PROBABILISTIC AND FRACTAL ASPECTS OF LEVY TREES

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Abstract

We investigate the random continuous trees called Lévy trees, which are obtained as scaling limits of discrete Galton-Watson trees. We give a mathematically precise definition of these random trees as random variables taking values in the set of equivalence classes of compact rooted $\mathbb{R}$-trees, which is equipped with the Gromov-Hausdorff distance. To construct Lévy trees, we make use of the coding by the height process which was studied in detail in previous work. We then investigate various probabilistic properties of Lévy trees. In particular we establish a branching property analogous to the well-known property for Galton-Watson trees: Conditionally given the tree below level a, the subtrees originating from that level are distributed as the atoms of a Poisson point measure whose intensity involves a local time measure supported on the vertices at distance a from the root. We study regularity properties of local times in the space variable, and prove that the support of local time is the full level set, except for certain exceptional values of a corresponding to local extinctions. We also compute several fractal dimensions of Lévy trees, including Hausdorff and packing dimensions, in terms of lower and upper indices for the branching mechanism function ψ which characterizes the distribution of the tree. We finally discuss some applications to super-Brownian motion with a general branching mechanism.

1 Introduction.

This work is devoted to the study of various properties of the so-called Lévy trees, which are continuous analogues of the discrete Galton-Watson trees. Our main contributions to the probabilistic analysis of Lévy trees include the construction of local time measures supported on level sets of the tree, the use of these local times to formulate and establish a branching property analogous to a well-known result in the discrete setting, and the proof of a "subtree" decomposition along the ancestral line of a typical vertex in the tree. Additionally, we study the fractal properties of Lévy trees and compute their Hausdorff and packing dimensions as well as that of particular subsets such as level sets, under broad assumptions on the branching mechanism characterizing the tree.

One major originality of the present article compared to our previous work [9],[20],[21] is to view Lévy trees as random variables taking values in the space of compact rooted $\mathbb{R}$-trees. The precise definition of an $\mathbb{R}$-tree is recalled in Section 2 below. Informally an $\mathbb{R}$-tree is a

metric space $(\mathcal{T}, d)$ such that for any two points σ and σ' in $\mathcal{T}$ there is a unique arc with endpoints σ and σ' and furthermore this arc is isometric to a compact interval of the real line. A rooted $\mathbb{R}$-tree is an $\mathbb{R}$-tree with a distinguished vertex called the root. We write $h(\mathcal{T})$ for the height of $\mathcal{T}$, that is the maximal distance from the root to a vertex in $\mathcal{T}$. Say that two rooted $\mathbb{R}$-trees are equivalent if there is a root-preserving isometry that maps one onto the other. It was noted in [13] that the set of equivalence classes of compact rooted $\mathbb{R}$-trees, equipped with the Gromov-Hausdorff distance [15] is a Polish space.

The study of $\mathbb{R}$-trees has been motivated by algebraic and geometric purposes. See in particular [26] and the survey [6]. One of our goals is to initiate a probabilistic theory of $\mathbb{R}$-trees, by starting with the fundamental case of Lévy trees. See [13] for another probabilistic application of $\mathbb{R}$-trees. We also mention the recent article [3], which discusses a different class of continuum random trees obtained as weak limits of birthday trees (instead of the Galton-Watson trees considered here), using ideas related to the present work.

To motivate our definition of Lévy trees, let us describe a simple approximation result, which is a special case of Theorem 4.1 below. Let μ be a probability measure on $\mathbb{Z}_+$, with $\mu(1) < 1$. Assume that μ has mean one and is in the domain of attraction of a stable distribution with index $\gamma \in (1, 2]$. When $\gamma = 2$, this holds as soon as μ has finite variance, and when $\gamma \in (1, 2)$, it is enough to assume that $\mu(k) \sim c\, k^{-1-\gamma}$ as $k \to \infty$. Denote by θ a Galton-Watson tree with offspring distribution μ, which describes the genealogy of a (discrete-time) Galton-Watson branching process with offspring distribution μ started initially with one ancestor. We can view θ as a (random) finite graph and equip it with the natural graph distance. If $r > 0$, the scaled tree $r\theta$ is obviously defined by requiring the distance between two neighboring vertices to be r instead of 1. Also let $h(\theta)$ stand for the maximal generation in θ. Then there is a σ-finite measure $\Theta(d\mathcal{T})$ on the space of (equivalence classes of) rooted compact $\mathbb{R}$-trees such that for every $a > 0$, the conditional law of the scaled tree $n^{-1}\theta$ knowing that $h(\theta) \geq an$ converges as $n \to \infty$ to the probability measure $\Theta(d\mathcal{T} \mid h(\mathcal{T}) \geq a)$, in the sense of weak convergence for the Gromov-Hausdorff distance on pointed metric spaces.

In a sense, the preceding result is not really new: See [2],[8] and especially Chapter 2 of [9] for related limit theorems with a different formalism. Still we believe that the formalism of $\mathbb{R}$-trees is useful both to formulate such results and to analyse the limiting objects as we do in the present work.

Let us turn to a more precise description of the class of random trees that will be considered here. A Lévy tree can be interpreted as the genealogical tree of a continuous-state branching process, whose law is characterized by a real function ψ defined on $[0, \infty)$, which is called the branching mechanism. Here we restrict our attention to the critical or subcritical case where ψ is nonnegative and of the form

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} \pi(dr)(e^{-\lambda r} - 1 + \lambda r)\,, \quad \lambda \geq 0,$$

where $\alpha, \beta \geq 0$ and π is a σ-finite measure on $(0, \infty)$ such that $\int_{(0,\infty)} \pi(dr)(r \wedge r^2) < \infty$. We assume throughout this work the condition

$$\int_1^\infty \frac{du}{\psi(u)} < \infty$$

which is equivalent to the a.s. extinction of the continuous-state branching process, and thus necessary for the compactness of the associated genealogical tree. Of particular importance

are the quadratic branching case $\psi(\lambda) = c\,\lambda^2$ and the stable case $\psi(\lambda) = c\,\lambda^\gamma$, $1 < \gamma < 2$, which both arise in the discrete approximation described above.

The precise definition of the ψ-Lévy tree then depends on the height process introduced by Le Gall and Le Jan [20] (see also Chapter 1 of [9]) in view of coding the genealogy of general continuous-state branching processes. The height process is obtained as a functional of the spectrally positive Lévy process X with Laplace exponent ψ. An important role is played by the excursion measure N of X above its minimum process. In the quadratic branching case $\psi(u) = c\,u^2$, X is a (scaled) Brownian motion, the height process H is a reflected Brownian motion and the "law" of H under N is just the Itô measure of positive excursions of linear Brownian motion: This is related to the fact that the contour process of Aldous' *Continuum Random Tree* is given by a normalized Brownian excursion (see [1] and [2]), or to the Brownian snake construction of superprocesses with quadratic branching mechanism (see e.g. [19]). In our more general setting, the height process can be defined informally as follows. For every $t \geq 0$, H_t measures the size of the set $\{s \leq t : X_s = \inf_{[s,t]} X_r\}$. A precise definition of H_t is recalled in Section 3 below. Under our assumptions, the process H has a continuous modification.

The claim is now that the sample path of H under N codes a random continuous tree called the ψ-Lévy tree. The precise meaning of the coding is explained in Section 2 in a deterministic setting, but let us immediately outline the construction of the tree. We write ζ for the duration of the excursion under N and define a random function d_H on $[0,\zeta]^2$ by setting

$$d_H(s,t) = H_s + H_t - 2m_H(s,t)\,,$$

where we have set $m_H(s,t) = \inf_{s\wedge t \leq r \leq s\vee t} H_r$. We introduce an associated equivalence relation by setting $s \sim_H t$ if and only if $d_H(s,t) = 0$. In particular, $0 \sim_H \zeta$. The function d_H obviously extends to the quotient set $\mathcal{T}_H := [0,\zeta]/\sim_H$ and defines a distance on this set. It is not hard to verify that $(\mathcal{T}_H, d_H)$ is a compact $\mathbb{R}$-tree, and its root is by definition the equivalence class of 0. Informally, each real number $s \in [0,\zeta]$ corresponds to a vertex at level H_s in the tree, and $d_H(s,t)$ is the distance between vertices corresponding to s and t (in particular s and t correspond to the same vertex if and only if $d_H(s,t) = 0$). The quantity $m_H(s,t)$ can be interpreted as the generation of the most recent common ancestor to s and t.

The law of the Lévy tree is by definition the distribution $\Theta(d\mathcal{T})$ of the compact rooted $\mathbb{R}$-tree $(\mathcal{T}_H, d_H)$ under the measure N. Notice that N is an infinite measure, and so is Θ. However, for every $a > 0$, $v(a) := \Theta(h(\mathcal{T}) > a) < \infty$, and more precisely $v(a)$ is determined by the equation

$$\int_{v(a)}^\infty \frac{du}{\psi(u)} = a.$$

Section 4 contains the proof of several important properties of Lévy trees. In particular, for every $a > 0$, we construct the local time ℓ^a at level a, which is a finite measure supported on the level set

$$\mathcal{T}(a) := \{\sigma \in \mathcal{T} : d(\rho(\mathcal{T}), \sigma) = a\}$$

where $\rho(\mathcal{T})$ denotes the root of $\mathcal{T}$. We then prove the fundamental "branching property": If $(\mathcal{T}^{(i),\circ}, i \in \mathcal{I})$ denote the connected components of the open set $\{\sigma \in \mathcal{T} : d(\rho(\mathcal{T}), \sigma) > a\}$, the closure $\mathcal{T}^{(i)}$ of each $\mathcal{T}^{(i),\circ}$ is a compact rooted $\mathbb{R}$-tree with root $\sigma_i \in \mathcal{T}(a)$ and, conditionally

on ℓ^a, the point measure

$$\sum_{i \in \mathcal{I}} \delta_{(\sigma_i, \mathcal{T}^{(i)})}$$

is Poisson with intensity $\ell^a(d\sigma)\,\Theta(d\mathcal{T})$ (see Theorem 4.2 for a slightly more precise result stating that this point measure is also independent of the part of the tree "below level a"). Up to some point, the branching property follows from a result of [9] (Proposition 1.3.1 or Proposition 4.2.3) showing that excursions of the height process above level a are distributed as the atoms of a Poisson point measure whose intensity is (a random multiple of) the law of H under N. In this form, the branching property has been recently used by Miermont [25] to investigate self-similar fragmentations of the stable tree.

Using the branching property, we investigate the regularity properties of local times. We show that the mapping $a \longrightarrow \ell^a$ has a càdlàg modification and that, except for a countable set of values of a (corresponding to local extinctions of the tree) the support of ℓ^a is the full level set $\mathcal{T}(a)$. This is used in Section 6 to extend to superprocesses with a general branching mechanism a continuity property of the support process that had been derived by Perkins [28] in the quadratic case.

In the final part of Section 4 we prove a Palm-like decomposition of the tree along the ancestor line of a typical vertex at level a (Theorem 4.5). This decomposition plays an important role in Section 5. We use it in Section 4 to analyse the multiplicity of vertices of the tree. By definition, the multiplicity $n(\sigma)$ of $\sigma \in \mathcal{T}$ is the number of connected components of $\mathcal{T}\backslash\{\sigma\}$. We prove that Θ a.e. $n(\sigma)$ takes values in the set $\{1, 2, 3, \infty\}$. We also characterize the branching mechanism functions ψ for which there exist binary ($n(\sigma) = 3$) or infinite ($n(\sigma) = \infty$) branching points. We then observe that infinite branching points are related to discontinuities of local times: Precisely, for any level b such that the mapping $a \longrightarrow \ell^a$ is discontinuous at b, there is a (unique) infinite branching point σ_b such that $\ell^b = \ell^{b-} + \lambda_b\,\delta_{\sigma_b}$ for some $\lambda_b > 0$. As a last application of our Palm decomposition, we prove an invariance property of the measure Θ under uniform re-rooting (Proposition 4.8).

Section 5 is mostly devoted to the computation of the Hausdorff and packing dimensions of various subsets of $\mathcal{T}$. For any subset A of $\mathcal{T}$, we denote by $\dim_h(A)$ the Hausdorff dimension of A and by $\dim_p(A)$ its packing dimension. Following [14], Section 3.1, we also consider the lower and upper box counting dimensions of A:

$$\underline{\dim}(A) = \liminf_{\delta \to 0} \frac{\log\left(\mathcal{N}(A, \delta)\right)}{\log(1/\delta)} \;,\quad \overline{\dim}(A) = \limsup_{\delta \to 0} \frac{\log\left(\mathcal{N}(A, \delta)\right)}{\log(1/\delta)} \;,$$

where $\mathcal{N}(A, \delta)$ is the minimal number of open balls with radius δ that are necessary to cover A. In order to state our main results, we need to introduce the lower and upper indices of ψ at infinity:

$$\gamma = \sup\{a \geq 0 \;:\; \lim_{\lambda \to \infty} \lambda^{-a}\psi(\lambda) = +\infty\} \quad,\quad \eta = \inf\{a \geq 0 \;:\; \lim_{\lambda \to \infty} \lambda^{-a}\psi(\lambda) = 0\}.$$

Note that $1 \leq \gamma \leq \eta$ and that $\eta = \gamma$ if ψ is regularly varying at infinity. Let E be a nonempty compact subset of the interval $(0, \infty)$ and assume that E is regular in the sense that its Hausdorff and upper box counting dimensions coincide: $\dim_h(E) = \overline{\dim}(E) = d(E) \in [0, 1]$ (here $\dim_h$ and $\overline{\dim}$ obviously refer to the usual metric on the real line). Set $a = \sup E$ and

$$\mathcal{T}(E) = \bigcup_{l \in E} \mathcal{T}(l).$$

Theorem 5.5 asserts that under the assumption $\gamma > 1$, we have Θ-a.e. on $\{h(\mathcal{T}) > a\}$,

$$\underline{\dim}(\mathcal{T}(E)) = \dim_h(\mathcal{T}(E)) = d(E) + \frac{1}{\eta - 1} \quad \text{and} \quad \overline{\dim}(\mathcal{T}(E)) = \dim_p(\mathcal{T}(E)) = d(E) + \frac{1}{\gamma - 1}.$$

In particular, we have Θ-a.e.

$$\dim_h(\mathcal{T}) = \frac{\eta}{\eta - 1} \quad , \quad \dim_p(\mathcal{T}) = \frac{\gamma}{\gamma - 1}$$

and, Θ-a.e. on $\{h(\mathcal{T}) > a\}$,

$$\dim_h(\mathcal{T}(a)) = \frac{1}{\eta - 1} \quad , \quad \dim_p(\mathcal{T}(a)) = \frac{1}{\gamma - 1}.$$

Note that in the stable branching case $\psi(u) = u^\gamma$, the Hausdorff dimension of $\mathcal{T}$ has been computed by Haas and Miermont [16] independently of the present work.

The proofs rely on the classical results linking upper and lower densities of a measure with the Hausdorff and packing dimensions of its support. Another useful ingredient is the following estimate for covering numbers of $\mathcal{T}$ (Proposition 5.2). We have Θ-a.e. for all sufficiently small δ,

$$\frac{v(2\delta)}{4\delta} \zeta \leq \mathcal{N}(\mathcal{T}, \delta) \leq \frac{12v(\delta/6)}{\delta} \zeta.$$

In Section 6, we give an application of Theorem 5.5 to the range of a superprocess $Z = (Z_l, l \geq 0)$ with branching mechanism ψ, whose spatial motion is standard Brownian motion in $\mathbb{R}^k$. To this end, we introduce the notion of a spatial tree, which allows us to combine the genealogical structure of $\mathcal{T}$ with independent spatial Brownian motions. Of course, this is more or less equivalent to the Lévy snake approach of [21] and Chapter 4 of [9]. Still the formalism of $\mathbb{R}$-trees makes this construction more tractable and more efficient for applications. Roughly speaking, spatial trees allow us to express the superprocess Z in terms of the occupation measure of a Gaussian process indexed by $\mathcal{T}$. It is therefore possible to use soft arguments to lift fractal properties of the index set $\mathcal{T}$ to the range of Z. We prove the following result (Theorem 6.3). Let $E \subset (0, \infty)$ and $a = \sup E$ be as above. Denote by $\mathcal{R}_E$ the range of Z over the time set E, defined by

$$\mathcal{R}_E = \overline{\bigcup_{l \in E} \operatorname{supp} Z_l}$$

where $\operatorname{supp} Z_l$ stands for the topological support of Z_l. If $\gamma > 1$, then a.s. on $\{\langle Z_a, 1 \rangle \neq 0\}$,

$$\dim_h(\mathcal{R}_E) = \left(2d(E) + \frac{2}{\eta - 1} \right) \wedge k. \tag{1}$$

In the quadratic branching case, this result was obtained earlier by Tribe [30] (see also Serlet [29]). For more general superprocesses, closely related results can be found in Theorem 2.1 of Delmas [7], whose proof depends on a subordination method which requires certain restrictions on the branching mechanism function ψ. See Dawson [4] for more references and results in the stable branching case.

This paper is intended to be as self-contained as possible. However, it is clear that many of our results depend on properties of the height process H that were derived in the monograph

[9]. For the reader's convenience, we have recalled most of the needed results in Section 3 below.

The paper is organized as follows. Section 2 explains the coding of trees by continuous functions in a deterministic setting, and also includes a brief discussion of the convergence of trees in the Gromov-Hausdorff metric. Section 3 recalls the basic facts about the height process and establishes an important preliminary result that is needed for the ancestral line decomposition of subsection 4.3. Section 4 is the core of this paper. It first contains the precise definition of the Lévy tree as the tree coded by the excursion of H under N, in the framework of Section 2. This definition is justified by limit theorems relating discrete and continuous trees. Section 4 then presents the basic probabilitic properties of Lévy trees, in particular the branching property, the existence and regularity of local times and the decomposition along an ancestral line. Fractal properties of Lévy trees are studied in Section 5. Finally, Section 6 discusses applications to superprocesses.

2 Deterministic trees

2.1 The $\mathbb{R}$-tree coded by a continuous function

We start with a basic definition (see e.g. [6]).

Definition 2.1 *A metric space $(\mathcal{T}, d)$ is an $\mathbb{R}$-tree if the following two properties hold for every $\sigma_1, \sigma_2 \in \mathcal{T}$.*

(i) *There is a unique isometric map f_{σ_1, σ_2} from $[0, d(\sigma_1, \sigma_2)]$ into $\mathcal{T}$ such that $f_{\sigma_1, \sigma_2}(0) = \sigma_1$ and $f_{\sigma_1, \sigma_2}(d(\sigma_1, \sigma_2)) = \sigma_2$.*

(ii) *If q is a continuous injective map from $[0, 1]$ into $\mathcal{T}$, such that $q(0) = \sigma_1$ and $q(1) = \sigma_2$, we have*
$$q([0, 1]) = f_{\sigma_1, \sigma_2}([0, d(\sigma_1, \sigma_2)]).$$

A rooted $\mathbb{R}$-tree is an $\mathbb{R}$-tree $(\mathcal{T}, d)$ with a distinguished vertex $\rho = \rho(\mathcal{T})$ called the root.

In what follows, $\mathbb{R}$-trees will always be rooted, even if this is not mentioned explicitly.

Let us consider a rooted $\mathbb{R}$-tree $(\mathcal{T}, d)$. The range of the mapping f_{σ_1, σ_2} in (i) is denoted by $[\![\sigma_1, \sigma_2]\!]$ (this is the line segment between σ_1 and σ_2 in the tree). In particular, for every $\sigma \in \mathcal{T}$, $[\![\rho, \sigma]\!]$ is the path going from the root to σ, which we will interpret as the ancestral line of vertex σ. More precisely we define a partial order on the tree by setting $\sigma \preccurlyeq \sigma'$ (σ is an ancestor of σ') if and only if $\sigma \in [\![\rho, \sigma']\!]$.

If $\sigma, \sigma' \in \mathcal{T}$, there is a unique $\eta \in \mathcal{T}$ such that $[\![\rho, \sigma]\!] \cap [\![\rho, \sigma']\!] = [\![\rho, \eta]\!]$. We write $\eta = \sigma \wedge \sigma'$ and call η the most recent common ancestor to σ and σ'.

By definition, the multiplicity of a vertex $\sigma \in \mathcal{T}$ is the number of connected components of $\mathcal{T} \backslash \{\sigma\}$. Vertices of $\mathcal{T} \backslash \{\rho\}$ which have multiplicity 1 are called leaves.

Our main goal in this section is to describe a method for constructing $\mathbb{R}$-trees, which is particularly well-suited to our forthcoming applications to random trees. We consider a (deterministic) continuous function $g : [0, \infty) \longrightarrow [0, \infty)$ with compact support and such that

$g(0) = 0$. To avoid trivialities, we will also assume that g is not identically zero. For every $s, t \geq 0$, we set

$$m_g(s,t) = \inf_{r \in [s \wedge t, s \vee t]} g(r),$$

and

$$d_g(s,t) = g(s) + g(t) - 2m_g(s,t).$$

Clearly $d_g(s,t) = d_g(t,s)$ and it is also easy to verify the triangle inequality

$$d_g(s,u) \leq d_g(s,t) + d_g(t,u)$$

for every $s, t, u \geq 0$. We then introduce the equivalence relation $s \sim t$ iff $d_g(s,t) = 0$ (or equivalently iff $g(s) = g(t) = m_g(s,t)$). Let $\mathcal{T}_g$ be the quotient space

$$\mathcal{T}_g = [0, \infty)/ \sim .$$

Obviously the function d_g induces a distance on $\mathcal{T}_g$, and we keep the notation d_g for this distance. We denote by $p_g : [0, \infty) \longrightarrow \mathcal{T}_g$ the canonical projection. Clearly p_g is continuous (when $[0, \infty)$ is equipped with the Euclidean metric and $\mathcal{T}_g$ with the metric d_g).

Theorem 2.1 *The metric space $(\mathcal{T}_g, d_g)$ is an $\mathbb{R}$-tree.*

We will view $(\mathcal{T}_g, d_g)$ as a rooted $\mathbb{R}$-tree with root $\rho = p_g(0)$. If $\zeta > 0$ is the supremum of the support of g, we have $p_g(t) = \rho$ for every $t \geq \zeta$. In particular, $\mathcal{T}_g = p_g([0, \zeta])$ is compact. We will call $\mathcal{T}_g$ the $\mathbb{R}$-tree coded by g.

Before proceeding to the proof of the theorem, we state and prove the following root change lemma.

Lemma 2.2 *Let $s_0 \in [0, \zeta)$. For any real $r \geq 0$, denote by $\bar{r}$ the unique element of $[0, \zeta)$ such that $r - \bar{r}$ is an integer multiple of ζ. Set*

$$g'(s) = g(s_0) + g(\overline{s_0 + s}) - 2m_g(s_0, \overline{s_0 + s}),$$

for every $s \in [0, \zeta]$, and $g'(s) = 0$ for $s > \zeta$. Then, the function g' is continuous with compact support and satisfies $g'(0) = 0$, so that we can define the metric space $(\mathcal{T}_{g'}, d_{g'})$. Furthermore, for every $s, t \in [0, \zeta]$, we have

$$d_{g'}(s,t) = d_g(\overline{s_0 + s}, \overline{s_0 + t}) \tag{2}$$

and there exists a unique isometry R from $\mathcal{T}_{g'}$ onto $\mathcal{T}_g$ such that, for every $s \in [0, \zeta]$,

$$R(p_{g'}(s)) = p_g(\overline{s_0 + s}). \tag{3}$$

Proof. It is immediately checked that g' satisfies the same assumptions as g, so that we can make sense of the tree $\mathcal{T}_{g'}$. Then the key step is to verify the relation (2). Consider first the case where $s, t \in [0, \zeta - s_0)$. Then two possibilities may occur.

If $m_g(s_0 + s, s_0 + t) \geq m_g(s_0, s_0 + s)$, then $m_g(s_0, s_0 + r) = m_g(s_0, s_0 + s) = m_g(s_0, s_0 + t)$ for every $r \in [s, t]$, and so

$$m_{g'}(s,t) = g(s_0) + m_g(s_0 + s, s_0 + t) - 2m_g(s_0, s_0 + s).$$

It follows that

$$
\begin{aligned}
d_{g'}(s,t) &= g'(s) + g'(t) - 2m_{g'}(s,t) \\
&= g(s_0 + s) - 2m_g(s_0, s_0 + s) + g(s_0 + t) - 2m_g(s_0, s_0 + t) \\
&\quad -2(m_g(s_0 + s, s_0 + t) - 2m_g(s_0, s_0 + s)) \\
&= g(s_0 + s) + g(s_0 + t) - 2m_g(s_0 + s, s_0 + t) \\
&= d_g(s_0 + s, s_0 + t).
\end{aligned}
$$

If $m_g(s_0 + s, s_0 + t) < m_g(s_0, s_0 + s)$, then the minimum in the definition of $m_{g'}(s,t)$ is attained at r_1 defined as the first $r \in [s,t]$ such that $g(s_0 + r) = m_g(s_0, s_0 + s)$ (because for $r \in [r_1, t]$ we will have $g(s_0 + r) - 2m_g(s_0, s_0 + r) \geq -m_g(s_0, s_0 + r) \geq -m_g(s_0, s_0 + r_1)$). Therefore,

$$
m_{g'}(s,t) = g(s_0) - m_g(s_0, s_0 + s),
$$

and

$$
\begin{aligned}
d_{g'}(s,t) &= g(s_0 + s) - 2m_g(s_0, s_0 + s) + g(s_0 + t) - 2m_g(s_0, s_0 + t) + 2m_g(s_0, s_0 + s) \\
&= d_g(s_0 + s, s_0 + t).
\end{aligned}
$$

The other cases are treated in a similar way and are left to the reader.

By (2), if $s, t \in [0, \zeta]$ are such that $d_{g'}(s,t) = 0$, we have $d_g(\overline{s_0 + s}, \overline{s_0 + t}) = 0$ so that $p_g(\overline{s_0 + s}) = p_g(\overline{s_0 + t})$. Noting that $\mathcal{T}_{g'} = p_{g'}([0, \zeta])$ (the supremum of the support of g' is less than or equal to ζ), we can define R in a unique way by the relation (3). From (2), R is an isometry, and it is also immediate that R takes $\mathcal{T}_{g'}$ onto $\mathcal{T}_g$. ∎

Proof of Theorem 2.1. Let us start with some preliminaries. For $\sigma, \sigma' \in \mathcal{T}_g$, we set $\sigma \preccurlyeq \sigma'$ if and only if $d_g(\sigma, \sigma') = d_g(\rho, \sigma') - d_g(\rho, \sigma)$. If $\sigma = p_g(s)$ and $\sigma' = p_g(t)$, it follows from our definitions that $\sigma \preccurlyeq \sigma'$ iff $m_g(s,t) = g(s)$. It is immediate to verify that this defines a partial order on $\mathcal{T}_g$.

For any $\sigma_0, \sigma \in \mathcal{T}_g$, we set

$$
[\![\sigma_0, \sigma]\!] = \{\sigma' \in \mathcal{T}_g : d_g(\sigma_0, \sigma) = d_g(\sigma_0, \sigma') + d_g(\sigma', \sigma)\}.
$$

If $\sigma = p_g(s)$ and $\sigma' = p_g(t)$, then it is easy to verify that $[\![\rho, \sigma]\!] \cap [\![\rho, \sigma']\!] = [\![\rho, \gamma]\!]$, where $\gamma = p_g(r)$, if r is any time which achieves the minimum of g between s and t. We then put $\gamma = \sigma \wedge \sigma'$.

We set $\mathcal{T}_g[\sigma] := \{\sigma' \in \mathcal{T}_g : \sigma \preccurlyeq \sigma'\}$. If $\mathcal{T}_g[\sigma] \neq \{\sigma\}$ and $\sigma \neq \rho$, then $\mathcal{T}_g \backslash \mathcal{T}_g[\sigma]$ and $\mathcal{T}_g[\sigma] \backslash \{\sigma\}$ are two nonempty disjoint open sets. To see that $\mathcal{T}_g \backslash \mathcal{T}_g[\sigma]$ is open, let s be such that $p_g(s) = \sigma$ and note that $\mathcal{T}_g[\sigma]$ is the image under p_g of the compact set $\{u \in [0, \zeta] : m_g(s,u) = g(s)\}$. The set $\mathcal{T}_g[\sigma] \backslash \{\sigma\}$ is open because if $\sigma' \in \mathcal{T}_g[\sigma]$ and $\sigma' \neq \sigma$, it easily follows from our definitions that the open ball centered at σ' with radius $d_g(\sigma, \sigma')$ is contained in $\mathcal{T}_g[\sigma] \backslash \{\sigma\}$.

We now prove property (i) of the definition of an $\mathbb{R}$-tree. By using Lemma 2.2 with s_0 such that $p_g(s_0) = \sigma_1$, we may assume that $\sigma_1 = \rho = p_g(0)$. If $\sigma \in \mathcal{T}_g$ is fixed, we have to prove that there exists a unique isometry f from $[0, d_g(\rho, \sigma)]$ into $\mathcal{T}_g$ such that $f(0) = \rho$ and $f(d_g(\rho, \sigma)) = \sigma$. Let $s \in p_g^{-1}(\{\sigma\})$, so that $g(s) = d_g(\rho, \sigma)$. Then, for every $a \in [0, d_g(\rho, \sigma)]$, we set

$$
w(a) = \inf\{r \in [0, s] : m_g(r, s) = a\}.
$$

Note that $g(w(a)) = a$. We put $f(a) = p_g(w(a))$. We have $f(0) = \rho$ and $f(d_g(\rho, \sigma)) = \sigma$, the latter because $m_g(w(g(s)), s) = g(s)$ implies $p_g(w(g(s))) = p_g(s) = \sigma$. It is also easy to verify that f is an isometry: If $a, b \in [0, d_g(\rho, \sigma)]$ with $a \leq b$, it is immediate that $m_g(w(a), w(b)) = a$, and so

$$d_g(f(a), f(b)) = g(w(a)) + g(w(b)) - 2a = b - a.$$

To get uniqueness, suppose that $\tilde{f}$ is an isometry satisfying the property in (i). Then, if $a \in [0, d_g(\rho, \sigma)]$,

$$d_g(\tilde{f}(a), \sigma) = d_g(\rho, \sigma) - a = d_g(\rho, \sigma) - d_g(\rho, \tilde{f}(a)).$$

Therefore, $\tilde{f}(a) \preccurlyeq \sigma$. Recall that $\sigma = p_g(s)$, and choose t such that $p_g(t) = \tilde{f}(a)$. Note that $g(t) = d_g(\rho, p_g(t)) = a$. Since $\tilde{f}(a) \preccurlyeq \sigma$ we have $g(t) = m_g(t, s)$. On the other hand, we also know that $a = g(w(a)) = m_g(w(a), s)$. It follows that we have $a = g(t) = g(w(a)) = m_g(w(a), t)$ and thus $d_g(t, w(a)) = 0$, so that $\tilde{f}(a) = p_g(t) = p_g(w(a)) = f(a)$. This completes the proof of (i).

As a by-product of the preceding argument, we see that $f([0, d_g(\rho, \sigma)]) = [\![\rho, \sigma]\!]$: Indeed, we have seen that for every $a \in [0, d_g(\rho, \sigma)]$, we have $f(a) \preccurlyeq \sigma$ and, on the other hand, if $\eta \preccurlyeq \sigma$, the end of the proof of (i) just shows that $\eta = f(d_g(\rho, \eta))$.

We turn to the proof of (ii). We let q be a continuous injective mapping from $[0, 1]$ into $\mathcal{T}_g$, and we aim at proving that $q([0, 1]) = f_{q(0), q(1)}([0, d_g(q(0), q(1))])$. From Lemma 2.2 again, we may assume that $q(0) = \rho$, and we set $\sigma = q(1)$. Then we have just noticed that $f_{0, \sigma}([0, d_g(\rho, \sigma)]) = [\![\rho, \sigma]\!]$.

We first argue by contradiction to prove that $[\![\rho, \sigma]\!] \subset q([0, 1])$. Suppose that $\eta \in [\![\rho, \sigma]\!] \backslash q([0, 1])$, and in particular, $\eta \neq \rho, \sigma$. Then $q([0, 1])$ is contained in the union of the two disjoint open sets $\mathcal{T}_g \backslash \mathcal{T}_g[\eta]$ and $\mathcal{T}_g[\eta] \backslash \{\eta\}$, with $q(0) = \rho \in \mathcal{T}_g \backslash \mathcal{T}_g[\eta]$ and $q(1) = \sigma \in \mathcal{T}_g[\eta] \backslash \{\eta\}$. This contradicts the fact that $q([0, 1])$ is connected.

Conversely, suppose that there exists $a \in (0, 1)$ such that $q(a) \notin [\![\rho, \sigma]\!]$. Set $\eta = q(a)$ and let $\gamma = \sigma \wedge \eta$. Note that $\gamma \in [\![\rho, \eta]\!] \cap [\![\eta, \sigma]\!]$ (from the definition of $\sigma \wedge \eta$, it is immediate to verify that $d_g(\eta, \sigma) = d_g(\eta, \gamma) + d_g(\gamma, \sigma)$). From the first part of the proof of (ii), $\gamma \in q([0, a])$ and, via a root change argument, $\gamma \in q([a, 1])$. Since q is injective, this is only possible if $\gamma = q(a) = \eta$, which contradicts the fact that $\eta \notin [\![\rho, \sigma]\!]$. ∎

Once we know that $(\mathcal{T}_g, d_g)$ is an $\mathbb{R}$-tree, it is straightforward to verify that the notation $\sigma \preccurlyeq \sigma'$, $[\![\sigma, \sigma']\!]$, $\sigma \wedge \sigma'$ introduced in the preceding proof is consistent with the definitions stated for a general $\mathbb{R}$-tree at the beginning of this section.

Let us briefly discuss multiplicities of vertices in the tree $\mathcal{T}_g$. If $\sigma \in \mathcal{T}_g$ is not a leaf then we must have $\ell(\sigma) < r(\sigma)$, where

$$\ell(\sigma) := \sup p_g^{-1}(\{\sigma\}), \quad r(\sigma) := \inf p_g^{-1}(\{\sigma\})$$

are respectively the smallest and the largest element in the equivalence class of σ in $[0, \zeta]$. Note that $m_g(\ell(\sigma), r(\sigma)) = g(\ell(\sigma)) = g(r(\sigma)) = d_g(\rho, \sigma)$. Denote by $(a_i, b_i), i \in \mathcal{I}$ the connected components of the open set $(\ell(\sigma), r(\sigma)) \cap \{t \in [0, \infty) : g(t) > d_g(\rho, \sigma)\}$ (the index set $\mathcal{I}$ is empty if σ is a leaf). Then we claim that the connected components of the open set $\mathcal{T}_g \backslash \{\sigma\}$ are the sets $p_g((a_i, b_i)), i \in \mathcal{I}$ and $\mathcal{T}_g \backslash \mathcal{T}_g[\sigma]$ (the latter only if σ is not the root). We have already noticed that $\mathcal{T}_g \backslash \mathcal{T}_g[\sigma]$ is open, and the argument used above for $\mathcal{T}_g[\sigma] \backslash \{\sigma\}$ also

shows that the sets $p_g((a_i, b_i))$, $i \in \mathcal{I}$ are open. Finally the sets $p_g((a_i, b_i))$ are connected as continuous images of intervals, and $\mathcal{T}_g \backslash \mathcal{T}_g[\sigma]$ is also connected because if $\sigma', \sigma'' \in \mathcal{T}_g \backslash \mathcal{T}_g[\sigma]$, $[\![\rho, \sigma']\!] \cup [\![\rho, \sigma'']\!]$ is a connected closed set contained in $\mathcal{T}_g \backslash \mathcal{T}_g[\sigma]$.

2.2 Convergence of trees

Two rooted $\mathbb{R}$-trees $\mathcal{T}_{(1)}$ and $\mathcal{T}_{(2)}$ are called equivalent if there is a root-preserving isometry that maps $\mathcal{T}_{(1)}$ onto $\mathcal{T}_{(2)}$. We denote by $\mathbb{T}$ the set of all equivalence classes of rooted compact $\mathbb{R}$-trees. The set $\mathbb{T}$ can be equipped with the (pointed) Gromov-Hausdorff distance, which is defined as follows.

If (E, δ) is a metric space, we use the notation $\delta_{Haus}(K, K')$ for the usual Hausdorff metric between compact subsets of E. Then, if $\mathcal{T}$ and $\mathcal{T}'$ are two rooted compact $\mathbb{R}$-trees with respective roots ρ and ρ', we define the distance $d_{GH}(\mathcal{T}, \mathcal{T}')$ as

$$d_{GH}(\mathcal{T}, \mathcal{T}') = \inf \Big(\delta_{Haus}(\varphi(\mathcal{T}), \varphi'(\mathcal{T}')) \vee \delta(\varphi(\rho), \varphi'(\rho')) \Big),$$

where the infimum is over all isometric embeddings $\varphi : \mathcal{T} \longrightarrow E$ and $\varphi' : \mathcal{T}' \longrightarrow E$ of $\mathcal{T}$ and $\mathcal{T}'$ into a common metric space (E, δ). Obviously $d_{GH}(\mathcal{T}, \mathcal{T}')$ only depends on the equivalence classes of $\mathcal{T}$ and $\mathcal{T}'$. Furthermore d_{GH} defines a metric on $\mathbb{T}$ (cf [15] and [13]).

According to Theorem 2 of [13], the metric space $(\mathbb{T}, d_{GH})$ is complete and separable. Furthermore, the distance d_{GH} can often be evaluated in the following way. First recall that if (E_1, d_1) and (E_2, d_2) are two compact metric spaces, a correspondence between E_1 and E_2 is a subset $\mathcal{R}$ of $E_1 \times E_2$ such that for every $x_1 \in E_1$ there exists at least one $x_2 \in E_2$ such that $(x_1, x_2) \in \mathcal{R}$ and conversely for every $y_2 \in E_2$ there exists at least one $y_1 \in E_1$ such that $(y_1, y_2) \in \mathcal{R}$. The distorsion of the correspondence $\mathcal{R}$ is defined by

$$\mathrm{dis}(\mathcal{R}) = \sup\{|d_1(x_1, y_1) - d_2(x_2, y_2)| : (x_1, x_2), (y_1, y_2) \in \mathcal{R}\}.$$

Then, if $\mathcal{T}$ and $\mathcal{T}'$ are two rooted $\mathbb{R}$-trees with respective roots ρ and ρ', we have

$$d_{GH}(\mathcal{T}, \mathcal{T}') = \frac{1}{2} \inf_{\mathcal{R} \in \mathcal{C}(\mathcal{T}, \mathcal{T}'), (\rho, \rho') \in \mathcal{R}} \mathrm{dis}(\mathcal{R}), \tag{4}$$

where $\mathcal{C}(\mathcal{T}, \mathcal{T}')$ denotes the set of all correspondences between $\mathcal{T}$ and $\mathcal{T}'$ (see Lemma 2.3 in [13]).

Lemma 2.3 *Let g and g' be two continuous functions with compact support from $[0, \infty)$ into $[0, \infty)$, such that $g(0) = g'(0) = 0$. Then,*

$$d_{GH}(\mathcal{T}_g, \mathcal{T}_{g'}) \leq 2\|g - g'\|,$$

where $\|g - g'\|$ stands for the uniform norm of $g - g'$.

Proof. We can construct a correspondence between $\mathcal{T}_g$ and $\mathcal{T}_{g'}$ by setting

$$\mathcal{R} = \{(\sigma, \sigma') : \sigma = p_g(t) \text{ and } \sigma' = p_{g'}(t) \text{ for some } t \geq 0\}.$$

In order to bound the distortion of $\mathcal{R}$, let $(\sigma, \sigma') \in \mathcal{R}$ and $(\eta, \eta') \in \mathcal{R}$. By our definition of $\mathcal{R}$ we can find $s, t \geq 0$ such that $p_g(s) = \sigma$, $p_{g'}(s) = \sigma'$ and $p_g(t) = \eta$, $p_{g'}(t) = \eta'$. Now recall that

$$d_g(\sigma, \eta) \quad = \quad g(s) + g(t) - 2m_g(s, t),$$

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$$d_{g'}(\sigma', \eta') \quad = \quad g'(s) + g'(t) - 2m_{g'}(s,t),$$

so that

$$|d_g(\sigma, \eta) - d_{g'}(\sigma', \eta')| \leq 4\|g - g'\|.$$

Thus we have $\mathrm{dis}(\mathcal{R}) \leq 4\|g - g'\|$ and the desired result follows from (4). ∎

3 The height process

3.1 The definition of the height process

We will now introduce the random process which codes, in the sense of subsection 2.1, the genealogical structure of a continuous-state branching process. Recall that a continuous-state branching process is a Markov process $(Y_t, t \geq 0)$ with values in the positive half-line $[0, \infty)$, with a Feller semigroup $(Q_t, t \geq 0)$ satisfying the following additivity (or branching) property: For every $t \geq 0$ and $x, x' \geq 0$,

$$Q_t(x, \cdot) * Q_t(x', \cdot) = Q_t(x + x', \cdot).$$

Informally, this is just saying that the union of two independent populations started respectively at x and x' will evolve like a single population started at $x + x'$.

We will consider only the critical or subcritical case, meaning that $\int_{[0,\infty)} y \, Q_t(x, dy) \leq x$ for every $t \geq 0$ and $x \geq 0$. Then the Laplace functional of the semigroup can be written in the following form:

$$\int_{[0,\infty)} e^{-\lambda y} \, Q_t(x, dy) = \exp(-x \, u_t(\lambda)), \tag{5}$$

where the function $(u_t(\lambda), t \geq 0, \lambda \geq 0)$ is determined by the differential equation

$$\frac{du_t(\lambda)}{dt} = -\psi(u_t(\lambda)) \,, \quad u_0(\lambda) = \lambda \,, \tag{6}$$

and ψ is a function of the type

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda r} - 1 + \lambda r) \, \pi(dr) \,,$$

where $\alpha, \beta \geq 0$ and π is a σ-finite measure on $(0, \infty)$ such that $\int_{(0,\infty)} (r \wedge r^2) \, \pi(dr) < \infty$. Conversely, for any function ψ of this type, there exists a (unique in law) continuous-state branching process Y whose transition kernel is determined from ψ by the preceding formulas. The process Y is called the ψ-continuous-state branching process (ψ-CSBP in short). It is well known that Y has only positive jumps (indeed Y can be obtained as a time change of a spectrally positive Lévy process, see Lamperti [17]).

In the present work, we will consider only the case where the ψ-CSBP becomes extinct almost surely, which is equivalent to the condition

$$\int_1^\infty \frac{du}{\psi(u)} < \infty. \tag{7}$$

Note that this implies that at least one of the following two conditions holds:

$$\beta > 0 \quad \text{or} \quad \int_{(0,1)} r\,\pi(dr) = \infty. \tag{8}$$

(8) is necessary and sufficient for the paths of Y to be of infinite variation a.s. The coding of the genealogy that is presented below remains valid under (8) even if (7) fails to hold, but the resulting tree is no longer compact (see Theorem 4.7 in [20]). On the other hand, if (8) is not satisfied (that is in the finite variation case), the underlying branching structure is basically discrete: See Section 3 of [20] and also [23] for a discussion with applications to queuing processes).

Special cases that satisfy our assumptions are the quadratic branching case $\psi(u) = c\,u^2$ and the stable branching case $\psi(u) = c\,u^\gamma$, for some $1 < \gamma < 2$.

It has been argued in [20] and [9] that the genealogy of the ψ-CSBP is coded by the so-called height process, which is itself a functional of the Lévy process with Laplace exponent ψ. We denote by X a (spectrally positive) Lévy process with Laplace exponent ψ, defined under the probability measure $\mathbf{P}$:

$$\mathbf{E}[\exp(-\lambda X_t)] = \exp(t\psi(\lambda))\,, \quad t, \lambda \geq 0\,.$$

The subcriticality assumption on Y and condition (8) are equivalent to saying respectively that X does no drift to $+\infty$ and has paths of infinite variation.

The height process $H = (H_t;\ t \geq 0)$ associated with X is defined in such a way that, for every $t \geq 0$, H_t measures the size of the set

$$\left\{ s \in [0, t]\ :\ X_{s-} \leq \inf_{s \leq r \leq t} X_r \right\}\,. \tag{9}$$

This is motivated by a discrete analogue for Galton-Watson trees (see Section 0.2 in [9]). To make the preceding definition precise, we use a time-reversal argument: For any $t > 0$, we define the Lévy process reversed at time t by

$$\widehat{X}_s^t = X_t - X_{(t-s)-}\,, \quad 0 \leq s < t \quad \text{and} \quad \widehat{X}_t^t = X_t.$$

Then $\widehat{X}^t$ is distributed as X up to time t. Let us set

$$S_s = \sup_{r \leq s} X_r \quad \text{and} \quad \widehat{S}_s^t = \sup_{r \leq s} \widehat{X}_r^t.$$

The set (9) is the image of

$$\left\{ s \in [0, t]\ :\ \widehat{S}_s^t = \widehat{X}_s^t \right\}\,.$$

under the time reversal operation $s \to t - s$. Recall that $S - X$ is a strong Markov process for which 0 is a regular point. So, we can consider its local time process at 0, which is denoted by $\Gamma(X) = (\Gamma_t(X), t \geq 0)$. We define the height process by

$$H_t = \Gamma_t(\widehat{X}^t)\,, \quad t \geq 0. \tag{10}$$

To complete the definition, we still need to specify the normalization of the local time $\Gamma(X)$. This can be done through the following approximation:

$$H_t = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t ds\,\mathbf{1}_{\{X_s \leq I_t^s + \varepsilon\}},$$

where $I_t^s = \inf_{s \le r \le t} X_r$ and the convergence holds in probability (this approximation follows from Lemma 1.1.3 in [9]). Thanks to condition (7), we know that the process H has a modification with continuous sample paths (Theorem 4.7 in [9]). From now on we consider only this modification. When $\beta > 0$, it is not hard to see that, for any $t \ge 0$,

$$H_t = \frac{1}{\beta}\mathrm{Leb}\left(\{\widehat{S}_s^t \; ; \; 0 \le s \le t\}\right),$$

where Leb stands for the Lebesgue measure on the real line. In particular when $\psi(u) = \beta u^2$ (X is a scaled Brownian motion), we see that $H_t = \beta^{-1}(S_t - X_t)$ is distributed as a (scaled) reflected Brownian motion.

For our purposes it will be crucial to define the height process also under the so-called excursion measure N. Set $I_t = \inf_{s \le t} X_s$ and recall that $X - I$ is a strong Markov process. Then for any $t \ge 0$, H_t only depends on the values taken by $X - I$ on the excursion interval that straddles t (at least informally this is obvious if we think of H_t as measuring the size of the set (9)). Under our assumptions, 0 is a regular point for $X - I$, and the process $-I$ can be chosen as the local time of $X - I$ at level 0. We denote by N the associated excursion measure, which plays a fundamental role throughout this work (as was already the case in [9]). The duration of the excursion under N is denoted by ζ. Let (g_i, d_i), $i \in \mathcal{I}$ be the excursion intervals of $X - I$ above 0. One easily verifies that $\mathbf{P}$ a.s.,

$$\bigcup_{i \in \mathcal{I}}(g_i, d_i) = \{s \ge 0 \; : \; X_s - I_s > 0\} = \{s \ge 0 \; : \; H_s > 0\}.$$

Denote by $H_i(s) = H_{g_i + s}$, $0 \le s \le \zeta_i = d_i - g_i$, $i \in \mathcal{I}$ the excursions of H away from 0. Then, each H_i can be written as a functional of the excursion of $X - I$ away from 0 corresponding to the interval (g_i, d_i). Consequently, if $C_+([0, \infty))$ denotes the space of all nonnegative continuous functions on $[0, \infty)$, the point measure

$$\sum_{i \in \mathcal{I}} \delta_{(-I_{g_i}, H_i)}(d\ell d\omega) \tag{11}$$

is a Poisson point measure on $\mathbb{R}_+ \times C_+([0, \infty))$ with intensity $d\ell\Delta(d\omega)$, where $\Delta(d\omega)$ is the σ-finite measure on $C_+([0, \infty))$ defined as the law of H under N. Note that in the Brownian case, Δ is the classical Itô measure of positive excursions of linear Brownian motion (up to a normalizing constant).

3.2   Local times of the height process

Let us start by the defining the local times under $\mathbf{P}$. For every $a \ge 0$, the local time of the height process at level a is the continuous increasing process $(L_s^a, s \ge 0)$ which can be characterized via the approximation

$$\lim_{\varepsilon \to 0} \mathbf{E}\left[\sup_{0 \le s \le t}\left|\frac{1}{\varepsilon}\int_0^s dr \mathbf{1}_{\{a < H_r \le a+\varepsilon\}} - L_s^a\right|\right] = 0$$

(see Section 1.3 of [9]). It is then easy to see that the support of the measure dL_s^a is contained in the closed set $\{s \ge 0 : H_s = a\}$. When $a > 0$, we have also

$$\lim_{\varepsilon \to 0} \mathbf{E}\left[\sup_{0 \le s \le t}\left|\frac{1}{\varepsilon}\int_0^s dr \mathbf{1}_{\{a-\varepsilon < H_r \le a\}} - L_s^a\right|\right] = 0 \,.$$

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Let us recall the "Ray-Knight theorem" for H ([9] Theorem 1.4.1, see also [20], Theorem 4.2), which can be viewed as a generalization of famous results about linear Brownian motion. For any $r \geq 0$, set: $T_r = \inf\{s \geq 0 \; : \; X_s = -r\}$. Then, the process $(L^a_{T_r} \; ; \; a \geq 0)$ is a ψ-CSBP started at r. In particular, this process has a càdlàg modification.

The local time at level a can also be used to describe the distribution of excursions of the height process above level a, and this will be very important for our applications. Let us fix $a > 0$ and denote by (α_j, β_j), $j \in \mathcal{J}$ the connected components of the open set $\{s \geq 0 : H_s > a\}$. For any $j \in \mathcal{J}$, denote by H^j the corresponding excursion of H defined by:

$$H^j_s = H_{(\alpha_j + s) \wedge \beta_j} - a \; , \quad s \geq 0.$$

Also set $\widetilde{H}^a_s = H_{\widetilde{\tau}^a_s}$, where for every $s \geq 0$,

$$\widetilde{\tau}^a_s = \inf\{t \geq 0 : \int_0^t dr \, \mathbf{1}_{\{H_r \leq a\}} > s\}.$$

Informally, $\widetilde{H}^a$ corresponds to the evolution of H "below level a".

The next result is a straightforward consequence of Proposition 1.3.1 in [9].

Proposition 3.1 *Under the probability* $\mathbf{P}$, *the point measure*

$$\sum_{j \in \mathcal{J}} \delta_{(L^a_{\alpha_j}, H^j)}(d\ell \, d\omega)$$

is independent of $\widetilde{H}^a$ *and is a Poisson point measure on* $\mathbb{R}_+ \times C_+([0, \infty))$ *with intensity* $d\ell \Delta(d\omega)$.

It will be important to define local times under the excursion measure N. This creates no additional difficulty thanks to the following simple remark. If $r > 0$, then for any $\delta > 0$, there is a positive probability under $\mathbf{P}$ that exactly one excursion of H away from zero hits level δ before time T_r. It easily follows that we can define for every $a > 0$ a continuous increasing process $(\Lambda^a_s, s \geq 0)$, such that, for every $\delta \in (0, a)$ and $t \geq 0$,

$$\lim_{\varepsilon \to 0} N \left(\mathbf{1}_{\{\sup H > \delta\}} \sup_{0 \leq s \leq t \wedge \zeta} \left| \frac{1}{\varepsilon} \int_0^s dr \mathbf{1}_{\{a - \varepsilon < H_r \leq a\}} - \Lambda^a_s \right| \right) = 0. \tag{12}$$

(see Section 1.3 in [9]). Again the support of the measure $d\Lambda^a_s$ is contained in $\{s : H_s = a\}$, N a.e. From the above-mentioned Ray-Knight theorem and elementary excursion theory for $X - I$ we get, for any $a > 0$ and any $\lambda \geq 0$,

$$N \left(1 - \exp(-\lambda \Lambda^a_\zeta) \right) = u_a(\lambda), \tag{13}$$

where $u_a(\lambda)$ is as in (5). We set $v(a) = \lim_{\lambda \to \infty} u_a(\lambda)$. By writing (6) in the form of integral equation and passing to the limit $\lambda \to \infty$ using (7), we see that the function v is finite on $(0, \infty)$ and determined by the equation

$$a = \int_{v(a)}^{\infty} \frac{du}{\psi(u)}.$$

Moreover, for every $a > 0$, we have

$$v(a) = N\left(\Lambda_\zeta^a > 0\right) = N\left(\sup_{s \leq \zeta} H_s > a\right). \tag{14}$$

The first equality follows from the definition of v. The second one can be deduced from Proposition 3.1, which implies that $\inf\{s \geq 0 : L_s^a > 0\} = \inf\{s \geq 0 : H_s > a\}$, $\mathbf{P}$ a.s.

We will need an analogue of Proposition 3.1 under the excursion measure N. To state it, fix $a > 0$ and denote by (α_j, β_j), $j \in \mathcal{J}$ the excursion intervals of H above level a (just as before, but we are now arguing under N) and for every $j \in \mathcal{J}$ let H^j be the corresponding excursion. Let the process $\widetilde{H}^a$ be defined as previously and let $\widetilde{\mathcal{H}}^a$ be the σ-field generated by $\widetilde{H}^a$ and the class of N-negligible measurable sets. From our approximation (12) it follows that Λ_ζ^a is measurable with respect to $\widetilde{\mathcal{H}}^a$.

Corollary 3.2 *Under the probability measure $N(\cdot \mid \sup H > a)$ and conditionally on $\widetilde{\mathcal{H}}^a$, the point measure*

$$\sum_{j \in \mathcal{J}} \delta_{(\Lambda_{\alpha_j}^a, H^j)}(d\ell\, d\omega)$$

is distributed as a Poisson point measure on $\mathbb{R}_+ \times C_+([0, \infty))$ with intensity $\mathbf{1}_{[0, \Lambda_\zeta^a]}(\ell) d\ell \Delta(d\omega)$.

This is really an immediate consequence of Proposition 3.1 if we notice that the law under $\mathbf{P}$ of the first excursion of H that hits level a is $N(\cdot \mid \sup H > a)$. Alternatively, the statement of Corollary 3.2 also appears as an intermediate result in the proof of Proposition 4.2.3 in [9].

We will need one additional property related to Corollary 3.2. First denote by $(\widetilde{\Lambda}_s^a, s \geq 0)$ the local time of $\widetilde{H}^a$ at level a, which may be defined either by an approximation similar to (12) or directly by the formula $\widetilde{\Lambda}_s^a = \Lambda_{\widetilde{\tau}_s^a}^a$. Then we have N a.e. on $\{\sup H > a\}$

$$\inf\{s \geq 0 : \widetilde{\Lambda}_s^a > \Lambda_{\alpha_j}^a\} = \int_0^{\alpha_j} ds\, \mathbf{1}_{\{H_s \leq a\}}, \text{ for every } j \in \mathcal{J}. \tag{15}$$

For a proof, see pages 108-109 of [9].

We conclude this section with an important regularity property of local times. Recall that a càdlàg process Y is said to have no fixed discontinuities if for every fixed $t > 0$, the sample path of Y is continuous at t outside a set of zero probability.

Lemma 3.3 *Set $\Lambda_s^0 = 0$ for every $s \geq 0$. Then the process $(\Lambda_\zeta^a, a \geq 0)$ has a càdlàg modification under N, and this modification has no fixed discontinuities.*

Proof. Let $r > 0$. Since the process $(L_{T_r}^a, a \geq 0)$ is a ψ-CSBP and thus a Feller process, it has a càdlàg modification with no fixed discontinuities under $\mathbf{P}$. Let H_i, $i \in \mathcal{I}$ be the excursions of H away from 0, as in (11), and for every $i \in \mathcal{I}$ let ζ_i be the duration of H_i. From our approximation of local times, it is easy to see that, for every $a > 0$,

$$L_{T_r}^a = \sum_{i \in \mathcal{I}, I_{g_i} > -r} \Lambda_{\zeta_i}^a(H_i), \quad N \text{ a.e.} \tag{16}$$

Using a previous remark about the existence of exactly one excursion of H hitting level δ before time T_r, we easily deduce from the previous formula and the càdlàg property of

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$(L^a_{T_r}, a \geq 0)$ that the process $(\Lambda^a_\zeta, a > 0)$ must have a càdlàg modification with no fixed discontinuities under N. Furthermore, if we use this modification in the right side of (16), for every $a > 0$, we will obviously obtain the càdlàg modification of the process $(L^a_{T_r}, a > 0)$.

It remains to verify that Λ^a_ζ converges to 0, N a.e. as $a \downarrow 0$ (we now consider the modification that has just been introduced). For this, we need a different argument. Let $\delta > 0$ and let H_{i_0} be the first excursion of H that reaches level δ. From properties of Poisson measures, the law under $\mathbf{P}$ of the point measure

$$\sum_{i \in \mathcal{I} \setminus \{i_0\}, I_{g_i} > -r} \delta_{(-I_{g_i}, H_i)}(dr d\omega)$$

is absolutely continuous with respect to that of

$$\sum_{i \in \mathcal{I}, I_{g_i} > -r} \delta_{(-I_{g_i}, H_i)}(dr d\omega).$$

In particular, the function

$$a \longrightarrow \sum_{i \in \mathcal{I} \setminus \{i_0\}, I_{g_i} > -r} \Lambda^a_{\zeta_i}(H_i)$$

must converge $\mathbf{P}$ a.s. to r as $a \downarrow 0$. Now note that, on the event $\{-I_{g_{i_0}} > -r\} = \{\sup_{[0, T_r]} H > \delta\}$, we have for every $a > 0$

$$\Lambda^a_{\zeta_{i_0}}(H_{i_0}) = \sum_{i \in \mathcal{I}, I_{g_i} > -r} \Lambda^a_{\zeta_i}(H_i) - \sum_{i \in \mathcal{I} \setminus \{i_0\}, I_{g_i} > -r} \Lambda^a_{\zeta_i}(H_i)$$

and use the fact that the distribution of H_{i_0} under $\mathbf{P}(\cdot \mid \sup_{[0, T_r]} H > \delta)$ coincides with the law of H under $N(\cdot \mid \sup H > \delta)$ to complete the proof. ∎

From now on, we assume that have chosen a modification of the collection $(\Lambda^a, a \geq 0)$ in such a way that the process $(\Lambda^a_\zeta, a \geq 0)$ is càdlàg. This will be important in the applications developed in Section 4 below.

Let us finally briefly comment on the use of the measures $\mathbf{P}$ and N for our purposes. As will be made precise in the next section, the height process under N codes a single (compact rooted) $\mathbb{R}$-tree, whereas under $\mathbf{P}$ it codes a Poissonnian collection of such trees, each excursion of H away from 0 corresponding to one tree.

3.3 A key lemma

In this subsection, we prove a basic preliminary lemma, which is a consequence of the results in [9]. We need to introduce some notation. Denote by M_f the space of all finite measures on $[0, \infty)$. If $\mu \in M_f$, we denote by $H(\mu) \in [0, \infty]$ the supremum of the (topological) support of μ. We also introduce a "killing operator" on measures defined as follows. For every $x \geq 0$, $k_x \mu$ is the element of M_f such that $k_x \mu([0, t]) = \mu([0, t]) \wedge (\mu([0, \infty)) - x)_+$ for every $t \geq 0$. Let M^*_f stand for the set of all measures $\mu \in M_f$ such that $H(\mu) < \infty$ and the topological support of μ is $[0, H(\mu)]$. If $\mu \in M^*_f$, we denote by Q_μ the law under $\mathbf{P}$ of the process H^μ defined by

$$\begin{aligned} H^\mu_t &= H(k_{-I_t} \mu) + H_t , & \text{if } t \leq T_{\langle \mu, 1 \rangle} , \\ H^\mu_t &= 0 , & \text{if } t > T_{\langle \mu, 1 \rangle} , \end{aligned}$$

where $T_{\langle\mu,1\rangle} = \inf\{t \geq 0 : X_t = -\langle\mu,1\rangle\}$. Our assumption $\mu \in M_f^*$ guarantees that H^μ has continuous sample paths, and we can therefore view Q_μ as a probability measure on the space $C_+([0,\infty))$ of nonnegative continuous functions on $[0,\infty)$.

Finally, let $\psi^*(u) = \psi(u) - \alpha\,u$, and let (U^1, U^2) be a two-dimensional subordinator with Laplace functional

$$E[\exp(-\lambda U_t^1 - \lambda' U_t^2)] = \exp\left(-\frac{\psi^*(\lambda) - \psi^*(\lambda')}{\lambda - \lambda'}\right).$$

(alternatively, (U^1, U^2) can be characterized by its drift and Lévy measure, see [9], p. 80). When $\lambda = \lambda'$, the ratio $\frac{\psi^*(\lambda) - \psi^*(\lambda')}{\lambda - \lambda'}$ should obviously be interpreted as $\psi'(\lambda) - \alpha$, so that we see that $U^1 + U^2$ is a subordinator with Laplace exponent $\psi' - \alpha$. For every $a \geq 0$, we let $\mathbb{M}_a$ be the probability measure on $(M_f^*)^2$ which is the distribution of $(\mathbf{1}_{[0,a]}(t)\,dU_t^1, \mathbf{1}_{[0,a]}(t)\,dU_t^2)$.

Lemma 3.4 *For any nonnegative measurable function F on $C_+([0,\infty))^2$,*

$$N\left(\int_0^\zeta ds\, F\Big((H_{(s-t)_+}, t \geq 0), (H_{(s+t)\wedge\zeta}, t \geq 0)\Big)\right)$$
$$= \int_0^\infty da\, e^{-\alpha a} \int \mathbb{M}_a(d\mu d\nu) \int Q_\mu(dh) Q_\nu(dh') F(h, h').$$

Remark. In the Brownian case $\psi(u) = u^2$, this lemma reduces to the well-known Bismut decomposition of the Brownian excursion.

Proof. We start by recalling some results from [9] (see Chapter 1 and Section 3.1 in [9]). We can define both under $\mathbf{P}$ and under N a càdlàg process $(\rho_t, \eta_t)_{t\geq 0}$ taking values in $(M_f^*)^2$ such that the following properties hold:

(i) We have $H(\rho_t) = H_t = H(\eta_t)$ for every $t \geq 0$, N a.e. and P a.e.

(ii) The process (ρ_t, η_t) is adapted with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the Lévy process X. Furthermore, if G is any nonnegative measurable functional on $C_+([0,\infty))$, we have for every $s > 0$, N a.e. on the event $\{s < \zeta\}$,

$$N\Big(G(H(\rho_{(s+t)\wedge\zeta}), t \geq 0)\,\Big|\,\mathcal{F}_s\Big) = Q_{\rho_s}(G). \tag{17}$$

(iii) The process $(\eta_{(\zeta-s)_-}, \rho_{(\zeta-s)_-})_{0\leq s<\zeta}$ has the same distribution as $(\rho_s, \eta_s)_{0\leq s<\zeta}$ under N.

(iv) For any nonnegative measurable function Φ on $(M_f^*)^2$,

$$N\left(\int_0^\zeta ds\, \Phi(\rho_s, \eta_s)\right) = \int_0^\infty da\, e^{-\alpha a}\, \mathbb{M}_a(G). \tag{18}$$

To make sense of the conditional expectation in (17), note that the event $\{s < \zeta\}$ has finite N-measure. We refer to [9] for a proof of properties (i) – (iv) above: See in particular Propositions 1.2.3 and 1.2.6 for (17), Corollary 3.1.6 for (iii) and Proposition 3.1.3 for (iv).

We now proceed to the proof of the lemma. We may and will assume that F is of the form $F(h, h') = F_1(h) F_2(h')$. Using (i) and then (ii), we have

$$N\left(\int_0^\zeta ds\, F_1\Big(H_{(s-t)_+}, t \geq 0\Big) F_2\Big(H_{(s+t)\wedge\zeta}, t \geq 0\Big)\right)$$

$$= N\Big(\int_0^\zeta ds\, F_1\Big(H(\rho_{(s-t)_+}), t \ge 0 \Big) F_2\Big(H(\rho_{(s+t)\wedge\zeta}), t \ge 0 \Big) \Big)$$

$$= N\Big(\int_0^\zeta ds\, F_1\Big(H(\rho_{(s-t)_+}), t \ge 0 \Big) Q_{\rho_s}(F_2) \Big).$$

From the time-reversal property (iii) we see that the last quantity is equal to

$$N\Big(\int_0^\zeta ds\, Q_{\eta_s}(F_2)\, F_1\Big(H(\rho_{(s+t)\wedge\zeta}), t \ge 0 \Big) \Big)$$

$$= N\Big(\int_0^\zeta ds\, Q_{\eta_s}(F_2)\, Q_{\rho_s}(F_1) \Big),$$

using (17) once again. The formula of the lemma now follows from (18). ∎

Corollary 3.5 *Let $a > 0$. Then, for any nonnegative measurable function F on $C_+([0,\infty))^2$,*

$$N\Big(\int_0^\zeta d\Lambda_s^a\, F\Big((H_{(s-t)_+}, t \ge 0), (H_{(s+t)\wedge\zeta}, t \ge 0) \Big) \Big)$$

$$= e^{-\alpha a} \int \mathbb{M}_a(d\mu d\nu) \int Q_\mu(dh) Q_\nu(dh') F(h, h').$$

Proof. This is a straightforward consequence of Lemma 3.4 and the approximation of local time given in (12). ∎

Remark. The case $F = 1$ of Corollary 3.5 gives $N(\Lambda_\zeta^a) = e^{-\alpha a}$, for every $a > 0$.

4 The Lévy tree

We have seen that N a.e. the function $s \to H_s$ satisfies the properties stated at the beginning of subsection 2.1, namely it is continuous with compact support and such that $H_0 = 0$.

Definition 4.1 *The ψ-Lévy tree is the tree $(\mathcal{T}_H, d_H)$ coded by the function $s \to H_s$ under the measure N.*

We will say the Lévy tree rather than the ψ-Lévy tree if there is no risk of confusion.

We denote by $\Theta(d\mathcal{T})$ the σ-finite measure on $\mathbb{T}$ which is the law of the Lévy tree, that is the law of the tree $\mathcal{T}_H$ under N. Note that the measurability of the random variable $\mathcal{T}_H$ follows from Lemma 2.3.

4.1 From discrete to continuous trees

In this subsection, we will state a result which justifies the definition of the ψ-Lévy tree by showing that it arises as the limit in the Gromov-Hausdorff distance of suitably rescaled discrete Galton-Watson trees.

We start by introducing some formalism for discrete trees. Let

$$\mathcal{U} = \bigcup_{n=0}^\infty \mathbb{N}^n$$

where $\mathbb{N} = \{1, 2, \ldots\}$ and by convention $\mathbb{N}^0 = \{\varnothing\}$. If $u = (u_1, \ldots u_m)$ and $v = (v_1, \ldots, v_n)$ belong to $\mathcal{U}$, we write $uv = (u_1, \ldots u_m, v_1, \ldots, v_n)$ for the concatenation of u and v. In particular $u\varnothing = \varnothing u = u$.

A (finite) rooted ordered tree θ is a finite subset of $\mathcal{U}$ such that:

(i) $\varnothing \in \theta$.

(ii) If $v \in \theta$ and $v = uj$ for some $u \in \mathcal{U}$ and $j \in \mathbb{N}$, then $u \in \theta$.

(iii) For every $u \in \theta$, there exists a number $k_u(\theta) \geq 0$ such that $uj \in \theta$ if and only if $1 \leq j \leq k_u(\theta)$.

We denote by $\mathbf{T}$ the set of all rooted ordered trees. In what follows, we see each vertex of the tree θ as an individual of a population whose θ is the family tree.

If θ is a tree and $u \in \theta$, we define the shift of θ at u by $\tau_u\theta = \{v \in U : uv \in \theta\}$. Note that $\tau_u\theta \in \mathbf{T}$. We also denote by $h(\theta)$ the height of $\mathcal{T}$, that is the maximal generation of a vertex in θ.
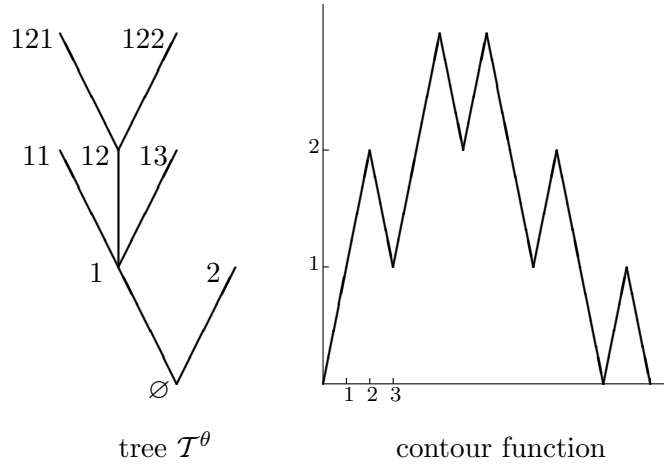

tree $\mathcal{T}^\theta$ contour function

Figure 1

For our purposes it will be convenient to view θ as an $\mathbb{R}$-tree: To this end, embed θ in the plane, in such a way that each edge corresponds to a line segment of length one, in the way suggested by the left part of Fig. 1. Denote by $\mathcal{T}^\theta$ the union of all these line segments and equip $\mathcal{T}^\theta$ with the obvious metric such that the distance between σ and σ' is the length of the shortest path from σ to σ' in $\mathcal{T}^\theta$. This construction leads to a (compact rooted) $\mathbb{R}$-tree whose equivalence class does not depend on the particular embedding.

To define now the contour function of $\mathcal{T}^\theta$, consider a particle that starts from the root and visits continuously all edges at speed one, going backwards as less as possible and respecting the lexicographical order of vertices. Then let $C^\theta(t)$ denote the distance to the root of the position of the particle at time t (for $t \geq 2(|\theta| - 1)$, we take $C^\theta(t) = 0$ by convention). Fig.1 explains the definition of the contour function better than a formal definition. Note that in the notation of Section 2, we have $\mathcal{T}^\theta = \mathcal{T}_{C^\theta}$, meaning that $\mathcal{T}^\theta$ coincides with the tree coded by the function C^θ.

Now let us turn to Galton-Watson trees. Let μ be a critical or subcritical offspring distribution. This means that μ is a probability measure on $\mathbb{Z}_+$ such that $\sum_{k=0}^{\infty} k\mu(k) \leq 1$. We exclude the trivial case where $\mu(1) = 1$. Then, there is a unique probability distribution Π_μ on $\mathbf{T}$ such that

(i) $\Pi_\mu(k_\varnothing = j) = \mu(j)$, $\quad j \in \mathbb{Z}_+$.

(ii) For every $j \geq 1$ with $\mu(j) > 0$, the shifted trees $\tau_1\theta, \ldots, \tau_j\theta$ are independent under the conditional probability $\Pi_\mu(\cdot \mid k_\varnothing = j)$ and their conditional distribution is Π_μ.

A random tree with distribution Π_μ is called a Galton-Watson tree with offspring distribution μ, or in short a μ-Galton-Watson tree. Obviously it describes the genealogy of the Galton-Watson process with offspring distribution μ started initially with one individual.

We can now state our result relating discrete and continuous trees. If $\mathcal{T}$ is a (compact rooted) $\mathbb{R}$-tree with metric d, and if $r > 0$, we slightly abuse notation by writing $r\mathcal{T}$ for the "same" tree equipped with the distance $r\,d$. Recall that the height of $\mathcal{T}$ is

$$h(\mathcal{T}) = \sup\{d(\rho(\mathcal{T}), \sigma) : \sigma \in \mathcal{T}\},$$

where $\rho(\mathcal{T})$ denotes the root of $\mathcal{T}$. For every real number x, $[x]$ denotes the integer part of x.

Theorem 4.1 *Let $(\mu_p)_{p\geq 1}$ be a sequence of critical or subcritical offspring distributions. For every $p \geq 1$ denote by Y^p a Galton-Watson branching process with offspring distribution μ_p, started at $Y_0^p = p$. Assume that there exists a nondecreasing sequence $(m_p)_{p\geq 1}$ of positive integers converging to $+\infty$ such that*

$$\left(p^{-1}Y_{[m_p t]}^p, \, t \geq 0\right) \xrightarrow[p\to\infty]{\text{(d)}} (Y_t, t \geq 0) \tag{19}$$

where the limiting process Y is a ψ-CSBP. Assume furthermore that for every $\delta > 0$,

$$\liminf_{p\to\infty} P[Y_{[m_p\delta]}^p = 0] > 0. \tag{20}$$

Then, for every $a > 0$, the law of the $\mathbb{R}$-tree $m_p^{-1}\mathcal{T}^\theta$ under $\Pi_{\mu_p}(d\theta \mid h(\theta) \geq [a\,m_p])$ converges as $p \to \infty$ to the probability measure $\Theta(d\mathcal{T} \mid h(\mathcal{T}) > a)$, in the sense of weak convergence of measures in the space $\mathbb{T}$.

Proof. We noted that the tree $\mathcal{T}^\theta$ is the tree coded by the function C^θ, in the sense of Section 2. Lemma 2.3 then shows that the convergence of the theorem follows from the weak convergence of the scaled contour function $(m_p^{-1}C^\theta(pm_p t), t \geq 0)$ under $\Pi_{\mu_p}(d\theta \mid h(\theta) \geq [a\,m_p])$ towards the height process H under $N(\cdot \mid \sup H > a)$. But this is precisely the contents of Proposition 2.5.2 in [9], which is itself a consequence of Theorem 2.3.1 in the same work. ∎

The technical assumption (20) guarantees that the Galton-Watson process Y^p dies out at a time of order m_p, as one expects from the convergence (19) (recall that Y dies out in finite time). See Chapter 2 of [9] for a discussion of this assumption, and note that it is always true in the case when $\mu_p = \mu$ for every p (Theorem 2.3.2 in [9]). In particular, the approximation result stated in the introduction above is easily seen to be a consequence of Theorem 4.1.

4.2 Local times and the branching property of Lévy trees

Let us start with a few simple observations. We recall that the generic element of $\mathbb{T}$ is denoted by $(\mathcal{T}, d)$. Then, for every $a > 0$,

$$\Theta(h(\mathcal{T}) > a) = N(\sup H > a) = v(a),$$

where the function v is determined by $\int_{v(a)}^{\infty} \psi(u)^{-1} du = a$.

The truncation of the tree $\mathcal{T}$ at level $a > 0$ is the new tree

$$\text{tr}_a(\mathcal{T}) = \{\sigma \in \mathcal{T} : d(\rho(\mathcal{T}), \sigma) \leq a\},$$

which is obviously equipped with the restriction of the distance d. It is easy to verify that the mapping $\mathcal{T} \to \text{tr}_a(\mathcal{T})$ from $\mathbb{T}$ into itself is measurable.

Let $\mathcal{T} \in \mathbb{T}$ and $a > 0$. Denote by $\mathcal{T}^{(i),\circ}$, $i \in \mathcal{I}$ the connected components of the open set

$$\mathcal{T}((a, \infty)) = \{\sigma \in \mathcal{T} : d(\rho(\mathcal{T}), \sigma) > a\}.$$

Notice that the index set $\mathcal{I}$ may be empty (if $h(\mathcal{T}) \leq a$), finite or countable. Let $i \in \mathcal{I}$. Then the ancestor of σ at level a must be the same for every $\sigma \in \mathcal{T}^{(i),\circ}$. We denote by σ_i this common ancestor and set $\mathcal{T}^{(i)} = \mathcal{T}^{(i),\circ} \cup \{\sigma_i\}$. Then $\mathcal{T}^{(i)}$ is a compact rooted $\mathbb{R}$-tree with root σ_i. The trees $\mathcal{T}^{(i)}$, $i \in I$ are called the subtrees of $\mathcal{T}$ originating from level a.

We set

$$\mathcal{N}_a^{\mathcal{T}} := \sum_{i \in \mathcal{I}} \delta_{(\sigma_i, \mathcal{T}^{(i)})},$$

which is a point measure on $\mathcal{T}(a) \times \mathbb{T}$. Also, for every $\delta > 0$, we let

$$Z(a, \delta) := |\{i \in \mathcal{I} : h(\mathcal{T}^{(i)}) \geq \delta\}|$$

be the number of subtrees of $\mathcal{T}$ originating from level a that hit level $a + \delta$.

Theorem 4.2 *For every $a \geq 0$ and for Θ a.e. $\mathcal{T} \in \mathbb{T}$ we can define a finite measure ℓ^a on $\mathcal{T}$, in such a way that the following properties hold:*

(i) *$\ell^0 = 0$ and, for every $a > 0$, ℓ^a is supported on $\mathcal{T}(a)$, $\Theta(d\mathcal{T})$ a.e.*

(ii) *For every $a > 0$, $\{\ell^a \neq 0\} = \{h(\mathcal{T}) > a\}$, $\Theta(d\mathcal{T})$ a.e.*

(iii) *For every $a > 0$, we have $\Theta(d\mathcal{T})$ a.e. for every bounded continuous function φ on $\mathcal{T}$,*

$$\begin{aligned}
\langle \ell^a, \varphi \rangle &= \lim_{\varepsilon \downarrow 0} \frac{1}{v(\varepsilon)} \int \mathcal{N}_a^{\mathcal{T}}(d\sigma d\mathcal{T}') \, \varphi(\sigma) \, \mathbf{1}_{\{h(\mathcal{T}') \geq \varepsilon\}} \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{v(\varepsilon)} \int \mathcal{N}_{a-\varepsilon}^{\mathcal{T}}(d\sigma d\mathcal{T}') \, \varphi(\sigma) \, \mathbf{1}_{\{h(\mathcal{T}') \geq \varepsilon\}} \qquad (21)
\end{aligned}$$

Furthermore, for every $a > 0$, the conditional distribution of the point measure $\mathcal{N}_a^{\mathcal{T}}(d\sigma d\mathcal{T}')$, under the probability measure $\Theta(d\mathcal{T} \mid h(\mathcal{T}) > a)$ and given $\text{tr}_a(\mathcal{T})$, is that of a Poisson point measure on $\mathcal{T}(a) \times \mathbb{T}$ with intensity $\ell^a(d\sigma)\Theta(d\mathcal{T}')$.

The last property is the most important one. It will be called the *branching property* of the Lévy tree as it is exactly analogous to the classical branching property for Galton-Watson trees (cf Property (ii) in the definition of Galton-Watson trees in subsection 4.1). The random measure ℓ^a will be called the *local time* of $\mathcal{T}$ at level a.

Remark. The reader may be a little puzzled by the mathematical meaning of the branching property as stated in the theorem, since our trees $\mathcal{T}$ are defined as equivalent classes of isometric objects, and $\mathcal{T}(a)$ does not seem to be a well-defined object. In the proof below, we will circumvent this difficulty by dealing with the tree $\mathcal{T}_H$ under N. A more intrinsinc way to state the branching property in a mathematically precise way is as follows. Consider first a fixed real tree $\mathcal{T}$, and assume that the local time measure ℓ^a of $\mathcal{T}$ can be defined via formula (21). Then let

$$\sum_{j \in \mathcal{J}} \delta_{(\eta_j, \mathcal{T}^{(j)})}$$

be a Poisson point measure on $\mathcal{T}(a) \times \mathbb{T}$ with intensity $\ell^a(d\eta)\Theta(d\mathcal{T}')$. Construct another real tree $\overline{\mathcal{T}}$ as the disjoint union

$$\overline{\mathcal{T}} = \mathrm{tr}_a(\mathcal{T}) \bigsqcup \left(\bigsqcup_{j \in \mathcal{J}} (\mathcal{T}^{(j)} \backslash \{\rho(\mathcal{T}^{(j)})\}) \right)$$

equipped with the obvious appropriate distance so that the sets $(\mathcal{T}^{(j)} \backslash \{\rho(\mathcal{T}^{(j)})\}) \sqcup \{\eta_j\}$ become the subtrees of $\overline{\mathcal{T}}$ originating from level a. Note that the distribution of $\overline{\mathcal{T}}$ only depends on the equivalence class of $\mathcal{T}$ in $\mathbb{T}$, and that this distribution is a measurable function of $\mathcal{T}$. The branching property can be restated by saying that if $\mathcal{T}$ is chosen according to the distribution $\Theta(d\mathcal{T} \mid h(\mathcal{T}) > a)$, then $\overline{\mathcal{T}}$ has the same distribution as $\mathcal{T}$.

Proof. From the definition of the measure Θ, it is enough to construct the measures ℓ^a and to verify the properties stated in the theorem, for the tree $\mathcal{T}_H$ associated with the height process H under the measure N. For every $a > 0$, we define the finite measure ℓ^a on $\mathcal{T}_H$ by setting

$$\langle \ell^a, \varphi \rangle = \int_0^\zeta d\Lambda_s^a \, \varphi(p_H(s)).$$

In other words, ℓ^a is the image of the measure $d\Lambda_s^a$ under the mapping p_H. The support property of local times guarantees that ℓ^a is N a.e. supported on $p_H(\{s : H_s = a\}) = \mathcal{T}_H(a)$.

Similarly we have seen that $\{\Lambda_\zeta^a > 0\} = \{\sup H > a\}$, N a.e., which gives property (ii). Before proving property (iii), we will discuss the branching property of the Lévy tree, which is basically a consequence of Corollary 3.2.

Recall the notation $\widetilde{H}^a$ for the process H truncated at level a, and note the easy identification

$$\mathcal{T}_{\widetilde{H}^a} = \mathrm{tr}_a(\mathcal{T}_H). \tag{22}$$

Indeed, if $A_s^a := \int_0^s dr \, \mathbf{1}_{\{H_r \leq a\}}$, there is a (unique) isometry mapping $\mathrm{tr}_a(\mathcal{T}_H)$ onto $\mathcal{T}_{\widetilde{H}^a}$ such that, for every $s \geq 0$ with $H_s \leq a$, $p_H(s)$ is mapped to $p_{\widetilde{H}^a}(A_s^a)$.

Recall the notation $\widetilde{\mathcal{H}}^a$ for the σ-field generated by $\widetilde{H}^a$ augmented with the class of N-negligible sets.

Claim. *The conditional distribution of the point measure $\mathcal{N}_a^{\mathcal{T}_H}$, under the probability measure $N(\cdot \mid \sup H > a)$ and given the σ-field $\widetilde{\mathcal{H}}^a$, is that of a Poisson point measure on $\mathcal{T}_H(a) \times \mathbb{T}$ with intensity $\ell^a(d\sigma)\Theta(d\mathcal{T})$.*

The claim is very close to the branching property (for the tree $\mathcal{T}_H$) as stated in the theorem, with the minor difference that we are conditioning with respect to the σ-field $\widetilde{\mathcal{H}}^a$ rather than with respect to the truncated tree $\mathcal{T}_{\tilde{H}^a}$ (which contains less information). We will see later that ℓ^a is N a.e. equal to a measurable function of the tree $\mathcal{T}_{\tilde{H}^a}$, and then the branching property will follow from the claim.

Let us first prove the claim. We recall the notation of Corollary 3.2: (α_j, β_j), $j \in \mathcal{J}$ are the connected components of the open set $\{s : H_s > a\}$ and, for every $j \in \mathcal{J}$, $H_s^j = H_{(\alpha_j+s)\wedge\beta_j} - a$. We also set $l_j = \Lambda_{\alpha_j}^a$. Corollary 3.2 asserts that, conditionally on $\widetilde{\mathcal{H}}^a$, the point measure

$$\sum_{j \in \mathcal{J}} \delta_{(l_j, H^j)}(d\ell\, d\omega)$$

is Poisson on $\mathbb{R}_+ \times C_+([0,\infty))$ with intensity $\mathbf{1}_{[0,\Lambda_\zeta^a)}(\ell)\, d\ell\, \Delta(d\omega)$. On the other hand, it is immediate from the construction of the tree $\mathcal{T}_H$ that

$$\mathcal{N}_a^{\mathcal{T}_H} = \sum_{j \in \mathcal{J}} \delta_{(p_H(\alpha_j), \mathcal{T}_{H^j})}.$$

To complete the proof of the claim, we will argue that conditionally given $\widetilde{H}^a$, each pair $(p_H(\alpha_j), \mathcal{T}_{H^j})$ is a function (not depending on j) of the pair (l_j, H^j). This is obvious for the second coordinate $\mathcal{T}_{H^j}$. Then, recalling the identification (22), we have for every $j \in \mathcal{J}$,

$$p_H(\alpha_j) = p_{\tilde{H}^a}(A_{\alpha_j}^a).$$

From (15), we get for every $j \in \mathcal{J}$,

$$p_H(\alpha_j) = p_{\tilde{H}^a}(\inf\{s : \widetilde{\Lambda}_s^a > l_j\}). \tag{23}$$

This is the formula we were aiming at. In view of applying Corollary 3.2, we still need to determine the image of the measure $\mathbf{1}_{[0,\Lambda_\zeta^a)}(l)\, dl$ under the mapping $l \to p_{\tilde{H}^a}(\inf\{s : \widetilde{\Lambda}_s^a > l\})$. Write $\gamma_s^a = \inf\{r : \Lambda_r^a > s\}$ and $\widetilde{\gamma}_s^a = \inf\{r : \widetilde{\Lambda}_r^a > s\}$, for every $s \in [0, \Lambda_\zeta^a)$. From the relation between Λ_s^a and $\widetilde{\Lambda}_s^a$, we see that $\widetilde{\gamma}_s^a = A_{\gamma_s^a}^a$ for every $s \in [0, \Lambda_\zeta^a)$. Hence, via the identification (22), we have also $p_H(\gamma_s^a) = p_{\tilde{H}^a}(\widetilde{\gamma}_s^a)$ for every $s \in [0, \Lambda_\zeta^a)$. Therefore, for any nonnegative measurable function φ on $\mathcal{T}_H$,

$$\langle \ell^a, \varphi \rangle = \int d\Lambda_s^a\, \varphi(p_H(s)) = \int_{[0,\Lambda_\zeta^a)} dr\, \varphi(p_H(\gamma_r^a)) = \int_{[0,\Lambda_\zeta^a)} dr\, \varphi(p_{\tilde{H}^a}(\widetilde{\gamma}_s^a)).$$

Thus ℓ^a is the image of the measure $\mathbf{1}_{[0,\Lambda_\zeta^a)}(l)\, dl$ under the mapping $l \to p_{\tilde{H}^a}(\inf\{s : \widetilde{\Lambda}_s^a > l\})$. Using (23) and recalling that Θ is the image of $\Delta(dh)$ under the mapping $h \longrightarrow \mathcal{T}_h$, we see that the claim follows from Corollary 3.2.

We now turn to the proof of (iii). Consider first the case $\varphi = 1$, where we have to prove

$$\langle \ell^a, 1 \rangle = \lim_{\varepsilon \downarrow 0} \frac{1}{v(\varepsilon)} Z(a - \varepsilon, \varepsilon) = \lim_{\varepsilon \downarrow 0} \frac{1}{v(\varepsilon)} Z(a, \varepsilon),$$

where Z refers to the tree $\mathcal{T}_H$. This easily follows from the preceding claim: To get the first equality, note that conditionally on $\langle \ell^{a-\varepsilon}, 1 \rangle$, $Z(a - \varepsilon, \varepsilon)$ is Poisson with parameter

$v(\varepsilon)\langle\ell^{a-\varepsilon},1\rangle$, and recall from subsection 3.2 that $\langle\ell^{a-\varepsilon},1\rangle = \Lambda_\zeta^{a-\varepsilon}$ converges N a.e. to $\langle\ell^a,1\rangle = \Lambda_\zeta^a$. Standard estimates for Poisson variables, together with the (obvious) monotonicity of the mapping $\varepsilon \to Z(a-\varepsilon,\varepsilon)$ then give the desired result. The case of the second equality is treated in a similar way.

Consider then a Lipschitz function φ on $\mathcal{T}_H$, with Lipschitz constant K. Let $\delta > 0$ be a rational number in $(0,a)$. Write $\mathcal{T}^{(l)}$, $l \in \mathcal{L}$ for the subtrees of $\mathcal{T}_H$ originating from level $a-\delta$, and (α_l,β_l) for the excursion interval of H above level $a-\delta$ that corresponds to $\mathcal{T}^{(l)}$. Again thanks to the claim, we can apply the case $\varphi = 1$ of (iii) to each tree $\mathcal{T}^{(l)}$, and we get that for every $l \in \mathcal{L}$,

$$\frac{1}{v(\varepsilon)} \mathcal{N}_{\delta-\varepsilon}^{\mathcal{T}^{(l)}}(\{(\sigma',\mathcal{T}') : h(\mathcal{T}') > \varepsilon\}) \underset{\varepsilon\to 0}{\longrightarrow} \Lambda_{\beta_l}^a - \Lambda_{\alpha_l}^a.$$

We have then

$$\liminf_{\varepsilon\to 0} \frac{1}{v(\varepsilon)} \int \mathcal{N}_{a-\varepsilon}^{\mathcal{T}_H}(d\sigma' d\mathcal{T}')\, \varphi(\rho(\mathcal{T}'))\, \mathbf{1}_{\{h(\mathcal{T}')\geq\varepsilon\}}$$

$$= \liminf_{\varepsilon\to 0} \frac{1}{v(\varepsilon)} \sum_{l\in\mathcal{L}} \int \mathcal{N}_{\delta-\varepsilon}^{\mathcal{T}^{(l)}}(d\sigma' d\mathcal{T}')\, \varphi(\rho(\mathcal{T}'))\, \mathbf{1}_{\{h(\mathcal{T}')\geq\varepsilon\}}$$

$$\geq \liminf_{\varepsilon\to 0} \sum_{l\in\mathcal{L}} \left(\frac{1}{v(\varepsilon)} \mathcal{N}_{\delta-\varepsilon}^{\mathcal{T}^{(l)}}(\{(\sigma',\mathcal{T}') : h(\mathcal{T}')\geq\varepsilon\}) \inf_{\mathcal{T}^{(l)}([0,\delta])} \varphi \right)$$

$$= \sum_{l\in\mathcal{L}} \left((\Lambda_{\beta_l}^a - \Lambda_{\alpha_l}^a) \inf_{s\in[\alpha_l,\beta_l],H_s\leq a} \varphi(p_H(s)) \right)$$

$$\geq \sum_{l\in\mathcal{L}} \int_{\alpha_l}^{\beta_l} d\Lambda_s^a \left(\varphi(p_H(s)) - 2K\delta \right)$$

$$= \langle\ell^a,\varphi\rangle - 2K\delta\langle\ell^a,1\rangle.$$

The fourth line in the previous calculation is an equality because the sum is in fact finite. In the last inequality, we used the Lipschitz property of φ, together with the fact that the distance between $p_H(s)$ and $p_H(s')$ is bounded above by 2δ whenever there exists $l \in \mathcal{L}$ such that $s,s' \in [\alpha_l,\beta_l]$ and $H_s \vee H_{s'} \leq a$.

Since δ was arbitrary we get

$$\liminf_{\varepsilon\to 0} \frac{1}{v(\varepsilon)} \int \mathcal{N}_{a-\varepsilon}^{\mathcal{T}_H}(d\sigma' d\mathcal{T}')\, \varphi(\rho(\mathcal{T}'))\, \mathbf{1}_{\{h(\mathcal{T}')\geq\varepsilon\}} \geq \langle\ell^a,\varphi\rangle.$$

The same method gives the analogous bound for the limsup behavior, and a similar argument applies to the other part of (iii). This completes the proof of properties (i)–(iii).

Finally, observing that for every $\varepsilon > 0$

$$\int \mathcal{N}_{a-\varepsilon}^{\mathcal{T}_H}(d\sigma' d\mathcal{T}')\, \varphi(\rho(\mathcal{T}'))\, \mathbf{1}_{\{h(\mathcal{T}')\geq\varepsilon\}} = \int \mathcal{N}_{a-\varepsilon}^{\mathcal{T}_{\tilde{H}^a}}(d\sigma' d\mathcal{T}')\, \varphi(\rho(\mathcal{T}'))\, \mathbf{1}_{\{h(\mathcal{T}')\geq\varepsilon\}}$$

we deduce from (iii) that ℓ^a coincides N a.e. with a measurable function of the truncated tree $\mathrm{tr}_a(\mathcal{T}_H) = \mathcal{T}_{\tilde{H}^a}$. Consequently in the claim above we may condition on $\mathrm{tr}_a(\mathcal{T}_H)$ rather than on the σ-field $\widetilde{\mathcal{H}}^a$. This gives the branching property for $\mathcal{T}_H$ and completes the proof of the theorem. ∎

Remark. Although our contruction of the measure ℓ^a for the tree $\mathcal{T}_H$ makes use of the coding via the height process, part (iii) of the theorem shows that ℓ^a is a function of the tree and does not depend on the particular coding that is used.

Theorem 4.3 *We can choose a modification of the collection $(\ell^a, a \geq 0)$ in such a way that the mapping $a \longrightarrow \ell^a$ is $\Theta(d\mathcal{T})$ a.e. càdlàg for the weak topology on finite measures on $\mathcal{T}$. We have then*

$$\inf\{a > 0 : \ell^a = 0\} = \sup\{a \geq 0 : \ell^a \neq 0\} = h(\mathcal{T}) , \quad \Theta \text{ a.e.}$$

Proof. Again, it is enough to prove this for the tree $\mathcal{T}_H$ under N. For every rational $q > 0$, denote by $\mathcal{T}^{(i)}$, $i \in \mathcal{I}_q$ the subtrees of $\mathcal{T}_H$ originating from level q (the index sets $\mathcal{I}_q$ are disjoint when q varies). Denote by $(\ell^{a,(i)}, a \geq 0)$ the local times of $\mathcal{T}^{(i)}$, which are well defined thanks to the branching property. Using the approximations of local time given in Theorem 4.2(iii), it is immediately checked that, for every $b > q$, we have

$$\ell^b = \sum_{i \in \mathcal{I}_q} \ell^{b-q,(i)} , \quad N \text{ a.e.} \tag{24}$$

Note that (24) holds for every rationals b, q with $0 < q < b$ outside a single set of zero N-measure. In addition, by Lemma 3.3, we can assume that outside the same set of zero N-measure, the mapping

$$\mathbb{Q} \cap (0, \infty) \ni b \longrightarrow \langle \ell^{b,(i)}, 1 \rangle = \ell^{b,(i)}(\mathcal{T}^{(i)})$$

has a càdlàg extension to $[0, \infty)$, for every rational $q > 0$ and every $i \in \mathcal{I}_q$.

We then show that the limit

$$\overline{\ell}^a := \lim_{\mathbb{Q} \ni b \downarrow a} \ell^b$$

exists for every $a \in [0, \infty)$. When $a = 0$, the result is immediate, with $\overline{\ell}^0 = 0$, since we already know that $\langle \ell^b, 1 \rangle \longrightarrow 0$ as $b \downarrow 0$, N a.e. (Lemma 3.3). Then, if $a > 0$ and if $\varphi : \mathcal{T}_H \longrightarrow \mathbb{R}$ is Lipschitz with constant K, we get from (24) that for every $b, b' \in (a, \infty) \cap \mathbb{Q}$ and $q \in (0, a) \cap \mathbb{Q}$

$$\langle \ell^b, \varphi \rangle \geq \sum_{i \in \mathcal{I}_q} \langle \ell^{b-q,(i)}, 1 \rangle \inf_{\mathcal{T}^{(i)}(b-q)} \varphi , \tag{25}$$

$$\langle \ell^{b'}, \varphi \rangle \leq \sum_{i \in \mathcal{I}_q} \langle \ell^{b'-q,(i)}, 1 \rangle \sup_{\mathcal{T}^{(i)}(b'-q)} \varphi . \tag{26}$$

Moreover, we have for every $i \in \mathcal{I}_q$,

$$\sup_{\mathcal{T}^{(i)}(b'-q)} \varphi \leq \inf_{\mathcal{T}^{(i)}(b-q)} \varphi + K(b + b' - 2q) , \tag{27}$$

because any point of $\mathcal{T}^{(i)}(u)$ is at distance u from the root of $\mathcal{T}^{(i)}$.

From the remarks of the beginning of the proof, we know that for every $i \in \mathcal{I}_q$,

$$\langle \ell^{b'-q,(i)}, 1 \rangle - \langle \ell^{b-q,(i)}, 1 \rangle$$

tends to 0 as $b, b' \downarrow a$ with $b, b' \in \mathbb{Q} \cap [a, \infty)$. Also note that in the sums appearing in (25) and (26) only finitely many terms can give a nonzero contribution, namely those for which $h(\mathcal{T}^{(i)}) > a - q$. Combining these facts with (25), (26) and (27) leads to

$$\limsup_{\mathbb{Q} \ni b, b' \downarrow a} \left(\langle \ell^{b'}, \varphi \rangle - \langle \ell^{b}, \varphi \rangle \right) \leq 2K(a - q) \sup_{x \in \mathbb{Q} \cap (0, \infty)} \langle \ell^{x}, 1 \rangle.$$

Since q can be taken arbitrarily close to a, this is enough to get the convergence of $\langle \ell^{b}, \varphi \rangle$ as $b \downarrow a$, $b \in \mathbb{Q}$, thus proving the existence of the right limit $\overline{\ell}^{a}$.

A similar argument gives the existence of left limits along rationals. The process $(\overline{\ell}^{a}, a \in [0, \infty))$ is thus càdlàg. In addition, it is easy to see that $\overline{\ell}^{a} = \ell^{a}$ N a.e., for every $a \in [0, \infty)$: If $a > 0$ is fixed, note that (26) will hold with b' replaced by a (outside a set of zero N-measure depending on a) and use the preceding argument to verify that the measures ℓ^{a} and $\overline{\ell}^{a}$ coincide N a.e.

It remains to prove that

$$\inf\{a > 0 : \overline{\ell}^{a} = 0\} = \sup\{a \geq 0 : \overline{\ell}^{a} \neq 0\} = h(\mathcal{T}_{H}) , \quad N \text{ a.e.} \tag{28}$$

We know that for every $\delta > 0$ the (càdlàg) process $(\langle \overline{\ell}^{\delta + r}, 1 \rangle, r \geq 0)$ is distributed under $N(\cdot \mid \sup H > \delta)$ as a ψ-CSBP. The strong Markov property of the ψ-CSBP then shows that if $T = \inf\{a \geq \delta : \overline{\ell}^{a} = 0\}$ we have $\overline{\ell}^{b} = 0$ for every $b \geq a$, N a.e. This gives the first equality in (28). The second one is immediate from the fact that $\{\sup H > q\} = \{\Lambda_{\zeta}^{q} \neq 0\}$, N a.e., for every rational q. ∎

Remark. We already noticed that the ψ-CSBP has only positive jumps, and the same holds for the total local time process $\langle \ell^{a}, 1 \rangle$ under Θ. A careful inspection of the previous proof then shows that Θ a.e. for every jump time b of the process $a \longrightarrow \ell^{a}$ we must have $\ell^{b} \geq \ell^{b-}$. As a consequence b is a jump time of the process $a \longrightarrow \ell^{a}$ if and only is it is a jump time of the total mass process $a \longrightarrow \langle \ell^{a}, 1 \rangle$. More information about these jumps will be given in Theorem 4.7.

From now on we consider only the càdlàg modification of the collection $(\ell^{a}, a \geq 0)$ obtained in Theorem 4.3. By combining the right-continuity of the mapping $a \to \ell^{a}$ with Theorem 4.2 (i), we get that Θ a.e. for every $a > 0$, ℓ^{a} is supported on the level set $\mathcal{T}(a)$. A more precise result will be derived below (Theorem 4.4).

We put

$$\mathbf{m} = \int_{0}^{\infty} da \, \ell^{a} \tag{29}$$

which defines a finite measure on the tree $\mathcal{T}$. When $\mathcal{T} = \mathcal{T}_{H}$ is the tree coded by the height process under N, the measure $\mathbf{m}$ coincides with the image of Lebesgue measure on $[0, \zeta]$ under the mapping $s \to p_{H}(s)$ (see formula (32) in [9]). However, formula (29) makes it clear that the measure $\mathbf{m}$ only depends on the tree $\mathcal{T}$ and not on a particular coding. We will write ζ for the total mass of $\mathbf{m}$, in agreement with the case of the tree $\mathcal{T}_{H}$. The next theorem will imply in particular that the topological support of $\mathbf{m}$ is $\mathcal{T}$, Θ a.e.

For every $\sigma \in \mathcal{T}$ and $\varepsilon > 0$, denote by $B(\sigma, \varepsilon)$ the open ball of radius ε centered at σ. We say that a vertex σ of $\mathcal{T}$ is an *extinction point* if there exists $\varepsilon > 0$ such that

$$d(\rho(\mathcal{T}), \sigma) = \sup_{\tau \in B(\sigma, \varepsilon)} d(\rho(\mathcal{T}), \tau).$$

Note that $p_H(s)$ is an extinction time of $\mathcal{T}_H$ iff $s \in [0, \zeta]$ is a local maximum of H. As a consequence there are at most countably many extinction points. We denote by $\mathcal{E}$ the set of all extinction levels, that is levels a such that $a = d(\rho(\mathcal{T}), \sigma)$ for some extinction point σ.

Theorem 4.4 *The following holds* $\Theta(d\mathcal{T})$ *a.e.:*

(i) *For every* $a \in (0, h(\mathcal{T})] \backslash \mathcal{E}$, *the topological support of* ℓ^a *is equal to the level set* $\mathcal{T}(a)$.

(ii) *For every* $a \in \mathcal{E}$, *the topological support of* ℓ^a *is* $\mathcal{T}(a) \backslash \{\sigma_a\}$, *where* σ_a *is the (unique) extinction point at level* a.

Proof. First observe that if σ is an extinction point at level b, the right-continuity of the mapping $a \to \ell^a$, together with the fact that $\mathrm{supp}\, \ell^a \subset \mathcal{T}(a)$, for every $a \geq 0$ implies that $\ell^b(B(\sigma, \varepsilon)) = 0$ for some $\varepsilon > 0$. Therefore $\sigma \notin \mathrm{supp}\, \ell^b$.

The proof of the theorem will then be complete if we can prove that that Θ a.e.:

(i) Extinction levels of two distinct extinction points are different.

(ii) For every $a > 0$ and $\sigma \in \mathcal{T}(a)$ which is not an extinction point, $\ell^a(B(\sigma, \varepsilon)) > 0$ for every $\varepsilon > 0$.

Recall the notation before Theorem 4.2: For every rational $q \geq 0$,

$$\mathcal{N}_q^{\mathcal{T}} = \sum_{i \in \mathcal{I}_q} \delta_{(\sigma_i, \mathcal{T}^{(i)})}$$

is the point measure associated with subtrees originating from level q (the index sets $\mathcal{I}_q$ are disjoint). Set $h_i = h(\mathcal{T}^{(i)})$ for every $i \in \mathcal{I}_q$ and every rational $q \geq 0$. Then it is easy to verify that

$$\mathcal{E} = \{q + h_i : q \in \mathbb{Q}_+, i \in \mathcal{I}_q\}.$$

Suppose that σ and σ' are two distinct extinction points such that $d(\rho(\mathcal{T}), \sigma) = d(\rho(\mathcal{T}), \sigma') = a$. Then by choosing a rational $q < a$ and sufficiently close to a, we see that there exist two distinct indices $i, i' \in \mathcal{I}_q$ such that $h(\mathcal{T}^{(i)}) = h(\mathcal{T}^{(i')})$. This is impossible by the branching property and the fact that the law of $h(\mathcal{T})$ under Θ has no atoms. This proves (i).

Then, if $(\ell^{b,(i)}, b \geq 0)$ denote the local times associated with the tree $\mathcal{T}^{(i)}$ (again this makes sense by the branching property), we get from Theorem 4.3 that $\ell^{b,(i)}(\mathcal{T}^{(i)}) > 0$ for every $b \in (0, h_i)$. Note that these properties hold simultaneously for all rationals $q \geq 0$ and $i \in \mathcal{I}_q$ outside a single set of zero Θ-measure. Furthermore, from the approximation of local time given in Theorem 4.2 (iii), we see that Θ a.e. for every $a > 0$ and every rational $q \in (0, a)$ we have

$$\ell^a = \sum_{i \in \mathcal{I}_q} \ell^{a-q,(i)}. \tag{30}$$

Note that in the preceding sum only finitely many terms can be nonzero. To derive this identity consider first the case when a is also rational and then use the right-continuity of the mapping $a \longrightarrow \ell^a$ (Theorem 4.3).

Finally, let $a > 0$ and $\sigma \in \mathcal{T}(a)$. Assume that σ is not an extinction point. Then for every rational $q < a$, the subtree $\mathcal{T}^{(i)}$ originating from level q and containing σ is such that $h(\mathcal{T}^{(i)}) > a - q$. Therefore $\ell^{a-q,(i)}(\mathcal{T}^{(i)}) > 0$ and by (30), $\ell^a(\mathcal{T}^{(i)} \cap \mathcal{T}(a)) = \ell^a(\mathcal{T}^{(i)}) > 0$. To complete the proof of (ii), simply note that $\mathcal{T}^{(i)} \cap \mathcal{T}(a)$ is contained in $B(\sigma, \varepsilon)$ if $2(a - q) < \varepsilon$. ∎

4.3 Decomposing the tree along an ancestral line

We shall need one more subtree decomposition under Θ, which is a consequence of Corollary 3.5. Before stating it, we introduce the relevant notation. Let $\mathcal{T} \in \mathbb{T}$ and $\sigma \in \mathcal{T}$. Denote by $\mathcal{T}^{(j),\circ}$, $j \in \mathcal{J}$ the connected components of the open set $\mathcal{T} \backslash [\![\rho(\mathcal{T}), \sigma]\!]$, and note that for every $j \in \mathcal{J}$, $\sigma_j := \sigma \wedge \tau$ does not depend on the choice of $\tau \in \mathcal{T}^{(j),\circ}$. Furthermore, $\mathcal{T}^{(j)} := \mathcal{T}^{(j),\circ} \cup \{\sigma_j\}$ is a (compact rooted) $\mathbb{R}$-tree with root σ_j. The trees $\mathcal{T}^{(j)}$, $j \in \mathcal{J}$ can be interpreted as the subtrees of $\mathcal{T}$ originating from the segment $[\![\rho(\mathcal{T}), \sigma]\!]$. We put

$$\mathcal{M}_\sigma = \sum_{j \in \mathcal{J}} \delta_{(d(\rho(\mathcal{T}),\sigma_j),\mathcal{T}^{(j)})},$$

thus defining a point measure on $[0,\infty) \times \mathbb{T}$.

Theorem 4.5 *For every $a > 0$ and every nonnegative measurable function Φ on $[0,\infty) \times \mathbb{T}$,*

$$\Theta\Big(\int \ell^a(d\sigma)\,\exp{-\langle \mathcal{M}_\sigma, \Phi \rangle}\Big) = \exp\Big(-\int_0^a dt\,\psi'\Big(\Theta(1 - \exp{-\Phi(t,\cdot)})\Big)\Big).$$

Proof. As previously, we argue on the tree $\mathcal{T}_H$ under N, and we abuse notation by writing $\mathcal{M}_\sigma$ and ℓ^a for the corresponding objects attached to the tree $\mathcal{T}_H$. If $s \in [0, \zeta]$, we also set

$$\begin{aligned}
\hat{H}_t^s &= H_{(s+t)\wedge\zeta}, & t \geq 0, \\
\check{H}_t^s &= H_{(s-t)_+}, & t \geq 0.
\end{aligned}$$

Then we observe that if $\sigma = p_H(s)$, we have

$$\mathcal{M}_\sigma = \mathcal{P}^{\hat{H}^s} + \mathcal{P}^{\check{H}^s} \tag{31}$$

where for any continuous function $h : [0,\infty) \longrightarrow [0,\infty)$ with compact support, the point measure $\mathcal{P}^h$ is defined as follows.

Let $\underline{h}(t) = \inf_{[0,t]} h$ and let (α_i, β_i), $i \in \mathcal{I}$ be the excursion intervals of $h - \underline{h}$ away from 0 (that is, the connected components of the set $\{h - \underline{h} > 0\}$). For every $i \in \mathcal{I}$, we set

$$h^i(t) = (h - \underline{h})((\alpha_i + t) \wedge \beta_i), \quad t \geq 0$$

and

$$\mathcal{P}^h = \sum_{i \in \mathcal{I}} \delta_{(h(\alpha_i), \mathcal{T}_{h^i})}.$$

The identity (31) is then a simple consequence of our definitions and the construction of the tree $\mathcal{T}_g$ in Section 2.

Using (31) and Corollary 3.5, we get

$$\Theta\Big(\int \ell^a(d\sigma)\,\exp{-\langle \mathcal{M}_\sigma, \Phi \rangle}\Big)$$

$$= N\Big(\int_0^\zeta d\Lambda_s^a\,\exp{-\langle \mathcal{P}^{\hat{H}^s} + \mathcal{P}^{\check{H}^s}, \Phi \rangle}\Big)$$

$$= e^{-\alpha a}\int \mathbb{M}_a(d\mu d\nu)\Big(\int Q_\mu(dh)\,e^{-\langle \mathcal{P}^h, \Phi \rangle}\Big)\Big(\int Q_\nu(dh)\,e^{-\langle \mathcal{P}^h, \Phi \rangle}\Big).$$

From Lemma 4.2.4 in [9], we immediately get

$$\int Q_\mu(dh)\, e^{-\langle \mathcal{P}^h, \Phi \rangle} = \exp -\int \mu(dt)\, N\Big(1-\exp-\Phi(t, \mathcal{T}_H)\Big) = \exp -\int \mu(dt)\, \Theta\Big(1-\exp-\Phi(t, \cdot)\Big).$$

Hence,

$$\Theta\Big(\int \ell^a(d\sigma)\, \exp-\langle \mathcal{M}_\sigma, \Phi \rangle\Big) = e^{-\alpha a}\int \mathbb{M}_a(d\mu d\nu)\, \exp\Big(-\int (\mu+\nu)(dt)\, \Theta\Big(1-\exp-\Phi(t, \cdot)\Big)\Big).$$

Recall from subsection 3.3 that the distribution of $\mu + \nu$ under $\mathbb{M}_a(d\mu d\nu)$ is the law of $\mathbf{1}_{[0,a]}(t)dU_t$, where U is a subordinator with Laplace exponent $\psi' - \alpha$. Therefore,

$$\begin{aligned}
\Theta\Big(\int \ell^a(d\sigma)\, \exp-\langle \mathcal{M}_\sigma, \Phi \rangle\Big) &= e^{-\alpha a}\, E\Big[\exp-\int_0^a dU_t\, \Theta(1-\exp-\Phi(t, \cdot))\Big] \\
&= \exp\Big(-\int_0^a dt\, \psi'\Big(\Theta(1-\exp-\Phi(t, \cdot))\Big)\Big).
\end{aligned}$$

$\blacksquare$

Remark. Combining the genealogical structure of the tree $\mathcal{T}$ with an independent spatial motion leads to a construction of superprocesses which will be explained in Section 6 below. In this setting, Theorem 4.5 is closely related to the representation for the historical Palm measure of superprocesses, which appears in Section 4.1 of [5] in the stable branching case.

We will now give a first application of Theorem 4.5 to properties of the Lévy tree. Recall from Section 2 that for every vertex σ of the tree $\mathcal{T}$, $n(\sigma)$ denotes the multiplicity of σ, defined as the number of connected components of $\mathcal{T}\backslash\{\sigma\}$. We write $\mathcal{L} = \{\sigma \in \mathcal{T}\backslash\{\rho(\mathcal{T})\} : n(\sigma) = 1\}$ for the set of leaves of $\mathcal{T}$.

Theorem 4.6 *We have:*

(i) *For every $a > 0$, $\ell^a(\mathcal{T}\backslash\mathcal{L}) = 0$, Θ a.e. Hence, $\mathbf{m}(\mathcal{T}\backslash\mathcal{L}) = 0$, Θ a.e.*

(ii) *$n(\sigma) \in \{1, 2, 3, \infty\}$ for all $\sigma \in \mathcal{T}$, Θ a.e.*

(iii) *The set $\{\sigma \in \mathcal{T} : n(\sigma) = 3\}$ of binary branching points is empty Θ a.e. if $\beta = 0$. If $\beta > 0$, the set of binary branching points is a countable dense subset of $\mathcal{T}$, Θ a.e.*

(iv) *The set $\{\sigma \in \mathcal{T} : n(\sigma) = \infty\}$ of infinite branching points is nonempty with positive Θ-measure iff $\pi \neq 0$. If $\langle \pi, 1 \rangle = \infty$, it is Θ a.e. a countable dense subset of $\mathcal{T}$. If $\langle \pi, 1 \rangle < \infty$ it is Θ a.e. a finite (possibly empty) subset of $\mathcal{T}$.*

Proof. We can reinterpret the result of Theorem 4.5 in the following way. Let $U = (U_t, t \geq 0)$ be a subordinator with Laplace exponent

$$\psi'(\lambda) = \alpha + 2\beta\lambda + \int \pi(dr)\, r(1 - e^{-\lambda r}).$$

Note in particular that U is killed at a time ξ which is exponentially distributed with parameter α. Write P for the probability measure under which U is defined.

Let $a > 0$, and note that the formula

$$\Sigma(A) = \Theta\Big(\int \ell^a(d\sigma)\, \mathbf{1}_A(\mathcal{M}_\sigma)\Big)$$

defines a (finite) measure on the set of point measures on $[0, a] \times \mathbb{T}$. Then Theorem 4.5 says that Σ is the law under $P(\cdot \cap \{a < \xi\})$ of a point measure $\mathcal{M}^*(dtd\mathcal{T})$ which conditionally given U is Poisson with intensity

$$\mathbf{1}_{[0,a]}(t)\, dU_t\, \Theta(d\mathcal{T}).$$

Since the measure dU_t a.s. gives no mass to the singleton $\{a\}$, $\mathcal{M}^*$ can have no atom of the form $(a, \mathcal{T}')$ for $\mathcal{T}' \in \mathbb{T}$. It follows that Θ a.e., $\ell^a(d\sigma)$ a.e., the point measure $\mathcal{M}_\sigma$ has no atom of this form. This means that ℓ^a-almost every σ is a leaf, and (i) follows.

Recalling that Θ is an infinite measure and using standard properties of Poisson measures, we see that only two possibilities may occur for instants $t \in [0, a)$:

- Either t is a time of jump of U and then the point measure $\mathcal{M}^*$ has infinitely many atoms of the form $(t, \mathcal{T}')$, $\mathcal{T}' \in \mathbb{T}$.

- Or t is not a time of jump of U and then the point measure $\mathcal{M}^*$ has at most one atom of the form $(t, \mathcal{T}')$, $\mathcal{T}' \in \mathbb{T}$, and may have one only if $\beta > 0$.

Using this and the relation between $\mathcal{M}^*$ and $\mathcal{M}_\sigma$, we see that for ℓ^a-almost every $\sigma \in \mathcal{T}$, the set $]\!]\rho(\mathcal{T}), \sigma[\![$ only contains vertices τ such that $n(\tau) \in \{2, 3, \infty\}$, and the value $n(\tau) = 3$ is only possible if $\beta > 0$. This property holds simultaneously for all rationals a, outside a single set of zero Θ-measure.

Now let σ be any vertex in the tree $\mathcal{T}$. If $n(\sigma) > 1$, then the set of descendants of σ is not empty and by Theorem 4.4 we can find a rational $a > d(\rho(\mathcal{T}), \sigma)$ such that this set has positive ℓ^a-measure. From the preceding property we deduce that $n(\sigma) \in \{2, 3, \infty\}$. This proves property (ii).

If $\beta = 0$, then we already noticed that the value $n(\tau) = 3$ is not achieved by any ancestor τ of σ, for ℓ^a-almost every $\sigma \in \mathcal{T}$. The same argument as in the proof of (ii) now shows that there is no vertex τ such that $n(\tau) = 3$, Θ a.e. This gives the first part of (iii).

On the contrary, if $\beta > 0$, we get from the relation between $\mathcal{M}_\sigma$ and $\mathcal{M}^*$ that the set of vertices τ with $n(\tau) = 3$ is dense in $]\!]\rho(\mathcal{T}), \sigma[\![$, for ℓ^a-almost every $\sigma \in \mathcal{T}$. If we apply this to all rationals a, we get the second part of (iii).

It remains to prove (iv). To prove this property it is convenient to argue on the tree $\mathcal{T}_H$ under N. If $\pi = 0$, then H is distributed under N as a (scaled) reflected Brownian motion with drift, and the fact that there are only binary branching points is clear since local minima of this process are distinct (alternatively, one can also use Theorem 4.5 in the same way as above). Suppose then that $\pi \neq 0$. We can then note that for every s such that $X_s - X_{s-} > 0$, $p_H(s)$ is an infinite branching point of $\mathcal{T}_H$. Indeed, if T is a stopping time such that $X_T - X_{T-} > 0$, and $X^{(T)}$ denotes the shifted process $X_s^{(T)} = X_{T+s} - X_T$, with associated minimum process $I_s^{(T)}$, any excursion interval of $X^{(T)} - I^{(T)}$ away from 0 before time $\inf\{s \geq 0 : X_s^{(T)} = -(X_T - X_{T-})\}$ will correspond to an excursion of H above level H_T, and will be associated under p_H with a connected component of $\mathcal{T}_H \backslash \{p_H(T)\}$ (recall the discussion at the end of subsection 2.1). Since there are infinitely many such excursions, we

see that $p_H(T)$ is an infinite branching point. When $\langle \pi, 1 \rangle = \infty$, the set of all jump times of X is dense in $[0, \zeta]$, N a.e., and it follows that the set of infinite branching points is dense in $\mathcal{T}_H$. On the other hand, when $\langle \pi, 1 \rangle < \infty$, X has discrete jumps and between jumps behaves like a (scaled) Brownian motion with drift. By analysing the behavior of the process H in that case, one easily obtains that the infinite branching points of $\mathcal{T}_H$ exactly correspond to jump times of X, so that the number of infinite branching points is finite N a.e. ∎

Remark. By using Theorem 4.5 as in the preceding proof, one easily gets the following additional property. If $\int r \, \pi(dr) = \infty$, then Θ a.e. for any vertex σ of $\mathcal{T} \backslash \{\rho(\mathcal{T})\}$, the "ancestral line" $[\![\rho(\mathcal{T}), \sigma]\!]$ contains infinitely many infinite branching points. On the other hand, if $\int r \, \pi(dr) < \infty$, then for every $a > 0$, the ancestral line of ℓ^a-almost every vertex contains finitely many infinite branching points.

We state another theorem relating discontinuities of local time to branching points of infinite multiplicity.

Theorem 4.7 *Let $\mathcal{I}(\mathcal{T}) = \{\sigma \in \mathcal{T} : n(\sigma) = \infty\}$. Then, Θ a.e. $\{d(\rho(\mathcal{T}), \sigma) : \sigma \in \mathcal{I}(\mathcal{T})\}$ coincides with the set of discontinuity times of the mapping $a \longrightarrow \ell^a$. Moreover, Θ a.e., for every discontinuity time b of the mapping $a \longrightarrow \ell^a$, there exists a unique $\sigma_b \in \mathcal{I}(\mathcal{T}) \cap \mathcal{T}(b)$, and we have*

$$\ell^b = \ell^{b-} + \lambda_b \, \delta_{\sigma_b}$$

where $\lambda_b > 0$ can be obtained via the approximation

$$\lambda_b = \lim_{\varepsilon \to 0} \frac{1}{v(\varepsilon)} \, n(\sigma_b, \varepsilon),$$

if $n(\sigma_b, \varepsilon)$ denotes the number of subtrees originating from σ_b with height greater than ε.

The number λ_b may be called the local time of the infinite branching point σ_b. As the proof will show, if the tree $\mathcal{T}$ is constructed as $\mathcal{T} = \mathcal{T}_H$ under the excursion measure N, then $\mathcal{I}(\mathcal{T})$ exactly consists of the vertices $p_H(s)$, where s varies in the set of discontinuity times of X. For any such s, the local time of the branching point $p_H(s)$ is just the jump ΔX_s of X at s.

Proof. We only sketch arguments. We assume that $\mathcal{T} = \mathcal{T}_H$ is the tree constructed from the height process H under N. First suppose that $\sigma \in \mathcal{I}(\mathcal{T})$ and let $b = d(\rho(\mathcal{T}), \sigma)$. Recall the discussion at the end of subsection 2.1. From the connection betwen the height process and the so-called exploration process (cf Chapter 1 of [9]) one easily sees that there must exist $r > 0$ with $p_H(r) = \sigma$ and $\Delta X_r > 0$. In a way similar to the end of the preceding proof, we may consider the path of X between r and $\inf\{t \geq r : X_t = X_{r-}\}$, and obtain that $\ell^b \geq \ell^{b-} + (\Delta X_r) \delta_\sigma$ (recall that we already know that $\ell^b \geq \ell^{b-}$).

Conversely, suppose that b is a discontinuity time of the mapping $a \longrightarrow \ell^a$. Let $r < b$ be a rational and write $(\mathcal{T}^{(i)}, i \in \mathcal{I})$ for the subtrees originating from level r. For every $i \in \mathcal{I}$, denote by $(\ell^{a,(i)}, a \geq 0)$ the local times of $\mathcal{T}^{(i)}$. From the branching property and the fact that the ψ-CSBP has no fixed discontinuities, we get that at most one (in fact exactly one) of the processes $a \longrightarrow \ell^{a,(i)}$ can be discontinuous at $b - r$. It easily follows that there exists σ_b with $\ell^b = \ell^{b-} + \lambda_b \delta_{\sigma_b}$, for some $\lambda_b > 0$. Using the branching property at level r_n, for a sequence of rationals (r_n) decreasing to b, we get that b must be a point of infinite multiplicity, and that the approximation formula of the theorem holds for λ_b. We leave details to the reader. ∎

As a last application of Theorem 4.5, we give a remarkable invariance property of the measure $\Theta(d\mathcal{T})$ under uniform re-rooting. If $\mathcal{T} \in \mathbb{T}$ and $\sigma \in \mathcal{T}$, we write $\mathcal{T}^{[\sigma]}$ for the "same" tree $\mathcal{T}$ with root σ.

Proposition 4.8 *The law of the tree $\mathcal{T}^{[\sigma]}$ under the measure $\Theta(d\mathcal{T}) \frac{\mathbf{m}(d\sigma)}{\mathbf{m}(\mathcal{T})}$ coincides with $\Theta(d\mathcal{T})$.*

Proof. It is enough to verify that, for any nonnegative measurable functional F on $\mathbb{T}$,

$$\Theta\Big(\int \mathbf{m}(d\sigma)\, F(\mathcal{T}^{[\sigma]})\Big) = \Theta\Big(\mathbf{m}(\mathcal{T})\, F(\mathcal{T})\Big). \tag{32}$$

Recall the notation $\mathcal{M}_\sigma$ before Theorem 4.5. We can easily find an explicit "reconstruction" functional Γ such that $\mathcal{T} = \Gamma(d(\rho(\mathcal{T}), \sigma), \mathcal{M}_\sigma)$ for every $\mathcal{T} \in \mathbb{T}$ and $\sigma \in \mathcal{T}$. For this functional Γ, we have also

$$\mathcal{T}^{[\sigma]} = \Gamma(d(\rho(\mathcal{T}), \sigma), \widetilde{\mathcal{M}}_\sigma),$$

provided we set

$$\widetilde{\mathcal{M}}_\sigma = \sum_{j \in \mathcal{J}} \delta_{(d(\rho(\mathcal{T}),\sigma)-d(\rho(\mathcal{T}),\sigma_j), \mathcal{T}^{(j)})},$$

with the notation preceding Theorem 4.5. Now Theorem 4.5 implies that for any $a > 0$ and any nonnegative measurable functional G,

$$\Theta\Big(\int \ell^a(d\sigma)\, G(\mathcal{M}_\sigma)\Big) = \Theta\Big(\int \ell^a(d\sigma)\, G(\widetilde{\mathcal{M}}_\sigma)\Big).$$

Apply this to $G(\mathcal{M}_\sigma) = F(\Gamma(a, \mathcal{M}_\sigma))$ and then integrate with respect to da to get (32). ∎

5 Fractal properties of $\mathcal{T}$

5.1 Covering numbers and box counting dimensions

Recall that for any subset A of $\mathcal{T}$ and any $\delta > 0$, we have set

$$\mathcal{N}(A, \delta) = \inf\Big\{ n \geq 1 \ : \ \exists \sigma_1, \ldots, \sigma_n \in \mathcal{T} \quad \text{s.t.} \quad A \subset \bigcup_{i=1}^{n} B(\sigma_i, \delta)\Big\}.$$

The following propositions give precise rates of growth for $\mathcal{N}(\mathcal{T}, \delta)$ and $\mathcal{N}(\mathcal{T}(a), \delta)$.

Proposition 5.1 *For any $a > 0$, Θ-a.e. on $\{h(\mathcal{T}) > a\}$,*

$$\lim_{\delta \to 0} \frac{\mathcal{N}(\mathcal{T}(a), \delta)}{v(\delta)} = \ell^a(\mathcal{T}).$$

Proposition 5.2 *We have Θ-a.e. for all $\delta > 0$ sufficiently small*

$$\frac{v(2\delta)}{4\delta} \zeta \leq \mathcal{N}(\mathcal{T}, \delta) \leq \frac{12\, v(\delta/6)}{\delta} \zeta.$$

We immediately deduce the following corollary.

Corollary 5.3 *For any $a > 0$, Θ-a.e. on $\{h(\mathcal{T}) > a\}$,*

$$\underline{\dim}(\mathcal{T}) = 1 + \underline{\dim}(\mathcal{T}(a)) = 1 + \liminf_{\delta \to 0} \frac{\log v(\delta)}{\log(1/\delta)}$$

and

$$\overline{\dim}(\mathcal{T}) = 1 + \overline{\dim}(\mathcal{T}(a)) = 1 + \limsup_{\delta \to 0} \frac{\log v(\delta)}{\log(1/\delta)}.$$

Proof of Proposition 5.1. Let $a > 0$ and $r \in (0, a)$. Following Section 4, we denote by $\mathcal{T}^{(i)}$, $i \in \mathcal{I}$ the subtrees of $\mathcal{T}$ originating from level $a - r$, and we set

$$\mathcal{I}_{(r)} := \{i \in \mathcal{I} : h(\mathcal{T}^{(i)}) \geq r\}$$

so that $|\mathcal{I}_{(r)}| = Z(a-r, r)$, with the notation introduced before Theorem 4.2. We then observe that, for every $r' > r$,

$$\mathcal{T}(a) \subset \bigcup_{i \in \mathcal{I}_{(r)}} \left(\mathcal{T}^{(i)} \cap \mathcal{T}(a) \right) \subset \bigcup_{i \in \mathcal{I}_{(r)}} \bar{B}(\rho(\mathcal{T}^{(i)}), r) \subset \bigcup_{i \in \mathcal{I}_{(r)}} B(\rho(\mathcal{T}^{(i)}), r'),$$

so that

$$\mathcal{N}(\mathcal{T}(a), r') \leq Z(a - r, r). \tag{33}$$

On the other hand, if σ and σ' are two vertices in $\mathcal{T}(a)$ that belong respectively to $\mathcal{T}^{(i)}$ and $\mathcal{T}^{(i')}$ for distinct indices i and i', we have $d(\sigma, \sigma') \geq 2r$. Therefore, σ and σ' must belong to distinct balls of any covering of $\mathcal{T}(a)$ by open balls with radius r. From this observation, we get

$$\mathcal{N}(\mathcal{T}(a), r) \geq Z(a - r, r). \tag{34}$$

By combining (33) and (34), we see that Proposition 5.1 follows from the case $\varphi = 1$ in Theorem 4.2(iii). ∎

Proof of Proposition 5.2. We start by proving the following lemma.

Lemma 5.4 *Θ-a.e. we can find a sequence $(D_n; n \geq 1)$ of finite subsets of $\mathcal{T}$ such that*

(i) $D_n \subset D_{n+1}$, $n \geq 1$.

(ii) *For every $n \geq 1$ and $\sigma \in \mathcal{T}$, there exists $\sigma' \in D_n$ such that $d(\sigma, \sigma') < 3.2^{-n}$.*

(iii) *For every $n \geq 1$ and every distinct $\sigma, \sigma' \in D_n$ we have $d(\sigma, \sigma') \geq 2^{-n}$.*

(iv) $\displaystyle \lim_{n \to \infty} 2^{-n} \frac{|D_n|}{v(2^{-n})} = \zeta.$

Proof of the lemma. For any $n \geq 1$ set $K_n = [2^n h(\mathcal{T})] - 1$. Let $n \geq 1$ and $k \in \{0, 1, \ldots, K_n\}$. Denote by

$$\mathcal{T}^{n,k,(i)}, \quad 1 \leq i \leq Z(k2^{-n}, 2^{-n})$$

the subtrees of $\mathcal{T}$ originating from level $k2^{-n}$ that hit level $(k+1)2^{-n}$. We can now use induction on n to select for every $n \geq 1$, $k \in \{0, 1, \ldots, K_n\}$ and $i \in \{1, \ldots, Z(k2^{-n}, 2^{-n})\}$, a vertex $\sigma_{k,i}^n \in \mathcal{T}^{n,k,(i)}(2^{-n}) \subset \mathcal{T}((k+1)2^{-n})$, in such a way that the following holds:

(P) Let $k \in \{0, 1, \ldots, K_n\}$ and $i \in \{1, \ldots, Z(k2^{-n}, 2^{-n})\}$. If $j \leq Z((2k+1)2^{-n-1}, 2^{-n-1})$ is the unique index such that $\sigma_{k,i}^n \in \mathcal{T}^{n+1,2k+1,(j)}$, then $\sigma_{2k+1,j}^{n+1} = \sigma_{k,i}^n$.

We then set

$$D_n := \{\sigma_{k,i}^n : 0 \leq k \leq K_n, 1 \leq i \leq Z(k2^{-n}, 2^{-n})\} \cup \{\rho(\mathcal{T})\}.$$

Property (i) is clear from (P). To prove (ii), let $\sigma \in \mathcal{T}$. If $d(\rho(\mathcal{T}), \sigma) \leq 2^{-n}$ the desired result is obvious since $\rho(\mathcal{T}) \in D_n$. So suppose that $d(\rho(\mathcal{T}), \sigma) > 2^{-n}$ and let $k \in \{0, 1, \ldots, K_n\}$ be such that $(k+1)2^{-n} \leq d(\rho(\mathcal{T}), \sigma) < (k+2)2^{-n}$. Clearly the ancestor of σ at generation $k2^{-n}$ must be the root of $\mathcal{T}^{n,k,(i)}$ for some i. Then simply write

$$d(\sigma, \sigma_{k,i}^n) \leq d(\rho(\mathcal{T}^{n,k,(i)}), \sigma) + d(\rho(\mathcal{T}^{n,k,(i)}), \sigma_{k,i}^n) < 2.2^{-n} + 2^{-n} = 3.2^{-n}.$$

The proof of (iii) is even simpler and is left to the reader.

It remains to prove (iv). For any $x > 0$, set

$$A_n(x) = 2^{-n} \sum_{k=1}^{[x2^n]} \left(\frac{Z(k2^{-n}, 2^{-n})}{v(2^{-n})} - \ell^{k2^{-n}}(\mathcal{T}) \right).$$

Let $k \geq 1$. We know from Theorem 4.2 that conditionally on the truncated tree $\mathrm{tr}_{k2^{-n}}(\mathcal{T})$, $Z(k2^{-n}, 2^{-n})$ is Poisson with mean $v(2^{-n})\ell^{k2^{-n}}(\mathcal{T})$. In particular,

$$\Theta \left(\frac{Z(k2^{-n}, 2^{-n})}{v(2^{-n})} - \ell^{k2^{-n}}(\mathcal{T}) \,\Big|\, \mathrm{tr}_{k2^{-n}}(\mathcal{T}) \right) = 0 \tag{35}$$

and

$$\Theta \left(\left| \frac{Z(k2^{-n}, 2^{-n})}{v(2^{-n})} - \ell^{k2^{-n}}(\mathcal{T}) \right|^2 \,\Big|\, \mathrm{tr}_{k2^{-n}}(\mathcal{T}) \right) = \frac{\ell^{k2^{-n}}(\mathcal{T})}{v(2^{-n})}. \tag{36}$$

(In both cases the conditional expectation should be understood with respect to the probability measure $\Theta(\cdot \mid h(\mathcal{T}) > k2^{-n})$.) It is also immediate to see that $Z(k2^{-n}, 2^{-n})$ is a measurable function of $\mathrm{tr}_{(k+1)2^{-n}}(\mathcal{T})$. It easily follows that for any $k' > k$,

$$\Theta \left(\left(\frac{Z(k2^{-n}, 2^{-n})}{v(2^{-n})} - \ell^{k2^{-n}}(\mathcal{T}) \right) \left(\frac{Z(k'2^{-n}, 2^{-n})}{v(2^{-n})} - \ell^{k'2^{-n}}(\mathcal{T}) \right) \right) = 0. \tag{37}$$

The combination of (37) and (36) gives

$$\Theta \left(A_n(x)^2 \right) = \frac{2^{-2n}}{v(2^{-n})} \sum_{k=1}^{[x2^n]} \Theta \left(\ell^{k2^{-n}}(\mathcal{T}) \right) \leq \frac{x2^{-n}}{v(2^{-n})},$$

since $\Theta(\ell^a(\mathcal{T})) = N(\Lambda_\zeta^a) = e^{-\alpha a} \leq 1$, for every $a > 0$ (cf the end of Section 3). Clearly the preceding estimate implies that $\sum_{n \geq 0} \Theta(A_n(x)^2) < \infty$ and thus, for any $x > 0$,

$$\Theta - \text{a.e.} \quad \lim_{n \to \infty} A_n(x) = 0. \tag{38}$$

Since the mapping $b \to \ell^b(\mathcal{T})$ is càdlàg, we have Θ-a.e.

$$\lim_{n \to \infty} 2^{-n} \sum_{k=1}^{[x2^n]} \ell^{k2^{-n}}(\mathcal{T}) = \int_0^x db \, \ell^b(\mathcal{T}).$$

Together with (38), this implies that for any $x > 0$, Θ-a.e.

$$\lim_{n \to \infty} \frac{2^{-n}}{v(2^{-n})} \sum_{k=1}^{[x2^n]} Z(k2^{-n}, 2^{-n}) = \int_0^x db \, \ell^b(\mathcal{T}).$$

Since the height $h(\mathcal{T})$ is finite, we can take $x = \infty$ in the preceding limit, which gives (iv) since $\int_0^\infty db \, \ell^b(\mathcal{T}) = \zeta$. $\blacksquare$

Proposition 5.2 is a simple consequence of Lemma 5.4. Let $\delta \in (0, 1/2)$ and $n \geq 1$ such that $2^{-n-1} \leq \delta < 2^{-n}$. From property (iii) in Lemma 5.4, we get that if δ is sufficiently small,

$$\mathcal{N}(\mathcal{T}, \delta) \geq \mathcal{N}(\mathcal{T}, 2^{-n-1}) \geq |D_n| \geq \frac{\zeta}{2} \frac{v(2^{-n})}{2^{-n}} \geq \frac{\zeta}{2} \frac{v(2\delta)}{2\delta}.$$

Similarly, if $3.2^{-n} \leq \delta < 3.2^{-n+1}$, we get from property (ii) in Lemma 5.4 that for δ sufficiently small,

$$\mathcal{N}(\mathcal{T}, \delta) \leq |D_n| \leq 2\zeta \frac{v(2^{-n})}{2^{-n}} \leq 2\zeta \frac{v(\delta/6)}{\delta/6}.$$

This completes the proof of Proposition 5.2. $\blacksquare$

5.2 Hausdorff and packing dimensions of subsets of $\mathcal{T}$

We first recall the well-known inequalities

$$\dim_h(B) \leq \underline{\dim}(B) \quad \text{and} \quad \dim_h(B) \leq \dim_p(B) \leq \overline{\dim}(B), \tag{39}$$

for any subset B of $\mathcal{T}$ (see e.g. Chapter 3 of [14]).

Let E be a compact subset of $(0, \infty)$, and set $A = \sup E$. We assume that the Hausdorff dimension and upper box counting dimension of E are equal and let $d(E) \in [0, 1]$ be their common value. Recall also the notation $\mathcal{T}(E) = \bigcup_{b \in E} \mathcal{T}(b)$. The lower and upper indices γ and η were defined in the introduction above. The aim of this subsection is to prove the following theorem.

Theorem 5.5 *Assume that $\gamma > 1$. Then, Θ a.e. on $\{h(\mathcal{T}) > A\}$,*

$$\underline{\dim}(\mathcal{T}(E)) = \dim_h(\mathcal{T}(E)) = d(E) + \frac{1}{\eta - 1} \quad \text{and} \quad \overline{\dim}(\mathcal{T}(E)) = \dim_p(\mathcal{T}(E)) = d(E) + \frac{1}{\gamma - 1}.$$

Proof: We first get upper bounds for the box counting dimensions of $\mathcal{T}(E)$. Let $\delta > 0$. Analogously to the above, we use the notation $\mathcal{N}(E, \delta)$ for the minimal number of open intervals of length 2δ that are needed to cover E. We can find real numbers a_i, $1 \leq i \leq \mathcal{N}(E, \delta)$ such that

$$E \subset \bigcup_{i=1}^{\mathcal{N}(E, \delta)} (a_i - \delta, a_i + \delta).$$

Observe that

$$\mathcal{T}(E \cap [0, 3\delta)) \subset \mathcal{T}([0, 3\delta)) \subset B(\rho(\mathcal{T}), 3\delta). \tag{40}$$

On the other hand, $E \cap [3\delta, \infty)$ is contained in the union of those intervals $(a_i - \delta, a_i + \delta)$ for which $a_i > 2\delta$.

Now let $b > 2\delta$. Denote by $\mathcal{T}_{(j)}$, $1 \le j \le Z(b - 2\delta, \delta)$ the subtrees of $\mathcal{T}$ originating from level $b - 2\delta$ that reach level $b - \delta$ and for every j, let $\sigma_{(j)}$ be the root of $\mathcal{T}_{(j)}$. Clearly any vertex in $\mathcal{T}((b - \delta, b + \delta))$ belongs to $\mathcal{T}_{(j)}$ for some index j and thus lies within distance 3δ from $\sigma_{(j)}$. Consequently,

$$\mathcal{N}(\mathcal{T}((b - \delta, b + \delta)), 3\delta) \le Z(b - 2\delta, \delta).$$

Now recall that $\Theta(Z(b - 2\delta, \delta)) = v(\delta)\Theta(\ell^{b-2\delta}(\mathcal{T})) \le v(\delta)$. Hence,

$$\Theta(\mathcal{N}(\mathcal{T}((b - \delta, b + \delta)), 3\delta)) \le v(\delta).$$

We apply this to $b = a_i$ for all indices i such that $a_i > 2\delta$. By summing over i, we get

$$\Theta(\mathcal{N}(\mathcal{T}(E \cap [3\delta, \infty)), 3\delta)) \le v(\delta)\mathcal{N}(E, \delta). \tag{41}$$

At this point, we need the following lemma.

Lemma 5.6 *Assume that $\gamma > 1$. Then,*

(i) $\quad \limsup\limits_{\delta \to 0} \dfrac{\log v(\delta)}{\log(1/\delta)} \; \le \; \dfrac{1}{\gamma - 1}$;

(ii) $\quad \liminf\limits_{\delta \to 0} \dfrac{\log v(\delta)}{\log(1/\delta)} \; \le \; \dfrac{1}{\eta - 1}$.

Proof of the lemma. Assertion (i) is easy from the definition of v and γ. Let us prove (ii). If $\eta = \gamma$, (ii) is a trivial consequence of (i). So, we assume that $\gamma < \eta$. Let $\eta' \in (\gamma, \eta)$ and $\gamma' \in (1, \gamma)$. There exists a sequence $u_n \uparrow \infty$ such that $\psi(u_n) \ge u_n^{\eta'}$, $n \ge 1$. Moreover, for all sufficiently large u, we have $\psi(u) \ge u^{\gamma'}$. Since ψ is convex, we get for n large enough and for any $u \ge u_n$,

$$\psi(u) \ge \max\left(\frac{u}{u_n} u_n^{\eta'}, u^{\gamma'} \right).$$

Set $F(a) = \int_a^\infty du/\psi(u)$. The previous inequality gives, for n large,

$$
\begin{aligned}
F(u_n) &\le \int_{u_n}^\infty \left(\max(u u_n^{\eta'-1}, u^{\gamma'}) \right)^{-1} du \\
&= \int_{u_n}^{u_n^{\frac{\eta'-1}{\gamma'-1}}} \frac{du}{u u_n^{\eta'-1}} + \int_{u_n^{\frac{\eta'-1}{\gamma'-1}}}^\infty \frac{du}{u^{\gamma'}} \\
&\le C \left(u_n^{1-\eta'} \log u_n + u_n^{1-\eta'} \right)
\end{aligned}
$$

for some positive constant C. Hence,

$$\left(\liminf_{\delta \to 0} \frac{\log v(\delta)}{\log 1/\delta} \right)^{-1} = \limsup_{a \to \infty} \frac{\log 1/F(a)}{\log a} \ge \eta' - 1$$

and (ii) follows by letting η' go to η. $\blacksquare$

We deduce from the previous lemma that for any $\varepsilon > 0$,

$$\liminf_{\delta \to 0} \delta^{\varepsilon + \frac{1}{\eta-1}} v(\delta) = \limsup_{\delta \to 0} \delta^{\varepsilon + \frac{1}{\gamma-1}} v(\delta) = 0. \tag{42}$$

Since $d(E) = \overline{\dim}(E)$, we also know that $\delta^{d(E)+\varepsilon} \mathcal{N}(E, \delta)$ tends to 0 as $\delta \to 0$. Thus, if (δ_n) is any sequence of positive reals decreasing to 0, it follows from (41), (42) and Fatou's lemma that

$$\Theta\left(\liminf_{n \to \infty} \delta_n^{2\varepsilon + d(E) + \frac{1}{\eta-1}} \mathcal{N}(\mathcal{T}(E \cap [3\delta_n, \infty)), 3\delta_n) \right) = 0.$$

Hence,

$$\liminf_{n \to \infty} \delta_n^{2\varepsilon + d(E) + \frac{1}{\eta-1}} \mathcal{N}(\mathcal{T}(E \cap [3\delta_n, \infty)), 3\delta_n) = 0 , \quad \Theta \text{ a.e.}$$

From (40) we have $\mathcal{N}(\mathcal{T}(E), 3\delta_n) \leq 1 + \mathcal{N}(\mathcal{T}(E \cap [3\delta_n, \infty)), 3\delta_n)$ and so we get $\underline{\dim}(\mathcal{T}(E)) \leq d(E) + 1/(\eta-1) + 2\varepsilon$. Since ε was arbitrary we conclude that $\underline{\dim}(\mathcal{T}(E)) \leq d(E) + 1/(\eta-1)$, Θ a.e.

To obtain an analogous upper bound for $\overline{\dim}(\mathcal{T}(E))$, we set $\delta_n = 2^{-n}$ and deduce from (41) and (42) that

$$\Theta\left(\sum_{n \geq 1} \delta_n^{3\varepsilon + d(E) + \frac{1}{\gamma-1}} \mathcal{N}\left(\mathcal{T}(E \cap [3\delta_n, \infty)), 3\delta_n \right) \right) < \infty.$$

Hence,

$$\lim_{n \to \infty} \delta_n^{3\varepsilon + d(E) + \frac{1}{\gamma-1}} \mathcal{N}\left(\mathcal{T}(E \cap [3\delta_n, \infty)), 3\delta_n \right) = 0 , \quad \Theta \text{ a.e.}$$

and the bound $\mathcal{N}(\mathcal{T}(E), 3\delta_n) \leq 1 + \mathcal{N}(\mathcal{T}(E \cap [3\delta_n, \infty)), 3\delta_n)$ allows us to replace $\mathcal{T}(E \cap [3\delta_n, \infty))$ with $\mathcal{T}(E)$. Then, a simple monotonicity argument implies that

$$\lim_{\delta \to 0} \delta^{3\varepsilon + d(E) + \frac{1}{\gamma-1}} \mathcal{N}(\mathcal{T}(E), \delta) = 0 , \quad \Theta \text{ a.e.}$$

It follows that $\overline{\dim}(\mathcal{T}(E)) \leq d(E) + 1/(\gamma-1)$, Θ a.e.

The proof of the theorem will be complete if we verify that for any $\varepsilon > 0$ we have Θ a.e. on $\{h(\mathcal{T}) > A\}$,

$$\dim_h(\mathcal{T}(E)) \geq d(E) + \frac{1}{\eta - 1 + \varepsilon} - 2\varepsilon \quad \text{and} \quad \dim_p(\mathcal{T}(E)) \geq d(E) + \frac{1}{\gamma - 1 + \varepsilon} - 2\varepsilon \tag{43}$$

We may assume that ε is small enough so that $1/(\gamma - 1 + \varepsilon) \geq 1/(\eta - 1 + \varepsilon) > 2\varepsilon$. Let us prove (43). Since $\dim_h(E) > d(E) - \varepsilon$, Frostman's lemma (see Corollary 4.12 in [14]) gives the existence of a non-trivial finite measure ν supported on E, such that

$$\forall x \in E, \ \forall \delta \in [0,1] \ : \ \nu([x - \delta, x + \delta]) \leq C \delta^{d(E)-\varepsilon} \tag{44}$$

where C is a positive constant independent of x and δ. Define the measure κ on $\mathcal{T}$ by

$$\kappa(d\sigma) = \int \nu(da) \ell^a(d\sigma).$$

Then κ is supported on $\mathcal{T}(E)$. Moreover κ is finite and non-trivial Θ a.e. on $\{\sup H > A\}$. We will prove that Θ-a.e. on $\{h(\mathcal{T}) > A\}$, we have

$$\limsup_{\delta \to 0} \delta^{-d(E)+2\varepsilon-\frac{1}{\eta-1+\varepsilon}} \kappa\left(B(\sigma,\delta)\right) < \infty \ , \qquad \kappa(d\sigma) \text{ a.e.} \tag{45}$$

and

$$\liminf_{\delta \to 0} \delta^{-d(E)+2\varepsilon-\frac{1}{\gamma-1+\varepsilon}} \kappa\left(B(\sigma,\delta)\right) < \infty \ , \qquad \kappa(d\sigma) \text{ a.e.} \tag{46}$$

Then the lower bounds (43) will follow from classical density results for packing and Hausdorff dimensions: See e.g. Theorems 6.9 and 6.11 in Mattila [24] ([24] deals with subsets of Euclidean space, but the arguments are easily adapted to our setting).

The proof of (45) and (46) will depend on a lower bound for the quantities

$$\mathcal{E}_{\delta,\lambda,b} := \Theta\left(\int \ell^b(d\sigma)e^{-\lambda\kappa(B(\sigma,\delta))}\right) \ , \qquad \lambda, b > 0, \ \delta \in (0,1].$$

We will apply Theorem 4.5 in order to get this bound. To this end, let us first fix $b > 0$ and $\sigma \in \mathcal{T}(b)$, and use the notation introduced before Theorem 4.5: $\mathcal{T}^{(j)}$, $j \in \mathcal{J}$ are the subtrees originating from the ancestral line $[\![\rho(\mathcal{T}),\sigma]\!]$, and for every $j \in \mathcal{J}$, $\sigma_j \in [\![\rho(\mathcal{T}),\sigma]\!]$ is the root of $\mathcal{T}^{(j)}$. Also set $d_j = d(\rho(\mathcal{T}),\sigma_j)$ to simplify notation.

If $\tau \in \mathcal{T}^{(j)}$ for some $j \in \mathcal{J}$, we have $d(\sigma,\tau) = b - d_j + d(\sigma_j,\tau)$. It follows that

$$B(\sigma,\delta)\backslash[\![\rho(\mathcal{T}),\sigma]\!] = \bigcup_{j\in\mathcal{J},d_j>b-\delta} \mathcal{T}^{(j)}((0,\delta+d_j-b)).$$

Notice that the union in the right side is disjoint. Also observe that, for every fixed $a > 0$, Θ a.e. the measure ℓ^a has no atoms. Indeed, if $\tau \in \mathcal{T}(a)$ were an atom of ℓ^a, the branching property of the Lévy tree would imply that τ is not a leaf, contradicting the fact that ℓ^a almost every vertex is a leaf (Theorem 4.6(i)). From this we get that Θ a.e., $\nu(da)$ a.e. ℓ^a has no atoms and since the set $[\![\rho(\mathcal{T}),\sigma]\!]$ has at most one point of intersection with each level set $\mathcal{T}(a)$ it follows that $\kappa([\![\rho(\mathcal{T}),\sigma]\!]) = 0$. Thus,

$$\kappa(B(\sigma,\delta)) = \kappa(B(\sigma,\delta)\backslash[\![\rho(\mathcal{T}),\sigma]\!]) = \sum_{j\in\mathcal{J},d_j>b-\delta} \kappa(\mathcal{T}^{(j)}((0,\delta+d_j-b))).$$

Now, if $a \leq b-\delta$ or $a \geq b+\delta$, the support property of ℓ^a implies that $\ell^a(\mathcal{T}^{(j)}((0,\delta+d_j-b))) = 0$ for every $j \in \mathcal{J}$ such that $d_j > b-\delta$. On the other hand, if $b-\delta < a < b+\delta$, the approximations of local time easily give

$$\ell^a(\mathcal{T}^{(j)}((0,\delta+d_j-b))) = \mathbf{1}_{\{0<a-d_j<\delta+d_j-b\}} \langle \ell^{(j),a-d_j}, 1\rangle,$$

where $\ell^{(j),a}$, $a > 0$ obviously denote the local time measures associated with the tree $\mathcal{T}^{(j)}$. We conclude that

$$\kappa(B(\sigma,\delta)) = \sum_{j\in\mathcal{J}} \mathbf{1}_{\{d_j>b-\delta\}} \int \nu(da) \, \mathbf{1}_{\{d_j<a<2d_j+\delta-b\}} \langle \ell^{(j),a-d_j}, 1\rangle.$$

In this form, we can apply the formula of Theorem 4.5 to get

$$\mathcal{E}_{\delta,\lambda,b} = \exp\left(-\int_0^b \psi'(\Phi_{\delta,\lambda,b}(t)) \, dt\right),$$

where
$$\Phi_{\delta,\lambda,b}(t) = \Theta\Big(1 - \exp{-\lambda}\int \nu(da)\,\mathbf{1}_{\{t<a<2t+\delta-b\}}\,\langle \ell^{a-t}, 1\rangle\Big).$$

Now set $\widetilde{\psi}(\lambda) = \frac{\psi(\lambda)}{\lambda}$ and note that for every $\lambda > 0$,

$$\psi'(\lambda) \leq \frac{\psi(2\lambda) - \psi(\lambda)}{\lambda} \leq 2\widetilde{\psi}(2\lambda). \tag{47}$$

If $t \in [0, b-\delta]$ we have $\Phi_{\delta,\lambda,b}(t) = 0$. On the other hand, if $t \in (b-\delta, b]$ then $(t, \delta+2t-b) \subset [b-\delta, b+\delta]$ and

$$\Phi_{\delta,\lambda,b}(t) \leq \lambda \int_{[(b-\delta)\vee t, b+\delta]} \nu(da)\,\Theta(\ell^{a-t}(\mathcal{T})) \leq \lambda\,\nu([b-\delta, b+\delta]).$$

Using this bound together with (47) we have

$$\int_0^b \psi'(\Phi_{\delta,\lambda,b}(t))\,dt \leq 2\delta\,\widetilde{\psi}(2\lambda\nu([b-\delta, b+\delta])).$$

and it follows that

$$\mathcal{E}_{\delta,\lambda,b} \geq 1 - 2\delta\,\widetilde{\psi}(2\lambda\nu([b-\delta, b+\delta])). \tag{48}$$

This is the lower bound we were aiming at.

If $r > 0$, (48) gives, for every $\delta \in (0, 1]$,

$$\begin{aligned}
\Theta\left(\int \kappa(d\sigma)\mathbf{1}_{\{\kappa(B(\sigma,\delta))>r\}}\right) &\leq \frac{e}{e-1}\Theta\left(\int \kappa(d\sigma)(1 - e^{-\frac{1}{r}\kappa(B(\sigma,\delta))})\right) \\
&= \frac{e}{e-1}\int \nu(db)(e^{-\alpha b} - \mathcal{E}_{\delta,1/r,b}) \\
&\leq \frac{2e}{e-1}\delta\int \nu(db)\widetilde{\psi}\left(2\nu([b-\delta, b+\delta])/r\right)
\end{aligned}$$

and by (44),

$$\Theta\left(\int \kappa(d\sigma)\mathbf{1}_{\{\kappa(B(\sigma,\delta))>r\}}\right) \leq C'\delta\widetilde{\psi}(2C\delta^{d(E)-\varepsilon}/r) \tag{49}$$

where C' is a positive constant depending on ν. By the definition of η, for all sufficiently large $\lambda > 0$, $\widetilde{\psi}(\lambda) \leq \lambda^{\eta-1+\varepsilon}$. Then, take $r = r(\delta) = 2C\delta^{d(E)-2\varepsilon+(1/(\eta-1+\varepsilon))}$ in (49) to get, for all sufficiently small $\delta > 0$

$$\Theta\left(\int \kappa(d\sigma)\mathbf{1}_{\{\kappa(B(\sigma,\delta))>r(\delta)\}}\right) \leq C''\delta^{\varepsilon(\eta-1+\varepsilon)}.$$

Set $\delta_n = 2^{-n}$. Since $\eta > 1$ we deduce from the previous inequality that

$$\Theta\Big(\int \kappa(d\sigma)\sum_{n\geq 1}\mathbf{1}_{\{\kappa(B(\sigma,\delta_n))>r(\delta_n)\}}\Big) < \infty$$

and this yields the estimate (45) for the upper density of κ.

It remains to prove (46). By the definition of γ, there exists an increasing sequence $u_n \uparrow \infty$ such that $\widetilde{\psi}(u_n) \leq u_n^{\gamma-1+\varepsilon}$. Define δ_n by $u_n = \delta_n^{\varepsilon-(1/(\gamma-1+\varepsilon))}$ and take $r(\delta_n) = 2C\delta_n^{d(E)-2\varepsilon+1/(\gamma-1+\varepsilon)}$ in (49) to get

$$\Theta\left(\int \kappa(d\sigma)\mathbf{1}_{\{\kappa(B(\sigma,\delta_n))>r(\delta_n)\}}\right) \leq C'''\delta_n^{\varepsilon(\gamma-1+\varepsilon)}.$$

Applying Fatou's lemma, we get that Θ-a.e. for κ-a.a. σ,

$$\liminf_{n\to\infty} \mathbf{1}_{\{\kappa(B(\sigma,\delta_n))>r(\delta_n)\}} = 0$$

which implies the estimate (46) and completes the proof of the theorem. ∎

5.3 Further results and open problems

In this section, we briefly discuss some extensions of the preceding results. For simplicity we restrict our attention to Hausdorff dimensions and measures. We start by weakening the condition $\gamma > 1$ in Theorem 5.5. As in Theorem 5.5 we let E be a (nonempty) compact subset of $(0,\infty)$ such that $\dim_h(E) = \overline{\dim}(E) = d(E)$, and we put $A = \sup E$. We use the standard convention $\frac{1}{0} = \infty$.

Proposition 5.7 *Suppose that for every integer $k \geq 1$,*

$$\int_a^\infty \frac{du}{\psi(u)} = o((\log a)^{-k}) \quad as\ a \to \infty. \tag{50}$$

Then, Θ a.e. on $\{h(\mathcal{T}) > A\}$,

$$\dim_h(\mathcal{T}(E)) = d(E) + \frac{1}{\eta - 1}.$$

Indeed, the proof of the estimate (45) does not depend on the assumption $\gamma > 1$, and this immediately gives the lower bound $\dim_h(\mathcal{T}(E)) \geq d(E) + \frac{1}{\eta-1}$. When $\eta = 1$, there is nothing more to prove. When $\eta > 1$, a slight modification of the proof of Lemma 5.6 shows that part (ii) of this lemma still holds under the condition (50). The first part of the proof of Theorem 5.5 then goes through without change.

Let us consider now the general case. From the preceding remarks, one easily gets the following statement.

Proposition 5.8 *We have Θ a.e. on $\{h(\mathcal{T}) > A\}$,*

$$d(E) + \frac{1}{\eta - 1} \leq \dim_h(\mathcal{T}(E)) \leq d(E) + \liminf_{\delta\to 0} \frac{\log v(\delta)}{\log(1/\delta)}.$$

In particular, if $\eta = \gamma = 1$, we have $\dim_h(\mathcal{T}(E)) = \infty$, Θ a.e. on $\{h(\mathcal{T}) > A\}$.

This leaves open the following question. Suppose that $1 = \gamma < \eta$ (and that (50) does not hold). Can one compute $\dim_h(\mathcal{T}(E))$, or simply $\dim_h(\mathcal{T})$?

Finally, let us discuss the stable case where more precise results are available. For any suitable function g, we write $\mathfrak{H}^g$ for the associated Hausdorff measure. The following theorem is proved in [10].

Theorem 5.9 (i) *Suppose that $\psi(u) = u^2$. Set*

$$g_1(r) = r \log \log(1/r) \, , \quad g_2(r) = r^2 \log \log(1/r).$$

There exist positive constants C_1 and C_2 such that Θ a.e.

$$C_1 \, \zeta \leq \mathfrak{H}^{g_2}(\mathcal{T}) \leq C_2 \, \zeta$$

and for every $a > 0$, Θ a.e. on $\{h(\mathcal{T}) > a\}$,

$$C_1 \, \langle \ell^a, 1 \rangle \leq \mathfrak{H}^{g_1}(\mathcal{T}(a)) \leq C_2 \, \langle \ell^a, 1 \rangle.$$

(ii) *Suppose that $\psi(u) = u^\gamma$ for some $\gamma \in (1,2)$. For every $s > 0$ set*

$$h_s(r) = r^{\frac{\gamma}{\gamma-1}} \left(\log(1/r) \right)^{\frac{1}{\gamma-1}} \left(\log \log(1/r) \right)^s.$$

Then, there exists a real number ξ such that, Θ a.e.,

$$\begin{aligned}
\mathfrak{H}^{h_s}(\mathcal{T}) &= \infty && \text{if } s > \tfrac{1}{\gamma-1}, \\
\mathfrak{H}^{h_s}(\mathcal{T}) &= 0 && \text{if } s < \xi.
\end{aligned}$$

The construction of superprocesses that will be developed in the next section shows that Theorem 5.9(i) is related to the very precise estimates which have been obtained for the Hausdorff measure of super-Brownian motion (see [27], [5], [22] and the references therein).

Theorem 5.9(ii) leaves open the question of determining the correct Hausdorff measure function for $\mathcal{T}$ in the stable case.

6   Some applications to super-Brownian motion

Denote by $M_f(\mathbb{R}^k)$ the set of all finite measures on $\mathbb{R}^k$ and by $C_{b+}(\mathbb{R}^k)$ the space of all nonnegative bounded continuous functions on $\mathbb{R}^k$. We also write $(P_t)_{t \geq 0}$ for the semigroup of standard Brownian motion in $\mathbb{R}^k$. Note that for every $t \geq 0$, P_t maps $C_{b+}(\mathbb{R}^k)$ into itself.

The super-Brownian motion with branching mechanism ψ (in short the ψ-super-Brownian motion) is the (time-homogeneous) Markov process $(Z_t, t \geq 0)$ with values in $M_f(\mathbb{R}^k)$ whose transition kernels can be characterized as follows. For every $\mu \in M_f(\mathbb{R}^k)$ and $\varphi \in C_{b+}(\mathbb{R}^k)$,

$$E[\exp(-\langle Z_t, \varphi \rangle) \mid Z_0 = \mu] = \exp(-\langle \mu, u_t \rangle),$$

where the function $(u_t(x); t \geq 0, x \in \mathbb{R}^k)$ is bounded and continuous and is the unique nonnegative solution of the integral equation

$$u_t(x) + \int_0^t P_{t-s}(\psi(u_s))(x) \, dx = P_t \varphi(x).$$

We will now explain how the genealogical structure given by the tree $\mathcal{T}$ under Θ can be combined with a spatial motion to give a construction of the ψ-super-Brownian motion. To present this construction in a way suitable for applications, it is convenient to introduce the notion of a spatial tree.

Informally, a (k-dimensional) spatial tree is a pair $(\mathcal{T}, W)$ where $\mathcal{T} \in \mathbb{T}$ and W is a continuous mapping from $\mathcal{T}$ into $\mathbb{R}^k$. Since we defined $\mathbb{T}$ as a space of equivalence classes of trees, we should be a little more precise at this point. If $\mathcal{T}$ and $\mathcal{T}'$ are two (rooted compact) $\mathbb{R}$-trees and W and W' are $\mathbb{R}^k$-valued continuous mappings defined respectively on $\mathcal{T}$ and $\mathcal{T}'$, we say that the pairs $(\mathcal{T}, W)$ and $(\mathcal{T}, W')$ are equivalent if there exists a root-preserving isometry Φ from $\mathcal{T}$ onto $\mathcal{T}'$ such that $W'_{\Phi(\sigma)} = W_\sigma$ for every $\sigma \in \mathcal{T}$. A spatial tree is then defined as an equivalent class for the preceding equivalence relation, and we denote by $\mathbb{T}_{sp}$ the space of all spatial trees. Needless to say we will often abuse notation and identify a spatial tree with an element of the corresponding equivalent class.

We denote by $\mathbb{T}_{sp}$ the set of all spatial trees. Recall the notation of subsection 2.2. We define a distance on $\mathbb{T}_{sp}$ by setting

$$d_{sp}((\mathcal{T}, W), (\mathcal{T}', W')) = \frac{1}{2} \inf_{\mathcal{R} \in \mathcal{C}(\mathcal{T}, \mathcal{T}'), (\rho, \rho') \in \mathcal{R}} \left(\mathrm{dis}(\mathcal{R}) + \sup_{(\sigma, \sigma') \in \mathcal{R}} |W_\sigma - W'_{\sigma'}| \right),$$

where ρ and ρ' obviously denote the respective roots of $\mathcal{T}$ and $\mathcal{T}'$. It is easy to verify that $(\mathbb{T}_{sp}, d_{sp})$ is a Polish space.

Let us fix $x \in \mathbb{R}^k$. Also let $\mathcal{T} \in \mathbb{T}$ be a compact rooted $\mathbb{R}$-tree with root $\varnothing$ and metric d. We may consider the $\mathbb{R}^k$-valued Gaussian process $(Y_\sigma, \sigma \in \mathcal{T})$ whose distribution is characterized by

$$E[Y_\sigma] = x \ ,$$
$$\mathrm{cov}(Y_\sigma, Y_{\sigma'}) = d(\varnothing, \sigma \wedge \sigma') \, \mathrm{Id} \ ,$$

where Id denotes the k-dimensional identity matrix. Note that

$$\mathrm{cov}(Y_\sigma - Y_{\sigma'}, Y_\sigma - Y_{\sigma'}) = d(\sigma, \sigma') \, \mathrm{Id}.$$

From Theorem 11.17 in [18], we know that under the condition

$$\int_0^1 (\log \mathcal{N}(\mathcal{T}, \varepsilon^2))^{1/2} \, d\varepsilon < \infty, \tag{51}$$

the process $(Y_\sigma, \sigma \in \mathcal{T})$ has a continuous modification. We keep the notation Y for this modification. Assuming that (51) holds, we denote by $Q_\mathcal{T}^x$ the law on $\mathbb{T}_{sp}$ of $(\mathcal{T}, (Y_\sigma, \sigma \in \mathcal{T}))$.

As a consequence of Proposition 5.2, condition (51) holds $\Theta(d\mathcal{T})$ a.e. if we assume that

$$\int_0^1 (\log v(\varepsilon^2))^{1/2} \, d\varepsilon < \infty. \tag{52}$$

From now on, we assume that (52) holds (this is automatic if $\gamma > 1$, by Lemma 5.6 (i)). The definition of $Q_\mathcal{T}^x$ then makes sense $\Theta(d\mathcal{T})$ a.e., and we may set

$$\mathbb{N}_x = \int \Theta(d\mathcal{T}) \, Q_\mathcal{T}^x,$$

which defines a σ-finite measure on $\mathbb{T}_{sp}$. We leave it to the reader to verify the needed measurability properties of the mapping $\mathcal{T} \to Q_\mathcal{T}^x$.

Remark. As a consequence of Theorem 4.5.2 in [9], a necessary and sufficient condition for the existence of a continuous modification of the process $(Y_\sigma, \sigma \in \mathcal{T})$, for Θ a.e. $\mathcal{T}$, should be

$$\int_1^\infty \Big(\int_0^t \psi(u)\, du \Big)^{-1/2} dt < \infty.$$

Note that this condition is stronger than (7). The proof of Theorem 4.5.2 in [9] strongly depends on connections between super-Brownian motion and partial differential equations. Condition (52) will be sufficient for our purposes in the present work.

We can now turn to connections with superprocesses. Under the measure $\mathbb{N}_x$, we may for every $a > 0$ define a measure $\mathcal{Z}_a = \mathcal{Z}_a(\mathcal{T}, W)$ on $\mathbb{R}^k$ by setting

$$\langle \mathcal{Z}_a, \varphi \rangle = \int_0^\zeta \ell^a(d\sigma)\, \varphi(W_\sigma).$$

The next proposition reformulates a special case of Theorem 4.2.1 in [9].

Proposition 6.1 *Let $\mu \in M_f(\mathbb{R}^k)$ and let*

$$\sum_{i \in \mathcal{I}} \delta_{(\mathcal{T}^i, W^i)}$$

be a Poisson point measure on $\mathbb{T}_{sp}$ with intensity $\int \mu(dx)\, \mathbb{N}_x$. Then the process $(Z_a, a \geq 0)$ defined by

$$\begin{aligned} Z_0 &= \mu\,, \\ Z_a &= \sum_{i \in \mathcal{I}} \mathcal{Z}_a(\mathcal{T}^i, W^i)\,, \quad a > 0\,, \end{aligned}$$

is a ψ-super-Brownian motion started at μ.

In the formula for Z_a, only finitely many terms can be nonzero, simply because finitely many trees in the collection $(\mathcal{T}^i, i \in \mathcal{I})$ are such that $h(\mathcal{T}^i) > a$. From Theorem 4.3, we see that the version of Z defined in the proposition is càdlàg on $(0, \infty)$ for the weak topology on finite measures on $\mathbb{R}^k$. By the known regularity properties of superprocesses (see e.g. the more general Theorem 2.1.3 in [5]), it must indeed be càdlàg on $[0, \infty)$. The fact that we obtain the "good" version of the superprocess is a nice feature of our construction in contrast with the Lévy snake approach of [21] or [9], where regularity properties of the resulting measure-valued process were not immediately apparent.

In view of Proposition 6.1, the measures $\mathbb{N}_x$ (or rather the distribution under $\mathbb{N}_x$ of the measure-valued process $(\mathcal{Z}_a, a \geq 0)$) are called the excursion measures of the ψ-super-Brownian motion. In the quadratic branching case, these measures play an important role in the study of connections between superprocesses and partial differential equations: See in particular [19]. In the case of a general branching branching mechanism, excursion measures are constructed via the Lévy snake in Chapter 4 of [9], and a different approach has been proposed recently by Dynkin and Kuznetsov [11].

As a simple application of the representation of Proposition 6.1, we use Theorem 4.4 to extend a result due to Perkins [28] in the case of the quadratic branching mechanism.

Proposition 6.2 *Let $Z = (Z_a, a \geq 0)$ be a ψ-super-Brownian motion in $\mathbb{R}^k$, and for every $a \geq 0$ let $\mathcal{S}_a$ denote the topological support of the random measure Z_a. Then the mapping $a \longrightarrow \mathcal{S}_a$ is càdlàg from $(0, \infty)$ into the set of all compact subsets of $\mathbb{R}^k$ equipped with the Hausdorff metric. Moreover, if a is a discontinuity time of this mapping there is a point $z_a \in \mathbb{R}^k$ such that $\mathcal{S}_{a-} = \mathcal{S}_a \cup \{z_a\}$.*

Remark. If we assume that the support of Z_0 is compact, it is also easy to prove that the mapping $a \longrightarrow \mathcal{S}_a$ is right-continuous at $a = 0$ for the Hausdorff metric.

Proof. We may assume that Z is given by the formula of Proposition 6.1. For every $i \in \mathcal{I}$, let $\mathcal{E}_i$ stand for the set of extinction times of $\mathcal{T}^i$. Then each set $\mathcal{E}_i$ is countable, and by arguments similar to the proof of Theorem 4.4 it is easy to prove that the sets $\mathcal{E}_i$ are pairwise disjoint. If $a \in \mathcal{E}_i$, write σ_a^i for the extinction point of $\mathcal{T}^i$ corresponding to the extinction time a. It now follows from Theorem 4.4 that a.s. for every $a > 0$:

- $\mathcal{S}_a = \bigcup_{i \in \mathcal{I}} \{W_\sigma^i : \sigma \in \mathcal{T}^i(a)\}$ if $a \notin \bigcup_{i \in \mathcal{I}} \mathcal{E}_i$;

- $\mathcal{S}_a = \{W_\sigma^j : \sigma \in \mathcal{T}^j(a) \backslash \{\sigma_a^j\}\} \cup \bigcup_{i \in \mathcal{I} \backslash \{j\}} \{W_\sigma^i : \sigma \in \mathcal{T}^i(a)\}$ if $a \in \mathcal{E}_j$ for some $j \in \mathcal{I}$.

As a straightforward consequence of these formulas, one can now verify that the mapping $a \longrightarrow \mathcal{S}_a$ is càdlàg on $(0, \infty)$, with

$$\mathcal{S}_{a-} = \bigcup_{i \in \mathcal{I}} \{W_\sigma^i : \sigma \in \mathcal{T}^i(a)\} \quad \text{for every } a > 0.$$

Furthermore the set of discontinuity times is contained in the union of the sets $\mathcal{E}_i$ over $i \in \mathcal{I}$, and if $a \in \mathcal{E}_j$ we have

$$\mathcal{S}_{a-} = \mathcal{S}_a \cup \{W_{\sigma_a^j}^j\}.$$

We leave details to the reader.   ∎

We can also apply Theorem 4.7 in connection with our construction of superprocesses. We recover the fact that for every discontinuity time s of Z there is a positive real number λ_s and a point $\omega_s \in \mathbb{R}^k$ such that $Z_s = Z_{s-} + \lambda_s \delta_{\omega_s}$. Precisely, there is an index $i \in \mathcal{I}$ and an infinite branching point σ_s of $\mathcal{T}^i$ at height s, such that $\omega_s = W^i(\sigma_s)$ and λ_s is the local time of the infinite branching point σ_s. We omit details since the preceding fact is known to hold in great generality: See Théorème 7 in [12].

We now proceed to investigate the Hausdorff dimension of the support of Z_a. From Proposition 6.1, it is enough to consider the random measures $\mathcal{Z}_a$ under $\mathbb{N}_x$. For every $a \geq 0$, we set

$$R_a = \operatorname{supp}(\mathcal{Z}_a)$$

and, if E is a subset of $\mathbb{R}_+$,

$$R_E = \overline{\bigcup_{a \in E} \operatorname{supp}(\mathcal{Z}_a)}.$$

Theorem 6.3 *Assume that $\gamma > 1$. Let E be a compact subset of $(0, \infty)$ whose Hausdorff dimension and upper box dimension are equal to $d(E) \in [0, 1]$, and set $A = \sup E$. Then, we have*

$$\dim_h R_E = \underline{\dim} R_E = \left(2d(E) + \frac{2}{\eta - 1}\right) \wedge k \ ,$$

$\mathbb{N}_0$ *a.e. on* $\{\mathcal{Z}_A \neq 0\}$.

We first state a simple continuity lemma.

Lemma 6.4 *Let $T \in \mathcal{T}$ be such that*

$$\limsup_{\varepsilon \to 0} \frac{\log \mathcal{N}(\mathcal{T}, \varepsilon)}{\log 1/\varepsilon} < \infty.$$

Then $Q_{\mathcal{T}}^x$ a.s., the mapping $\sigma \to W_\sigma$ is Hölder continuous with exponent $\frac{1}{2} - \delta$ for any $\delta \in (0, \frac{1}{2})$.

Proof. Standard chaining arguments show that, for every integer $m \geq 1$ and every $u > 0$,

$$Q_{\mathcal{T}}^x\left(\sup_{d(\sigma,\sigma')<u} |W_\sigma - W_{\sigma'}|\right) \ \leq \ k\Big(u^{1/2}\sqrt{\log(1 + \mathcal{N}(\mathcal{T}, 2^{-2m})^2)}$$

$$+ 16 \sum_{p=m+1}^{\infty} 2^{-p}\sqrt{\log(1 + \mathcal{N}(\mathcal{T}, 2^{-2p}))}\Big).$$

See e.g. formula (11.6) in [18] and note that a correct choice of the distance on $\mathcal{T}$ in order to apply this formula is $d'(\sigma, \sigma') = 2\sqrt{d(\sigma, \sigma')}$ (see the comments on page 320 of [18]).

From the assumption of the lemma we now get the existence of a constant $C(\mathcal{T})$ such that, for every $m \geq 1$ and $u \in (0, 1)$,

$$Q_{\mathcal{T}}^x\left(\sup_{d(\sigma,\sigma')<u} |W_\sigma - W_{\sigma'}|\right) \leq k\, C(\mathcal{T})\, (u^{1/2}m^{1/2} + m2^{-m}).$$

Choosing m so that $u^{-2r} < m \leq 2u^{-2r}$ we get, for every $r \in (0, 1/2)$,

$$Q_{\mathcal{T}}^x\left(\sup_{d(\sigma,\sigma')<u} |W_\sigma - W_{\sigma'}|\right) \leq C'(r, k, \mathcal{T})\, u^{\frac{1}{2}-r}.$$

An application of the Borel-Cantelli lemma now completes the proof. ∎

Proof of Theorem 6.3. From the support properties of the measures ℓ^a, we have $\mathbb{N}_x$ a.e. for every $a > 0$,

$$\operatorname{supp} \mathcal{Z}_a \subset \{W_\sigma : \sigma \in \mathcal{T}(a)\}.$$

Therefore,

$$R_E \subset \{W_\sigma : \sigma \in \mathcal{T}(E)\}$$

(note that $\{W_\sigma : \sigma \in \mathcal{T}(E)\}$ is closed as the image of the compact set $\mathcal{T}(E)$ under the continuous mapping $\sigma \to W_\sigma$). The upper bound

$$\underline{\dim} R_E \leq \left(2d(E) + \frac{2}{\eta - 1}\right)$$

is then an immediate consequence of Theorem 5.5 and Lemma 6.4. Note that the assumption of Lemma 6.4 holds $\Theta(d\mathcal{T})$ a.e. by Proposition 5.2 and Lemma 5.6.

Since the bound $\dim_h R_E \leq \underline{\dim} R_E$ is always true, the proof of Theorem 6.3 will be complete if we can verify that

$$\dim_h R_E \geq \left(2d(E) + \frac{2}{\eta - 1}\right) \wedge k, \tag{53}$$

$\mathbb{N}_x$ a.e. on $\{\mathcal{Z}_A \neq 0\}$. To this end, let $\varepsilon > 0$ and

$$b(\varepsilon) = d(E) - 2\varepsilon + \frac{1}{\eta - 1 + \varepsilon}.$$

As in the proof of Theorem 5.5 we can consider a finite measure ν supported on E such that, if κ denotes the measure

$$\kappa(d\sigma) = \int \nu(da)\, \ell^a(d\sigma),$$

we have

$$\limsup_{\delta \to 0} \kappa\left(B(\sigma, \delta)\right) \delta^{-b(\varepsilon)} < \infty, \qquad \kappa(d\sigma) \text{ a.e.}$$

Furthermore κ is nonzero Θ a.e. on $\{h(\mathcal{T}) > A\} = \{\mathcal{Z}_A \neq 0\}$. Notice that the measure M defined as the image of κ under the mapping $\sigma \to W_\sigma$ is supported on R_E, simply because

$$M = \int \nu(da)\, \mathcal{Z}_a$$

and ν is supported on E.

Now, for any positive integer q, set

$$F_q = \{\sigma \in \mathcal{T}(E) \ : \ \forall \delta \in (0, 1/q]\,, \ \kappa(B(\sigma, \delta)) \leq q\delta^{b(\varepsilon)}\}$$

On the event $\{h(\mathcal{T}) > A\}$, we can find q_0 such that $\kappa(F_{q_0}) > 0$. We denote by $\widetilde{\kappa}$ the restriction of κ to F_{q_0}. It is then immediate to verify that, for any $b < b(\varepsilon)$,

$$\int \widetilde{\kappa}(d\sigma) \int \widetilde{\kappa}(d\sigma') d(\sigma, \sigma')^{-b} < \infty. \tag{54}$$

Finally, if $\widetilde{M}$ denotes the image of $\widetilde{\kappa}$ under $\sigma \to W_\sigma$, we have for any $b < k/2 \wedge b(\varepsilon)$

$$\begin{aligned}
Q_{\mathcal{T}}^x \left(\int \widetilde{M}(dz) \int \widetilde{M}(dy)|z - y|^{-2b}\right) &= Q_{\mathcal{T}}^x \left(\int \widetilde{\kappa}(d\sigma) \int \widetilde{\kappa}(d\sigma')|W_\sigma - W_{\sigma'}|^{-2b}\right) \\
&= \int \widetilde{\kappa}(d\sigma) \int \widetilde{\kappa}(d\sigma') Q_{\mathcal{T}}^x \left(|W_\sigma - W_{\sigma'}|^{-2b}\right) \\
&= C \int \widetilde{\kappa}(d\sigma) \int \widetilde{\kappa}(d\sigma') d(\sigma, \sigma')^{-b},
\end{aligned}$$

where C is a finite constant. The latter integral is finite by (54). Therefore, $\Theta(d\mathcal{T})$ a.e. on $\{h(\mathcal{T}) > A\}$ we have $Q_{\mathcal{T}}^x$ a.s.

$$\int \widetilde{M}(dz) \int \widetilde{M}(dy)|z - y|^{-2b} < \infty.$$

Furthermore $\widetilde{M}$ is supported on R_E because $\widetilde{M} \leq M$. Frostman's lemma now yields the desired result (53). $\blacksquare$

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