The Hausdorff measure of stable trees

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Abstract

We study fine properties of the so-called stable trees, which are the scaling limits of critical Galton-Watson trees conditioned to be large. In particular we derive the exact Hausdorff measure function for Aldous' continuum random tree and for its level sets. It follows that both the uniform measure on the tree and the local time measure on a level set coincide with certain Hausdorff measures. Slightly less precise results are obtained for the Hausdorff measure of general stable trees.

1 Introduction

The purpose of the present work is to study the Hausdorff measure properties of the continuous random trees called stable trees. Roughly speaking, stable trees are the continuous random trees that arise as scaling limits of Galton-Watson trees with critical offspring distribution, which are conditioned to be large in some sense. In the most important case where the offspring distribution also has a finite variance, this leads to Aldous' continuum random tree (the CRT, see [1] and [2]) and variants of the CRT. Alternatively, stable trees can be viewed as describing the genealogical structure of continuous-state branching processes with a stable branching mechanism of the type $\psi(u) = u^\alpha$ for $1 < \alpha \leq 2$. Thus they also encode the genealogy of superprocesses with stable branching mechanism, which have been studied by many authors. The case $\alpha = 2$ yields the so-called quadratic branching mechanism, corresponding to finite variance superprocesses.

Stable trees are particular instances of the more general Lévy trees studied in [7]. In the formalism of [7], Lévy trees are random variables taking values in the space of all (compact) rooted $\mathbb{R}$-trees. Informally an $\mathbb{R}$-tree is a metric space $(\mathcal{T}, d)$ such that for any two points σ and σ' in $\mathcal{T}$ there is a unique arc with endpoints σ and σ' and furthermore this arc is isometric to a compact interval of the real line. A rooted $\mathbb{R}$-tree is an $\mathbb{R}$-tree with a distinguished vertex ρ called the root. We write $H(\mathcal{T})$ for the height of $\mathcal{T}$, that is the maximal distance from the root to a vertex in $\mathcal{T}$. Two rooted $\mathbb{R}$-trees are called equivalent if there is a root-preserving isometry that maps one onto the other. It was noted in [8] that the set $\mathbb{T}$ of equivalence classes of compact rooted $\mathbb{R}$-trees, equipped with the Gromov-Hausdorff distance [9], is a Polish space.

It is shown in [7] that with every critical or subcritical branching mechanism function ψ such that the corresponding branching process dies out a.s. one can associate a σ-finite measure Θ

on $\mathbb{T}$ which is called the "law" of the Lévy tree with branching mechanism ψ. Although Θ is an infinite measure, the quantity

$$v(\varepsilon) = \Theta(H(\mathcal{T}) > \varepsilon)$$

is finite for every $\varepsilon > 0$ and is determined by the equation $\int_{v(\varepsilon)}^{\infty} \psi(u)^{-1} du = \varepsilon$. Lévy trees enjoy the important "branching property", which is analogous to a classical result for Galton-Watson trees: For every $a > 0$, under the probability measure $\Theta(\cdot \mid H(\mathcal{T}) > a)$ and conditionally given the part of the tree below level a, the subtrees above that level are distributed as the atoms of a Poisson point measure whose intensity is a random multiple of Θ (the random factor is the total mass of the local time measure at level a that will be discussed below). It has recently been shown by Weill [19] that this branching property characterizes Lévy trees.

When $\psi(u) = u^{\alpha}$ for some $\alpha \in (1, 2]$ we write $\Theta_\alpha = \Theta$ and call Θ_α the law of the stable tree with index α. In addition to the branching property, stable trees possess the following scaling property. For every $r > 0$ and every tree $\mathcal{T} \in \mathbb{T}$, denote by $r\mathcal{T}$ the "same" tree $\mathcal{T}$ with metric d replaced by $r\,d$. Then, for every $r > 0$, the law of $r\mathcal{T}$ under $\Theta_\alpha(d\mathcal{T})$ is $r^{\frac{1}{\alpha-1}}\Theta_\alpha$.

An explicit construction of Θ_α may be given through the coding of real trees from the height process studied in [14] and [6] (see also Theorem 2.1 of [7] for the coding of real trees). This construction is especially simple in the case $\alpha = 2$, since the height process is then just a Brownian excursion, and this approach essentially reduces to Aldous' construction of the CRT from the normalized Brownian excursion (Corollary 22 in [2]). Alternatively, we may use the following approximation by discrete trees. Let π be a probability distribution on $\{0, 1, \ldots\}$. Assume that π has mean 1 and is in the domain of attraction of a stable distribution with index α, in the sense that there exists an increasing sequence $(a_n)_{n=1,2,\ldots}$ of positive integers such that, if $\xi_1, \xi_2, \ldots$ are i.i.d. with distribution π, $(a_n)^{-1}(\xi_1 + \cdots + \xi_n - n)$ converges in distribution to a stable distribution with index α. Let $c > 0$ be a constant and for every $n \geq 1$ let θ_n be a Galton-Watson tree with offspring distribution π conditioned to have height greater than cn. Notice that θ_n can be viewed as a random $\mathbb{R}$-tree by affecting length 1 to each edge. Then the distribution of $n^{-1}\theta_n$ converges as $n \to \infty$ to the probability measure $\Theta_\alpha(\cdot \mid H(\mathcal{T}) > c)$. This result follows from a special case of Proposition 2.5.2 in [7]. See also Aldous [2] and Duquesne [5] for related statements.

Before stating our main results, we still need to introduce important random measures associated with stable trees. For every $a > 0$, we can define $\Theta_\alpha(d\mathcal{T})$ a.e. a random measure ℓ^a on the level set $\mathcal{T}(a) := \{\sigma \in \mathcal{T} : d(\rho, \sigma) = a\}$, which is in a sense uniformly spread over that level set: For every $\varepsilon > 0$, write $\mathcal{T}_\varepsilon(a)$ for the finite subset of $\mathcal{T}(a)$ consisting of those vertices which have descendants at level $a + \varepsilon$, then for every bounded continuous function φ on $\mathcal{T}$,

$$\langle \ell^a, \varphi \rangle = \lim_{\varepsilon \downarrow 0} \frac{1}{v(\varepsilon)} \sum_{\sigma \in \mathcal{T}_\varepsilon(a)} \varphi(\sigma).$$

We refer to Section 4.2 of [7] for the construction and main properties of these "local time" measures. The uniform measure $\mathbf{m} = \mathbf{m}_{(\mathcal{T})}$ on the tree $\mathcal{T}$ is then defined by

$$\mathbf{m} = \int_0^\infty da\, \ell^a \,. \tag{1}$$

We start with the case $\alpha = 2$ where we can identify the exact Hausdorff measure function for the tree $\mathcal{T}$ and its level sets. The notation $h - m$ stands for the Hausdorff measure associated with the function h.

Theorem 1.1 *For every $r \in (0, 1/2)$, set*

$$h(r) = r^2 \log\log \frac{1}{r}.$$

There exists a positive constant C_0 such that Θ_2 a.e., for every Borel subset A of $\mathcal{T}$,

$$h - m(A) = C_0\, \mathbf{m}(A).$$

According to this theorem, the measure $\mathbf{m}$ coincides with a certain Hausdorff measure on $\mathcal{T}$. This justifies the fact that $\mathbf{m}$ is called the *uniform* measure on the tree.

The law $\Theta_2^{(1)}$ of the CRT is informally defined by $\Theta_2^{(1)} = \Theta_2(d\mathcal{T} \mid \mathbf{m}(\mathcal{T}) = 1)$. More precisely, the CRT is coded by a Brownian excursion conditioned to have duration 1 (in the sense explained below in Section 3), whereas Θ_2 is the law of the tree coded by a Brownian excursion under the Itô measure. Since the excursion normalized to have duration 1 and the Itô measure are related by simple scaling transformations, the following corollary is an immediate consequence of Theorem 1.1.

Corollary 1.2 *Theorem 1.1 remains valid, with the same constant C_0, if Θ_2 is replaced by the law $\Theta_2^{(1)}$ of the CRT.*

Let us now discuss level sets. The next theorem shows that the local time measure ℓ^a coincides with a certain Hausdorff measure on the level set $\mathcal{T}(a)$.

Theorem 1.3 *For every $r \in (0, 1/2)$, set*

$$\widetilde{h}(r) = r \log\log \frac{1}{r}.$$

There exists a positive constant $\widetilde{C}_0$ such that for every $a > 0$, one has Θ_2 a.e. for every Borel subset A of $\mathcal{T}(a)$,

$$\widetilde{h} - m(A) = \widetilde{C}_0\, \ell^a(A).$$

When $1 < \alpha < 2$, we are unable to identify an exact Hausdorff measure function for the tree, but we still get rather precise information.

Theorem 1.4 *Suppose that $1 < \alpha < 2$. For every $u \in \mathbb{R}$ and $r \in (0, e^{-1})$, set*

$$h_u(r) = r^{\frac{\alpha}{\alpha-1}} (\log \frac{1}{r})^{\frac{1}{\alpha-1}} (\log\log \frac{1}{r})^u.$$

Then,

(i) $h_u - m(\mathcal{T}) = \infty$ *if* $u > \frac{1}{\alpha-1}$, Θ_α *a.e.*

(ii) $h_u - m(\mathcal{T}) = 0$ *if* $u < 0$, Θ_α *a.e.*

The preceding results were announced, in a less precise form, in Theorem 5.9 of [7]. Finally, we also have an analogue of Theorem 1.3 in the stable case.

Theorem 1.5 *Suppose that* $1 < \alpha < 2$ *and let* $a > 0$. *For every* $u \in \mathbb{R}$ *and* $r \in (0, e^{-1})$, *set*

$$\widetilde{h}_u(r) = r^{\frac{1}{\alpha-1}} (\log \frac{1}{r})^{\frac{1}{\alpha-1}} (\log\log \frac{1}{r})^u.$$

Then,

(i) $\widetilde{h}_u - m(\mathcal{T}(a)) = \infty$ *if* $u > \frac{1}{\alpha-1}$, Θ_α *a.e. on* $\{H(\mathcal{T}) > a\}$.

(ii) $\widetilde{h}_u - m(\mathcal{T}(a)) = 0$ *if* $u < 0$, Θ_α *a.e.*

Let us briefly comment on the relation between these theorems and earlier results. The Hausdorff dimension of stable trees was computed independently in [7] and in [10]. It is remarkable that the exact Hausdorff measure function of the tree under Θ_2 (or of the CRT) is the same as the one for a transient Brownian path, which was derived by Ciesielski and Taylor [3] following earlier work of Lévy. As we will see, some results from [3] play a role in the proof of Theorem 1.1. The preceding theorems are also reminiscent of the very precise results about the Hausdorff measure of the support and range of super-Brownian motion, which have been obtained by Perkins and his co-authors (see [16], [4], [15] and references therein). This should not come as a surprise since superprocesses with a stable branching mechanism are easily constructed by combining the genealogical structure of stable trees with independent spatial motions (see e.g. Proposition 6.1 in [7]).

The paper is organized as follows. Section 2 gives the basic comparison results for Hausdorff measures that are used in the proofs. Section 3 contains the proof of Theorems 1.1 and 1.3. Here we rely on the coding of trees by Brownian excursions, which has been exploited in other contexts, and in particular in the Brownian snake approach to superprocesses [13]. Section 4 gives a few preliminary results about stable trees, which are used in Section 5 to prove Theorems 1.4 and 1.5. In contrast with Section 3, we rely on general properties of Lévy trees that have been derived in [7], and in particular on the subtree decomposition along the ancestral line of a typical vertex (Theorem 4.2 below). Section 5 also formulates conjectures for the exact Hausdorff measure of stable trees and their level sets.

2 Comparison results for Hausdorff measures

In this section, we give a comparison result for Hausdorff measures that will be used in the proofs below. For subsets of Euclidean space, this result can be found as Lemmas 2 and 3 of Rogers and Taylor [18] (see also Theorem 1.4 in Perkins [16] for a more precise formulation).

For the reader's convenience, and also because the arguments of [18] do not extend immediately to the general setting which is considered here, we provide a short proof below.

We consider a compact metric space E. For every $x \in E$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x with radius r. If $c > 1$ is fixed, we let $\mathcal{H}_c$ be the set of all monotone increasing continuous functions $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$ and $g(2r) \leq cg(r)$ for every $r \geq 0$. As in the introduction, $g - m$ stands for the Hausdorff measure associated with g. For any subset A of E,

$$g - m(A) = \lim_{\varepsilon \downarrow 0} \Big(\inf_{(U_i)_{i\in I} \in \mathcal{V}_\varepsilon(A)} \sum_{i\in I} g(\mathrm{diam}(U_i)) \Big), \tag{2}$$

where $\mathcal{V}_\varepsilon(A)$ is the collection of all countable coverings of A by subsets of E with diameter less than ε, and $\mathrm{diam}(U)$ denotes the diameter of U.

Lemma 2.1 *Let $c > 0$. There exist two positive constants M_1 and M_2 that depend only on c, such that the following holds for every function $g \in \mathcal{H}_c$. Let μ be a finite Borel measure on E and let A be a Borel subset of E.*

(i) *If*

$$\limsup_{n\to\infty} \frac{\mu(B(x,2^{-n}))}{g(2^{-n})} \leq 1 \ , \ \textit{for every } x \in A,$$

then,

$$g - m(A) \geq M_1\, \mu(A).$$

(ii) *If*

$$\limsup_{n\to\infty} \frac{\mu(B(x,2^{-n}))}{g(2^{-n})} \geq 1 \ , \ \textit{for every } x \in A,$$

then,

$$g - m(A) \leq M_2\, \mu(A).$$

Proof: (i) For every integer $n \geq 1$, set

$$A_n := \{x \in A : \mu(B(x,2^{-k})) \leq 2\, g(2^{-k}) \text{ for every } k \geq n\}.$$

By assumption, $A = \lim \uparrow A_n$ and so $\mu(A) = \lim \uparrow \mu(A_n)$. Now fix $n \geq 1$ and consider a countable covering $(U_i)_{i\in I}$ of A_n by sets of positive diameter strictly less than 2^{-n}. For every $i \in I$, denote by $r_i > 0$ the diameter of U_i and pick $x_i \in U_i \cap A_n$. Let $k_i \geq n$ be the unique integer such that $2^{-k_i-1} \leq r_i < 2^{-k_i}$. Then, for every $i \in I$, we have

$$U_i \cap A_n \subset \bar{B}(x_i, r_i) \subset B(x_i, 2^{-k_i}).$$

Recalling the definition of A_n it follows that

$$\sum_{i\in I} g(r_i) \geq c^{-1} \sum_{i\in I} g(2^{-k_i}) \geq (2c)^{-1} \sum_{i\in I} \mu(B(x_i, 2^{-k_i})) \geq (2c)^{-1} \mu(A_n)$$

since the balls $B(x_i, 2^{-k_i})$ cover A_n. From the definition of Hausdorff measure we now get $g-m(A) \geq g-m(A_n) \geq (2c)^{-1}\mu(A_n)$ and the desired result follows by letting $n \uparrow \infty$.

(ii) From the general theory of Hausdorff measures (cf Corollary 2, p.99 in [17]), we know that

$$g-m(A) = \sup\{g-m(K) : K \subset A,\ K \text{ compact}\}.$$

Hence we may assume in the proof that A is compact.

Then let $\varepsilon > 0$. By assumption, for every $x \in A$, we may find $r_x \in (0, \varepsilon/8)$ such that

$$\mu(B(x, r_x)) \geq \frac{1}{2}\, g(r_x).$$

By compactness, we may then find $x_1, \ldots, x_n \in A$, such that

$$A \subset \bigcup_{i=1}^{n} B(x_i, r_{x_i})$$

and we may assume that $r_{x_1} \geq r_{x_2} \geq \cdots \geq r_{x_n}$. We can then construct a finite subset $1 = m_1 < m_2 < \cdots < m_\ell$ of $\{1, 2, \ldots, n\}$ in such a way that if $y_j = x_{m_j}$ we have

$$A \subset \bigcup_{j=1}^{\ell} B(y_j, 4r_{y_j})$$

and the balls $B(y_j, r_{y_j})$ and $B(y_{j'}, r_{y_{j'}})$ are disjoint if $j \neq j'$. In fact we start with $m_1 = 1$, and we proceed by induction. Suppose that we have constructed $m_1 < m_2 < \cdots < m_{p-1}$ in such a way that

$$\bigcup_{i=1}^{m_{p-1}} B(x_i, r_{x_i}) \subset \bigcup_{j=1}^{p-1} B(y_j, 4r_{y_j}).$$

and the balls $B(y_j, r_{y_j})$, $1 \leq j \leq p-1$ are disjoint. If

$$A \subset \bigcup_{i=1}^{p-1} B(y_j, 4r_{y_j})$$

then the construction is complete. Otherwise we let $k > m_{p-1}$ be the first integer such that $B(x_k, r_{x_k})$ is not contained in the union of the balls $B(y_j, 4r_{y_j})$ for $j \leq p-1$, and we put $m_p = k$. Plainly, $B(y_p, r_{y_p}) \cap B(y_q, r_{y_q}) = \varnothing$ if $1 \leq q \leq p-1$, because otherwise this would contradict the fact that the ball $B(y_p, r_{y_p}) = B(x_k, r_{x_k})$ contains a point that does not belong to $B(y_q, 4r_{y_q})$. This completes the construction by induction.

Now the balls $B(y_j, 4r_{y_j})$ provide a covering of A by sets of diameter less than ε, and

$$\sum_{j=1}^{\ell} g(8r_{y_j}) \leq c^3 \sum_{j=1}^{\ell} g(r_{y_j}) \leq 2c^3 \sum_{j=1}^{\ell} \mu(B(y_j, r_{y_j})) \leq 2c^3\, \mu(A_\varepsilon)$$

where A_ε stands for the ε-neighborhood of A. Let ε go to 0 to get the desired result. □

3 The Brownian tree

In this section, we prove Theorem 1.1 and Theorem 1.3. We will make an extensive use of the coding by Brownian excursions. Denote by $\mathbf{n}(de)$ the Itô measure of positive excursions of linear Brownian motion normalized so that $\mathbf{n}(\sup e > \varepsilon) = \varepsilon^{-1}$, and by $\zeta = \zeta(e)$ the duration of excursion e. For every $s, t \in [0, \zeta]$, we set

$$d_e(s,t) = e(s) + e(t) - 2\, m_e(s,t)$$

where

$$m_e(s,t) = \inf_{s \wedge t \leq r \leq s \vee t} e(r).$$

We define an equivalence relation on $[0, \zeta]$ by setting $s \sim t$ if $d_e(s,t) = 0$. Then the quotient set $\mathcal{T}_e := [0,\zeta]/\sim$ equipped with the metric d_e is a random real tree ([7] Theorem 2.1), whose root is by convention the equivalence class of 0, and the distribution of $\mathcal{T}_e$ under $\mathbf{n}(de)$ is Θ_2. Furthermore, up to an unimportant multiplicative factor 2 which we will ignore, the uniform measure $\mathbf{m}$ on $\mathcal{T}_e$ is just the image of Lebesgue measure on $[0, \zeta]$ under the canonical projection from $[0,\zeta]$ onto $[0,\zeta]/\sim$, and similarly the local time measure ℓ^a is the image of the usual Brownian local time measure at level a. Therefore in proving Theorem 1.1 and Theorem 1.3, we may and will deal with the tree $\mathcal{T}_e$ under $\mathbf{n}(de)$.

Proof of Theorem 1.1. We first establish the existence of two positive constants c_1 and c_2 such that, $\mathbf{n}(de)$ a.e. for every Borel subset A of $\mathcal{T}_e$,

$$c_1\, \mathbf{m}(A) \leq h - m(A) \leq c_2\, \mathbf{m}(A). \tag{3}$$

Lower bound. By abuse of notation we will often identify an element s of $[0,\zeta]$ with its equivalence class in $\mathcal{T}_e = [0,\zeta]/\sim$. We first prove that, $\mathbf{n}(de)$ a.e., for $\mathbf{m}$-almost all $s \in \mathcal{T}_e$, one has

$$\limsup_{\varepsilon \to 0} \frac{\mathbf{m}(\{t \in \mathcal{T}_e : d_e(s,t) \leq \varepsilon\})}{h(\varepsilon)} \leq C_1 \tag{4}$$

for some finite constant C_1.

To prove (4), we need a simple decomposition lemma for the Brownian excursion. Assume that, on a certain probability space, we are given two processes $(B_t, t \geq 0)$ and $(B'_t, t \geq 0)$ and for every $a \geq 0$ a probability measure Π_a such that B and B' are under Π_a two independent Brownian motions started at a. Also set

$$T = \inf\{t \geq 0 : B_t = 0\}\ ,\ T' = \inf\{t \geq 0 : B'_t = 0\}$$

and write $C(\mathbb{R}_+, \mathbb{R})$ for the space of all continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$.

Lemma 3.1 *For every nonnegative measurable function F on $C(\mathbb{R}_+,\mathbb{R})^2$,*

$$\int \mathbf{n}(de) \int_0^\zeta ds F((e(s+t))_{t\geq 0}, (e((s-t)_+))_{t \geq 0}) = 2 \int_0^\infty da\, \Pi_a[F((B_{t\wedge T})_{t\geq 0}, (B'_{t \wedge T'})_{t\geq 0})].$$

This is basically Bismut's decomposition of the Brownian excursion. See [12], Lemma 1 for a simple proof (notice that our normalization of Itô's measure differs by a factor 2 from the one in [12]).

Note that by the definition of the distance d_e, and the preceding identification of $\mathbf{m}$,

$$\mathbf{m}(\{t \in \mathcal{T}_e : d_e(s,t) \le \varepsilon\}) = \int_0^s dt\, 1_{\{e(s)+e(t)-2m_e(s,t)\le\varepsilon\}} + \int_s^\zeta dt\, 1_{\{e(s)+e(t)-2m_e(s,t)\le\varepsilon\}}.$$

From Lemma 3.1, we see that our claim (4) will follow if we can prove that for every $a > 0$,

$$\limsup_{\varepsilon\to 0} \frac{\int_0^\infty dt\, 1_{\{a+B_t-2I_t\le\varepsilon\}} + \int_0^\infty dt\, 1_{\{a+B'_t-2I'_t\le\varepsilon\}}}{h(\varepsilon)} \le C_1, \qquad \Pi_a \text{ a.s.} \tag{5}$$

where

$$I_t = \inf_{s\le t} B_s \ , \ I'_t = \inf_{s\le t} B'_s.$$

By translation invariance, it is enough to consider the case $a = 0$ in (5). A famous theorem of Pitman states that the process $R_t := B_t - 2I_t$ is under Π_0 a three-dimensional Bessel process started at 0, that is, it has the same distribution as the modulus of a three-dimensional Brownian motion started from the origin. From estimates due to Ciesielski and Taylor [3], there exists a finite constant C_2 such that

$$\limsup_{\varepsilon\to 0} \frac{\int_0^\infty dt\, 1_{\{R_t\le\varepsilon\}}}{h(\varepsilon)} = C_2, \qquad \Pi_0 \text{ a.s.}$$

From this and the analogous statement for $R'_t := B'_t - 2I'_t$, we deduce (5), which completes the proof of (4). The lower bound in (3) then follows from Lemma 2.1 (i).

Upper bound. From Lemma 2.1 (ii), the upper bound in (3) will follow if we can prove the existence of a constant $K_1 > 0$ such that, $\mathbf{n}(de)$ a.e.,

$$h - m(\{s \in \mathcal{T}_e : \limsup_{\varepsilon\to 0} \frac{\mathbf{m}(\{t \in \mathcal{T}_e : d_e(s,t) \le \varepsilon\})}{h(\varepsilon)} \le K_1\}) = 0. \tag{6}$$

For every integer $n \ge 0$, set $\varepsilon_n = 2^{-n}$. We will prove the existence of a constant K_2 such that, for every integer $n_0 \ge 0$, $\mathbf{n}(de)$ a.e.,

$$h - m(\{s \in \mathcal{T}_e : e(s) > 2^{-n_0} \text{ and } \mathbf{m}(\{t \in \mathcal{T}_e : d_e(s,t) \le \varepsilon_p\}) \le K_2 h(\varepsilon_p), \forall p \ge n_0\}) = 0. \tag{7}$$

Clearly, (6) follows from (7). To prove (7), we will need to introduce suitable coverings of the sets

$$F_{n_0}^A := \{s \in \mathcal{T}_e : 2^{-n_0} < e(s) < A \text{ and } \mathbf{m}(\{t \in \mathcal{T}_e : d_e(s,t) \le \varepsilon_p\}) \le K_2 h(\varepsilon_p), \forall p \ge n_0\},$$

where A is a positive integer. For every $n \ge 0$, consider the sequence of stopping times defined inductively as follows

$$T_0^n = 0,\ T_1^n = \inf\{s \ge 0 : e(s) = 2^{-n}\},\ T_{k+1}^n = \inf\{s \ge T_k^n : |e(s) - e(T_k^n)| = 2^{-n}\},$$

where $\inf\varnothing=\infty$ by convention. The sequence $(2^n e(T^n_k)1_{\{T^n_k<\infty\}}, k\geq 0)$ is distributed under $\mathbf{n}(de \mid T^n_1<\infty)$ as a positive excursion of simple random walk. In particular, for every integer $j\geq 1$,

$$\mathbf{n}\Big(\sum_{k=0}^{\infty} 1_{\{T^n_k<\infty,\, e(T^n_k)=j2^{-n}\}}\Big) = 2\,\mathbf{n}(T^n_1<\infty) = 2^{n+1}. \tag{8}$$

Let $s\in F^A_{n_0}$ and $n\geq n_0+2$. There exists a unique integer $k>0$ such that $s\in[T^n_k,T^n_{k+1})$. From our definitions, we have then

$$d_e(T^n_k,s)\leq 3\cdot 2^{-n}.$$

As a consequence, for every $p\in\{n_0,n_0+1,\ldots,n-2\}$, we have

$$\{t\in\mathcal{T}_e : d_e(s,t)\leq\varepsilon_p\}\supset\{t\in\mathcal{T}_e : d_e(T^n_k,t)\leq\varepsilon_p/4\}.$$

It follows that

$$F^A_{n_0}\subset\bigcup_{k\in I_{n_0,n}}[T^n_k,T^n_{k+1}) \tag{9}$$

where

$$I_{n_0,n}=\{k\geq 0 : T^n_k<\infty,\ 2^{-n_0}\leq e(T^n_k)\leq A \\ \text{and } \int_{T^n_k}^{\zeta} dt\,1_{\{d_e(T^n_k,t)\leq\varepsilon_p/4\}}\leq K_2 h(\varepsilon_p),\ \forall p\in\{n_0,\ldots,n-2\}\}.$$

To bound the cardinality $\#I_{n_0,n}$ of the set $I_{n_0,n}$, we use the strong Markov property under the excursion measure to write, for every $k\geq 1$,

$$\mathbf{n}(k\in I_{n_0,n})=\mathbf{n}\Big(\mathbf{1}_{\{T^n_k<\infty,\,2^{-n_0}\leq e(T^n_k)\leq A\}} \\ \times\Pi_{e(T^n_k)}\Big[\int_0^T dt\,1_{\{e(T^n_k)+B_t-2I_t\leq\varepsilon_p/4\}}\leq K_2h(\varepsilon_p),\ \forall p\in\{n_0,\ldots,n-2\}\Big]\Big).$$

Using again Pitman's theorem recalled above, we have for every $a\geq 2^{-n_0}$,

$$\Pi_a\Big[\int_0^T dt\,1_{\{a+B_t-2I_t\leq\varepsilon_p/4\}}\leq K_2h(\varepsilon_p),\ \forall p\in\{n_0,\ldots,n-2\}\Big] \\ =\Pi_0\Big[\int_0^\infty dt\,1_{\{R_t\leq\varepsilon_p/4\}}\leq K_2h(\varepsilon_p),\ \forall p\in\{n_0,\ldots,n-2\}\Big]$$

where R is under Π_0 a three-dimensional Bessel process started at 0. It follows from Theorem 1.2 in [11] that, provided n_0 is large enough, we can choose K_2 sufficiently small so that the last probability is bounded above by

$$\exp(-c(n-n_0)^{1/2})$$

for some positive constant c. Hence,

$$\mathbf{n}(k\in I_{n_0,n})\leq\exp(-c(n-n_0)^{1/2})\,\mathbf{n}(T^n_k<\infty,\ 2^{-n_0}\leq e(T^n_k)\leq A)$$

and by summing over k, and using (8)

$$\mathbf{n}(|I_{n_0,n}|) \leq \exp(-c(n-n_0)^{1/2})\, A\, 2^{2n+1}.$$

In particular, by Fatou's lemma,

$$\liminf_{n\to\infty} 2^{-2n}\, \exp(c(n-n_0)^{1/2})\, |I_{n_0,n}| < \infty$$

$\mathbf{n}(de)$ a.e. Now recall (9) and note that the diameter (with respect to the distance d_e) of each interval $[T^n_k, T^n_{k+1}]$ is bounded above by $4\,2^{-n}$. Our claim (7) then follows from the definition of Hausdorff measures. This completes the proof of (3).

Theorem 1.1 can be deduced from the bounds (3) and an appropriate zero-one law. This is similar to the argument used in Section 7 of [15], but there are some differences.

Let us write p_e for the canonical projection from $[0,\zeta]$ onto $\mathcal{T}_e = [0,\zeta]/\sim$. We first observe that, for every $0 \leq s \leq t \leq \zeta$, the quantity $h-m(p_e([s,t]))$ is a mesurable function of e. To see this, note that in the definition (2) of $h-m(p_e([s,t]))$, we may restrict our attention to finite coverings with balls (use compactness and the fact that any subset of a real tree is contained in a closed ball with the same diameter). Moreover, it is enough to consider balls with rational diameter, and with a center of the form $p_e(r)$ for some rational number $r \in [0,\zeta]$. The desired measurability property then follows easily.

We then define a finite measure ν on $[0,\zeta]$ by setting, for every $t \in [0,\zeta]$,

$$\nu([0,t]) = h-m(p_e([0,t])).$$

Plainly, the mapping $t \longrightarrow h-m(p_e([0,t]))$ is continuous and so ν is nonatomic. Then we have also, for every $0 \leq s \leq t \leq \zeta$,

$$\nu([s,t]) = h-m(p_e([s,t])).$$

Indeed, this is a consequence of the following observation: If $0 \leq u < v \leq s < t \leq \zeta$, the set $p_e([u,v]) \cap p_e([s,t])$ is contained in the ancestral line of $p_e(s)$, and so we must have

$$h-m(p_e([u,v]) \cap p_e([s,t])) = 0.$$

Since $\mathbf{m}$ is obtained as the image of Lebesgue measure under p_e, it is easy to verify that $\mathbf{m}(p_e([s,t])) = t-s$ for every $0 \leq s \leq t \leq \zeta$, $\mathbf{n}(de)$ a.e. From the bounds (3), we get $\mathbf{n}(de)$ a.e. for every $0 \leq s \leq t \leq \zeta$,

$$c_1(t-s) \leq \nu([s,t]) \leq c_2(t-s).$$

Hence the measure ν is absolutely continuous with respect to Lebesgue measure on $[0,\zeta]$, and by a standard differentiation theorem its density is equal almost everywhere to

$$\frac{d\nu}{dt} = \lim_{\epsilon\to 0} \frac{\nu([t-\epsilon,t+\epsilon])}{2\epsilon} = \lim_{\epsilon\to 0} \frac{h-m(p_e([t-\epsilon,t+\epsilon]))}{2\epsilon}. \tag{10}$$

It is easy to see that the quantity $h-m(p_e([t-\epsilon,t+\epsilon]))$ is a measurable function of the path $(e(t+u)-e(t), -\epsilon \leq u \leq \epsilon)$. Hence we can use Lemma 3.1 and the standard $0-1$ law for Brownian motion to get that the last limit in (10) must be equal to a constant $C_0 \in [0,\infty]$, dt a.e., $\mathbf{n}(de)$ a.e. Obviously, $C_0 \in [c_1,c_2]$ and in particular $0 < C_0 < \infty$.

We have thus $h-m(A)=C_0\,\mathbf{m}(A)$ for every subset of the tree of the form $p_e([s,t])$, or for any finite union of such sets. However, every open subset U of the tree is the increasing limit of a sequence of such unions (note that $p_e^{-1}(U)$ is a countable union of open intervals). Hence $h-m(A)=C_0\,\mathbf{m}(A)$ for every open subset A of $\mathcal{T}_e$, which is enough to complete the proof. □

Proof of Theorem 1.3. We now turn to the Hausdorff measure of level sets of $\mathcal{T}_e$. Recall that

$$\mathcal{T}_e(a)=\{s\in\mathcal{T}_e : d_e(0,s)=a\}=\{s\in\mathcal{T}_e : e(s)=a\}.$$

We also denote by $(\ell^a_s, s\geq 0)$ the (Brownian) local time process of e at level a. Then the measure $\ell^a(ds)$ associated with the increasing function $s\to\ell^a_s$ can be interpreted as a measure on $\mathcal{T}_e(a)$ and indeed coincides with the one discussed in the introduction (up to a multiplicative factor 2 which is irrelevant for our purposes). Moreover, for every nonnegative measurable function F on $C(\mathbb{R}_+,\mathbb{R})^2$,

$$\int \mathbf{n}(de)\int_0^\zeta \ell^a(ds)\,F((e(s+t))_{t\geq 0},(e((s-t)_+))_{t\geq 0}) = 2\,\Pi_a[F((B_{t\wedge T})_{t\geq 0},(B'_{t\wedge T'})_{t\geq 0})]. \qquad (11)$$

This formula is easily derived from Lemma 3.1 and the usual approximations of Brownian local time.

As in the proof of Theorem 1.1, we first establish the existence of two positive constants $\widetilde{c}_1$ and $\widetilde{c}_2$ such that, $\mathbf{n}(de)$ a.e. for every Borel subset A of $\mathcal{T}(a)$,

$$\widetilde{c}_1\,\ell^a(A)\leq \widetilde{h}-m(A)\leq \widetilde{c}_2\,\ell^a(A). \qquad (12)$$

Lower bound. Similarly as in the proof of the lower bound in (3), it is enough to show that there exists a constant C_1' such that, $\mathbf{n}(de)$ a.e., for ℓ^a-almost all $s\in\mathcal{T}_e(a)$,

$$\limsup_{\varepsilon\to 0}\frac{\ell^a(\{t\in\mathcal{T}_e(a): d_e(s,t)\leq\varepsilon\})}{\widetilde{h}(\varepsilon)}\leq C_1'. \qquad (13)$$

If $0<\varepsilon<a$ and $s,t\in\mathcal{T}_e(a)$, we have $d_e(s,t)\leq\varepsilon$ if and only if $m_e(s,t)\leq\varepsilon/2$. From this observation and (11), we see that (13) will follow if we can verify that, for every $a>0$, Π_a a.s.,

$$\limsup_{\varepsilon\to 0}\frac{L^a_{T_{a-\varepsilon}}(B)+L^a_{T'_{a-\varepsilon}}(B')}{\widetilde{h}(\varepsilon)}\leq C_1', \qquad (14)$$

where $(L^a_t(B), t\geq 0)$ is the local time process of B at level a, and $T_{a-\varepsilon}=\inf\{t: B_t=a-\varepsilon\}$, with a similar notation for $L^a_t(B')$ and $T'_{a-\varepsilon}$.

It is well known that the distribution of $L^a_{T_{a-\varepsilon}}(B)$ under Π_a is exponential with mean 2ε. Therefore an application of the Borel-Cantelli lemma immediately shows that, for $\varepsilon_n=2^{-n}$,

$$\limsup_{n\to\infty}\frac{L^a_{T_{a-\varepsilon_n}}(B)}{\widetilde{h}(\varepsilon_n)}\leq 1.$$

It readily follows that (14) holds with $C_1'=8$.

Upper bound. This is similar to the proof of the upper bound in (3). It is enough to show that there is a constant K'_1, not depending on a, such that, $\mathbf{n}(de)$ a.e.,

$$\widetilde{h} - m(\{s \in \mathcal{T}_e(a) : \limsup_{\varepsilon \to 0} \frac{\ell^a(\{t \in \mathcal{T}_e(a) : d_e(s,t) \leq \varepsilon\})}{\widetilde{h}(\varepsilon)} \leq K'_1\}) = 0. \tag{15}$$

This requires finding good coverings for the sets

$$G_{n_0} = \{s \in \mathcal{T}_e(a) : \ \ell^a(\{t \in \mathcal{T}_e(a) : d_e(s,t) \leq \varepsilon_p\}) \leq K'\widetilde{h}(\varepsilon_p), \, \forall p \geq n_0\},$$

where K' is a positive constant. Fix $a > 0$ and $n_0 \geq 1$ such that $2^{-n_0} < a$. To cover G_{n_0}, introduce the stopping times defined for every $n \geq n_0$,

$$T_0^n = \inf\{s : e(s) = a\} \ , \ T_1^n = \inf\{s > T_0^n : |e(s) - a| = 2^{-n}\}$$

and by induction,

$$T_{2k}^n = \inf\{s > T_{2k-1}^n : e(s) = a\} \ , \ T_{2k+1}^n = \inf\{s > T_{2k}^n : |e(s) - a| = 2^{-n}\}.$$

It is easy to verify that

$$\mathbf{n}\Big(\sum_{k=0}^{\infty} 1_{\{T_{2k}^n < \infty\}}\Big) \leq C_{(a)} 2^n \tag{16}$$

where the constant $C_{(a)}$ only depends on a.

In a way very similar to the proof of Theorem 1.1, we have

$$G_{n_0} \subset \bigcup_{k \in J_{n_0,n}} [T_{2k}^n, T_{2k+1}^n] \tag{17}$$

where

$$J_{n_0,n} = \{k : T_{2k}^n < \infty, \ \int_{T_{2k}^n}^{\zeta} d\ell_t^a \, 1_{\{d_e(T_{2k}^n, t) \leq \varepsilon_p/4\}} \leq K'\widetilde{h}(\varepsilon_p), \, \forall p \in \{n_0, \ldots, n-2\}\}.$$

By the strong Markov property at time T_{2k}^n,

$$\mathbf{n}(k \in J_{n_0,n}) = \mathbf{n}\Big(T_{2k}^n < \infty, \, \Pi_a[L^a_{T_{a-\varepsilon_p/8}} \leq K'\widetilde{h}(\varepsilon_p), \forall p \in \{n_0, \ldots, n-2\}]\Big).$$

Now note that the variables $L^a_{T_{a-\varepsilon_p/8}} - L^a_{T_{a-\varepsilon_p/16}}$, $p \in \{n_0, \ldots, n-2\}$ are independent under Π_a. Moreover, conditionally on the event that it is strictly positive, which has probability $1/2$, the variable $L^a_{T_{a-\varepsilon_p/8}} - L^a_{T_{a-\varepsilon_p/16}}$ is exponentially distributed with mean $\varepsilon_p/4$. It follows that

$$\begin{aligned}
&\Pi_a[L^a_{T_{a-\varepsilon_p/8}} \leq K'\widetilde{h}(\varepsilon_p), \forall p \in \{n_0, \ldots, n-2\}] \\
&\quad \leq \Pi_a[L^a_{T_{a-\varepsilon_p/8}} - L^a_{T_{a-\varepsilon_p/16}} \leq K'\widetilde{h}(\varepsilon_p), \forall p \in \{n_0, \ldots, n-2\}] \\
&\quad = \prod_{p=n_0}^{n-2} (1 - \frac{1}{2}\exp(-4K' \log\log 2^p)).
\end{aligned}$$

If $K' < 1/8$, the latter quantity is bounded above by $\exp(-n^{1/2})$ for n large. Therefore we get for all n sufficiently large,

$$\mathbf{n}(k \in J_{n_0,n}) \leq \exp(-n^{1/2})\,\mathbf{n}(T^n_{2k} < \infty).$$

By combining this with (16), and using Fatou's lemma, we arrive at

$$\liminf_{n\to\infty} 2^{-n} \exp(n^{1/2})\,\#J_{n_0,n} < \infty, \tag{18}$$

$\mathbf{n}(de)$ a.e. Since by construction the d_e-diameter of each interval $[T^n_{2k}, T^n_{2k+1}]$ is bounded above by $4\,2^{-n}$, (18) and (17) lead to $\widetilde{h} - m(G_{n_0}) = 0$. The estimate (15) now follows for a suitable choice of K'_1, and this completes the proof of the bounds (12).

The end of the proof is now similar to the final part of the proof of Theorem 1.1. We introduce the random measure $\widetilde{\nu}$ on $[0,\zeta]$ defined by

$$\widetilde{\nu}([0,t]) = \widetilde{h} - m(p_e([0,t]) \cap \mathcal{T}(a)).$$

The bounds (12) imply that $\widetilde{\nu}(dt)$ is absolutely continuous with respect to $\ell^a(dt)$, and that its density is bounded below and above by $\widetilde{c}_1$ and $\widetilde{c}_2$ respectively. A zero-one law argument, now relying on (11), shows that this density is equal to a constant $\widetilde{C}_0$, $\ell^a(dt)$ a.e., $\mathbf{n}(de)$ a.e. Moreover this constant does not depend on a. We leave details to the reader. □

4 Preliminaries about stable trees

In this section we collect the basic facts about stable trees that will be needed in the proof of Theorem 1.4 and Theorem 1.5. We refer to [7] for additional details.

We fix $\alpha \in (1,2)$. As in the introduction above, we write Θ_α for the distribution of the stable tree with index α. In the terminology of [7], this corresponds to the measure Θ associated with the branching mechanism function $\psi(u) = u^\alpha$. Note that Θ_α is a σ-finite measure on the space $\mathbb{T}$ of (rooted) $\mathbb{R}$-trees, which puts no mass on the trivial tree consisting only of the root.

In the same way as in the previous section, Θ_α can be defined and studied in terms of its coding function. However, the role of the Brownian excursion in the case $\alpha = 2$ is now played by the stable height process, which is a less tractable probabilistic object. For this reason, rather than using the coding function as we did in the case $\alpha = 2$, we will state here the key properties of the stable tree that are relevant to our study, and that can be found in [7].

We already mentioned the scaling invariance property of Θ_α: For every $r > 0$, the distribution of the scaled tree $r\mathcal{T}$ under Θ_α is $r^{\frac{1}{\alpha-1}}\Theta_\alpha$. We can also express the local times $\ell^a_{(r\mathcal{T})}$ and uniform measure $\mathbf{m}_{(r\mathcal{T})}$ of the scaled tree $r\mathcal{T}$ in terms of the local times $\ell^a_{(\mathcal{T})}$ and uniform measure $\mathbf{m}_{(\mathcal{T})}$ of the tree $\mathcal{T}$. Precisely, considering only the total masses of these random measures, we have Θ_α a.e.,

$$\langle \ell^a_{(r\mathcal{T})}, 1\rangle = r^{\frac{1}{\alpha-1}} \langle \ell^{a/r}_{(\mathcal{T})}, 1\rangle\ ,\ \langle \mathbf{m}_{(r\mathcal{T})}, 1\rangle = r^{\frac{\alpha}{\alpha-1}} \langle \mathbf{m}_{(\mathcal{T})}, 1\rangle. \tag{19}$$

This can be checked from the approximation of local time recalled in the introduction above.

Informally, the tree $\mathcal{T}$ under Θ_α describes the genealogy of descendants of a single individual in a continuous-state branching process with branching mechanism $\psi(u) = u^\alpha$. The total mass $\langle \ell^a, 1\rangle$ then corresponds to the population at time (or level) a. To make this more precise, we can state the following "Ray-Knight property" of local times. Let $x > 0$ and let

$$\sum_{i\in I} \delta_{\mathcal{T}_i}$$

be a Poisson point measure on $\mathbb{T}$ with intensity $x\Theta_\alpha$. The real-valued process $(X_t)_{t\geq 0}$ defined by

$$\begin{cases} X_t = \sum_{i\in I} \langle \ell^t_{(\mathcal{T}_i)}, 1\rangle & \text{if } t > 0\,, \\ X_0 = x \end{cases}$$

is a continuous-state branching process with branching mechanism $\psi(u) = u^\alpha$, started at $X_0 = x$. This means that $(X_t)_{t\geq 0}$ is a Feller Markov process on $\mathbb{R}_+$ and that the Laplace transform of its semigroup is determined as follows: For every $\lambda > 0$,

$$E[\exp(-\lambda X_t) \mid X_0 = x] = \exp(-x\, u_t(\lambda))$$

where $(u_t(\lambda))_{t\geq 0}$ is determined from the integral equation

$$u_t(\lambda) + \int_0^t ds\, u_s(\lambda)^\alpha = \lambda,$$

so that

$$u_t(\lambda) = (\lambda^{1-\alpha} + (\alpha - 1)t)^{\frac{1}{1-\alpha}}. \tag{20}$$

Note that we have also

$$u_t(\lambda) = \Theta_\alpha(1 - \exp -\lambda\langle \ell^t, 1\rangle)$$

from the exponential formula for Poisson measures. Using the Markov property of X, one easily derives similar integral equations for finite-dimensional marginal distributions of $(X_t)_{t\geq 0}$: See e.g. Section II.3 of [13] where the more general setting of superprocesses is considered. We will need the following particular case: For every $\gamma, \lambda > 0$, the function

$$v_t(\gamma, \lambda) = \Theta_\alpha\Big(1 - \exp\Big(-\gamma\int_0^t ds\,\langle \ell^s, 1\rangle - \lambda\langle \ell^t, 1\rangle\Big)\Big)$$

solves the integral equation

$$v_t + \int_0^t ds\,(v_s)^\alpha = \gamma t + \lambda.$$

Recall that $\mathcal{T}(r)$ denotes the level set of $\mathcal{T}$ at level r, and $H(\mathcal{T})$ stands for the height of $\mathcal{T}$. We also use the notation $\mathcal{T}_{\leq r}$ for the set $\{\sigma \in \mathcal{T} : d(\rho, \sigma) \leq r\}$.

Lemma 4.1 *There exist two positive constants c_α and C_α such that, for every $b > 0$,*

$$\Theta_\alpha\Big(\mathbf{m}(\mathcal{T}_{\leq 1}) \leq b\,,\ H(\mathcal{T}) \geq 1\Big) \leq C_\alpha\, \exp(-c_\alpha b^{-\frac{\alpha-1}{\alpha}}).$$

Proof: With the preceding notation, set

$$v_t^0(\gamma) = v_t(\gamma, 0) = \Theta_\alpha\Big(1 - \exp\Big(-\gamma \int_0^t ds\, \langle \ell^s, 1\rangle\Big)\Big)$$

so that $v_t^0 = v_t^0(\gamma)$ solves the integral equation

$$v_t^0 + \int_0^t ds\, (v_s^0)^\alpha = \gamma t.$$

It follows, that, for every $t \geq 0$, $v_t^0(\gamma) \in [0, \gamma^{1/\alpha})$ is determined by

$$\int_0^{v_t^0(\gamma)} \frac{dy}{\gamma - y^\alpha} = t. \tag{21}$$

Similarly, if $v_t^\infty(\gamma) = \lim \uparrow v_t(\gamma, \lambda)$ as $\lambda \uparrow \infty$, we have

$$v_t^\infty(\gamma) = \Theta_\alpha\Big(1 - \mathbf{1}_{\{\ell^t=0\}} \exp\Big(-\gamma \int_0^t ds\, \langle \ell^s, 1\rangle\Big)\Big)$$

and $v_t^\infty(\gamma) \in (\gamma^{1/\alpha}, \infty)$ is determined from the equation

$$\int_{v_t^\infty(\gamma)}^\infty \frac{dy}{y^\alpha - \gamma} = t.$$

Simple analytic arguments show that

$$v_t^0(\gamma) = \gamma^{1/\alpha}(1 - e^{-\varphi_t(\gamma)})$$

where

$$\lim_{\gamma\to\infty} \gamma^{\frac{1}{\alpha}-1} \varphi_t(\gamma) = \alpha t.$$

Similarly,

$$v_t^\infty(\gamma) = \gamma^{1/\alpha}(1 + e^{-\phi_t(\gamma)})$$

where

$$\lim_{\gamma\to\infty} \gamma^{\frac{1}{\alpha}-1} \phi_t(\gamma) = \alpha t.$$

Now observe that

$$\Theta_\alpha\Big(\mathbf{1}_{\{\ell^1\neq 0\}} \exp\Big(-\gamma \int_0^1 ds\, \langle \ell^s, 1\rangle\Big)\Big) = v_1^\infty(\gamma) - v_1^0(\gamma) = \gamma^{1/\alpha}(e^{-\varphi_1(\gamma)} + e^{-\phi_1(\gamma)}).$$

Furthermore, by construction, $\mathbf{m}(\mathcal{T}_{\leq 1}) = \int_0^1 ds\, \langle \ell^s, 1\rangle$ and $\{\ell^1 \neq 0\} = \{H(\mathcal{T}) \geq 1\}$, Θ_α a.e. (cf Theorem 4.2 in [7]). We get

$$\lim_{\gamma\to\infty} \frac{\log \Theta_\alpha(\mathbf{1}_{\{H(\mathcal{T})\geq 1\}} \exp(-\gamma \mathbf{m}(\mathcal{T}_{\leq 1})))}{\gamma^{1-\frac{1}{\alpha}}} = -\alpha.$$

The estimate of the lemma now follows. □

An important role in the next section will be played by a subtree decomposition along the ancestral line of a randomly chosen vertex. For the reader's convenience, we now recall this result.

We first introduce the relevant notation. Let $\mathcal{T} \in \mathbb{T}$ and $\sigma \in \mathcal{T}$. Denote by $[\![\rho(\mathcal{T}), \sigma]\!]$ the line segment from the root ρ to σ, that is the ancestral line of σ. If $\sigma, \sigma' \in \mathcal{T}$, the notation $\sigma \wedge \sigma'$ stands for the most recent common ancestor to σ and σ' (equivalently, $[\![\rho, \sigma]\!] \cap [\![\rho, \sigma']\!] = [\![\rho, \sigma \wedge \sigma']\!]$). Denote by $\mathcal{T}^{(j),\circ}$, $j \in \mathcal{J}$ the connected components of the open set $\mathcal{T} \backslash [\![\rho, \sigma]\!]$, and note that for every $j \in \mathcal{J}$, $\sigma_j := \sigma \wedge \tau$ does not depend on the choice of $\tau \in \mathcal{T}^{(j),\circ}$. Furthermore, $\mathcal{T}^{(j)} := \mathcal{T}^{(j),\circ} \cup \{\sigma_j\}$ is a (compact rooted) $\mathbb{R}$-tree with root σ_j. The trees $\mathcal{T}^{(j)}$, $j \in \mathcal{J}$ can be interpreted as the subtrees of $\mathcal{T}$ originating from the segment $[\![\rho, \sigma]\!]$. We put

$$\mathcal{M}_\sigma = \sum_{j \in \mathcal{J}} \delta_{(d(\rho(\mathcal{T}), \sigma_j), \mathcal{T}^{(j)})},$$

thus defining a point measure on $[0, \infty) \times \mathbb{T}$.

Theorem 4.2 *For every $a > 0$ and every nonnegative measurable function Φ on $[0, \infty) \times \mathbb{T}$,*

$$\Theta_\alpha\Big(\int \ell^a(d\sigma) \, \exp -\langle \mathcal{M}_\sigma, \Phi \rangle \Big) = \exp\Big(-\alpha \int_0^a dt \, \Big(\Theta_\alpha(1 - \exp -\Phi(t, \cdot))\Big)^{\alpha-1} \Big).$$

This is the case $\psi(u) = u^\alpha$ in Theorem 4.5 of [7].

5 The Hausdorff measure of the stable tree

In this section we prove Theorem 1.4 and Theorem 1.5. We keep the notation and assumptions of the preceding section.

Proof of Theorem 1.4. Part (i) is an immediate consequence of the following proposition.

Proposition 5.1 *Suppose that $h : [0, \infty) \to [0, \infty)$ is a monotone increasing function that can be written in the form $h(r) = r^{\frac{\alpha}{\alpha-1}} g(r)$ where g is monotone decreasing in a neighborhood of the origin. Then the condition*

$$\sum_{n=1}^{\infty} g(2^{-n})^{-(\alpha-1)} < \infty \tag{22}$$

implies that $h - m(\mathcal{T}) = \infty$, Θ_α a.e.

Conjecture. If (22) fails, then $h - m(\mathcal{T}) = 0$, Θ_α a.e.

Proof: Let $a > 0$. If $\sigma \in \mathcal{T}$ and $\varepsilon > 0$, denote by $B(\sigma, \varepsilon)$ the closed ball of radius ε centered at σ. Then Theorem 4.2 implies that, for every $\lambda > 0$ and $\varepsilon \in (0, a]$,

$$\Theta_\alpha\Big(\int \ell^a(d\sigma) \, \exp -\lambda \mathbf{m}(B(\sigma, \varepsilon)) \Big) = \exp\Big(-\alpha \int_{a-\varepsilon}^a \Phi_{\varepsilon,\lambda,a}(t)^{\alpha-1} \, dt \Big), \tag{23}$$

where for $t \in [a-\varepsilon, a]$,

$$\Phi_{\varepsilon,\lambda,a}(t) = \Theta_\alpha\Big(1 - \exp -\lambda \mathbf{m}(B(\rho, \varepsilon - (a-t)))\Big).$$

Details of the derivation of (23) can be found on p.593-594 of [7], where a similar formula is derived in greater generality.

In agreement with the notation of the preceding section, we put

$$v_r^0(\lambda) = \Theta_\alpha(1 - \exp -\lambda \mathbf{m}(B(\rho, r))) = \Theta_\alpha(1 - \exp -\lambda \mathbf{m}(\mathcal{T}_{\leq r})).$$

Then, (23) can be rewritten in the form

$$\Theta_\alpha\Big(\int \ell^a(d\sigma)\, \exp -\lambda \mathbf{m}(B(\sigma,\varepsilon))\Big) = \exp\Big(-\alpha \int_0^\varepsilon dr\, v_r^0(\lambda)^{\alpha-1}\Big). \tag{24}$$

Recall that $v_r^0(\lambda)$ is determined from equation (21). From this equation one immediately derives the following scaling property: For every $\varepsilon > 0$,

$$v_{\varepsilon r}^0(\lambda) = \varepsilon^{-\frac{1}{\alpha-1}}\, v_r^0(\varepsilon^{\frac{\alpha}{\alpha-1}}\lambda). \tag{25}$$

Hence,

$$\int_0^\varepsilon dr\, v_r^0(\lambda)^{\alpha-1} = \int_0^1 dr\, v_r^0(\varepsilon^{\frac{\alpha}{\alpha-1}}\lambda)^{\alpha-1}.$$

If we substitute this identity into (24) and replace λ by $\varepsilon^{-\frac{\alpha}{\alpha-1}}\lambda$, we arrive at

$$\Theta_\alpha\Big(\int \ell^a(d\sigma)\, \exp(-\lambda\varepsilon^{-\frac{\alpha}{\alpha-1}} \mathbf{m}(B(\sigma,\varepsilon)))\Big) = \exp\Big(-\alpha \int_0^1 dr\, v_r^0(\lambda)^{\alpha-1}\Big). \tag{26}$$

The local time measure ℓ^a satisfies $\Theta_\alpha(\langle \ell^a, 1\rangle) = 1$ (take $\lambda = 0$ in (26)). Thus, $\Theta_\alpha(d\mathcal{T})\ell^a(d\sigma)$ defines a probability measure on the set of "pointed $\mathbb{R}$-trees", that is pairs consisting of an $\mathbb{R}$-tree $\mathcal{T}$ and a distinguished point $\sigma \in \mathcal{T}$ (in addition to the root). Denote by μ the law of $\varepsilon^{-\frac{\alpha}{\alpha-1}}\mathbf{m}(B(\sigma,\varepsilon))$ under the probability measure $\Theta_\alpha(d\mathcal{T})\ell^a(d\sigma)$. By (26), this law does not depend on the choice of ε and a, provided that $0 < \varepsilon \leq a$. Furthermore, the Laplace transform of μ is given by

$$\int \mu(dx)\, e^{-\lambda x} = \exp\Big(-\alpha \int_0^1 dr\, v_r^0(\lambda)^{\alpha-1}\Big).$$

By monotone convergence,

$$\frac{v_r^0(\lambda)}{\lambda} = \Theta\Big(\frac{1 - \exp(-\lambda \mathbf{m}(B(\rho,r)))}{\lambda}\Big) \xrightarrow[\lambda\downarrow 0]{} \Theta(\mathbf{m}(B(\rho,r))) = r$$

and

$$\lambda^{1-\alpha} \int_0^1 dr\, v_r^0(\lambda)^{\alpha-1} \xrightarrow[\lambda\downarrow 0]{} \int_0^1 dr\, r^{\alpha-1} = \frac{1}{\alpha}.$$

It follows that

$$\int \mu(dx)\, e^{-\lambda x} = 1 - \lambda^{\alpha-1} + o(\lambda^{\alpha-1})$$

as $\lambda \to 0$. Consequently, there exists a constant C such that, for every $y > 0$,

$$\mu([y,\infty)) \leq C\, y^{-(\alpha-1)}. \tag{27}$$

Let h and g be as in the statement of the proposition, and let N be an integer such that $2^{-N} \leq a$. Then, using (27),

$$\sum_{n=N}^{\infty} \Theta_\alpha\Big(\int \ell^a(d\sigma)\,\mathbf{1}_{\{\mathbf{m}(B(\sigma,2^{-n}))\geq h(2^{-n})\}}\Big) = \sum_{n=N}^{\infty} \mu([g(2^{-n}),\infty)) \leq C \sum_{n=N}^{\infty} g(2^{-n})^{-(\alpha-1)} < \infty$$

by our assumption (22). Hence,

$$\sum_{n=N}^{\infty} \mathbf{1}_{\{\mathbf{m}(B(\sigma,2^{-n}))\geq h(2^{-n})\}} < \infty\ , \quad \ell^a(d\sigma) \text{ a.e., } \Theta_\alpha \text{ a.e.}$$

and so

$$\limsup_{n\to\infty} \frac{\mathbf{m}(B(\sigma,2^{-n}))}{h(2^{-n})} \leq 1\ , \quad \ell^a(d\sigma) \text{ a.e., } \Theta_\alpha \text{ a.e.}$$

Since this holds for every $a > 0$, we can replace $\ell^a(d\sigma)$ a.e. by $\mathbf{m}(d\sigma)$ a.e. in the last display. By Lemma 2.1 (i), this implies

$$h - m(\mathcal{T}) > 0\ , \quad \Theta_\alpha \text{ a.e.}$$

Finally, we may find a function $\widetilde{h}(r) = r^{\frac{\alpha}{\alpha-1}}\widetilde{g}(r)$, such that $\widetilde{h}$ and $\widetilde{g}$ satisfy the same assumptions as h and g, and $\widetilde{g}(r)/g(r) \longrightarrow 0$ as $r \to 0$. We have $\widetilde{h} - m(\mathcal{T}) > 0$ which implies $h - m(\mathcal{T}) = \infty$. This completes the proof of Proposition 5.1. $\square$

We now turn to the proof of part (ii) of Theorem 1.4. We thus fix $u < 0$, and we aim at proving that $h_u - m(\mathcal{T}) = 0$, Θ_α a.e. We also fix $\delta \in (0,1/2)$ and an integer $n_0 \geq 1$ such that $2^{-n_0} < \delta$. The main step of the proof is to control the Hausdorff measure $h_u - m(B_{n_0})$ of the "bad set"

$$B_{n_0} = \{\sigma \in \mathcal{T} : 2\delta \leq d(\rho,\sigma) \leq (2\delta)^{-1} \text{ and } \mathbf{m}(B(\sigma,2^{-n})) \leq h_u(2^{-n}) \text{ for every } n \geq n_0\}.$$

Let $p \geq n_0 + 3$ be an integer. For every integer $k \geq 1$ denote by $(\mathcal{T}_j^{k,p})_{1\leq j\leq N_{k,p}}$ the subtrees of $\mathcal{T}$ above level $k2^{-p}$ with height greater than 2^{-p} (cf Section 4.2 in [7]). Also set

$$\widetilde{\mathcal{T}}_j^{k,p} = \{\sigma \in \mathcal{T}_j^{k,p} : d(\rho,\sigma) \in [k2^{-p},(k+2)2^{-p})\}.$$

To simplify notation, we put

$$I_p = \{(k,j) : k \geq 1, 1 \leq j \leq N_{k,p}\}.$$

Suppose that $\widetilde{\mathcal{T}}_j^{k,p} \cap B_{n_0} \neq \varnothing$ for some $(k,j) \in I_p$, and let $\sigma_0 \in \widetilde{\mathcal{T}}_j^{k,p} \cap B_{n_0}$. Then, for every $\sigma \in \widetilde{\mathcal{T}}_j^{k,p}$,

$$B(\sigma,2^{-n}) \subset B(\sigma_0,2^{-n}+4\,2^{-p}) \subset B(\sigma_0,2^{-n+1})$$

provided that $n \leq p-2$. Since $\sigma_0 \in B_{n_0}$, we have, for every $\sigma \in \widetilde{\mathcal{T}}_j^{k,p}$,

$$\mathbf{m}(B(\sigma, 2^{-n})) \leq h_u(2^{-n+1}) \ , \text{ for every } n \in \{n_0+1, n_0+2, \ldots, p-2\}.$$

Thus if we set

$$\widetilde{B}_{n_0,p} = \{\sigma \in \mathcal{T} : \delta \leq d(\rho,\sigma) \leq \delta^{-1} \text{ and } \mathbf{m}(B(\sigma,2^{-n})) \leq h_u(2^{-n+1}) \\ \text{for every } n \in \{n_0+1, n_0+2, \ldots, p-2\}\},$$

we see that the condition $\widetilde{\mathcal{T}}_j^{k,p} \cap B_{n_0} \neq \varnothing$ implies $\widetilde{\mathcal{T}}_j^{k,p} \subset \widetilde{B}_{n_0,p}$.

It follows that, for every real $b > 0$,

$$\begin{aligned}
&\#\{(k,j) \in I_p : \widetilde{\mathcal{T}}_j^{k,p} \cap B_{n_0} \neq \varnothing\} \\
&\leq \#\{(k,j) \in I_p : k \leq 2^p\delta^{-1} \text{ and } \mathbf{m}(\widetilde{\mathcal{T}}_j^{k,p}) < b\} + b^{-1} \sum_{k=1}^{\infty} \sum_{j=1}^{N_{k,p}} \int_{\widetilde{\mathcal{T}}_j^{k,p}} \mathbf{m}(d\sigma)\, \mathbf{1}_{\widetilde{B}_{n_0,p}}(\sigma) \\
&\leq \#\{(k,j) \in I_p : k \leq 2^p\delta^{-1} \text{ and } \mathbf{m}(\widetilde{\mathcal{T}}_j^{k,p}) < b\} + 2b^{-1} \int \mathbf{m}(d\sigma)\, \mathbf{1}_{\widetilde{B}_{n_0,p}}(\sigma). \qquad (28)
\end{aligned}$$

In the last bound we used the fact that, for every $\sigma \in \mathcal{T}$, there are at most two pairs $(k,j) \in I_p$ such that $\sigma \in \widetilde{\mathcal{T}}_j^{k,p}$.

We will apply the bound (28) with

$$b = b_p = 2^{-p\frac{\alpha}{\alpha-1}}\, (\log p)^{-\kappa},$$

where $\kappa > \frac{\alpha}{\alpha-1}$ is arbitrary. We use different arguments to bound the two terms in the right-hand side of (28). To bound the first term, we apply Lemma 4.1. From this lemma and the scaling properties of the stable tree recalled in the preceding section, we have, for every $b > 0$ and $r > 0$,

$$\Theta_\alpha\Big(\mathbf{m}(\mathcal{T}_{\leq r}) \leq b \ , \ H(\mathcal{T}) \geq r\Big) \leq C_\alpha\, r^{-\frac{1}{\alpha-1}} \exp(-c_\alpha r\, b^{-\frac{\alpha-1}{\alpha}}). \qquad (29)$$

On the other hand, the branching property of the stable tree (cf Theorem 4.2 in [7]) guarantees that for every $k \geq 1$, under $\Theta_\alpha(\cdot \mid H(\mathcal{T}) \geq k2^{-p})$ and conditionally given $\langle \ell^{k2^{-p}}, 1\rangle$, the trees $\mathcal{T}_1^{k,p}, \ldots, \mathcal{T}_{N_{k,p}}^{k,p}$ are distributed as the atoms of a Poisson point measure with intensity $\langle \ell^{k2^{-p}}, 1\rangle\, \Theta_\alpha(\cdot \cap \{H(\mathcal{T}) \geq 2^{-p})$. Recalling that $\Theta_\alpha(\langle \ell^{k2^{-p}}, 1\rangle) = 1$, we get

$$\begin{aligned}
&\Theta_\alpha\Big(\#\{(k,j) \in I_p : k \leq 2^p\delta^{-1} \text{ and } \mathbf{m}(\widetilde{\mathcal{T}}_j^{k,p}) < b_p\}\Big) \\
&= \sum_{k=1}^{[\delta^{-1}2^p]} \Theta_\alpha\Big(\#\{j : 1 \leq j \leq N_{k,p} \text{ and } \mathbf{m}(\widetilde{\mathcal{T}}_j^{k,p}) < b_p\}\Big) \\
&= \sum_{k=1}^{[\delta^{-1}2^p]} \Theta_\alpha(\mathbf{m}(\mathcal{T}_{\leq 2\cdot 2^{-p}}) \leq b_p,\, H(\mathcal{T}) \geq 2^{-p}) \\
&\leq \delta^{-1} 2^p\, \Theta_\alpha(\mathbf{m}(\mathcal{T}_{\leq 2^{-p}}) \leq b_p,\, H(\mathcal{T}) \geq 2^{-p}) \\
&\leq C_\alpha\, \delta^{-1}\, 2^{p\frac{\alpha}{\alpha-1}} \exp(-c_\alpha 2^{-p} b_p^{-\frac{\alpha-1}{\alpha}})
\end{aligned}$$

using (29) in the last bound. Recalling our choice of b_p, we deduce from this bound that

$$\sum_{p=n_0+3}^{\infty} h_u(2^{-p})\,\Theta_\alpha\Big(\#\{(k,j)\in I_p : k\leq 2^p\delta^{-1} \text{ and } \mathbf{m}(\widetilde{\mathcal{T}}_j^{k,p})<b_p\}\Big)<\infty.$$

It follows that

$$\lim_{p\to\infty} h_u(2^{-p})\,\#\{(k,j)\in I_p : k\leq 2^p\delta^{-1} \text{ and } \mathbf{m}(\widetilde{\mathcal{T}}_j^{k,p})<b_p\}=0\;,\;\; \Theta_\alpha \text{ a.e.} \qquad (30)$$

We now turn to the second term in the right-hand side of (28). From the definition of $\widetilde{B}_{n_0,p}$ and the identity $\mathbf{m}=\int_0^\infty da\,\ell^a$, we have

$$\Theta_\alpha\Big(\int \mathbf{m}(d\sigma)\,\mathbf{1}_{\widetilde{B}_{n_0,p}}(\sigma)\Big)=\Theta_\alpha\Big(\int_\delta^{\delta^{-1}} da\int \ell^a(d\sigma)\,\mathbf{1}_{\{\mathbf{m}(B(\sigma,2^{-n}))\leq h_u(2^{-n+1})\;,\;\forall n\in\{n_0+1,\ldots,p-2\}}\Big).$$

For every $n\geq n_0+1$, set

$$\mathcal{C}(\sigma,2^{-n})=\{\sigma'\in\mathcal{T} : 2^{-n-2}\leq d(\sigma\wedge\sigma',\sigma)<2^{-n-1} \text{ and } 0<d(\sigma\wedge\sigma',\sigma')\leq 2^{-n-1}\}.$$

Clearly, $\mathcal{C}(\sigma,2^{-n})\subset B(\sigma,2^{-n})$ and so

$$\Theta_\alpha\Big(\int \mathbf{m}(d\sigma)\,\mathbf{1}_{\widetilde{B}_{n_0,p}}(\sigma)\Big)\leq\int_\delta^{\delta^{-1}} da\,\Theta_\alpha\Big(\int \ell^a(d\sigma)\,\mathbf{1}_{\{\mathbf{m}(\mathcal{C}(\sigma,2^{-n}))\leq h_u(2^{-n+1})\;,\;\forall n\in\{n_0+1,\ldots,p-2\}}\Big).$$

Let us fix $a\in[\delta,\delta^{-1}]$. It follows from Theorem 4.2 that under the probability measure $\Theta_\alpha(d\mathcal{T})\ell^a(d\sigma)$, the random variables

$$\mathbf{m}(\mathcal{C}(\sigma,2^{-n}))\;,\;\; n=n_0+1,n_0+2,\ldots$$

are independent, and furthermore the law of $\mathbf{m}(\mathcal{C}(\sigma,2^{-n}))$ is determined by

$$\Theta_\alpha\Big(\int \ell^a(d\sigma)\,\exp(-\lambda\mathbf{m}(\mathcal{C}(\sigma,2^{-n})))\Big)=\exp(-\alpha 2^{-n-2}v^0_{2^{-n-1}}(\lambda)^{\alpha-1})$$

where $v^0_r(\lambda)$ is as previously. From the scaling property (25), we see that the law ν of

$$2^{n\frac{\alpha}{\alpha-1}}\,\mathbf{m}(\mathcal{C}(\sigma,2^{-n}))$$

under $\Theta_\alpha(d\mathcal{T})\ell^a(d\sigma)$ does not depend on a nor on n (this is indeed true provided $2^{-n-1}\leq a$, which holds here since $2^{-n-1}\leq 2^{-n_0-1}\leq\delta\leq a$). Furthermore, the Laplace transform of ν is

$$\int \nu(dx)\,e^{-\lambda x}=\exp\Big(-\frac{\alpha}{4}\,v^0_{1/2}(\lambda)^{\alpha-1}\Big).$$

Since $\lambda^{-1}v^0_{1/2}(\lambda)\uparrow 1/2$ as $\lambda\downarrow 0$, we get

$$\int \nu(dx)\,e^{-\lambda x}=1-\frac{\alpha}{2}2^{-\alpha}\,\lambda^{\alpha-1}+o(\lambda^{\alpha-1})$$

as $\lambda \downarrow 0$. From a standard Tauberian theorem, it follows that there exists a constant $c_0 > 0$ such that, for every $y \geq 1$,

$$\nu([y,\infty)) \geq c_0\, y^{1-\alpha}.$$

Using this bound together with the previously mentioned independence, we get

$$\begin{aligned}
&\Theta_\alpha\Big(\int \ell^a(d\sigma)\,\mathbf{1}_{\{\mathbf{m}(\mathcal{C}(\sigma,2^{-n}))\leq h_u(2^{-n+1})\ ,\ \forall n\in\{n_0+1,\dots,p-2\}\}}\Big)\\
&= \prod_{n=n_0+1}^{p-2}\Big(1-\nu([2^{n\frac{\alpha}{\alpha-1}}h_u(2^{-n+1}),\infty))\Big)\\
&\leq \prod_{n=n_0+1}^{p-2}\Big(1-c_0 2^{-n\alpha}h_u(2^{-n+1})^{1-\alpha}\Big)\\
&= \prod_{n=n_0+1}^{p-2}\Big(1-c_0 2^{-\alpha}((n-1)\log 2)^{-1}(\log(n-1)+\log\log 2)^{(1-\alpha)u}\Big)\\
&\leq \exp\Big(-\overline{c}_0\Big((\log(p-2))^{1-(1-\alpha)u}-(\log n_0)^{1-(1-\alpha)u}\Big)\Big)
\end{aligned}$$

where $\overline{c}_0$ is a positive constant and the last bound follows from simple analytic estimates.

By integrating with respect to a, we arrive at

$$\Theta_\alpha\Big(\int \mathbf{m}(d\sigma)\,\mathbf{1}_{\widetilde{B}_{n_0,p}}(\sigma)\Big) \leq \delta^{-1}\,\exp\Big(-\overline{c}_0\Big((\log(p-2))^{1-(1-\alpha)u}-(\log n_0)^{1-(1-\alpha)u}\Big)\Big).$$

Notice that $1-(1-\alpha)u > 1$ since $u < 0$. It then follows from the preceding bound that

$$\sum_{p=n_0+3}^{\infty} h_u(2^{-p})b_p^{-1}\,\Theta_\alpha\Big(\int \mathbf{m}(d\sigma)\,\mathbf{1}_{\widetilde{B}_{n_0,p}}(\sigma)\Big) < \infty$$

and thus

$$\lim_{p\to\infty} h_u(2^{-p})b_p^{-1}\int \mathbf{m}(d\sigma)\,\mathbf{1}_{\widetilde{B}_{n_0,p}}(\sigma) = 0\ ,\ \ \Theta_\alpha \text{ a.e.} \tag{31}$$

By (28), (30) and (31), we have

$$\lim_{p\to\infty} h_u(2^{-p})\#\{(k,j)\in I_p : \widetilde{\mathcal{T}}_j^{k,p}\cap B_{n_0}\neq\varnothing\} = 0\ ,\ \ \Theta_\alpha \text{ a.e.}$$

Since the sets $\widetilde{\mathcal{T}}_j^{k,p}\cap B_{n_0}$ provide a covering of B_{n_0} by sets with diameter less than $4\,2^{-p}$, the definition of Hausdorff measure gives

$$h_u - m(B_{n_0}) = 0\ ,\ \ \Theta_\alpha \text{ a.e.}$$

By passing to the limit $n_0 \uparrow \infty$ and $\delta \downarrow 0$, we obtain

$$h_u - m\Big(\Big\{\sigma\in\mathcal{T} : \limsup_{n\to\infty}\frac{\mathbf{m}(B(\sigma,2^{-n}))}{h_u(2^{-n})} < 1\Big\}\Big) = 0\ ,\ \ \Theta_\alpha \text{ a.e.}$$

On the other hand, Lemma 2.1 (ii) yields

$$h_u - m\Big(\Big\{\sigma \in \mathcal{T} : \limsup_{n\to\infty} \frac{\mathbf{m}(B(\sigma, 2^{-n}))}{h_u(2^{-n})} \geq 1\Big\}\Big) \leq M_2\, \mathbf{m}(\mathcal{T}) < \infty\ ,\ \ \Theta_\alpha \text{ a.e.}$$

We conclude that $h_u - m(\mathcal{T}) < \infty$ and since this holds for every $u < 0$, we must indeed have $h_u - m(\mathcal{T}) = 0$, Θ_α a.e. □

Proof of Theorem 1.5. Many arguments here are similar to the preceding proof, and we will only sketch details. Without loss of generality we may take $a = 1$. Part (i) is a consequence of the following proposition.

Proposition 5.2 *Suppose that $\widetilde{h} : [0, \infty) \to [0, \infty)$ is a monotone increasing function that can be written in the form $\widetilde{h}(r) = r^{\frac{1}{\alpha-1}}\widetilde{g}(r)$ where $\widetilde{g}$ is monotone decreasing in a neighborhood of the origin. Then the condition*

$$\sum_{n=1}^{\infty} \widetilde{g}(2^{-n})^{-(\alpha-1)} < \infty \tag{32}$$

implies that $\widetilde{h} - m(\mathcal{T}(1)) = \infty$, Θ_α a.e. on $\{H(\mathcal{T}) > 1\}$.

Conjecture. If (32) fails, then $\widetilde{h} - m(\mathcal{T}(1)) = 0$, Θ_α a.e.

Proof: Using Theorem 4.2 in the same way as in the derivation of (24), we have for $\varepsilon \in (0, 1]$ and $\lambda > 0$,

$$\Theta_\alpha\Big(\int \ell^1(d\sigma)\, \exp -\lambda \ell^1(B(\sigma, \varepsilon))\Big) = \exp -\alpha \int_0^{\varepsilon/2} dr\, u_r(\lambda)^{\alpha-1}$$

where $u_r(\lambda) = \Theta(1 - \exp -\lambda\langle \ell^r, 1\rangle)$ is given by (20). Straightforward calculations now give

$$\Theta_\alpha\Big(\int \ell^1(d\sigma)\, \exp -\lambda \ell^1(B(\sigma, \varepsilon))\Big) = \Big(1 + \frac{(\alpha-1)\lambda^{\alpha-1}\varepsilon}{2}\Big)^{-\frac{\alpha}{\alpha-1}}.$$

Hence the law $\widetilde{\mu}$ of $\varepsilon^{-\frac{1}{\alpha-1}}\ell^1(B(\sigma, \varepsilon))$ under $\Theta_\alpha(d\mathcal{T})\ell^1(d\sigma)$ does not depend on $\varepsilon \in (0, 1]$ and as in the proof of Proposition 5.1, there is a constant $\widetilde{C}$ such that, for every $y > 0$,

$$\widetilde{\mu}([y, \infty)) \leq \widetilde{C}\, y^{-(\alpha-1)}.$$

If $\widetilde{h}$ is as in the statement of the proposition, it follows that

$$\limsup_{n\to\infty} \frac{\ell^1(B(\sigma, 2^{-n}))}{\widetilde{h}(2^{-n})} \leq 1\ , \quad \ell^1(d\sigma) \text{ a.e., } \Theta_\alpha \text{ a.e.}$$

The end of the proof is now similar to that of Proposition 5.1. □

Let us now turn to the proof of (ii). The outline is again similar to the proof of Theorem 1.4 (ii) but there are a few minor differences. We fix $u < 0$ and an integer $n_0 \geq 1$. The "bad set" is now defined by

$$B_{n_0} = \{\sigma \in \mathcal{T}(1) : \ell^1(B(\sigma, 2^{-n})) \leq \widetilde{h}_u(2^{-n}) \text{ for every } n \geq n_0\}.$$

If $p \geq n_0+2$ is an integer, we denote by $\mathcal{T}_j^p$, $1 \leq j \leq N_p$ the subtrees of $\mathcal{T}$ above level $1-2^{-p}$ that intersect $\mathcal{T}(1)$. Arguing in the proof of Theorem 1.4, we can check that if $\mathcal{T}_j^p \cap B_{n_0} \neq \varnothing$, then $\mathcal{T}_j^p \cap \mathcal{T}(1) \subset \widetilde{B}_{n_0,p}$, where

$$\widetilde{B}_{n_0,p} = \{\sigma \in \mathcal{T}(1) : \ell^1(B(\sigma, 2^{-n})) \leq h_u(2^{-n+1}) \text{ for every } n \in \{n_0+1, \ldots, p-1\}\}.$$

It follows that, for every $b > 0$,

$$\#\{j \leq N_p : \mathcal{T}_j^p \cap B_{n_0} \neq \varnothing\} \leq \#\{j \leq N_p : \ell^1(\mathcal{T}_j^p) < b\} + b^{-1} \int \ell^1(d\sigma)\, \mathbf{1}_{\widetilde{B}_{n_0,p}}(\sigma). \tag{33}$$

We apply this estimate with $b = b_p = 2^{-\frac{p}{\alpha-1}} p^{-\kappa}$, where $\kappa > \alpha(\alpha-1)^{-2}$.

To bound the first term in the right-hand side of (33), we use (20) to get for every $\lambda > 0$ and $r > 0$,

$$\Theta_\alpha(\exp -\lambda \langle \ell^r, 1\rangle \mid \langle \ell^r, 1\rangle > 0) = 1 - \Big(\frac{(\alpha-1)r + \lambda^{1-\alpha}}{(\alpha-1)r}\Big)^{\frac{1}{1-\alpha}}.$$

It follows that there is a constant C_α such that, for every $r > 0$ and $b > 0$,

$$\Theta_\alpha(0 < \langle \ell^r, 1\rangle \leq b) \leq C_\alpha\, r^{-\frac{\alpha}{\alpha-1}}\, b^{\alpha-1}.$$

Using the branching property as in the proof of Theorem 1.4, we get

$$\Theta_\alpha(\#\{j \leq N_p : \ell^1(\mathcal{T}_j^p) < b_p\}) = \Theta_\alpha(0 < \langle \ell^{2^{-p}}, 1\rangle < b_p) \leq C_\alpha\, 2^{\frac{p}{\alpha-1}}\, p^{-\kappa(\alpha-1)},$$

and from the choice of κ, we have

$$\lim_{p\to\infty} \widetilde{h}_u(2^{-p})\, \#\{j \leq N_p : \ell^1(\mathcal{T}_j^p) < b_p\} = 0\ ,\ \ \Theta_\alpha \text{ a.e.} \tag{34}$$

In order to bound the second term in the right-hand side of (33), we set for every $\sigma \in \mathcal{T}(1)$ and every integer $n \geq 1$,

$$\widetilde{\mathcal{C}}(\sigma, 2^{-n}) = \{\sigma' \in \mathcal{T}(1) : 1 - 2^{-n-1} < d(\rho, \sigma \wedge \sigma') \leq 1 - 2^{-n-2}\}$$

in such a way that $\widetilde{\mathcal{C}}(\sigma, 2^{-n}) \subset B(\sigma, 2^{-n})$. It easily follows from Theorem 4.2 that the random variables $\ell^1(\widetilde{\mathcal{C}}(\sigma, 2^{-n}))$, $n \geq 1$ are independent under the probability measure $\Theta_\alpha(d\mathcal{T})\ell^1(d\sigma)$. Furthermore, simple calculations give for every $\lambda > 0$,

$$\begin{aligned}\Theta_\alpha\Big(\int \ell^1(d\sigma)\, \exp -\lambda \ell^1(\widetilde{\mathcal{C}}(\sigma, 2^{-n}))\Big) &= \exp\Big(-\alpha \int_{2^{-n-2}}^{2^{-n-1}} dr\, u_r(\lambda)^{\alpha-1}\Big)\\ &= \Big(\frac{\lambda^{1-\alpha} + (\alpha-1)2^{-n-1}}{\lambda^{1-\alpha} + (\alpha-1)2^{-n-2}}\Big)^{-\frac{\alpha}{\alpha-1}}.\end{aligned}$$

Hence the law $\widetilde{\nu}$ of $2^{\frac{n}{\alpha-1}} \ell^1(\widetilde{\mathcal{C}}(\sigma, 2^{-n}))$ under $\Theta_\alpha(d\mathcal{T})\ell^1(d\sigma)$ does not depend on n. From the preceding Laplace transform, we also get the existence of a constant $\widetilde{c}_0 > 0$ such that, for every $y \geq 1$,

$$\widetilde{\nu}([y, \infty)) \geq \widetilde{c}_0\, y^{1-\alpha}.$$

Using this lower bound and the previously mentioned independence, the same calculations as in the proof of Theorem 1.4 lead to

$$\Theta_\alpha\Big(\int \ell^1(d\sigma)\,\mathbf{1}_{\widetilde{B}_{n_0,p}}(\sigma)\Big) \leq \exp\Big(-c_0'\Big((\log(p-1))^{1-(1-\alpha)u} - (\log n_0)^{1-(1-\alpha)u}\Big)\Big)$$

where c_0' is a positive constant. Since $1-(1-\alpha)u > 1$, it easily follows that

$$\lim_{p\to\infty} \widetilde{h}_u(2^{-p})b_p^{-1}\int \ell^1(d\sigma)\,\mathbf{1}_{\widetilde{B}_{n_0,p}}(\sigma) = 0\ ,\ \Theta_\alpha \text{ a.e.} \tag{35}$$

Thanks to (33), (34) and (35), the remaining part of the proof is now similar to the end of the proof of Theorem 1.4. □

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