepsilon=\varepsilon_k$ and note that we can then choose $\delta_k>0$ sufficiently small so that the right-hand side of (8) is strictly less than $2^{-k}\eta$. Therefore, there exists an integer n_k such that, for every $n\ge n_k$,

$$P\left(\sup_{|s-s'|\le\delta_k} n^{-1/4}\, d_n^\circ(pns,pns')\ge\varepsilon_k\right)\le 2^{-k}\eta. \tag{9}$$

By choosing δ_k even smaller if necessary, we may assume that (9) holds for every $n\ge 1$. It follows that, for every $n\ge 1$,

$$P\left(\bigcap_{k\ge 1}\left\{\sup_{|s-s'|\le\delta_k} n^{-1/4}\, d_n^\circ(pns,pns')\le\varepsilon_k\right\}\right)\ge 1-\eta. \tag{10}$$

Let K denote the set of all functions $\omega \in C([0,1]^2, \mathbb{R})$ such that $\omega(0,0) = 0$ and, for every $k \geq 1$,

$$\sup\{|\omega(s,t) - \omega(s',t')| : |s - s'| \leq \delta_k, |t - t'| \leq \delta_k\} \leq 2\,\varepsilon_k.$$

Then K is a compact subset of $C([0,1]^2, \mathbb{R})$. By (7) and (10), the probability that the random function $(s,t) \longrightarrow n^{-1/4} d_n(pns, pnt)$ belongs to K is bounded below by $1 - \eta$, for every $n \geq 1$. Since η was arbitrary, this completes the proof of the first assertion.

The second assertion is an easy consequence of the first one and the Gromov compactness criterion (Theorem 7.4.15 in [10]). We omit details, since this result is not really needed in what follows. $\square$

From (4) and Proposition 3.2, there exists a strictly increasing sequence $(n_k)_{k \geq 1}$ such that along this sequence we have the joint convergence in distribution

$$\left(\frac{1}{2}\sqrt{\frac{p}{p-1}}\, n^{-1/2}\, C^n_{pnt}, \left(\frac{9}{4p(p-1)}\right)^{1/4} n^{-1/4} V^n_{pnt}, \left(\frac{9}{4p(p-1)}\right)^{1/4} n^{-1/4}\, d_n(pns, pnt)\right)_{0 \leq s \leq 1, 0 \leq t \leq 1} \underset{n \to \infty}{\longrightarrow} \left(\overline{\mathbf{e}}_t, \overline{Z}_t, D(s,t)\right)_{0 \leq s \leq 1, 0 \leq t \leq 1}. \quad (11)$$

Here the limiting triple $\left(\overline{\mathbf{e}}_t, \overline{Z}_t, D(s,t)\right)$ is defined on a suitable probability space, the pair $(\overline{\mathbf{e}}, \overline{Z})$ obviously has the same distribution as before, and D is a continuous process indexed by $[0,1]^2$ and taking values in $\mathbb{R}_+$. In the remaining part of this work, we restrict our attention to values of n belonging to the sequence $(n_k)_{k \geq 1}$. In particular, when we pass to the limit as $n \to \infty$, this always means along the sequence $(n_k)_{k \geq 1}$.

Thanks to the Skorokhod representation theorem (see e.g. Theorem 3.1.8 in [16]), we may and will assume that the convergence (11) holds almost surely, in the sense of uniform convergence over $[0,1]^2$. Strictly speaking, we should replace for every $n \geq 1$ the random mobile θ_n (respectively the random map M_n) with another random mobile $\widetilde{\theta}_n$ (resp. another random map $\widetilde{M}_n$) having the same distribution, but we do not keep track of this replacement in the notation.

The next proposition records some properties of the random function $D(s,t)$. We write $\simeq$ instead of $\simeq_{\overline{\mathbf{e}}}$ for the equivalence relation defining the tree $\mathcal{T}_{\overline{\mathbf{e}}}$: $\mathcal{T}_{\overline{\mathbf{e}}} = [0,1] \,/\! \simeq$ as was explained in subsection 2.3.

Proposition 3.3 *The following properties hold almost surely.*
(i) *For every $s, t, u \in [0,1]$,*

$$D(s,s) = 0$$
$$D(s,t) = D(t,s)$$

and

$$D(s,u) \leq D(s,t) + D(t,u).$$

(ii) *For every $s, t \in [0,1]$,*

$$D(s,t) \leq D^\circ(s,t).$$

(iii) *For every $s, t \in [0,1]$, the property $s \simeq t$ implies $D(s,t) = 0$.*
(iv) *For every $s \in [0,1]$, $D(0,s) = \overline{Z}_s$.*

Proof: Except for the first one, the properties in (i) are immediate from the analogous properties for d_n and the (almost sure) convergence (11). Similarly, (ii) follows from Lemma 3.1 and the convergence (6), which holds a.s. along the sequence $(n_k)_{k\geq 1}$ if (11) also holds a.s. along this sequence. The first property in (i) then readily follows from (ii).

Let us prove (iii). Let $s,t \in [0,1]$ with $s<t$. If $s \simeq t$, we have

$$\overline{\mathbf{e}}_s = \overline{\mathbf{e}}_t = \inf_{s\leq r\leq t} \overline{\mathbf{e}}_r.$$

Suppose first that $\overline{\mathbf{e}}_r > \overline{\mathbf{e}}_s$ for every $r \in]s,t[$. From the uniform convergence of the function $\frac{1}{2}\sqrt{p/(p-1)}\, n^{-1/2}\, C^n_{pnt}$ towards $\overline{\mathbf{e}}_t$, an elementary argument yields the existence of two sequences (i_n) and (j_n) of integers in $[pn]$ such that:

- $\dfrac{i_n}{pn} \longrightarrow s$ and $\dfrac{j_n}{pn} \longrightarrow t$ as $n \to \infty$.
- For n sufficiently large, $j_n \geq i_n + 2$ and $C^n_{i_n} = C^n_{j_n} < \inf\limits_{i_n<k<j_n} C^n_k$.

As we already noticed in subsection 2.2, the last property ensures that $i_n \sim_n j_n$ and thus $d_n(i_n, j_n) = 0$. By passing to the limit $n \to \infty$, we get $D(s,t) = 0$.

If $\overline{\mathbf{e}}_r = \overline{\mathbf{e}}_s$ for some $r \in]s,t[$, then r is necessarily unique, because otherwise the tree $\mathcal{T}_{\overline{\mathbf{e}}}$, which is isometric to $\mathcal{T}_{\mathbf{e}}$, would have a point with multiplicity strictly greater than 3. By the preceding argument, $D(s,r) = D(r,t) = 0$ and thus $D(s,t) = 0$ by the triangle inequality in (i).

Let us finally prove (iv). Let $s \in [0,1]$ and let (i_n) be a sequence of integers such that $i_n/(pn) \longrightarrow s$ as $n \to \infty$. From the properties of the Bouttier-Di Francesco-Guitter bijection, we know that $d_n(0,i_n) = V^n_{i_n}$. On the other hand, (11) ensures that $(9/(4p(p-1)))^{1/4} n^{-1/4} d_n(0,i_n)$ converges to $D(0,s)$, and that $(9/(4p(p-1)))^{1/4} n^{-1/4} V^n_{i_n}$ converges to $\overline{Z}_s$. The desired result follows. □

We define an equivalence relation $\approx$ on $[0,1]$ by setting

$$s \approx t \quad \text{if and only if} \quad D(s,t) = 0.$$

Clearly, D induces a metric, which we still denote by D, on the quotient set $[0,1]\,/\approx$. The bound $D \leq D^\circ$ ensures that the canonical projection from $[0,1]$ onto $[0,1]\,/\approx$ is continuous when $[0,1]\,/\approx$ is equipped with the metric D. In particular the metric space $([0,1]\,/\approx, D)$ is compact.

For our purposes, it will be convenient to view this metric space as a quotient of the real tree $\mathcal{T}_{\overline{\mathbf{e}}}$. By property (iii) of the previous proposition, we may define $D(a,b)$ for $a,b \in \mathcal{T}_{\overline{\mathbf{e}}} = [0,1]\,/\simeq$ simply by setting $D(a,b) = D(s,t)$ where s, resp. t, is any representative of a, resp. b, in $[0,1]$. The equivalence relation $\approx$ then makes sense on $\mathcal{T}_{\overline{\mathbf{e}}}$, and the quotient space $(\mathcal{T}_{\overline{\mathbf{e}}}\,/\approx, D)$ is obviously isometric to $([0,1]\,/\approx, D)$. As a consequence of Proposition 3.3 (iv) and the triangle inequality, for every $a,b \in \mathcal{T}_{\overline{\mathbf{e}}}$, the condition $D(a,b) = 0$ implies $\overline{Z}_a = \overline{Z}_b$.

Before stating the main result, we need to introduce some additional notation. For every $a,b \in \mathcal{T}_{\overline{\mathbf{e}}}$, we set

$$D^\circ(a,b) = \inf\{D^\circ(s,t) : s,t \in [0,1], a \simeq s, b \simeq t\}.$$

Suppose that neither of the relations $a \prec b$ and $b \prec a$ holds, and assume for definiteness that $a < b$. Then the infimum in the definition of $D^\circ(a,b)$ is attained when $[s,t]$ is the minimal subinterval of $[0,1]$ such that $a \simeq s$ and $b \simeq t$, and it follows that

$$D^\circ(a,b) = \overline{Z}_a + \overline{Z}_b - 2 \inf_{c \in [a,b]} \overline{Z}_c$$

where $[a,b]$ is the lexicographical interval between a and b in $\mathcal{T}_{\mathbf{\overline{e}}}$, as defined in subsection 2.3. On the other hand, if $a \prec b$, then the preceding formula does not necessarily hold: We have instead

$$D^\circ(a,b) = \overline{Z}_a + \overline{Z}_b - 2 \sup\left(\inf_{c \in [a,b]} \overline{Z}_c, \inf_{c \in [b,a]} \overline{Z}_c \right).$$

The function $D^\circ(a,b)$, $a,b \in \mathcal{T}_{\mathbf{\overline{e}}}$ needs not satisfy the triangle inequality. For this reason, we set for every $a,b \in \mathcal{T}_{\mathbf{\overline{e}}}$,

$$D^*(a,b) = \inf \left\{ \sum_{i=1}^{q} D^\circ(a_{i-1}, a_i) \right\}$$

where the infimum is over all choices of the integer $q \geq 1$ and of the finite sequence $a_0, a_1, \ldots, a_q$ in $\mathcal{T}_{\mathbf{\overline{e}}}$ such that $a_0 = a$ and $a_q = b$.

Since $D \leq D^\circ$, and D satisfies the triangle inequality, it is clear that we have

$$0 \leq D(a,b) \leq D^*(a,b) \leq D^\circ(a,b)$$

for every $a,b \in \mathcal{T}_{\mathbf{\overline{e}}}$.

We can now state our main result. Recall that we are restricting our attention to values of n belonging to the sequence $(n_k)_{k\geq 1}$, and that we assume that the convergence (11) holds a.s. along this sequence.

Theorem 3.4 *We have almost surely*

$$\left(\mathbf{m}_n, \left(\frac{9}{4p(p-1)}\right)^{1/4} n^{-1/4} d_n \right) \underset{n\to\infty}{\longrightarrow} (\mathcal{T}_{\mathbf{\overline{e}}} / \approx, D)$$

in the sense of the Gromov-Hausdorff distance on compact metric spaces. In addition, a.s. for every $a,b \in \mathcal{T}_{\mathbf{\overline{e}}}$, the relation $a \approx b$ holds if and only if one of the following equivalent properties holds:

(i) $D(a,b) = 0$.

(ii) $D^*(a,b) = 0$.

(iii) $D^\circ(a,b) = 0$.

Remarks. (a) Although the process D may depend on the sequence $(n_k)_{k\geq 1}$, the equivalence relation $\approx$ does not, since it can be defined by either (ii) or (iii) in Theorem 3.4. As was already observed in the introduction, this guarantees that the limiting compact metric space $(\mathcal{T}_{\mathbf{\overline{e}}} / \approx, D)$ is homeomorphic to $(\mathcal{T}_{\mathbf{\overline{e}}} / \approx, D^*)$, and thus that its topology does not depend on the choice of the sequence $(n_k)_{k\geq 1}$ (nor on the value of p). Still it is tempting to conjecture that

$D(a,b) = D^*(a,b)$, for every $a,b \in \mathcal{T}_{\overline{\mathbf{e}}}$. If this conjecture is correct, the convergence (11), or that of Theorem 3.4, does not require the use of a subsequence.

(b) It is not hard to prove that equivalence classes in $\mathcal{T}_{\overline{\mathbf{e}}}$ for the equivalence relation $\approx$ can contain only 1, 2 or 3 points. For every fixed $s \in [0,1]$, it is easy to verify that the equivalence class of $p_{\overline{\mathbf{e}}}(s)$ is a singleton a.s. Furthermore, one can check that a.s. for every rational numbers r,s,t,u such that $0 < r < s < t < u < 1$ one has

$$\inf_{r \le x \le s} \overline{Z}_x \neq \inf_{t \le x \le u} \overline{Z}_x.$$

(The easiest way to derive this property is to use the Brownian snake approach that is presented below in Section 5.) It follows that an equivalence class cannot contain more than 3 points. Conversely, if we are given two rationals $0 < r < s < 1$, there exists an a.s. unique $y \in]r,s[$ such that

$$\overline{Z}_y = \inf_{r \le x \le s} \overline{Z}_x,$$

and the vertex of $\mathcal{T}_{\overline{\mathbf{e}}}$ corresponding to y is a leaf of $\mathcal{T}_{\overline{\mathbf{e}}}$. Set $t_1 = \sup\{u \le r : \overline{Z}_u = \overline{Z}_y\}$ and $t_2 = \inf\{u \ge s : \overline{Z}_u = \overline{Z}_y\}$. Then $t_1 \approx y \approx t_2$, and t_1, y and t_2 correspond to different vertices of the tree $\mathcal{T}_{\overline{\mathbf{e}}}$. To summarize, the equivalence class of a typical vertex $a \in \mathcal{T}_{\overline{\mathbf{e}}}$ is a singleton, but there is a continuum of equivalence classes consisting of pairs, and there are countably many equivalence classes containing three elements. These properties are not used below. They are derived in greater detail in the subsequent paper [24] where they play an important role (see Lemma 3.1 in [24]).

Proof of Theorem 3.4 (first part): The main difficulty in the proof of Theorem 3.4 comes from the implication (i)⇒(iii). Notice that the other implications (iii)⇒(ii)⇒(i) are trivial. The implication (i)⇒(iii) is established in the next section. We now prove the first assertion of Theorem 3.4.

Recall that the metric spaces $(\mathcal{T}_{\overline{\mathbf{e}}} / \approx, D)$ and $([0,1] / \approx, D)$ are isometric. For every integer n, consider the equivalence relation $\approx_n$ defined on $[0,1]$ by setting

$$s \approx_n t \quad \text{if and only if} \quad d_n(\lfloor pns \rfloor, \lfloor pnt \rfloor) = 0.$$

Clearly, the quotient space $E_n := [0,1] /\!\approx_n$ equipped with the metric $\delta_n(s,t) = d_n(\lfloor pns \rfloor, \lfloor pnt \rfloor)$ is isometric to (τ_n°, d_n) or equivalently to $(\mathbf{m}_n \backslash \{\partial_n\}, d_n)$.

Since

$$d_{GH}((\mathbf{m}_n \backslash \{\partial_n\}, n^{-1/4} d_n), (\mathbf{m}_n, n^{-1/4} d_n)) \underset{n\to\infty}{\longrightarrow} 0$$

the first part of Theorem 3.4 reduces to checking that we have a.s.

$$d_{GH}\left(\left(E_n, \left(\frac{9}{4p(p-1)}\right)^{1/4} n^{-1/4} \delta_n\right), (E_\infty, D)\right) \underset{n\to\infty}{\longrightarrow} 0 \tag{12}$$

where $E_\infty := [0,1] / \approx$.

To this end, we construct a correspondence between the metric spaces E_n and E_∞ by setting

$$\mathcal{C}_n = \{(a,b) \in E_n \times E_\infty : \text{there exists } t \in [0,1] \text{ such that } a \approx_n t \text{ and } b \approx t\}.$$

In order to bound the distortion of this correspondence, consider two pairs $(a,b),(a',b') \in \mathcal{C}_n$. By definition, there exist $s,t \in [0,1]$ such that $a \approx_n s, b \approx s$ and $a' \approx_n t, b' \approx t$. Then we have

$$\begin{aligned}\delta_n(a,a') &= d_n(\lfloor pns \rfloor, \lfloor pnt \rfloor)\\ D(b,b') &= D(s,t).\end{aligned}$$

Thus, when E_n is equipped with the distance $(9/(4p(p-1)))^{1/4}\, n^{-1/4}\delta_n$, and E_∞ with the distance D, the distortion of $\mathcal{C}_n$ is

$$\begin{aligned}&\sup_{(a,b),(a',b')\in\mathcal{C}_n} \left| \left(\frac{9}{4p(p-1)}\right)^{1/4} n^{-1/4}\delta_n(a,a') - D(b,b') \right|\\ &\quad\le \sup_{s,t\in[0,1]} \left| \left(\frac{9}{4p(p-1)}\right)^{1/4} n^{-1/4} d_n(\lfloor pns \rfloor, \lfloor pnt \rfloor) - D(s,t) \right|,\end{aligned}$$

which tends to 0 a.s. by (11). The first assertion of Theorem 3.4 now follows from the known result connecting the Gromov-Hausdorff distance between two compact metric spaces with the infimum of the distortion of correspondences between these two spaces (Theorem 7.3.25 in [10]). □

Before proceeding to the second part of the proof of Theorem 3.4, let us state and prove a closely related result.

Proposition 3.5 *Let $k \ge 1$ be an integer. For every $n \ge 1$, let $Y_1^n, \ldots, Y_k^n$ be k random variables which conditionally given M_n are independent and uniformly distributed over $\mathbf{m}_n$. Also, given the triple $(\overline{\mathbf{e}}, \overline{Z}, D)$, let $Y_1^\infty, \ldots, Y_k^\infty$ be random variables with values in $\mathcal{T}_{\overline{\mathbf{e}}}$ which are independent and distributed according to λ. Then,*

$$\left(\left(\frac{9}{4p(p-1)}\right)^{1/4} n^{-1/4}\, d_n(Y_i^n, Y_j^n) \right)_{1\le i\le k, 1\le j\le k} \xrightarrow[n\to\infty]{\text{(d)}} \left(D(Y_i^\infty, Y_j^\infty) \right)_{1\le i\le k, 1\le j\le k}.$$

Remarks. (a) Informally, Proposition 3.5 means that the convergence in Theorem 3.4 can be reinforced in the sense of convergence of measured metric spaces, provided $\mathbf{m}_n$ is equipped with the uniform probability measure and $\mathcal{T}_{\overline{\mathbf{e}}}/\approx$ is equipped with the image of λ under the canonical projection. We could give other versions of this reinforcement: See Chapter $3\frac{1}{2}$ of the book [17] for various notions of convergence of measured metric spaces. Here we content ourselves with the preceding proposition, which will be useful in Section 6 below.
(b) The reader may be puzzled by our assumption on $Y_1^\infty, \ldots, Y_k^\infty$, since we seem to be dealing with random variables taking values in a *random* state space. It is however a straightforward matter to give a mathematically rigorous (although less intuitive) version of the statement of the proposition.

Proof: Recall that $\mathbf{m}_n = \tau_n^\circ \cup \{\partial_n\}$. We may and will assume that $Y_1^n, \ldots, Y_k^n$ are uniformly distributed over τ_n° rather than over $\mathbf{m}_n$.

Then let $U_1, \ldots, U_k$ be k independent random variables which are uniformly distributed over $[0,1]$ and independent of all other random quantities we have considered until now. We may then take $Y_i^\infty = p_{\mathbf{e}}(U_i)$ for every $1 \le i \le k$. Also, for every $1 \le i \le k$, we let $\widetilde{Y}_i^n$ be the equivalent class of $\lfloor pnU_i \rfloor$ in the quotient set $[pn] / \sim_n = \tau_n^\circ$. The (almost sure) convergence (11) implies that

$$\left(\Big(\frac{9}{4p(p-1)}\Big)^{1/4} n^{-1/4}\, d_n(\widetilde{Y}_i^n, \widetilde{Y}_j^n)\right)_{1\le i\le k, 1\le j\le k} \underset{n\to\infty}{\overset{\text{(a.s.)}}{\longrightarrow}} \big(D(Y_i^\infty, Y_j^\infty)\big)_{1\le i\le k, 1\le j\le k}.$$

This does not immediately give us the desired result, because the variables $\widetilde{Y}_i^n$ are not uniformly distributed over τ_n°. Still we will see that in a sense they are close enough to variables that have the desired uniform distribution. To this end, for every n and every $1 \le i \le k$, set

$$k_i^n = \lceil ((p-1)n + 2)U_i \rceil$$

and let Y_i^n be the k_i^n-th element in the sequence of vertices of τ_n° listed in lexicographical order. Clearly, the variables Y_i^n have the properties stated in the proposition. To complete the proof, it is therefore enough to check that, for every $1 \le i \le k$,

$$n^{-1/4} d_n(Y_i^n, \widetilde{Y}_i^n) \underset{n\to\infty}{\overset{\text{a.s.}}{\longrightarrow}} 0. \tag{13}$$

Note that a.s. for every $t \in]0,1]$, the number of distinct vertices of τ_n° that appear in the depth-first search sequence before rank $\lfloor pnt \rfloor$ behaves as $(p-1)nt$ when $n \to \infty$. To see this, observe that in the evolution of the contour function of τ_n° each step which is not downwards corresponds in the depth-first search sequence to a vertex of τ_n° that has not been visited before, and then use (11) to see that the number of downward steps before time $\lfloor pnt \rfloor$ behaves like nt when $n \to \infty$ (indeed the difference between the numbers of upward and downward steps is $O(n^{1/2})$ as $n \to \infty$).

From the preceding remarks, we get that a.s. for every $s, t \in [0,1]$ such that $s < U_i < t$, if n is large enough, the vertex Y_i^n is visited by the depth-first search sequence during the time interval $[\lfloor pns \rfloor, \lfloor pnt \rfloor]$. Thus, for n sufficiently large,

$$n^{-1/4} d_n(Y_i^n, \widetilde{Y}_i^n) \le n^{-1/4} \sup_{\lfloor pns \rfloor \le j \le \lfloor pnt \rfloor} d_n(j, \lfloor pnU_i \rfloor).$$

The right-hand side can be made arbitrarily small when n is large by choosing s and t sufficiently close to U_i. This completes the proof of (13) and of Proposition 3.5. □

4 The key step

This section is devoted to the second part of the proof of Theorem 3.4, that is to the proof of the implication (i)⇒(iii) in this theorem. We start with a lemma.

Lemma 4.1 *Almost surely, for every $a, b \in \mathcal{T}_{\mathbf{e}}$, the condition $D(a,b) = 0$ implies $\overline{Z}_c \ge \overline{Z}_a = \overline{Z}_b$ for every $c \in [\![a, b]\!]$.*

Proof: We already noticed that the condition $D(a,b) = 0$ forces $\overline{Z}_a = \overline{Z}_b$. We can immediately exclude the case $a = \overline{\rho}$ since this would imply $\overline{Z}_b = \overline{Z}_a = 0$ and $b = \overline{\rho} = a$. Then we can assume without loss of generality that $a < b$. We argue by contradiction, assuming that there exists $c \in]\!]a, b[\![$ such that $\overline{Z}_c < \overline{Z}_a$. For definiteness, we assume that $c \in [\![a \wedge b, a[\![$. The symmetric case $c \in [\![a \wedge b, b[\![$ is treated in a similar manner.

Let $s < t$ be such that $a \simeq s$ and $b \simeq t$. We can then find $r \in]s, t[$ such that $c \simeq r$. Choose $i_n, j_n, k_n \in [pn]$, with $i_n \leq k_n \leq j_n$, such that $i_n/(pn) \longrightarrow s$, $j_n/(pn) \longrightarrow t$ and $k_n/(pn) \longrightarrow r$. Denote by a_n, b_n, c_n the vertices in τ_n° corresponding respectively to i_n, j_n, k_n. Since $c \in [\![a \wedge b, a[\![$, a simple argument using the convergence of the first components in (11), and the remarks of the beginning of subsection 2.2, shows that k_n can be chosen in such a way that $c_n \in]\!]a_n \wedge b_n, a_n[\![$ for every n sufficiently large. Denote by $\tau_n^\circ(c_n)$ the set of all descendants of c_n in τ_n°. Then $a_n \in \tau_n^\circ(c_n)$ but $b_n \notin \tau_n^\circ(c_n)$.

By (11) and our assumption $D(a, b) = 0$ we know that $d_n(a_n, b_n) = o(n^{1/4})$ as $n \to \infty$. Let $\gamma_n = (\gamma_n(i), 0 \leq i \leq d_n(a_n, b_n))$ be a geodesic path from a_n to b_n in the map M_n. When n is large, the path γ_n must lie entirely in τ_n°, because if ∂_n belongs to this path the equality $d_n(a_n, b_n) = d_n(a_n, \partial_n) + d_n(\partial_n, b_n) = V_{i_n}^n + V_{j_n}^n$ yields a contradiction with the property $d_n(a_n, b_n) = o(n^{1/4})$.

Denote by g_n the last point on the geodesic γ_n that belongs to $\tau_n^\circ(c_n)$. Since g_n is a point of the geodesic γ_n and $d_n(a_n, b_n) = o(n^{1/4})$, we have

$$\ell_{g_n}^n = d_n(\partial_n, g_n) = d_n(\partial_n, a_n) + o(n^{1/4}) = n^{1/4}\overline{Z}_a + o(n^{1/4})$$

as $n \to \infty$. On the other hand, since $k_n/(pn) \longrightarrow r$ and $c \simeq r$,

$$\ell_{c_n}^n = V_{k_n}^n = n^{1/4}\overline{Z}_c + o(n^{1/4})$$

as $n \to \infty$. Hence, for n large we must have $\ell_{g_n}^n > \ell_{c_n}^n$.

Using the way edges of the map M_n are reconstructed from the mobile θ_n, we now see that any edge starting from g_n in M_n connects g_n with another point of $\tau_n^\circ(c_n)$. Indeed, any successor of the vertex g_n must clearly lie in $\tau_n^\circ(c_n)$ because in the depth-first search sequence of τ_n°, a vertex with label $\ell_{g_n}^n - 1$ will be visited after the last visit of g_n before coming back to c_n and exiting the tree $\tau_n^\circ(c_n)$. Similarly, g_n cannot be a successor of a vertex $h \notin \tau_n^\circ(c_n)$: If this were the case we would have $\ell_h^n - 1 = \ell_{g_n}^n > \ell_{c_n}^n$, and the depth-first search sequence of τ_n° would visit a vertex with label $\ell_h^n - 1$ after visiting h before entering the set $\tau_n^\circ(c_n)$. Finally, the fact that g_n is not connected to any point outside $\tau_n^\circ(c_n)$ gives a contradiction with our choice of g_n. □

Proposition 4.2 *Almost surely, for every pair (a, b) in $\mathcal{T}_{\overline{\mathbf{e}}}$ such that a is an ancestor of b and $a \neq b$, we have $D(a, b) > 0$.*

Proof: We argue by contradiction, assuming that there exists a pair (a, b) in $\mathcal{T}_{\overline{\mathbf{e}}}$ such that a is an ancestor of b, $a \neq b$ and $D(a, b) = 0$. Notice that the case $a = \overline{\rho}$ is excluded since we already know from Proposition 3.3 (iv) that $D(\overline{\rho}, b) = \overline{Z}_b > 0$ for every $b \neq \overline{\rho}$. So we assume that $a \neq \overline{\rho}$. Recall that we have automatically $\overline{Z}_a = \overline{Z}_b$.

Let $s, t \in [0, 1]$ be such that $a \simeq s$ and $b \simeq t$. Since a is an ancestor of b we can choose s and t such that $s < t$ and $\overline{\mathbf{e}}_r > \overline{\mathbf{e}}_s$ for every $r \in]s, t]$. Since $D(s, t) = D(a, b) = 0$, (11) gives

$$n^{-1/4} d_n(pns, pnt) \underset{n \to \infty}{\longrightarrow} 0.$$

So, for every n, we can find $i_n^\circ, j_n \in [pn]$ such that $i_n^\circ \le j_n$, $|i_n^\circ - pns| \le 1$, $|j_n - pnt| \le 1$ and

$$n^{-1/4} d_n(i_n^\circ, j_n) \underset{n\to\infty}{\longrightarrow} 0.$$

Let $i_n = \sup\{k \in [i_n^\circ, j_n] \cap \mathbb{Z} : C_k^n = C_{i_n^\circ}^n\}$. By (11) and the condition $\overline{\mathbf{e}}_r > \overline{\mathbf{e}}_s$ for every $r \in]s,t]$, we must have $n^{-1}(i_n - i_n^\circ) \longrightarrow 0$ as $n \to \infty$. As a consequence, $n^{-1/4} d_n(i_n^\circ, i_n) \longrightarrow 0$. Let a_n and b_n be the vertices in τ_n° such that $a_n \sim_n i_n$ and $b_n \sim_n j_n$. Then provided n is sufficiently large, the remarks of subsection 2.2 show that a_n is an ancestor of b_n. Moreover we have

$$n^{-1/4} d_n(a_n, b_n) = n^{-1/4} d_n(i_n, j_n) \underset{n\to\infty}{\longrightarrow} 0. \tag{14}$$

By Lemma 4.1, we also know that $\overline{Z}_c \ge \overline{Z}_a$ for every $c \in [\![a,b]\!]$. Recall that the conditioned tree $(\mathcal{T}_{\overline{\mathbf{e}}}, \overline{Z})$ is obtained by re-rooting the unconditioned tree $(\mathcal{T}_{\mathbf{e}}, Z)$ at the vertex corresponding to the minimal spatial position, and that along a given line segment of $\mathcal{T}_{\mathbf{e}}$, Z evolves like linear Brownian motion. Since local minima of linear Brownian motion are distinct, a simple argument then shows that the equality $\overline{Z}_c = \overline{Z}_a$ can hold for at most one value of $c \in]\!]a,b[\![$. Hence, we can find $\eta > 0$ such that the properties $c \in]\!]a,b[\![$ and $d_{\overline{\mathbf{e}}}(a,c) < \eta$ imply $\overline{Z}_c > \overline{Z}_a$.

Since $\overline{\mathbf{e}}_r > \overline{\mathbf{e}}_s$ for every $r \in]s,t]$, Lemma 2.2 implies that for every $\varepsilon > 0$,

$$\inf_{r \in [s, s+\varepsilon]} \overline{Z}_r < \overline{Z}_s.$$

It follows that there exists one (in fact infinitely many) subtree $\mathcal{T}^1$ from the left side of $[\![a,b]\!]$, with root $\rho^1 \in]\!]a,b[\![$, such that $d_{\overline{\mathbf{e}}}(a,\rho^1) < \eta$ and

$$\inf_{c \in \mathcal{T}^1} \overline{Z}_c < \overline{Z}_a.$$

We denote by $[\alpha, \beta]$ the interval coding $\mathcal{T}^1$: The elements of $\mathcal{T}^1$ are exactly the equivalence classes in $\mathcal{T}_{\overline{\mathbf{e}}}$ of the reals in $[\alpha,\beta]$, and in particular $\rho^1 \simeq \alpha \simeq \beta$. In a similar way, using a time reversal argument, we can construct a subtree $\mathcal{T}^2$ from the right side of $[\![a,b]\!]$, with root $\rho^2 \in]\!]a,b[\![$, such that $d_{\overline{\mathbf{e}}}(a,\rho^2) < \eta$ and

$$\inf_{c \in \mathcal{T}^2} \overline{Z}_c < \overline{Z}_a.$$

We can always choose $\mathcal{T}^1$ and $\mathcal{T}^2$ in such a way that $\rho^1 \prec \rho^2$. From our choice of η, we have then

$$\inf_{c \in [\![\rho^1, \rho^2]\!]} \overline{Z}_c > \overline{Z}_a.$$

We now exploit the convergence (11) to get similar properties for the discrete trees (τ_n°, ℓ^n). We can find a positive number κ such that the following holds for n sufficiently large. There exists a subtree τ_n^1 from the left side of $[\![a_n, b_n]\!]$ with root $\rho_n^1 \in]\!]a_n, b_n[\![$ such that

$$\inf_{x \in \tau_n^1} \ell_x^n \le \ell_{a_n}^n - \kappa n^{1/4}. \tag{15}$$

The subtree τ_n^1 is coded by an interval $[\alpha_n, \beta_n] \cap \mathbb{Z}$ (via the identification $\tau_n^\circ = [pn]/\sim_n$) such that $\alpha_n/(pn) \longrightarrow \alpha$ and $\beta_n/(pn) \longrightarrow \beta$. Similarly, there exists a subtree τ_n^2 from the right side of $[\![a_n, b_n]\!]$ with root $\rho_n^2 \in]\!]a_n, b_n[\![$ such that

$$\inf_{x \in \tau_n^2} \ell_x^n \le \ell_{a_n}^n - \kappa n^{1/4}. \tag{16}$$

Furthermore, $\rho_n^1 \prec \rho_n^2$ and

$$\inf_{x\in[\![\rho_n^1,\rho_n^2]\!]} \ell_x^n \geq \ell_{a_n}^n + \kappa n^{1/4}. \tag{17}$$

Let $\gamma_n = (\gamma_n(i), 0 \leq i \leq d_n(a_n, b_n))$ be a geodesic path from a_n to b_n in M_n. As in the proof of Lemma 4.1, we know that the path γ_n lies in τ_n° when n is large. Denote by u_n the last point on the geodesic γ_n that does not belong to the set

$$\{x \in \tau_n^\circ : \rho_n^1 \leq x \ll \rho_n^2\}.$$

This definition makes sense because $a_n < \rho_n^1$. Also $u_n \neq b_n$ since $\rho_n^1 \leq b_n \ll \rho_n^2$. Denote by v_n the point following u_n on the geodesic γ_n.

Since $n^{-1/4}d_n(a_n, b_n) \longrightarrow 0$ as $n \to \infty$, we know that

$$n^{-1/4} \sup_{0\leq i\leq d_n(a_n,b_n)} d_n(a_n, \gamma_n(i)) \underset{n\to\infty}{\longrightarrow} 0,$$

and therefore

$$n^{-1/4} \sup_{0\leq i\leq d_n(a_n,b_n)} |\ell_{a_n}^n - \ell_{\gamma_n(i)}^n| \underset{n\to\infty}{\longrightarrow} 0. \tag{18}$$

The preceding properties imply that $v_n \in \tau_n^1 \cup \tau_n^2$ for n sufficiently large. Indeed, we have $\rho_n^1 \leq v_n \ll \rho_n^2$ by construction and we also know that $\ell_{v_n}^n > \ell_{a_n}^n - \kappa n^{1/4}$ if n is large, by (18). Suppose that $v_n \notin \tau_n^1 \cup \tau_n^2$. Then, by (15) and (16), v_n can be connected to a point y that does not belong to $\{x \in \tau_n^\circ : \rho_n^1 \leq x \ll \rho_n^2\}$ only if $y \in [\![\rho_n^1, \rho_n^2]\!]$. Thus we get that $u_n \in [\![\rho_n^1, \rho_n^2]\!]$, but this is impossible by (17) and (18), if n is large enough.

So, for n sufficiently large, we have either $v_n \in \tau_n^1$ or $v_n \in \tau_n^2$. One of these two cases has to occur infinitely often. For definiteness, we assume that the property $v_n \in \tau_n^1$ occurs infinitely often and from now on until the final part of the proof we restrict our attention to integers n such that this property holds.

Then the following properties hold for n large:

(i) $v_n \in \tau_n^1$ and $v_n \neq \rho_n^1$.

(ii) $u_n \leq \rho_n^1$.

(iii) Every point w that comes after v_n on the geodesic γ_n satisfies $\rho_n^1 \leq w$.

The property $v_n \neq \rho_n^1$ is clear from (18) and (17). To get (ii), recall that by construction we have either $u_n \leq \rho_n^1$ or $\rho_n^2 \ll u_n$ (or both together). Suppose that $\rho_n^2 \ll u_n$. If n is large, the fact that u_n is connected with a point of $\tau_n^1\backslash\{\rho_n^1\}$ and the property (16) then imply that $u_n \in [\![\varnothing, \rho_n^2]\!]$. However $u_n \in [\![\rho_n^1, \rho_n^2]\!]$ is excluded by (17) and (18), and thus we get $u_n \in [\![\varnothing, \rho_n^1]\!]$, so that in particular $u_n \leq \rho_n^1$. Finally, (iii) is clear from the definition of u_n.

Thanks to (i)–(iii), we can apply Lemma 2.1, and we get that if n is large enough, for every point y of $\tau_n^1\backslash\{\rho_n^1\}$ such that

$$\ell_x^n > \sup_{0\leq i\leq d_n(a_n,b_n)} \ell_{\gamma_n(i)}^n \quad \text{for every } x \in [\![\rho_n^1, y]\!] \tag{19}$$

we have

$$d_n(y,b_n) \le d_n(u_n,b_n) + \ell^n_y - \inf_{0\le i\le d_n(a_n,b_n)} \ell^n_{\gamma_n(i)}. \tag{20}$$

For every $\varepsilon > 0$, denote by $\mathcal{U}^\varepsilon_n$ the set of all vertices $y \in \tau^1_n$ such that:

- $\ell^n_y \le \ell^n_{a_n} + \frac{3\varepsilon}{2} n^{1/4}$;
- $\ell^n_x \ge \ell^n_{a_n} + \frac{\varepsilon}{16} n^{1/4}$, for every $x \in [\![\rho^1_n, y]\!]$.

Recall that $[\alpha,\beta]$ is the interval coding $\mathcal{T}^1$ and that $s \simeq a$. We denote by $\mathcal{U}^\varepsilon_\infty$ the set of all $r \in [\alpha,\beta]$ such that

- $\overline{Z}_r < \overline{Z}_s + (\frac{9}{4p(p-1)})^{1/4}\,\varepsilon$;
- $\overline{Z}_{r'} > \overline{Z}_s + (\frac{9}{4p(p-1)})^{1/4}\,\frac{\varepsilon}{8}$, for every $r' \in [\![\rho^1, r]\!]$.

(When writing $[\![\rho^1, r]\!]$ we slightly abuse notation by identifying r with the corresponding vertex in $\mathcal{T}_{\overline{\mathbf{e}}}$.) Notice that $\mathcal{U}^\varepsilon_\infty$ is open.

Moreover, let $]u,v[$ be a connected component of $\mathcal{U}^\varepsilon_\infty$, and let $[u',v']$ be a compact subinterval of $]u,v[$. We claim that for every n sufficiently large, we must have

$$[pnu', pnv'] \cap \mathbb{Z} \subset \mathcal{U}^\varepsilon_n \tag{21}$$

in the sense that every vertex y of τ°_n such that $y \sim_n k$ for some $k \in [pnu', pnv'] \cap \mathbb{Z}$ belongs to $\mathcal{U}^\varepsilon_n$. To see this, first note that the property $[pnu', pnv'] \cap \mathbb{Z} \subset \tau^1_n$ holds for n sufficiently large because $]u,v[\subset [\alpha,\beta]$. Then suppose that for every n belonging to a subsequence converging to ∞ we can find a vertex $y_n \in \tau^1_n$ such that $y_n \sim_n k_n$ for some $k_n \in [pnu', pnv'] \cap \mathbb{Z}$ and at least one of the two conditions

(a) $\ell^n_{y_n} \le \ell^n_{a_n} + \frac{3\varepsilon}{2} n^{1/4}$,

(b) $\ell^n_x \ge \ell^n_{a_n} + \frac{\varepsilon}{16} n^{1/4}$, for every $x \in [\![\rho^1_n, y_n]\!]$,

does not hold. By compactness we can assume that $k_n/(pn) \longrightarrow r \in [u',v']$. If condition (a) fails for infinitely many values of n, (11) gives

$$\overline{Z}_r \ge \overline{Z}_s + \Big(\frac{9}{4p(p-1)}\Big)^{1/4} \frac{3\varepsilon}{2},$$

which contradicts the fact that $[u',v'] \subset \mathcal{U}^\varepsilon_\infty$. If (b) fails for infinitely values of n, then for these values of n we can find $\overline{k}_n \in [\alpha_n, k_n] \cap \mathbb{Z}$ such that

$$C^n_{\overline{k}_n} = \inf_{\overline{k}_n \le k \le k_n} C^n_k$$

and

$$V^n_{\overline{k}_n} < \ell^n_{a_n} + \frac{\varepsilon}{16} n^{1/4}.$$

Again by compactness, we can assume that $\overline{k}_n/(pn) \longrightarrow \overline{r} \in [\alpha, r]$. We have then

$$\overline{\mathbf{e}}_{\overline{r}} = \inf_{\overline{r} \le r' \le r} \overline{\mathbf{e}}_{r'}$$

so that $\overline{r} \in [\![\rho^1, r]\!]$, and

$$\overline{Z}_{\overline{r}} \le \overline{Z}_s + \Big(\frac{9}{4p(p-1)}\Big)^{1/4} \frac{\varepsilon}{16}$$

thus contradicting the fact that $[u', v'] \subset \mathcal{U}^\varepsilon_\infty$. This completes the proof of our claim (21).

If I is a finite union of closed subintervals of $[0,1]$ the number of vertices of $\tau_n^\circ = [pn]/\sim_n$ for which the first representative in $[pn]$ belongs to pnI behaves like $(p-1)n|I|$ as $n \to \infty$, where $|I|$ denotes the Lebesgue measure of I. When I is of the type $[0,t]$, this was observed in the proof of Proposition 3.5, and the general case follows by a simple argument. Thus (21) implies that

$$\liminf_{n\to\infty} \frac{1}{(p-1)n} \, \#\mathcal{U}^\varepsilon_n \ge \lambda(\mathcal{U}^\varepsilon_\infty). \tag{22}$$

We can now use (14), (18) and (20) to see that for n sufficiently large, for every $y \in \mathcal{U}^\varepsilon_n \backslash \{\rho^1_n\}$, we have

$$d_n(y, b_n) \le 2\varepsilon\, n^{1/4}.$$

(Notice that condition (19) is satisfied for every $y \in \mathcal{U}^\varepsilon_n$ when n is large enough.) Hence, for every $y, y' \in \mathcal{U}^\varepsilon_n$ we have also

$$d_n(y, y') \le 4\varepsilon\, n^{1/4}.$$

Recall that $B_n(y,R)$ denotes the closed ball with radius R centered at y in the metric space $(\mathbf{m}_n, d_n)$. We have thus $\#B_n(y, 4\varepsilon n^{1/4}) \ge \#\mathcal{U}^\varepsilon_n$ for every $y \in \mathcal{U}^\varepsilon_n$.

Let (ε_k) be any fixed sequence monotonically decreasing to 0. By Lemma 2.4, we can find $\delta_0 > 0$ and an integer k_0 such that for every $k \ge k_0$,

$$\lambda(\mathcal{U}^{\varepsilon_k}_\infty) \ge 2\delta_0\, \varepsilon_k^2.$$

From (22) we then see that for every $k \ge k_0$, if n is sufficiently large, we have

$$\#\mathcal{U}^{\varepsilon_k}_n \ge \delta_0\, \varepsilon_k^2\, (p-1)n.$$

By preceding remarks, this entails that for every $k \ge k_0$, if n is sufficiently large,

$$\#\{y \in \mathbf{m}_n : \#B_n(y, 4\varepsilon_k n^{1/4}) \ge \delta_0 \varepsilon_k^2 (p-1)n\} \ge \delta_0 \varepsilon_k^2 (p-1)n. \tag{23}$$

Since we restricted our attention to integers n such that $v_n \in \tau^1_n$, the bound (23) only holds for those integers. However, a symmetric argument shows that (23) also holds for all (sufficiently large) integers n such that $v_n \in \tau^2_n$, possibly with different values of δ_0 and k_0. Thus by changing δ_0 and k_0 if necessary, we can assume that (23) holds for all sufficiently large integers n.

On the other hand, (5) shows that for every $\delta > 0$ and every k,

$$E\Big[\frac{1}{(p-1)n} \#\{y \in \mathbf{m}_n : \#B_n(y, 4\varepsilon_k n^{1/4}) \ge \delta\varepsilon_k^2 (p-1)n\}\Big]$$
$$\underset{n\to\infty}{\longrightarrow} P\Big[\overline{\mathcal{I}}\Big(\Big[0, 4\Big(\frac{9}{4p(p-1)}\Big)^{1/4} \varepsilon_k\Big]\Big) \ge \delta\varepsilon_k^2\Big].$$

By Lemma 2.3, we have for every $\delta > 0$,

$$P\Big[\overline{\mathcal{I}}\Big(\Big[0, 4\Big(\frac{9}{4p(p-1)}\Big)^{1/4}\varepsilon_k\Big]\Big) \geq \delta\varepsilon_k^2\Big] = o(\varepsilon_k^2)$$

as $k \to \infty$. Hence, Fatou's lemma gives

$$E\Big[\liminf_{n\to\infty} \frac{1}{n}\#\{y \in \mathbf{m}_n : \#B_n(y, 4\varepsilon_k n^{1/4}) \geq \delta\varepsilon_k^2(p-1)n\}\Big] = o(\varepsilon_k^2)$$

as $k \to \infty$. Another application of Fatou's lemma yields that

$$E\Big[\liminf_{k\to\infty}\Big(\liminf_{n\to\infty} \frac{1}{\varepsilon_k^2 n}\#\{y \in \mathbf{m}_n : \#B_n(y, 4\varepsilon_k n^{1/4}) \geq \delta\varepsilon_k^2(p-1)n\}\Big)\Big] = 0.$$

By applying the above to a sequence of values of δ decreasing to 0, we obtain that a.s. for every $\delta > 0$,

$$\liminf_{k\to\infty}\Big(\liminf_{n\to\infty} \frac{1}{\varepsilon_k^2 n}\#\{y \in \mathbf{m}_n : \#B_n(y, 4\varepsilon_k n^{1/4}) \geq \delta\varepsilon_k^2(p-1)n\}\Big) = 0.$$

This contradicts (23), thus completing the proof of Proposition 4.2. □

Proposition 4.3 *Almost surely, for every pair (a, b) in $\mathcal{T}_{\overline{\mathbf{e}}}$ such that a is not an ancestor of b and b is not an ancestor of a, the condition $D(a, b) = 0$ implies $D^\circ(a, b) = 0$.*

Proof: The proof is similar to that of Proposition 4.2 but the fact that we already know the property stated in this proposition makes the argument a little simpler. We again argue by contradiction, assuming that there exists a pair (a, b) satisfying the condition of the proposition, such that $D(a, b) = 0$ and $D^\circ(a, b) > 0$. Without loss of generality we may and will assume that $a < b$. Recall that we have automatically $\overline{Z}_a = \overline{Z}_b$.

Let $[s, t]$ be the smallest subinterval of $[0, 1]$ such that $a \simeq s$ and $b \simeq t$. As in the proof of Proposition 4.2, we can find $a_n, b_n \in \tau_n^\circ$ in such a way that there exist $i_n, j_n \in [pn]$ with $i_n \leq j_n$, $a_n \sim_n i_n$, $b_n \sim_n j_n$ and $i_n/(pn) \longrightarrow s$, $j_n/(pn) \longrightarrow t$ as $n \to \infty$. We have then

$$n^{-1/4} d_n(a_n, b_n) \underset{n\to\infty}{\longrightarrow} D(a, b) = 0.$$

From Lemma 4.1, we also know that $\overline{Z}_c \geq \overline{Z}_a$ for every $c \in [\![a, b]\!]$. Recall that we assumed

$$\overline{Z}_a + \overline{Z}_b - 2\inf_{c\in[a,b]} \overline{Z}_c = D^\circ(a, b) > 0.$$

It follows that

$$\inf_{c\in[a,b]\setminus[\![a,b]\!]} \overline{Z}_c < \overline{Z}_a = \overline{Z}_b.$$

Since the minimum of $\overline{\mathbf{e}}$ over $[s, t]$ is attained at a unique time corresponding to the vertex $a \wedge b$ (otherwise the tree $\mathcal{T}_{\overline{\mathbf{e}}}$ would have a point with multiplicity strictly greater than 3), we have

$$[a, b]\setminus[\![a, b]\!] = ([a, a\wedge b]\setminus[\![a, a\wedge b]\!]) \cup ([a\wedge b, b]\setminus[\![a\wedge b, b]\!]).$$

Thus at least one of the following two conditions holds:

$$\inf_{c\in[a,a\wedge b]\setminus[\![a,a\wedge b]\!]} \overline{Z}_c < \overline{Z}_a, \tag{24}$$

or

$$\inf_{c\in[a\wedge b,b]\setminus[\![a\wedge b,b]\!]} \overline{Z}_c < \overline{Z}_a. \tag{25}$$

For definiteness, we assume that (25) holds. The symmetric case where (24) holds is treated in a similar manner.

Under (25), there exists a subtree $\mathcal{T}^1$ from the left side of $[\![a\wedge b,b]\!]$, with root $\rho^1 \in]\!]a\wedge b,b[\![$, such that

$$\inf_{c\in\mathcal{T}^1} \overline{Z}_c < \overline{Z}_a.$$

We let $[\alpha,\beta]$ be the interval coding $\mathcal{T}^1$.

As in the proof of Proposition 4.2, we can find a positive number κ such that the following holds for n sufficiently large. There exists a subtree τ_n^1 of τ_n°, from the left side of $[\![a_n\wedge b_n,b_n]\!]$, with root $\rho_n^1 \in]\!]a_n\wedge b_n,b_n[\![$ and such that

$$\inf_{x\in\tau_n^1} \ell_x^n \leq \ell_{a_n}^n - \kappa n^{1/4}. \tag{26}$$

Furthermore, τ_n^1 is coded by an interval $[\alpha_n,\beta_n]\cap\mathbb{Z}$, with $i_n \leq \alpha_n \leq \beta_n \leq j_n$, and $\alpha_n/(pn) \longrightarrow \alpha$, $\beta_n/(pn) \longrightarrow \beta$ as $n\to\infty$.

Let $\gamma_n = (\gamma_n(i), 0\leq i\leq d_n(a_n,b_n))$ be a geodesic path from a_n to b_n. As previously, we know that γ_n lies entirely in τ_n° when n is large. Furthermore, as in the proof of Proposition 4.2, we have

$$n^{-1/4} \sup_{0\leq i\leq d_n(a_n,b_n)} |\ell_{a_n}^n - \ell_{\gamma_n(i)}^n| \underset{n\to\infty}{\longrightarrow} 0. \tag{27}$$

We first claim that for n sufficiently large the path γ_n does not intersect $[\![\varnothing,\rho_n^1]\!]$. Indeed, suppose that γ_n intersects $[\![\varnothing,\rho_n^1]\!]$ for infinitely many values of n, and for such values write g_n for one of the intersection points. Let $k_n\in[pn]$ be such that $g_n \sim_n k_n$, and let r be any accumulation point of $k_n/(pn)$ in $[0,1]$. If $c\in\mathcal{T}_{\overline{\mathbf{e}}}$ is such that $c\simeq r$, the property $d_n(g_n,b_n)\leq d_n(a_n,b_n) = o(n^{1/4})$ ensures that $D(c,b)=0$. However, the fact that $g_n\in[\![\varnothing,\rho_n^1]\!]$ easily implies that $c\in[\![\overline{\rho},\rho^1]\!]$. Hence we have both $D(c,b)=0$ and $c\in[\![\overline{\rho},b]\!]$ with $c\neq b$. By Proposition 4.2 this cannot occur.

Now let u_n be the last point on the geodesic γ_n that belongs to $\{x\in\tau_n^\circ : x<\rho_n^1\}$. This makes sense since a_n belongs to the latter set. Also $u_n\neq b_n$ since $\rho_n^1\leq b_n$. Let v_n be the point following u_n on the geodesic γ_n. We claim that $v_n\in\tau_n^1$ if n is sufficiently large. Indeed, the property (26) warrants that a vertex y belonging to the set

$$\{x\in\tau_n^\circ : \rho_n^1\leq x\}\setminus\tau_n^1$$

and such that $\ell_y^n > \ell_{a_n}^n - \kappa n^{1/4}$ cannot be connected to u_n, except possibly if $u_n\in[\![\varnothing,\rho_n^1]\!]$. However we just saw that this case does not occur for n sufficiently large. By applying the preceding considerations to $y=v_n$, using (27), we get our claim.

Then the following properties hold for n large:

(i) $v_n \in \tau_n^1$ and $v_n \neq \rho_n^1$.

(ii) $u_n \leq \rho_n^1$.

(iii) Every point w that comes after v_n on the geodesic γ_n satisfies $\rho_n^1 \leq w$.

From Lemma 2.1, we get that if n is large enough, for every point y of $\tau_n^1 \backslash \{\rho_n^1\}$ such that

$$\ell_x^n > \sup_{0 \leq i \leq d_n(a_n, b_n)} \ell_{\gamma_n(i)}^n \quad \text{for every } x \in [\![\rho_n^1, y]\!]$$

we have

$$d_n(y, b_n) \leq d_n(u_n, b_n) + \ell_y^n - \inf_{0 \leq i \leq d_n(a_n, b_n)} \ell_{\gamma_n(i)}^n.$$

The end of the argument is now entirely similar to the end of the proof of Proposition 4.2: We use (11), Lemma 2.3 and Lemma 2.4 to show that the preceding properties lead to a contradiction. This completes the proof of Proposition 4.3. □

The implication (i)⇒(iii) in Theorem 3.4 is a consequence of Propositions 4.2 and 4.3. This completes the proof of Theorem 3.4.

5 Proof of the technical estimates

In this section, we prove the three lemmas that were stated at the end of subsection 2.4. We first need to recall some basic properties of the Brownian snake. More information can be found in the monograph [21].

The (one-dimensional) Brownian snake is a Markov process taking values in the space $\mathcal{W}$ of finite paths in $\mathbb{R}$. Here a finite path is simply a continuous mapping $\mathrm{w} : [0, \zeta] \longrightarrow \mathbb{R}$, where $\zeta = \zeta_{(\mathrm{w})}$ is a nonnegative real number called the lifetime of w. The set $\mathcal{W}$ is a Polish space when equipped with the distance

$$d(\mathrm{w}, \mathrm{w}') = |\zeta_{(\mathrm{w})} - \zeta_{(\mathrm{w}')}| + \sup_{t \geq 0} |\mathrm{w}(t \wedge \zeta_{(\mathrm{w})}) - \mathrm{w}'(t \wedge \zeta_{(\mathrm{w}')})|.$$

The endpoint (or tip) of the path w is denoted by $\widehat{\mathrm{w}} = \mathrm{w}(\zeta_{(\mathrm{w})})$.

Let $\Omega := C(\mathbb{R}_+, \mathcal{W})$ be the space of all continuous functions from $\mathbb{R}_+$ into $\mathcal{W}$, which is equipped with the topology of uniform convergence on every compact subset of $\mathbb{R}_+$. The canonical process on Ω is then denoted by $W_s(\omega) = \omega(s)$ for $\omega \in \Omega$, and we write $\zeta_s = \zeta_{(W_s)}$ for the lifetime of W_s.

Let $\mathrm{w} \in \mathcal{W}$. The law of the Brownian snake started from w is the probability measure $\mathbb{P}_{\mathrm{w}}$ on Ω which can be characterized as follows. First, the process $(\zeta_s)_{s \geq 0}$ is under $\mathbb{P}_{\mathrm{w}}$ a reflected Brownian motion in $[0, \infty[$ started from $\zeta_{(\mathrm{w})}$. Secondly, the conditional distribution of $(W_s)_{s \geq 0}$ knowing $(\zeta_s)_{s \geq 0}$, which is denoted by $\Theta_{\mathrm{w}}^{\zeta}$, is characterized by the following properties:

(i) $W_0 = \mathrm{w}$, $\Theta_{\mathrm{w}}^{\zeta}$ a.s.

(ii) The process $(W_s)_{s \geq 0}$ is time-inhomogeneous Markov under $\Theta_{\mathrm{w}}^{\zeta}$. Moreover, if $0 \leq s \leq s'$,

- $W_{s'}(t) = W_s(t)$ for every $t \le m(s,s') := \inf_{[s,s']} \zeta_r$, $\Theta^\zeta_{\rm w}$ a.s.
- Under $\Theta^\zeta_{\rm w}$, $(W_{s'}(m(s,s')+t) - W_{s'}(m(s,s')))_{0\le t\le \zeta_{s'}-m(s,s')}$ is independent of W_s and distributed as a one-dimensional Brownian motion started at 0 and stopped at time $\zeta_{s'} - m(s,s')$.

Informally, the value W_s of the Brownian snake at time s is a random path with a random lifetime ζ_s evolving like reflecting Brownian motion in $[0,\infty[$. When ζ_s decreases, the path is erased from its tip, and when ζ_s increases, the path is extended by adding "little pieces" of Brownian paths at its tip.

We denote by $n(de)$ the Itô measure of positive Brownian excursions, which is a σ-finite measure on the space $C(\mathbb{R}_+,\mathbb{R}_+)$, and we write

$$\sigma(e) = \inf\{s > 0 : e(s) = 0\}$$

for the duration of excursion e. For $s > 0$, $n_{(s)}$ will denote the conditioned measure $n(\cdot \mid \sigma = s)$. In particular $n_{(1)}(de)$ is the law of the normalized excursion $\mathbf{e}$, or more precisely of $(\mathbf{e}_{t\wedge 1})_{t\ge 0}$. Our normalization of the excursion measure is fixed by the relation

$$n = \int_0^\infty \frac{ds}{2\sqrt{2\pi s^3}}\, n_{(s)}, \tag{28}$$

and we have then $n(\sup_{s\ge 0} e(s) > \varepsilon) = (2\varepsilon)^{-1}$ for every $\varepsilon > 0$.

If $x \in \mathbb{R}$, the excursion measure $\mathbb{N}_x$ of the Brownian snake from x is given by

$$\mathbb{N}_x = \int_{C(\mathbb{R}_+,\mathbb{R}_+)} n(de)\, \Theta^e_{\overline{x}}$$

where $\overline{x}$ denotes the trivial element of $\mathcal{W}$ with lifetime 0 and initial point x. With a slight abuse of notation we also write $\sigma(\omega) = \inf\{s > 0 : \zeta_s(\omega) = 0\}$ for $\omega \in \Omega$. We can then consider the conditioned measures

$$\mathbb{N}^{(s)}_x = \mathbb{N}_x(\cdot \mid \sigma = s) = \int_{C(\mathbb{R}_+,\mathbb{R}_+)} n_{(s)}(de)\, \Theta^e_{\overline{x}}\,.$$

We can now relate the Brownian snake to the Brownian trees of subsection 2.4: We may define the pair $(\mathbf{e}, Z)$ under the probability measure $\mathbb{N}^{(1)}_0$ by taking $\mathbf{e}_s = \zeta_s$ and $Z_s = \widehat{W}_s$, for every $0 \le s \le 1$. Furthermore the path $(W_s(t), 0 \le t \le \zeta_s)$ is then interpreted in terms of the labels attached to the ancestors of $p_{\mathbf{e}}(s)$: If $a = p_{\mathbf{e}}(s)$ is a vertex of the tree $\mathcal{T}_{\mathbf{e}}$, and $c \in [\![\rho, a]\!]$ is the ancestor of a at generation $t = d_{\mathbf{e}}(\rho, c)$, we have $Z_c = W_s(t)$. These identifications follow very easily from the properties of the Brownian snake.

For future reference, we state a crude bound on the increments of the process $(\widehat{W}_s)_{s\ge 0}$ under $\mathbb{N}_0$. In a way analogous to subsection 2.3 we set, for every $s, t \ge 0$,

$$d_\zeta(s,t) = \zeta_s + \zeta_t - 2 \inf_{s\wedge t \le r \le s\vee t} \zeta_r\,.$$

Lemma 5.1 *Let $b \in]0, 1/2[$. Then $\mathbb{N}_0(d\omega)$ a.e. there exists $\varepsilon_0(\omega) > 0$ such that for every $s, t \ge 0$ with $d_\zeta(s,t) \le \varepsilon_0$, one has*

$$|\widehat{W}_s - \widehat{W}_t| \le (d_\zeta(s,t))^b.$$

Proof: Conditionally on $(\zeta_r)_{r\geq 0}$, the process $(\widehat{W}_r)_{r\geq 0}$ is Gaussian with mean 0 and such that $E[(\widehat{W}_s-\widehat{W}_t)^2]=d_\zeta(s,t)$ for every $s,t\geq 0$. The bound of the lemma then follows from standard chaining arguments. We leave details to the reader. $\square$

Proof of Lemma 2.3: Set

$$\underline{W}=\inf_{s\geq 0}\widehat{W}_s,$$

and, for every $\varepsilon>0$,

$$\mathcal{J}(\varepsilon)=\int_0^\sigma ds\,\mathbf{1}\{\widehat{W}_s-\underline{W}\leq\varepsilon\}.$$

Thanks to the remarks preceding Lemma 5.1, the quantity $\overline{\mathcal{I}}([0,\varepsilon])$ in Lemma 2.3 has the same distribution as $\mathcal{J}(\varepsilon)$ under $\mathbb{N}_0^{(1)}$. Therefore, the statement of Lemma 2.3 reduces to checking that

$$\mathbb{N}_0^{(1)}(\mathcal{J}(\varepsilon)\geq\alpha\varepsilon^2)=o(\varepsilon^2)$$

as $\varepsilon\to 0$. From (28) and simple scaling arguments, it is enough to verify that

$$\mathbb{N}_0(\mathcal{J}(\varepsilon)\geq\alpha\varepsilon^2\,,\,\sigma>1/2)=o(\varepsilon^2) \tag{29}$$

as $\varepsilon\to 0$. For every $\delta>0$, we have $\mathcal{J}(\varepsilon)=\mathcal{J}_\delta(\varepsilon)+\mathcal{J}'_\delta(\varepsilon)$, where

$$\mathcal{J}_\delta(\varepsilon)=\int_0^\sigma ds\,\mathbf{1}\{\widehat{W}_s-\underline{W}\leq\varepsilon,\zeta_s<\delta\}\;,\;\mathcal{J}'_\delta(\varepsilon)=\int_0^\sigma ds\,\mathbf{1}\{\widehat{W}_s-\underline{W}\leq\varepsilon,\zeta_s\geq\delta\}.$$

Let us fix $\beta>0$. By Lemma 3.2 in [25], we can choose $\delta>0$ small enough so that, for every $\varepsilon\in]0,1[$,

$$\mathbb{N}_0\Big(\mathbf{1}_{\{\sigma>1/2\}}\,\mathcal{J}_\delta(\varepsilon)\Big)\leq\beta\,\varepsilon^4.$$

On the other hand, Lemma 3.3 in [25] yields the existence of a constant K_δ such that, for every $\varepsilon\in]0,1[$,

$$\mathbb{N}_0\Big((\mathcal{J}'_\delta(\varepsilon))^2\Big)\leq K_\delta\,\varepsilon^8.$$

Then,

$$\begin{aligned}
\mathbb{N}_0(\mathcal{J}(\varepsilon)\geq\alpha\varepsilon^2\,,\,\sigma>1/2) &\leq \mathbb{N}_0(\mathcal{J}_\delta(\varepsilon)\geq\frac{\alpha}{2}\,\varepsilon^2\,,\,\sigma>1/2)+\mathbb{N}_0(\mathcal{J}'_\delta(\varepsilon)\geq\frac{\alpha}{2}\,\varepsilon^2)\\
&\leq \frac{2}{\alpha\varepsilon^2}\mathbb{N}_0\Big(\mathbf{1}_{\{\sigma>1/2\}}\,\mathcal{J}_\delta(\varepsilon)\Big)+\frac{4}{\alpha^2\varepsilon^4}\,\mathbb{N}_0\Big((\mathcal{J}'_\delta(\varepsilon))^2\Big)\\
&\leq \frac{2\beta}{\alpha}\,\varepsilon^2+\frac{4K_\delta}{\alpha^2}\,\varepsilon^4.
\end{aligned}$$

It follows that

$$\limsup_{\varepsilon\to 0}\varepsilon^{-2}\mathbb{N}_0(\mathcal{J}(\varepsilon)\geq\alpha\varepsilon^2\,,\,\sigma>1/2)\leq\frac{2\beta}{\alpha}.$$

Since β was arbitrary, this completes the proof of (29) and of the lemma. $\square$

Proof of Lemma 2.2: We first explain why it is enough to prove the statement concerning the pair $(\mathbf{e},Z)$. This follows from a re-rooting argument. Recall the notation of subsection 2.4. For every fixed $s\in[0,1[$, set

- $\mathbf{e}^{[s]}_t = \mathbf{e}_s + \mathbf{e}_{s\oplus t} - 2\, m_{\mathbf{e}}(s, s\oplus t)$;
- $Z^{[s]}_t = Z_{s\oplus t} - Z_s$,

for every $t \in [0,1]$. By construction, $(\overline{\mathbf{e}}, \overline{Z}) = (\mathbf{e}^{[s_*]}, Z^{[s_*]})$. Also, $(\mathbf{e}^{[s]}, Z^{[s]}) \overset{\rm (d)}{=} (\mathbf{e}, Z)$ for every fixed $s \in [0,1[$: See Proposition 4.9 in [27] or Theorem 2.3 in [25]. Hence, if U is uniformly distributed over $[0,1[$ and independent of $(\mathbf{e}, Z)$, we have also $(\mathbf{e}^{[U]}, Z^{[U]}) \overset{\rm (d)}{=} (\mathbf{e}, Z)$.

Suppose there exists an increase point $r \in]0,1[$ of the pair $(\overline{\mathbf{e}}, \overline{Z}) = (\mathbf{e}^{[s_*]}, Z^{[s_*]})$. Then for every s sufficiently close to s_*, $r + s_* - s$ will be an increase point of the pair $(\mathbf{e}^{[s]}, Z^{[s]})$ (this can be verified by direct inspection of the formulas defining the pair $(\mathbf{e}^{[s]}, Z^{[s]})$, keeping in mind that s_* corresponds to a leaf of the tree $\mathcal{T}_{\mathbf{e}}$, so that immediately after or immediately before s_*, $\mathbf{e}_s$ takes values strictly less than $\mathbf{e}_{s_*}$). In particular, the pair $(\mathbf{e}^{[U]}, Z^{[U]})$ will have an increase point with positive probability, which contradicts the first assertion of the lemma.

Let us now prove the statement concerning the pair $(\mathbf{e}, Z)$. In terms of the Brownian snake, we need to check that $\mathbb{N}^{(1)}_0$ a.s. the pair $(\zeta_s, \widehat{W}_s)_{0\le s\le 1}$ has no increase point. By a simple scaling argument, it is enough to verify that the same property holds for the pair $(\zeta_s, \widehat{W}_s)_{0\le s\le \sigma}$ under the excursion measure $\mathbb{N}_0$ (obviously time 1 is now replaced by σ in the definition of an increase point). To this end, we will use the following lemma.

Lemma 5.2 *Let $\delta > 0$. Let $\mathrm{w} \in \mathcal{W}$ with $\mathrm{w}(0) = 0$ and $\zeta_{(\mathrm{w})} = a > 0$, and let $\varepsilon \in]0, a]$. Consider the stopping times*

$$T = \inf\{s \ge 0 : \zeta_s = a + \delta\},$$
$$T' = \inf\{s \ge 0 : \zeta_s = a - \varepsilon\}.$$

On the event $\{T < T'\}$, we also define

$$L = \sup\{s \le T : \zeta_s = a\}.$$

Then there exists a constant C_δ, which only depends on δ, such that, for every $\eta \in]0,1]$,

$$\mathbb{P}_{\mathrm{w}}(T < T' \text{ and } \widehat{W}_s > \widehat{W}_L - \eta \text{ for every } s \in [L,T]) \le C_\delta\, \varepsilon\, \eta^3.$$

Remark. The exponent 3 in η^3 is sharp and related to the fact that the bound of the lemma is a "one-sided" estimate. This should be compared with the exponent 4 that appears in similar two-sided estimates derived in [25]. This observation also explains why seven-dimensional Bessel processes appear in the proof below, rather than nine-dimensional Bessel processes as in [25].

Proof: Under $\mathbb{P}_{\mathrm{w}}$, $(\zeta_s)_{s\ge 0}$ is distributed as a reflected linear Brownian motion started from a. In particular,

$$\mathbb{P}_{\mathrm{w}}(T < T') = \frac{\varepsilon}{\varepsilon + \delta}.$$

Moreover, from standard connections between linear Brownian motion and the three-dimensional Bessel process, we know that under the conditional probability $\mathbb{P}_{\mathrm{w}}(\cdot \mid T < T')$, the shifted process

$$Y_s := \zeta_{(L+s)\wedge T} - a\ ,\ \ s \ge 0$$

is distributed as a three-dimensional Bessel process started from 0 and stopped when it first hits δ. At this point, it is convenient to introduce the future infimum process of Y,

$$J_s := \inf_{r \geq s} Y_r \ , \ s \geq 0$$

and the excursions of $Y - J$ away from 0: Let $]\alpha_i, \beta_i[$, $i \in I$, be the connected components of the open set $\{s \geq 0 : Y_s > J_s\}$, and for every $i \in I$ set

$$e_i(s) = Y_{(\alpha_i+s)\wedge\beta_i} - Y_{\alpha_i}.$$

Then the point measure

$$\sum_{i\in I} \delta_{(Y_{\alpha_i}, e_i)}(dt\, de)$$

is Poisson with intensity

$$2\,\mathbf{1}_{[0,\delta]}(t)\,\mathbf{1}_{\{\sup_{s\geq 0} e(s)<\delta-t\}}\,dt\;n(de).$$

The last property follows from standard facts of excursion theory. See e.g. Lemma 1 in [1] for a detailed derivation.

We can then combine the preceding excursion decomposition of the paths of Y with the spatial displacements of the Brownian snake, in a way similar to the proof of Lemma V.5 in [21]. Let $H = \sup_{s\geq 0} \zeta_s$ denote the maximum of the lifetime process. It follows that

$$\begin{aligned}&\mathbb{P}_{\mathrm{w}}(T < T' \text{ and } \widehat{W}_s > \widehat{W}_L - \eta \text{ for every } s \in [L, T])\\ &= \frac{\varepsilon}{\varepsilon+\delta}\, E_0\Big[\mathbf{1}_{\{\xi[0,\delta]\subset]-\eta,\infty[\}}\, \exp\Big(-2\int_0^\delta dt\, \mathbb{N}_{\xi_t}(H < \delta - t, \underline{W} \leq -\eta)\Big)\Big] \qquad (30)\end{aligned}$$

where $(\xi_t)_{t\geq 0}$ is a linear Brownian motion started from x under the probability measure P_x, and we use the notation $\xi[0, \delta]$ for the range of ξ over the time interval $[0, \delta]$.

From this point, the argument is very similar to the end of the proof of Proposition 4.2 in [25], to which we refer the reader for more details. For every $x > 0$, we set

$$f(x) = \mathbb{N}_0(\underline{W} > -x \mid H = 1)$$

and

$$G(x) = 4\int_0^x u(1 - f(u))\, du.$$

Note that $G(+\infty) = 6$ (see Section 4 in [25]). By conditioning with respect to H and then using a scaling argument, we get

$$\int_0^\delta dt\, \mathbb{N}_{\xi_t}(H < \delta - t, \underline{W} \leq -\eta) = \int_0^\delta dt \int_0^{\delta-t} \frac{du}{2u^2}\,(1 - f(\frac{\xi_t+\eta}{\sqrt{u}})).$$

Hence, the right-hand side of (30) can be written as

$$\begin{aligned}&\frac{\varepsilon}{\varepsilon+\delta}\, E_0\Big[\mathbf{1}_{\{\xi[0,\delta]\subset]-\eta,\infty[\}}\, \exp\Big(-\int_0^\delta dt\int_0^{\delta-t} \frac{du}{u^2}\,(1 - f(\frac{\xi_t+\eta}{\sqrt{u}}))\Big)\Big]\\ &= \frac{\varepsilon}{\varepsilon+\delta}\, E_\eta\Big[\mathbf{1}_{\{\xi[0,\delta]\subset]0,\infty[\}}\, \exp\Big(-\int_0^\delta dt\int_0^{\delta-t} \frac{du}{u^2}\,(1 - f(\frac{\xi_t}{\sqrt{u}}))\Big)\Big].\end{aligned}$$

From the definition of G, the property $G(+\infty)=6$ and a change of variables, we have

$$\int_0^{\delta-t}\frac{du}{u^2}\,(1-f(\frac{\xi_t}{\sqrt{u}}))=(\xi_t)^{-2}\Big(3-\frac{1}{2}G(\frac{\xi_t}{\sqrt{\delta-t}})\Big).$$

Hence we get

$$\begin{aligned}&\mathbb{P}_{\rm w}(T<T' \text{ and } \widehat{W}_s>\widehat{W}_L-\eta \text{ for every } s\in[L,T])\\ &=\frac{\varepsilon}{\varepsilon+\delta}\,E_\eta\Big[\mathbf{1}_{\{\xi[0,\delta]\subset]0,\infty[\}}\,\exp\Big(-3\int_0^\delta\frac{dt}{\xi_t^2}+\frac{1}{2}\int_0^\delta\frac{dt}{\xi_t^2}\,G(\frac{\xi_t}{\sqrt{\delta-t}})\Big)\Big].\end{aligned}\tag{31}$$

Proposition 2.6 of [25], which reformulates absolute continuity relations between Bessel processes due to Yor, implies that the right-hand side of (31) is equal to

$$\frac{\varepsilon}{\varepsilon+\delta}\,\eta^3\,E^{(7)}_\eta\Big[(R_\delta)^{-3}\,\exp\Big(\frac{1}{2}\int_0^\delta\frac{dt}{R_t^2}\,G(\frac{R_t}{\sqrt{\delta-t}})\Big)\Big],$$

where $(R_t)_{t\geq0}$ is a seven-dimensional Bessel process started from η under the probability measure $P^{(7)}_\eta$. Finally, we can argue as in the end of the proof of Proposition 4.2 in [25] to verify the existence of a constant C'_δ such that, for every $\eta>0$,

$$E^{(7)}_\eta\Big[(R_\delta)^{-3}\,\exp\Big(\frac{1}{2}\int_0^\delta\frac{dt}{R_t^2}\,G(\frac{R_t}{\sqrt{\delta-t}})\Big)\Big]\leq C'_\delta.$$

Lemma 5.2 follows with $C_\delta=\delta^{-1}C'_\delta$. □

We come back to the proof of Lemma 2.2. We fix $\delta\in]0,1[$. For every $\varepsilon\in]0,1[$, we introduce the sequence of stopping times defined inductively by

$$T^\varepsilon_0=0\ ,\ T^\varepsilon_{i+1}=\inf\{s>T^\varepsilon_i:|\zeta_s-\zeta_{T^\varepsilon_i}|=\varepsilon\},$$

with the usual convention $\inf\varnothing=\infty$. For every index i such that $T^\varepsilon_i<\infty$, we also set

$$\begin{aligned}S^\varepsilon_i&=\inf\{s>T^\varepsilon_i:\zeta_s=\zeta_{T^\varepsilon_i}+\delta\},\\ \widetilde{T}^\varepsilon_i&=\inf\{s>T^\varepsilon_i:\zeta_s=\zeta_{T^\varepsilon_i}-2\varepsilon\}.\end{aligned}$$

On the event $\{T^\varepsilon_i=\infty\}$ simply set $S^\varepsilon_i=\widetilde{T}^\varepsilon_i=\infty$. Finally, on the event $\{T^\varepsilon_i<\infty\}\cap\{S^\varepsilon_i<\infty\}$, we put

$$L^\varepsilon_i=\sup\{s<S^\varepsilon_i:\zeta_s=\zeta_{T^\varepsilon_i}\}.$$

Fix an integer $A>1$, and let $\mathcal{A}_{\varepsilon,\eta}$ be the event that there exists $i\geq1$ such that $T^\varepsilon_i<\infty$, $\zeta_{T^\varepsilon_i}\in]0,A]$, $S^\varepsilon_i<\widetilde{T}^\varepsilon_i<\infty$ and

$$\widehat{W}_s>\widehat{W}_{L^\varepsilon_i}-\eta$$

for every $s\in[L^\varepsilon_i,S^\varepsilon_i]$.

From Lemma 5.2 and the strong Markov property for the Brownian snake, we have

$$\begin{aligned}\mathbb{N}_0(\mathcal{A}_{\varepsilon,\eta})&\leq\mathbb{N}_0\Big(\sum_{i=1}^\infty\mathbf{1}\{T^\varepsilon_i<\infty,\zeta_{T^\varepsilon_i}\in]0,A]\}\,\mathbf{1}\{S^\varepsilon_i<\widetilde{T}^\varepsilon_i<\infty\}\,\mathbf{1}\{\widehat{W}_s>\widehat{W}_{L^\varepsilon_i}-\eta\,,\,\forall s\in[L^\varepsilon_i,S^\varepsilon_i]\}\Big)\\ &\leq 2C_\delta\varepsilon\,\eta^3\,\mathbb{N}_0\Big(\sum_{i=1}^\infty\mathbf{1}\{T^\varepsilon_i<\infty,\zeta_{T^\varepsilon_i}\in]0,A]\}\Big).\end{aligned}$$

Standard properties of linear Brownian motion give

$$\mathbb{N}_0\Big(\sum_{i=1}^{\infty}\mathbf{1}\{T_i^\varepsilon<\infty,\zeta_{T_i^\varepsilon}\in]0,A]\}\Big)=\frac{1}{\varepsilon}\lfloor\frac{A}{\varepsilon}\rfloor.$$

Therefore we have obtained the bound

$$\mathbb{N}_0(\mathcal{A}_{\varepsilon,\eta})\leq 2C_\delta A\,\varepsilon^{-1}\eta^3.$$

We apply this estimate with $\varepsilon=\varepsilon_p=2^{-p}$, for every integer $p\geq 1$, and $\eta=(2\varepsilon_p)^b$, where $b\in]\frac{1}{3},\frac{1}{2}[$. It follows that $\mathbb{N}_0$ a.e. for all p sufficiently large the event $\mathcal{A}_{\varepsilon_p,(2\varepsilon_p)^b}$ does not occur.

To complete the argument, notice that it is enough to prove that there cannot exist $r>0$ such that $\zeta_r\leq A-1$, $\inf\{u\geq r:\zeta_u=\zeta_r+2\delta\}<\infty$ and

$$\zeta_s\geq\zeta_r \text{ and } \widehat{W}_s\geq\widehat{W}_r\ ,\ \text{ for every } s\in[r,\inf\{u\geq r:\zeta_u=\zeta_r+2\delta\}].$$

We argue by contradiction and suppose that there is such a value of r. Let $i\geq 1$ be such that $r\in]T_{i-1}^{\varepsilon_p},T_i^{\varepsilon_p}]$. If p has been taken large enough, we have $S_i^{\varepsilon_p}<\widetilde{T}_i^{\varepsilon_p}\wedge\inf\{u\geq r:\zeta_u=\zeta_r+2\delta\}$, and for every $s\in[T_i^{\varepsilon_p},S_i^{\varepsilon_p}]$,

$$\widehat{W}_s\geq\widehat{W}_r>\widehat{W}_{L_i^{\varepsilon_p}}-(2\varepsilon_p)^b\ ,$$

where the last inequality follows from Lemma 5.1 since $d_\zeta(r,L_i^{\varepsilon_p})<2\varepsilon_p$. We thus get a contradiction with the fact that $\mathcal{A}_{\varepsilon_p,(2\varepsilon_p)^b}$ does not occur when p is large. This contradiction completes the proof. □

Proof of Lemma 2.4: We first observe that it is enough to prove the statement of Lemma 2.4 when the pair $(\mathcal{T}_{\overline{\mathbf{e}}},\overline{Z})$ is replaced by $(\mathcal{T}_{\mathbf{e}},Z)$, and of course $\overline{\rho}$ is also replaced by the root ρ of $\mathcal{T}_{\mathbf{e}}$. This follows from a re-rooting argument analogous to the one we used at the beginning of the proof of Lemma 2.2. Let us only sketch the argument. We assume that the property of Lemma 2.4 has been derived when the pair $(\mathcal{T}_{\overline{\mathbf{e}}},\overline{Z})$ is replaced by $(\mathcal{T}_{\mathbf{e}},Z)$. Suppose that the conclusion of this lemma fails for some subtree of $\mathcal{T}_{\overline{\mathbf{e}}}$. Then it will also fail for some subtree of the re-rooted tree $\mathcal{T}_{\mathbf{e}^{[s]}}$, provided that s is sufficiently close to s_*. Hence with positive probability it will fail for some subtree of $\mathcal{T}_{\mathbf{e}^{[U]}}$, where U is uniformly distributed over $[0,1]$. Since we saw that $(\mathbf{e}^{[U]},Z^{[U]})\overset{\text{(d)}}{=}(\mathbf{e},Z)$, this leads to a contradiction.

Then, we notice that by a symmetry argument we need only consider subtrees of $\mathcal{T}_{\mathbf{e}}$ from the right side of $[\![\rho,a]\!]$. Furthermore, as we already observed, the pair $(\mathbf{e}_s,Z_s)_{0\leq s\leq 1}$ has the same distribution as $(\zeta_s,\widehat{W}_s)_{0\leq s\leq 1}$ under $\mathbb{N}_0^{(1)}$. By scaling, it is then enough to prove that the analogue of Lemma 2.4 holds for the pair $(\zeta_s,\widehat{W}_s)_{0\leq s\leq\sigma}$ under $\mathbb{N}_0$. We can thus reformulate the desired property in the following way. Let us fix $s>0$, and argue on the event $\{s<\sigma\}$. Denote by $a=p_\zeta(s)$ the vertex corresponding to s in the tree $\mathcal{T}_\zeta$. The subtrees of $\mathcal{T}_\zeta$ from the right side of $[\![\rho,a]\!]$ exactly correspond to the excursions of the shifted process $(\zeta_{s+r})_{r\geq 0}$ above its past minimum process. More precisely, set

$$\zeta_r^{(s)}=\zeta_{s+r}\ ,\ \check{\zeta}_r^{(s)}=\inf_{0\leq u\leq r}\zeta_u^{(s)}$$

for every $r\geq 0$. Denote by $]\alpha_i,\beta_i[$, $i\in I$, the connected components of the open set $\{r\geq 0:\zeta_r^{(s)}>\check{\zeta}_r^{(s)}\}$. Then for each $i\in I$, the set $\mathcal{T}_i^1:=p_\zeta([s+\alpha_i,s+\beta_i])$ is a subtree of $\mathcal{T}_\zeta$ from the right side of $[\![\rho, a]\!]$ with root $p_\zeta(s+\alpha_i) = p_\zeta(s+\beta_i)$, and conversely all subtrees from the right side of $[\![\rho, a]\!]$ are obtained in this way. Recall the interpretation of the path $(W_{s+r}(t), 0 \le t \le \zeta_{s+r})$ as giving the labels of the ancestors of the vertex $p_\zeta(s+r)$ in the tree $\mathcal{T}_\zeta$. In order to get the statement of Lemma 2.4, it is enough to prove the following claim.

Claim. $\mathbb{N}_0$ *a.e. on the event* $\{s < \sigma\}$, *for every* $\mu > 0$ *and every* $i \in I$ *such that*

$$\inf_{\alpha_i \le r \le \beta_i} \widehat{W}_{s+r} < \widehat{W}_{s+\alpha_i} - \mu \tag{32}$$

we have

$$\liminf_{\varepsilon \to 0} \varepsilon^{-2} \int_{\alpha_i}^{\beta_i} dr\, \mathbf{1}\{\widehat{W}_{s+r} \le \widehat{W}_{s+\alpha_i} - \mu + \varepsilon\} \mathbf{1}\{W_{s+r}(t) \ge \widehat{W}_{s+\alpha_i} - \mu + \frac{\varepsilon}{8}, \forall t \in [\zeta_{s+\alpha_i}, \zeta_{s+r}]\} > 0. \tag{33}$$

Note that the preceding claim is concerned with subtrees from the right side of one particular vertex $a = p_\zeta(s)$, whereas the statement of Lemma 2.4 holds simultaneously for all choices of the vertex a. However, assuming that the claim is proved, it immediately follows that the desired property holds for all subtrees from the right side of $[\![\rho, p_\zeta(s)]\!]$, for all rational numbers $s > 0$, outside a single set of zero $\mathbb{N}_0$-measure. Recall that a subtree from the right side of $[\![\rho, p_\zeta(s)]\!]$ cannot be rooted at $p_\zeta(s)$, by our definitions, and thus any subtree from the right side of $[\![\rho, p_\zeta(s)]\!]$ is also a subtree from the right side of $[\![\rho, p_\zeta(s')]\!]$ as soon as s' is close enough to s. It follows from this observation that the desired result holds simultaneously for all choices of $a = p_\zeta(s)$.

Let us now discuss the proof of the claim. Recall that $s > 0$ is fixed and that we argue on the event $\{s < \sigma\}$. For every $i \in I$ and every $r \ge 0$ set

$$\begin{aligned}
&\zeta^i_r = \zeta_{(s+\alpha_i+r)\wedge(s+\beta_i)} - \zeta_{s+\alpha_i} \\
&W^i_r(t) = W_{(s+\alpha_i+r)\wedge(s+\beta_i)}(\zeta_{s+\alpha_i} + t) - \widehat{W}_{s+\alpha_i}\ , \quad \text{for every } t \in [0, \zeta^i_r]
\end{aligned}$$

and view W^i_r as a finite path with lifetime ζ^i_r, so that $W^i = (W^i_r)_{r\ge0}$ is a random element of $\Omega = C(\mathbb{R}_+, \mathcal{W})$. Also set $\sigma_i = \beta_i - \alpha_i$, which corresponds to the duration of the "excursion" ζ^i. By combining the Markov property at time s with Lemma V.5 in [21], we get that under the probability measure $\mathbb{N}_0(\cdot \mid s < \sigma)$ and conditionally on W_s, the point measure

$$\sum_{i\in I} \delta_{W^i}(d\omega)$$

is Poisson on Ω with intensity $2\,\zeta_s\,\mathbb{N}_0(d\omega)$. Now observe that condition (32) reduces to

$$\inf_{r\ge0} \widehat{W}^i_r < -\mu$$

and that the integral in (33) is equal to

$$\int_0^{\sigma_i} dr\, \mathbf{1}\{\widehat{W}^i_r \le -\mu + \varepsilon\}\, \mathbf{1}\{W^i_r(t) \ge -\mu + \frac{\varepsilon}{8}\ ,\ \forall t \in [0, \zeta^i_r]\}.$$

Thanks to these observations and to our previous description of the conditional distribution of the point measure $\sum_{i\in I} \delta_{W^i}$, we see that our claim follows from the next lemma.

Lemma 5.3 $\mathbb{N}_0$ *a.e. for every* $\mu \in]0, -\underline{W}[$, *we have*

$$\liminf_{\varepsilon \to 0} \varepsilon^{-2} \int_0^\sigma dr\, \mathbf{1}\{\widehat{W}_r \le -\mu + \varepsilon\}\, \mathbf{1}\{W_r(t) \ge -\mu + \frac{\varepsilon}{8}\ ,\ \forall t \in [0, \zeta_r]\} > 0.$$

Proof of Lemma 5.3. We fix an integer $N \ge 2$. Without loss of generality, we may and will restrict our attention to values $\mu \in [N^{-1}, N]$. We also consider another integer $n \ge N$. If j is the integer such that $(j-1)2^{-n-3} < \mu \le j2^{-n-3}$, and if $2^{-n-1} \le \varepsilon \le 2^{-n}$, we have the following simple inequalities:

$$\begin{aligned} &\mathbf{1}\{\widehat{W}_r \le -\mu + \varepsilon\}\, \mathbf{1}\{W_r(t) \ge -\mu + \frac{\varepsilon}{8}\ ,\ \forall t \in [0, \zeta_r]\} \\ &\quad \ge \mathbf{1}\{\widehat{W}_r \le -\mu + 2^{-n-1}\}\, \mathbf{1}\{W_r(t) \ge -\mu + 2^{-n-3}\ ,\ \forall t \in [0, \zeta_r]\} \\ &\quad \ge \mathbf{1}\{\widehat{W}_r \le -j2^{-n-3} + 2^{-n-1}\}\, \mathbf{1}\{W_r(t) \ge -j2^{-n-3} + 2^{-n-2}\ ,\ \forall t \in [0, \zeta_r]\}. \end{aligned}$$

So, for every integer j such that $N^{-1}2^{n+3} \le j \le N2^{n+3}$, we set

$$U_{n,j} = \int_0^\sigma dr\, \mathbf{1}\{\widehat{W}_r \le -j2^{-n-3} + 2^{-n-1}\}\, \mathbf{1}\{W_r(t) \ge -j2^{-n-3} + 2^{-n-2}\ ,\ \forall t \in [0, \zeta_r]\}.$$

For every $r > 0$, denote by L^r the total mass of the exit measure of the Brownian snake from the open set $]-r, \infty[$ (see e.g. Chapter 6 of [21] for the definition and main properties of exit measures). Note that $\{\underline{W} < -r\} = \{L^r > 0\}$, $\mathbb{N}_0$ a.e. Put

$$r_{n,j} := -j2^{-n-3} + 2^{-n-1} \le -N^{-1}/2 < 0$$

to simplify notation. By the special Markov property (cf Section 2.4 in [25]), conditionally on $\{L^{r_{n,j}} = \ell\}$, the variable $U_{n,j}$ is distributed as

$$\int \mathcal{N}(d\omega)\, X_n(\omega)$$

where $\mathcal{N}$ is a Poisson point measure with intensity $\ell\mathbb{N}_0$, and

$$X_n = \int_0^\sigma dr\, \mathbf{1}\{\widehat{W}_r \le 0\}\, \mathbf{1}\{W_r(t) \ge -2^{-n-2}\ ,\ \forall t \in [0, \zeta_r]\}.$$

From scaling properties of $\mathbb{N}_0$,

$$\int \mathcal{N}(d\omega)\, X_n(\omega) \overset{\text{(d)}}{=} 2^{-4n} \int \mathcal{N}_n(d\omega)\, X_0(\omega),$$

where $\mathcal{N}_n$ is a Poisson point measure with intensity $\ell 2^{2n}\mathbb{N}_0$. Note that the quantity

$$2^{-2n} \int \mathcal{N}_n(d\omega)\, X_0(\omega)$$

is the mean of 2^{2n} independent nonnegative random variables distributed as $\int \mathcal{N}(d\omega)\, X_0(\omega)$.

We can then use standard large deviations estimates for sums of i.i.d. random variables to derive the following. If $\eta > 0$ is fixed, we can find two positive constants ν and κ such that, for every n large enough, for every integer $j \geq N^{-1}2^{n+3}$,

$$\mathbb{N}_0(\{2^{2n}U_{n,j} \leq \nu\} \cap \{L^{r_{n,j}} \geq \eta\}) \leq \exp(-\kappa 2^{2n})\,\mathbb{N}_0(L^{r_{n,j}} \geq \eta) \leq c_0 \exp(-\kappa 2^{2n}),$$

where $c_0 = \mathbb{N}_0(\underline{W} \leq -N^{-1}/2)$ is a positive constant. In the last inequality we use the fact that $\mathbb{N}_0(L^{r_{n,j}} \geq \eta) \leq \mathbb{N}_0(L^{r_{n,j}} > 0) = \mathbb{N}_0(\underline{W} \leq r_{n,j})$. We can sum the preceding estimate over values of $j \in [N^{-1}2^{n+3}, N2^{n+3}]$, and then use the Borel-Cantelli lemma to get that $\mathbb{N}_0$ a.e. for all n sufficiently large and all $j \in [N^{-1}2^{n+3}, N2^{n+3}]$ we have either $L^{r_{n,j}} < \eta$ or $U_{n,j} > \nu\, 2^{-2n}$.

Now recall the elementary inequalities of the beginning of the proof. It follows that $\mathbb{N}_0$ a.e., for all $\mu \in [N^{-1}, N]$ we have either

$$\inf_{r\in[-\mu,-(2N)^{-1}]\cap\mathbb{Q}} L^r \leq \eta \tag{34}$$

or, for ε small enough,

$$\int_0^\sigma dr\,\mathbf{1}\{\widehat{W}_r \leq -\mu+\varepsilon\}\,\mathbf{1}\{W_r(t) \geq -\mu + \frac{\varepsilon}{8}\ ,\ \forall t \in [0,\zeta_r]\} \geq \nu\varepsilon^2. \tag{35}$$

A simple application of the special Markov property shows that under the probability measure $\mathbb{N}_0(\cdot \mid \underline{W} \leq -(2N)^{-1})$ the process $(L^{-(2N)^{-1}-a})_{a\geq 0}$ is a continuous-state branching process, hence a Feller Markov process which is absorbed at the origin. Thus, for every $a > 0$, we have

$$\inf_{r\in[-(2N)^{-1}-a,-(2N)^{-1}]\cap\mathbb{Q}} L^r > 0, \qquad \mathbb{N}_0 \text{ a.e. on the event } \{\underline{W} < -(2N)^{-1} - a\}. \tag{36}$$

We now take $\eta = \eta_k = 2^{-k}$, for every integer $k \geq 1$ (then $\nu = \nu_k$ also depends on k). If

$$\mu_k = \inf\{a \in [(2N)^{-1}, \infty[\cap\mathbb{Q} : L^{-a} \leq \eta_k\}$$

condition (34) fails for all $\mu \in [N^{-1}, N \wedge \mu_k[$, and so (35) must hold for the same values of μ. Since (36) shows that $\mu_k \uparrow -\underline{W}$ as $k \uparrow \infty$, $\mathbb{N}_0$ a.e. on $\{\underline{W} < -(2N)^{-1}\}$, this completes the proof of Lemma 5.3 and Lemma 2.4. □

6 Hausdorff dimension

In this section we compute the Hausdorff dimension of the limiting metric space appearing in Theorem 3.4. Although the metric D is not known explicitly, it turns out that we have enough information to determine this Hausdorff dimension.

Theorem 6.1 *We have a.s.*

$$\dim(\mathcal{T}_{\overline{\mathbf{e}}}/\!\approx, D) = 4.$$

Proof: We first derive the upper bound $\dim(\mathcal{T}_{\overline{\mathbf{e}}}/\!\approx, D) \leq 4$. Recall that the process $(Z_t)_{t\in[0,1]}$ is Gaussian conditionally given $(\mathbf{e}_t)_{t\geq 0}$, and that the conditional second moment of $Z_t - Z_s$ is

$m_{\mathbf{e}}(s,t)$. Also recall that the function $t \longrightarrow \mathbf{e}_t$ is a.s. Hölder continuous with exponent $\frac{1}{2} - \varepsilon$, for any $\varepsilon > 0$. From this fact and an application of the classical Kolmogorov lemma, we get that the mapping $t \longrightarrow Z_t$ is a.s. Hölder continuous with exponent $\frac{1}{4} - \varepsilon$, for any $\varepsilon \in]0, \frac{1}{4}[$. Clearly the same holds if Z is replaced by $\overline{Z}$. Hence, if $\varepsilon \in]0, \frac{1}{4}[$ is fixed, there exists a (random) constant C_1 such that, for every $s, t \in [0,1]$,

$$|\overline{Z}_s - \overline{Z}_t| \leq C_1 \, |s-t|^{\frac{1}{4}-\varepsilon}.$$

It immediately follows that, for every $s, t \in [0,1]$,

$$D^\circ(s,t) \leq 2C_1 \, |s-t|^{\frac{1}{4}-\varepsilon}.$$

Since $D \leq D^\circ$, we see that the canonical projection from $[0,1]$ onto $[0,1]/\approx$ (equipped with the metric D) is Hölder continuous with exponent $\frac{1}{4} - \varepsilon$. It follows that $\dim([0,1]/\approx, D) \leq (\frac{1}{4} - \varepsilon)^{-1}$ and since ε was arbitrary, we get $\dim(\mathcal{T}_{\overline{\mathbf{e}}}/\approx, D) = \dim([0,1]/\approx, D) \leq 4$.

The proof of the corresponding lower bound requires the following lemma. Recall that λ denotes the uniform probability measure on $\mathcal{T}_{\overline{\mathbf{e}}}$ (cf subsection 2.4). For every $a \in \mathcal{T}_{\overline{\mathbf{e}}}$ and every $\varepsilon > 0$, we set $B_D(a,\varepsilon) = \{b \in \mathcal{T}_{\overline{\mathbf{e}}} : D(a,b) < \varepsilon\}$.

Lemma 6.2 *There exists a constant C such that, for every $r \in]0,1]$,*

$$E\Big[\int_{\mathcal{T}_{\overline{\mathbf{e}}}} \lambda(da)\, \lambda(B_D(a,r))\Big] \leq C\, r^4.$$

Assume that the result of the lemma holds, and fix $\varepsilon \in]0,1]$. From the bound of the lemma, we get that, for every integer $k \geq 1$,

$$E[\lambda(\{a \in \mathcal{T}_{\overline{\mathbf{e}}} : \lambda(B_D(a, 2^{-k})) \geq 2^{-k(4-\varepsilon)}\})] \leq C\, 2^{-k\varepsilon}.$$

By summing this estimate over k, we obtain

$$\limsup_{k\to\infty} \frac{\lambda(B_D(a,2^{-k}))}{2^{-k(4-\varepsilon)}} \leq 1 \;, \quad \lambda(da) \text{ a.e., a.s.}$$

By standard density theorems for Hausdorff measures, this implies that $\dim(\mathcal{T}_{\overline{\mathbf{e}}}/\approx, D) \geq 4 - \varepsilon$, a.s., which completes the proof of Theorem 6.1. It only remains to prove Lemma 6.2. $\square$

Proof of Lemma 6.2: We rely on the case $k = 2$ of Proposition 3.5. With the notation of this proposition, we have

$$\begin{aligned} E\Big[\int_{\mathcal{T}_{\overline{\mathbf{e}}}} \lambda(da)\, \lambda(B_D(a,r))\Big] &= E\Big[\int_{\mathcal{T}_{\overline{\mathbf{e}}} \times \mathcal{T}_{\overline{\mathbf{e}}}} \lambda(da)\lambda(db)\, \mathbf{1}_{\{D(a,b)<r\}}\Big] \\ &= P[D(Y_1^\infty, Y_2^\infty) < r] \\ &\leq \liminf_{n\to\infty} P\Big[d_n(Y_1^n, Y_2^n) < (4p(p-1)/9)^{1/4} n^{1/4} r\Big]. \end{aligned}$$

On the other hand, it follows from Proposition 2.6 that

$$\begin{aligned} P\Big[d_n(Y_1^n, Y_2^n) < (4p(p-1)/9)^{1/4} n^{1/4} r\Big] &= E\Big[\frac{1}{(p-1)n+2} \# B_n(Y_n^1, (4p(p-1)/9)^{1/4} n^{1/4} r)\Big] \\ &\underset{n\to\infty}{\longrightarrow} E[\overline{\mathcal{I}}([0,r])]. \end{aligned}$$

Therefore we have obtained the bound

$$E\Big[\int_{\mathcal{T}_{\overline{\mathbf{e}}}} \lambda(da)\,\lambda(B_D(a,r))\Big] \leq E[\overline{\mathcal{I}}([0,r])].$$

Recall the notation of the proof of Lemma 2.3 in Section 5. We know that $\overline{\mathcal{I}}([0,r])$ has the same distribution as $\mathcal{J}(r)$ under $\mathbb{N}_0^{(1)}$. Furthermore the estimates recalled in the proof of Lemma 2.3 imply that, for every $r \in]0,1]$,

$$\mathbb{N}_0\Big(\mathbf{1}_{\{\sigma>1/2\}}\mathcal{J}(r)\Big) \leq C'\,r^4$$

for a certain constant C'. A simple scaling argument then gives, with another constant C,

$$E[\overline{\mathcal{I}}([0,r])] = \mathbb{N}_0^{(1)}(\mathcal{J}(r)) \leq C\,r^4.$$

This completes the proof of Lemma 6.2. □

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