On Gabber's refined uniformization [Ga1]

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1. Statement of refined uniformization and refined de Jong's theorems

1.0. Recall [EGA IV 7] that a scheme X is called *quasi-excellent* if X is locally noetherian, its formal fibers are geometrically regular (i. e. the fibers of $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x} \to \operatorname{Spec} \mathcal{O}_{X,x}$ are regular and remain so after any finite extension), and for any scheme X' of finite type over X, the set of regular points of X' is open. This last condition is implied by the others when X is local. A scheme X is called *excellent* if it is quasi-excellent and universally catenary, i. e. any ring of finite type over a local ring of X satisfies the chain condition. Any scheme of finite type over a quasi-excellent (resp. excellent) scheme is quasi-excellent (resp. excellent). The spectrum of a complete noetherian local ring, or of a Dedekind ring of fraction field of characteristic zero is excellent. Quasi-excellent schemes are universally Japanese. If a noetherian scheme X has the property that any scheme X' integral and of finite type over X is the target of a proper birational map whose source is regular, then X is quasi-excellent.

Theorem 1.1 (Gabber). *Let X be a noetherian, quasi-excellent scheme, and let $Z \subset X$ be a nowhere dense closed subset. Let ℓ be a prime number invertible on X. Then there exists a finite family of maps of finite type $p_i : X'_i \to X$ with the following properties :*

(a) X'_i is regular, connected and the inverse image Z'_i of Z in X'_i is the support of a strict ncd, or the empty space ;

(b) if η'_i is the maximal point of X'_i, $\eta_i = p_i(\eta'_i)$ is a maximal point of X and the fibers

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at the maximal points of X *of the morphism* $p : X' = \coprod X'_i \to X$ *defined by the* p_i*'s are finite ;*

(c) the morphism p *admits a section after base change by a map* $g = g_1g_2g_3g_4$*, where* $g_1 : X_1 \to X_0 = X$ *is a nilpotent immersion,* $g_2 : X_2 \to X_1$ *is a modification,* $g_3 : X_3 \to X_2$ *is finite, flat, of degree prime to* ℓ*, and* $g_4 : X_4 \to X_3$ *is a Nisnevich étale cover.*

By a *modification* we mean a proper, surjective morphism, sending each maximal point to a maximal point, and which is an isomorphism above a dense open subset. A *Nisnevich cover* is an étale covering family such that each point downstairs has a preimage with trivial residual extension.

1.2. Let S be a noetherian scheme and ℓ a prime number. Let $\mathcal{R}_S$ be the category whose objects are reduced S-schemes of finite type whose all maximal points map to maximal points of S with a finite residue field extension and morphisms are S-morphisms (note that because of the condition on residue field extensions, S-morphisms send maximal points to maximal points). The ℓ'*-topology* on $\mathcal{R}_S$ is the topology generated by the covering families of the following two types :

(i) Nisnevich covers

(ii) proper surjective morphisms $X' \to X$ with the property that for any maximal point η of X there exists a maximal point η' of X' above η such that $k(\eta')/k(\eta)$ is finite and of degree prime to ℓ.

Fiber products in $\mathcal{R}_S$ are defined by taking the reduced closure of fiber products at maximal points. Typical examples of surjective maps for the ℓ'-topology are ℓ'*-alterations*, i. e. proper surjective maps in $\mathcal{R}_S$ with prime to ℓ residue field extensions at the maximal points. One can show, using Gruson-Raynaud's flattening theorem, that condition 1.1 (c) is equivalent to saying that p is covering for the ℓ'-topology. One can also show that in 1.1 (c) one can permute g_3 and g_4. See [ILO] for details.

Theorem 1.1 is a *local* resolution theorem. However, as a by-product of the proof, Gabber obtained the following *global* resolution theorems, which improve classical theorems of de Jong [dJ1] :

Theorem 1.3 (Gabber). *Let* X *be a scheme separated and of finite type over a field* k*,* $Z \subset X$ *a nowhere dense closed subset,* ℓ *a prime* $\neq \mathrm{char}(k)$*. Then there exists a finite extension* k' *of* k *of order prime to* ℓ *and an* ℓ'*-alteration* $p : X' \to X$ *over* $\mathrm{Spec}\, k' \to \mathrm{Spec}\, k$*, with* X' *smooth and quasi-projective over* k' *and* $p^{-1}(Z)$ *the support of a strict normal crossings divisor.*

Theorem 1.4 (Gabber). *Let* S *be an excellent trait,* X *a scheme separated and of finite type over* S*,* ℓ *a prime invertible on* S*,* $Z \subset X$ *a nowhere dense closed subset. Then there exists a finite extension* S'/S *of degree prime to* ℓ *and an* ℓ'*-alteration* $p : X' \to X$ *over* $S' \to S$*, a sncd divisor* T' *in* X' *such that* X' *is regular and quasi-projective over* S'*,* $Z' = p^{-1}(Z)$ *is a subdivisor of* $(X'_{s'})_{\mathrm{red}} \cup T'$ *(*s' *denoting the closed point of* S'*), with* (X', T') *étale locally given by* $X' = S'[t_1, \cdots, t_n]/(t_1^{a_1} ... t_r^{a_r} - \pi)$*,* π *a uniformizing parameter in* S'*,* $\gcd(p, a_1, \cdots, a_r) = 1$*,* $p = \mathrm{char}(k)$*,* k *the residue field of* S*, and* $T' = S'[t_1, \cdots, t_n]/(t_{r+1} \cdots t_m)$*.*

2. Reduction to the local complete case

2.0. As the problem of finding a uniformizing morphism p is local for the Nisnevich topology, and henselization preserves quasi-excellency [EGA IV 18.7.6], by a standard limit argument we may assume that the scheme X of 1.1 is *local henselian.* For the convenience of further notations, we will denote it by S. The goal of this section is to show that from uniformization data for the pair consisting of the completion $\widehat{S}$ of S at the closed point and the trace $\widehat{Z}$ of Z on $\widehat{S}$ (the completion of a closed subscheme of S having Z as underlying subspace) one can deduce similar data for (S, Z). This relies on Artin-Popescu's algebraization theorem and a method of approximation *à la Artin-Rees* developed by Gabber, for which we give some algebraic preliminaries in 2.1-2.3. The main geometric result needed for the algebraization of uniformization data is 2.6.

2.1. Les $\mathcal{A}$ be an abelian category having countable sums. If M is an object of $\mathcal{A}$ endowed with a (decreasing) filtration $F = (F^n)_{n\in\mathbb{Z}}$, we define, as usual, $M(n)$ to be M endowed with the filtration $F^r(M(n)) = F^{r+n}M$. We set

$$M(*) = \oplus_{n\in\mathbb{Z}} M(n). \tag{2.1.1}$$

This is a $\mathbb{Z}$-graded object of $\mathcal{A}$, equipped with a (compatible) filtration $F^rM(*) = \oplus_{n\in\mathbb{Z}} F^rM(n)$. For an interval $[a, b]$ of $\mathbb{Z}$, we define the *associated graded* of M *of width* $[a, b]$ (or the $[a, b]$-graded of M, for short) to be

$$\mathrm{gr}^{\cdot}_{[a,b]}(M) = (F^a/F^b)M(*) = \oplus_{n\in\mathbb{Z}} F^{a+n}M/F^{b+n}M. \tag{2.1.2}$$

We sometimes write $\mathrm{gr}_{[a,b]}$ instead of $\mathrm{gr}^{\cdot}_{[a,b]}$. For $[a, b] = [0, k]$, we simply say *of width k* (or k-graded) and write $\mathrm{gr}^{\cdot}_k$ (or gr_k) instead of $\mathrm{gr}^{\cdot}_{[0,k]}$. For $k = 1$, $\mathrm{gr}_1(M)$ is the usual associated graded $\mathrm{gr}\, M = \oplus F^nM/F^{n+1}M$. In general, $\mathrm{gr}_{[a,b]} M$ is a succesive extension of shifted gr : the filtration induced by F on $\mathrm{gr}_{[a,b]} M$, for $b > a$, has for associated graded

$$\mathrm{gr}_F(\mathrm{gr}_{[a,b]}M) = \oplus_{a\le r\le b-1} \mathrm{gr}M(r).$$

2.2. We'll be mostly interested in the following situation : A is a comutative ring of a topos (in practice, the structural sheaf of rings of a noetherian scheme or formal scheme), I an ideal of A, and the I-adic filtration on A-modules M, defined by $F^nM = I^nM$ for $n \geq 0$ and $F^nM = M$ for $n \leq 0$. Then, for $k \geq 1$,

$$\mathrm{gr}_{I,k}A = A/I \oplus A/I^2 \oplus \cdots \oplus A/I^k \oplus I/I^{k+1} \oplus I^2/I^{k+2} \oplus \cdots$$

(with A/I^k in degree zero) is a graded A/I^k-algebra, which is a $(k-1)$-*thickening* of the usual graded algebra $\mathrm{gr}_I A$. For example, for $k = 2$, $\mathrm{gr}_{I,2} A$ is an A/I^2-extension

$$0 \to \mathrm{gr}_I A(1) \to \mathrm{gr}_{I,2}(A) \to \mathrm{gr}_I(A) \to 0$$

of $\mathrm{gr}_I A$ by $\mathrm{gr}_I A(1)$ viewed as an ideal of square zero. If M is an A-module,

$$\mathrm{gr}_{I,k}M = M/IM \oplus M/I^2M \oplus \cdots \oplus M/I^kM \oplus IM/I^{k+1}M \oplus I^2M/I^{k+2}M \oplus \cdots$$

(with M/I^kM in degree zero) is a graded $\mathrm{gr}_{I,k} A$-module, generated over $\mathrm{gr}_{I,k} A$ by its degree zero part, i. e. the natural map

$$M/I^kM \otimes_{A/I^k} \mathrm{gr}_{I,k}A \to \mathrm{gr}_{I,k}M$$

is surjective. If B is a commutative A-algebra, then $\mathrm{gr}_k B$ is a graded $\mathrm{gr}_k A$-algebra (generated, as a module, by its degree zero part B/I^kB). We will drop the index I when no confusion can arise.

Let L, M be A-modules, and k a positive integer. Following Gabber, we define a (I,k)-*morphism* from L to M (or k-morphism if there is no ambiguity on I) as a morphism $u : \mathrm{gr}_k L \to \mathrm{gr}_k M$ of $\mathrm{gr}_k A$-graded modules. If B, C are A-algebras, an (I,k)-morphism from B to C (or k-morphism, for short) is a morphism $u : \mathrm{gr}_k B \to \mathrm{gr}_k C$ of $\mathrm{gr}_k A$-graded algebras. Composition of k-morphisms is defined in the obvious way. We say that L and M (resp. B, C) are (I,k)-*close* (or k-close, for short) if there exists a k-isomorphism from L to M (resp. B to C).

We have similar definitions for complexes. If L, M are complexes of A-modules, and k is a positive integer, an (I,k)-*morphism* (or k-morphism) from L to M is a morphism $u : \mathrm{gr}_k L \to \mathrm{gr}_k M$ of complexes of $\mathrm{gr}_k A$-graded modules, and we say that L and M are (I,k)-*close* (or k-close) if there exists a k-isomorphism from L to M. If L and M are k-close, the cohomology sheaves H^iL and H^iM are not k-close in general. We have, however, the following result (due to Gabber), which plays a key role in the subsequent approximation of formal geometric data.

Lemma 2.3. *Let A be a noetherian ring and I an ideal contained in the radical of A. Let L be a complex of finitely generated A-modules, concentrated in degrees in $[-2,0]$, and such that $H^{-1}L = 0$. Then there exist integers $k_0 > c \geq 0$ such that for all $k \geq k_0$, if M is a complex of finitely generated A-modules, concentrated in degrees in $[-2,0]$, which is (I,k)-close to L, then, $H^{-1}M = 0$ and for all $i \in \mathbb{Z}$, H^iM (resp. Z^iM, B^iM) is $(I, k-c)$-close to H^iL (resp. Z^iL, B^iL).*

The proof relies on a lemma of Serre [Se, II 15] on the degeneration of the spectral sequence of a filtered complex along a diagonal $p+q=m$, in the presence of Artin-Rees assumptions. The hypothesis that $H^{-1}L = 0$ is essential, as the case where $L = [\, A \xrightarrow{0} A \,]$ trivially shows. Recall that an *Artin-Rees constant* for a submodule F of a finitely generated A-module E (and the I-adic filtration) is an integer $c \geq 0$ such that $I^nE \cap F \subset I^{n-c}F$ for all $n \geq c$. Serre's result is that if K is a bounded complex of finitely generated A-modules, then c is an Artin-Rees constant for the images of the differentials of the spectral sequence of K filtered by the I-adic filtration if and only if this spectral sequence degenerates at E_{c+1}. When the components of L and M are free, this easily yields the conclusion of 2.3 for B^{-1}, B^0 and H^0. This partial result suffices for the geometric application in 2.6. (See [ILO] for details.)

2.4. Let A be a henselian, noetherian, local ring, with maximal ideal $\mathbf{m}$, $S = \operatorname{Spec} A$, $\widehat{S} = \operatorname{Spec} \widehat{A}$. We have a chain of morphisms

$$s = S_0 \to S_1 \to \cdots \to S_n \to \cdots \to \widehat{S} \to S,$$

where $S_n = \operatorname{Spec} A_n$, $A_n = A/\mathbf{m}^{n+1}$. The formal scheme $\mathcal{S} = \operatorname{Spf} \widehat{A}$ defined by $\widehat{S}$ is the direct limit of the S_n's.

Let $\mathcal{X}$ be a noetherian formal scheme, of finite type over $\mathcal{S}$ (in particular, $\widehat{\mathbf{m}}\mathcal{O}_{\mathcal{X}}$ is an ideal of definition of $\mathcal{X}$, and $\mathcal{X} = \text{ind.lim}\, \mathcal{X}_n$, where $\mathcal{X}_n = \mathcal{X} \times_{\mathcal{S}} S_n$ is a scheme of finite type over S_n). Consider the $\widehat{\mathbf{m}}$-adic filtration on $\mathcal{O}_{\mathcal{X}}$. For $k > 0$, the graded ring $\operatorname{gr}_k \mathcal{O}_{\mathcal{X}}$ (2.2) is a sheaf, on $\mathcal{X}_0$, of graded algebras over the graded ring $\operatorname{gr}_k \mathcal{O}_{\mathcal{S}}$. It is quasi-coherent on $\mathcal{X}_{k-1}$ as a sheaf of algebras over $\mathcal{O}_{\mathcal{X}_{k-1}} = \mathcal{O}_{\mathcal{X}}/\widehat{\mathbf{m}}^k\mathcal{O}_{\mathcal{X}}$. The *$k$-extended normal cone*

$$C_k(\mathcal{X}) := \operatorname{Spec} \operatorname{gr}_k \mathcal{O}_{\mathcal{X}}$$

is an $k-1$-thickening of the (usual) normal cone (of $\mathcal{X}_0$ in $\mathcal{X}$),

$$C_1(\mathcal{X}) = \operatorname{Spec}(\oplus \widehat{\mathbf{m}}^n \mathcal{O}_{\mathcal{X}}/\widehat{\mathbf{m}}^{n+1}\mathcal{O}_{\mathcal{X}}),$$

extending the thickening $\mathcal{X}_0 \to \mathcal{X}_{k-1}$ of their vertices. If $\mathcal{M}$ is a coherent sheaf on $\mathcal{X}$, $\operatorname{gr}_k \mathcal{M}$ (calculated for the $\widehat{\mathbf{m}}$-adic filtration) is a graded module over $\operatorname{gr}_k \mathcal{O}_{\mathcal{X}}$, quasi-coherent over $\mathcal{X}_{k-1}$ (and even over $C_k(\mathcal{X})$).

Let now $\mathcal{X}$, $\mathcal{Y}$ be formal schemes of finite type over $\mathcal{S}$, and $k > 0$. Assume we are given a morphism $\varphi : \mathcal{X}_{k-1} \to \mathcal{Y}_{k-1}$ of S_{k-1}-schemes. We define an *$(\mathbf{m}, k)$-morphism* (or *k-morphism*) *from $\mathcal{X}$ to $\mathcal{Y}$ above* φ (or *extending* φ) as a (homogeneous) morphism of cones $f : C_k(\mathcal{X}) \to C_k(\mathcal{Y})$ over $C_k(\mathcal{S})$ extending φ, i. e. a morphism of graded $\operatorname{gr}_k \mathcal{O}_{\mathcal{S}}$-algebras $\varphi^{-1} \operatorname{gr}_k \mathcal{O}_{\mathcal{Y}} \to \operatorname{gr}_k \mathcal{O}_{\mathcal{X}}$ over $\varphi^{-1}\mathcal{O}_{\mathcal{Y}_{k-1}} \to \mathcal{O}_{\mathcal{X}_{k-1}}$. If $\mathcal{M}$ (resp. $\mathcal{N}$) is a coherent sheaf on $\mathcal{X}$ (resp. $\mathcal{Y}$), a *k-morphism from $\mathcal{N}$* to $\mathcal{M}$ *above* f is a morphism of graded modules $u : \varphi^{-1}(\operatorname{gr}_k \mathcal{N}) \to \operatorname{gr}_k \mathcal{M}$ above f. Composition of k-morphisms f, or pairs (f, u), is defined in the obvious way. In particular, we have a notion of k-isomorphism (above an isomorphism φ). We say that $\mathcal{X}$ and $\mathcal{Y}$ (resp. $(\mathcal{X}, \mathcal{M})$ and $(\mathcal{Y}, \mathcal{N})$) are *$k$-close* if there exists a k-isomorphism from $\mathcal{X}$ to $\mathcal{Y}$ (resp. from $(\mathcal{X}, \mathcal{M})$ to $(\mathcal{Y}, \mathcal{N})$) (above an isomorphism $\mathcal{X}_{k-1} \to \mathcal{Y}_{k-1}$). Finally, if X, Y are *schemes* of finite type over $\widehat{S}$ (or S), we define k-morphisms from X to Y as k-morphisms from $\widehat{X}$ to $\widehat{Y}$, their formal completions along their special fibers X_s, Y_s. We have a similar definition for k-morphisms from (X, M) to (Y, N), where M (resp. N) is a coherent sheaf on X (resp. Y).

2.5. Assume from now on that S is *quasi-excellent* (or, equivalently, *excellent* , as A is henselian). Consider a commutative diagram

$$\begin{tikzcd} & T \arrow[d, "f"] \\ \widehat{S} \arrow[ur, "g"] \arrow[r, "i"'] & S \end{tikzcd} ,$$

with T an S-scheme of finite type. As A is excellent, the morphism $\widehat{S} \to S$ is *regular.* By Popescu's theorem ([Po1], [Po2]), $\widehat{A}$ is therefore a filtering inductive limit of smooth A-algebras of finite type. As A is moreover henselian, it follows that, for all $n \in \mathbb{N}$, there exists a *section* u of f such that

$$ui \equiv g \quad \text{mod. } \mathbf{m}^{n+1},$$

which means that the compositions $S_n \to \widehat{S} \to S \to T$ and $S_n \to \widehat{S} \to T$ (where the last maps are respectively u and g) are equal. Such a section u is said to be $n+1$-*close* to g. It depends on n, and is not unique.

Write $\widehat{A}$ as a filtering inductive limit of A-algebras of finite type B_α, indexed by a filtering ordered set E. Let $T_\alpha = \operatorname{Spec} B_\alpha$, so that we get commutative diagrams

$$\begin{tikzcd} & T_\alpha \arrow[d, "f_\alpha"] \\ \widehat{S} \arrow[ur, "g_\alpha"] \arrow[r, "i"] & S \end{tikzcd} ,$$

with $\widehat{S} = \text{inv.lim}\, T_\alpha$. Let X be an $\widehat{S}$-scheme of finite type. Then there exists $\alpha_0 \in E$ and a *model* of X over T_{α_0}, i. e. a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_{\alpha_0} \arrow[d] \\ \widehat{S} \arrow[r, "g_{\alpha_0}"] & T_{\alpha_0} \end{tikzcd} ,$$

with X_{α_0} of finite type over T_{α_0}. Such a model is not unique. However, if $X_{\alpha_0}/T_{\alpha_0}$ and $X_{\alpha_1}/T_{\alpha_1}$ are two such models, then there exists α_2 greater than or equal to α_0 and α_1 such that X_{α_0} and X_{α_1} become isomorphic on T_{α_2}. For $\alpha \geq \alpha_0$, denote by X_α/T_α the model of X deduced by pull-back of $X_{\alpha_0}/T_{\alpha_0}$.

Let $n \in \mathbb{N}$ and $u : S \to T_\alpha$ be a section of f_α which is $n+1$-close to g_α, and consider the pull-back X_u of X_α by u, i. e. the S-scheme of finite type defined by the cartesian diagram

$$\begin{tikzcd} X_\alpha \arrow[d] & X_u \arrow[l] \arrow[d] \\ T_\alpha & S \arrow[l, "u"'] \end{tikzcd} .$$

Let $j_n : S_n \to \widehat{S}$ be the inclusion. As $g_\alpha j_n = u i j_n$, X_α, the pull-back of X_u to S_n is identified with X_n, i. e. we have a cartesian square

(2.5.1)
$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & X_u \arrow[d] \\ S_n \arrow[r] & S \end{tikzcd} .$$

If Y is a second $\widehat{S}$-scheme of finite type, and $h : X \to Y$ an $\widehat{S}$-morphism, one can similarly find an α_0, a model Y_{α_0} of Y over T_{α_0}, and a model h_{α_0} of h. Then, given u as above, we get an S-morphism $h_u : X_u \to Y_u$ inducing $h_n : X_n \to Y_n$.

The goal is to show that if X (resp. h) enjoys certain properties, e. g. reducedness, regularity, etc. (resp. smoothness, regular immersion, etc.), then, for α and n sufficiently large, X_u (resp. h_u) enjoys the same properties in an open neighborhood of the special fiber X_s. The main technical tool to do this is the following result (of Gabber) :

Theorem 2.6. *Let S and the projective system $(T_\alpha)_{\alpha\in E}$ be as in 2.5. Let X be an $\widehat{S}$-scheme of finite type, X_{α_0} a model of X over T_{α_0}, X_α the model over T_α deduced by base change. Then there exist $\alpha_1 \in E$, $\alpha_1 \geq \alpha_0$, and integers $n_0 \geq c > 0$ satisfying the following property :*

For any $n \in \mathbb{N}$ with $n \geq n_0$, $\alpha \in E$, $\alpha \geq \alpha_1$, and any section u of f_α which is $(n+1)$-close to g_α, there exists a unique $(n+1-c)$-isomorphism (2.4)

$$a : X \to X_u$$

extending the isomorphism $X_{n-c} \to (X_u)_{n-c}$ deduced from (2.5.1).

As $\mathrm{gr}_k\, \mathcal{O}_X$ is generated over $\mathrm{gr}_k\, \mathcal{O}_S$ by $\mathcal{O}_{X_{k-1}}$, the uniqueness is clear. By the uniqueness, one is reduced to the case where X is affine. Embedding X in a standard affine space $\mathbb{A}^m_S$, one is reduced to a problem of approximation of coherent *modules* on $Z = \mathbb{A}^m_S$, or equivalently, of finitely generated modules over the ring $B = \widehat{A}[t_1, \cdots, t_m]$, equipped with the I-adic filtration, where $I = \widehat{\mathbf{m}}B$. Given a finitely generated B-module M, choose a presentation of M of length 2 by free finitely generated B-modules :

$$(*) \qquad L^{-2} \to L^{-1} \to L^0 \to M \to 0.$$

In the case we are interested in, $M = \Gamma(X, \mathcal{O}_X)$, and M is the pull-back of the B_{α_1}-module $M_{\alpha_1} = \Gamma(X_{\alpha_1}, \mathcal{O})$, where $B_{\alpha_1} = A_{\alpha_1}[t_1, \cdots, t_m]$, with $\alpha_1 \geq \alpha_0$ such that $X_{\alpha_1} \subset \mathbb{A}^m_{T_{\alpha_1}}$ is a model of $X/\widehat{A}$ dominating $X_{\alpha_0}/T_{\alpha_0}$, and $T_{\alpha_1} = \mathrm{Spec}\, A_{\alpha_1}$, $L^0 = B$ and the last map is a surjective homomorphism of rings. Enlarging α_1 we may assume that $(*)$ comes by base change from a similar presentation of M_{α_1},

$$(*)_{\alpha_1} \qquad L^{-2}_{\alpha_1} \to L^{-1}_{\alpha_1} \to L^0_{\alpha_1} \to M_{\alpha_1} \to 0.$$

with $L^i_{\alpha_1}$ free of the same rank as L^i, $L^0_{\alpha_1} = B_{\alpha_1} = A_{\alpha_1}[t_1, \cdots, t_m]$, and the last map a surjective homomorphism of rings. We denote by $L_\alpha \to M_\alpha$ $(= \Gamma(X_\alpha, \mathcal{O}))$ the pull-back of $(*)_{\alpha_1}$ to B_α. Choose constants $k_0 = n_0 + 1 > c$ as in 2.3 for the complex $L = g^*_\alpha L_\alpha$, where $g_\alpha : \widehat{S} \to T_\alpha$. Let $n \geq n_0$ and $u : S \to T_\alpha$ be a section of T_α/S which is $(n+1)$-close to g_α. Then the complex of free B-modules $K_u := (ui)^* L_\alpha$ (where $i : \widehat{S} \to S$) can be identified componentwise with L. As u is $(n+1)$-close to g_α, the matrix coefficients of the differentials of K_u are congruent to those of L modulo $\widehat{\mathbf{m}}^{n+1}$, and therefore we have an $(n+1)$-isomorphism from $\mathrm{gr}_{n+1}\, L$ to $\mathrm{gr}_{n+1}\, K_u$ given by the identity componentwise. By 2.3, $H^0 L$ and $H^0 K_u$ are $(n+1-c)$-close as B-modules, and *a fortiori* as $\widehat{A}$-algebras, by an isomorphism compatible with the isomorphism $\Gamma(X_{n-c}, \mathcal{O}) \to \Gamma((X_u)_{n-c}, \mathcal{O})$ deduced from (2.5.1).

The problem with approximation is that given α, n, and the section u of f_α which is $(n+1)$-close to g_α, the fact that $(X_u)_n$ and X_n are isomorphic doesn't tell us much on X_u, as compared to X. For example, if we know the dimension of X at a point x of its special fiber, we can't even say anything about the dimension of X_u at x. The notion of k-closeness repairs this. The following lemma (applied to localizations of the affine space $\mathbb{A}^m_{\widehat{S}}$ above at points of the special fiber, and the ideal generated by $\widehat{\mathbf{m}}$), combined with 2.6, will allow to reduce uniformisation to the complete local case :

Lemma 2.7. *Let A be a noetherian local ring, with maximal ideal* $\mathbf{m}$*, and* $I \subset \mathbf{m}$ *an ideal. Let $B = A/J$ and $B' = A/J'$ be quotients of A.*

(i) If B and B' are $(I,1)$-close, then $\dim B = \dim B'$*.*

(ii) If B and B' are $(I,2)$-close, and B is regular, then B' is regular.

Assertion (i) follows from the formula

$$\dim B = \dim \mathrm{gr}_{IB} B$$

(itself a consequence, for $\operatorname{codim} V(I) \geq 1$, of $\dim B = \dim \operatorname{Proj}(\oplus_{n\in\mathbb{N}} I^n B) = 1 + \dim \operatorname{Proj}(\oplus_{n\in\mathbb{N}}(I^n B/I^{n+1}B))$. For assertion (ii), denote by ω_B the Zariski cotangent space of B at its closed point. Then, B is regular if and only if $\dim B = \dim \omega_B$. In view of (i), assertion (ii) follows from the fact that $B \to B/I^2B$ induces an isomorphism on ω.

Gabber shows that reducedness and normality are similarly preserved by (I,n)-closeness, for n sufficiently large. Is it the case, more generally, for the R_k and S_k conditions ?

2.8. Let $Z \subset S$ be a closed, nowhere dense subsscheme, $\widehat{Z} \subset \widehat{S}$ its trace on $\widehat{S}$, and suppose we are given ℓ'-uniformization data for $\widehat{Z} \subset \widehat{S}$, i. e. a commutative diagram with cartesian square

(2.8.1)
$$\begin{tikzcd} D' \arrow[r] \arrow[d] & X' \arrow[d, "p'"] & \\ \widehat{Z} \arrow[r] & \widehat{S} & Y' \arrow[l, "q'"'] \arrow[ul] \end{tikzcd},$$

where X' is the disjoint sum of a finite number of regular schemes X'_i, D'_{red} cuts out on each X'_i a strict normal crossings divisor $(D'_{\mathrm{red}})_i$ (or the empty space), p' satisfies condition (b) of 1.1, and q' is a composition $q'_1q'_2q'_3q'_4$, with q'_1 a thickening, q'_2 a proper modification, q'_3 a finite flat surjective map of generic degree prime to ℓ, q'_4 a Nisnevich cover. We want to deduce from (2.8.1), by approximation, an analogous diagram

(2.8.2)
$$\begin{tikzcd} D \arrow[r] \arrow[d] & X \arrow[d, "p"] & \\ Z \arrow[r] & S & Y \arrow[l, "q"'] \arrow[ul] \end{tikzcd}.$$

Finding the triangle is not so hard, from 2.6 and 2.7. The square demands extra work (see [ILO] for details).

3. Cohen-Gabber

3.1. *Glimpses on the strategy of the proof of 1.1.*

As we are reduced to the local henselian (or even complete) case, we may in particular assume the scheme X of 1.1 of finite Krull dimension. We prove 1.1 by induction on this dimension. We assume that it holds for all pairs (X,Z) with $\dim X \leq d-1$ and we prove it for X of dimension d. By the previous reduction it is enough to prove it for X local, complete, of dimension d. A refinement, due to Gabber, of Cohen's structure theorem, explained in this section, will then allow us to fiber X in curves over a quasi-excellent scheme of dimension $d-1$. This will be done in the next section. Then the inductive hypothesis and de Jong's nodal curve theorem reduce the problem to a problem of log geometry : this reduction is done in section 5. Finally, this last problem is solved in sections 6, 7, 8.

A classical theorem of Cohen [EGA 0_{IV} 19.8.8] asserts that if A is an integral, noetherian, complete local ring of equicharacteristic and residue field k, and of dimension d, then A is finite over a subring of the form $k[[t_1, \cdots, t_d]]$. Gabber improves this, both in equal characteristic $p > 0$ and in mixed characteristic $(0, p)$. We start with the equal characteristic p theorem, whose statement, in its equivariant form, is used in the proof of the mixed characteristic theorem.

Theorem 3.2 (Gabber). *Let A be an equicharacteristic, reduced, complete, noetherian local ring, of dimension d, with residue field $k = A/\mathbf{m}$ of characteristic $p > 0$, endowed with an action of a finite group G of order invertible in k. Then there exists an injective G-equivariant homomorphism $h : k[[t_1, \cdots, t_d]] \to A$, with $k \to A$ lifting the identity of k, G fixing the t_i's, and h finite and generically étale.*

3.3. **Step 1** : *The case A equidimensional, and $G = \{1\}$.*

This is the crucial case. The main bulk of the proof consists in showing the existence of a field of representatives κ of k in A (i. e. a subfield of A going isomorphically to k) such that the generic rank of $\widehat{\Omega}^1_{A/\kappa}$ is equal to d. If $b = (b_i)_{i \in I}$ is a p-basis of k, fields of representatives κ correspond to liftings $\widetilde{b}$ of b in A (there is a unique κ containing the $\widetilde{b}_i$). One makes a preliminary choice of $\widetilde{b}$ and of a system of parameters $y = (y_1, \cdots, y_d)$ of A, so that A is finite over its subring (of formal power series) $\kappa[[y_1, \cdots, y_d]]$, κ corresponding to $\widetilde{b}$. A technical difficulty is that the p-base b is not necessarily finite. Using a standard stabilization result for differentials [Ma, §30, Lemma 6], one first finds a cofinite subfield κ' of κ such that the generic rank of $\widehat{\Omega}^1_{A/\kappa'}$ is equal to $d + [\kappa : \kappa']$. One then modifies a finite number of the $\widetilde{b}_i$'s (and consequently κ) in order to ensure that the generic rank of $\widehat{\Omega}^1_{A/\kappa}$ is d. Finally, one replaces the system of parameters y by the system of parameters t defined by $t_i = y_i^p(1 + f_i)$, where f_i $(1 \leq i \leq d)$ is a system of elements of $\mathbf{m}$ such that the $df_i \in \widehat{\Omega}^1_{A/\kappa}$'s are linearly independent at each maximal point. Then A is finite and generically étale over $\kappa[[t_1, \cdots, t_d]]$.

3.4. **Step 2** : *The general case*

We first need the following lemma, for which no reference could be found. A proof (due to Gabber) is given in [ILO].

Lemma 3.4.1. *Let A be a noetherian ring endowed with an action of a finite group G of order invertible on A, and let A^G be its subring of invariants. Then A^G is noetherian and A is finite over A^G. If A is reduced, so is A^G, and A is generically étale over A^G.*

The trace operator $\mathrm{Tr} = (1/|G|) \sum_{g \in G} g$ shows that, if J is an ideal of A^G, $J = (JA) \cap A^G$, hence A^G is noetherian, and reduced if A is. *A priori*, A is only integral over A^G. To show that A is actually finite over A^G, one first reduces to the case where A is reduced, then using the form $\mathrm{Tr}(xy)$, one reduces to the case where A is a product of fields, and eventually a field, which is classical (and shows the last assertion).

To show 3.2 in the equidimensional case, apply 3.3 to A^G, which is of the same type as A, with residue field k^G : A^G is finite over $k^G[[t_1, \cdots, t_d]]$. As k is étale over k^G, there is a unique, hence G-equivariant, k^G-homomorphism $k \to A$ lifting the identity of k. The non equidimensional case requires a little additional work.

We now come to the mixed characteristic case :

Theorem 3.5 (Gabber). *Let A be an integral, normal, complete noetherian local ring of dimension $d \geq 2$, with residue field k of characteristic $p > 0$ and fraction field of characteristic zero. Let ℓ be a prime number invertible in k. Then there exist the following data :*

- a finite extension B of A, endowed with an action of an ℓ-group H, with B integral, normal, complete local, with residue field k', such that the degree of B^H over A is prime to ℓ ;

- a mixed characteristic complete discrete valuation ring C with residue field k', endowed with an action of H compatible with that of H on k' ;

- an H-equivariant local morphism $u : C[[t_1, \cdots, t_{d-1}]] \to B$, where the t_i's are fixed by H, such that $\mathrm{Spec}\, u$ *is finite and étale over an open subset whose intersection with the special fiber is dense.*

The scheme $X = \mathrm{Spec}\, A$ is over $\mathbb{Z}_p$. When its fiber $X_p = X \otimes \mathbb{F}_p$ is reduced, it suffices to apply the case $G = \{1\}$ of 3.2 to X_p : X_p is finite and generically étale over a subring $\mathrm{Spec}\, k[[x_1, \cdots, x_{d-1}]]$; if $C(k)$ is a Cohen ring for k, lifting the x_i's to t_i's in A and lifting $C(k) \to k$ to $C(k) \to A$ yields a homomorphism $C(k)[[t_1, \cdots, t_{d-1}]] \to A$ which has the desired properties (finite, and generically étale on the special fiber). No extension of A is needed. The general case requires the full force of 3.2 and the following theorem of Epp (see [ILO] and [Ga2], proof of th. 2.2 for details) :

Theorem 3.6 (Epp [Ep]). *Let S (resp. T) be a complete trait, with closed point s (resp. t) of characteristic $p > 0$, $g : T \to S$ be a dominant, local morphism. Assume that $k(s)$ is perfect, and that the maximal perfect subfield k_0 of $k(t)$ is algebraic over $k(s)$. Then there exists a finite extension of traits $S' \to S$ such that if T' denotes the normalization of $(T \times_S S')_{\mathrm{red}}$, the special fiber $T'_{s'}$ (where s' is the closed point of S') is reduced.*

Remark 3.7. As T. Saito observes, the special fiber of B^H in 3.5 is not necessarily reduced, and in particular, $\mathrm{Spec}\, B^H$ is not in general étale over an open subset of $\mathrm{Spec}\, C^H[[t_1, \cdots, t_{d-1}]]$ having a dense intersection with the special fiber. He gives the following example. Let k be an algebraically closed field of characteristic $p > 0$, $W = W(k)$ the ring of Witt vectors on k, ℓ a prime different from p, $A = W[[x, y]]/(x^\ell y - p)$. Let B be the normalization of $A \otimes_W W[t]/(t^\ell - p)$. Then $H = \mu_\ell(k)$ acts on B (via its action on $W[t]/(t^\ell - p)$) and B satisfies the properties of 3.5. However, B^H is A, whose special fiber is not reduced.

4. Refined partial algebraization

Theorem 4.1 (Gabber). *(a) Let $X = \operatorname{Spec} A$, where A is a reduced, complete, noetherian local ring of dimension $d \geq 1$, and of equicharacteristic $p > 0$. Then there exists a diagram*

(4.1.1)
$$\begin{tikzcd} X \arrow[r, "g"] & X' \arrow[d, "h"] \\ & S \end{tikzcd},$$

and a closed point x' in X', where S is regular, complete, noetherian, local of equicharacteristic p and of dimension $d-1$, h is a morphism of finite type sending x' to the closed point of S, and g induces an isomorphism from X to the completion of X' at x'. Moreover, for any finite collection Z_i $(1 \leq i \leq n)$ of closed subschemes of X, there exists a diagram (4.1.1) and closed subschemes Z'_i of X' such that the pull-back of Z'_i by g is Z_i for all i.

(b) Let $X = \operatorname{Spec} A$, where A is a normal, complete, noetherian local ring, of mixed characteristic $(0,p)$ and dimension $d \geq 2$. Let ℓ be a prime number invertible on X. Then there exists a diagram

(4.1.2)
$$\begin{tikzcd} X & X_1 \arrow[l, "q"'] \arrow[r, "g"] & X' \arrow[d, "h"] \\ & & S \end{tikzcd}$$

and a closed point x' in X', where S is regular, complete, noetherian, local of mixed characteristic $(0,p)$ and of dimension $d-1$, h is a morphism of finite type sending x' to the closed point of S, q is a finite, surjective, local morphism of generic degree prime to ℓ, with X_1 normal, local, and g induces an isomorphism of X_1 with the completion of X' at x'. Moreover, for any finite collection Z_i $(1 \leq i \leq n)$ of closed subschemes of X, there exists a choice of (4.1.2) and closed subschemes Z'_i of X' such that the pull-back of Z'_i by g is equal to the pull-back of Z_i by q for all i.

The proof of (a) (resp. (b)) will combine Cohen-Gabber theorem 3.2 (resp. 3.5) with the following algebraization result of Elkik :

Theorem 4.2 (Elkik, [El, th. 5]). *Consider a cartesian diagram of affine schemes*

$$\begin{tikzcd} \widehat{Y} \arrow[r] \arrow[d] & Y \arrow[d] \\ \widehat{X} \arrow[r] & X \end{tikzcd},$$

where X is an affine noetherian scheme, Y a closed subscheme, and $\widehat{X}$ is the completion of X along Y (hence $\widehat{Y} \xrightarrow{\sim} Y$). Assume that the pair (X, Y) is henselian. Then the pull-back functor from the category of finite X-schemes étale over $X - Y$ to that of finite $\widehat{X}$-schemes étale over $\widehat{X} - \widehat{Y}$ is an equivalence.

Recall that the fact that the pair (X, Y) is henselian means that any étale map $X' \to X$ which is an isomorphism above Y admits a section. Equivalently, if $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} A/I$, for any polynomial $f \in A[T]$, any root b of the reduction $\overline{f}$ of f mod I such that $\overline{f}'(b)$ is a unit can be lifted to a root of f.

4.3. *Sketch of proof of 4.1 (a)*

By 3.2 (with $G = \{1\}$) A is finite and generically étale over $k[[t_1, \cdots, t_d]]$. We may assume that A is étale outside $V(f)$ for a non zero $f \in k[[t_1, \cdots, t_d]]$. By Weierstrass preparation theorem, after a change of coordinates, we may assume that f is a Weierstrass polynomial $t_d^N + \sum_{i<N} a_i t_d^i$, with a_i in the maximal ideal of $R = k[[t_1, \cdots, t_{d-1}]]$. Let $R\{t_d\}$ the the henselization of $R[t_d]$ at the origin. Using Weierstrass division theorem, one sees that the (f)-adic completion of $R\{t_d\}$ is $R[[t_d]]$. One checks moreover that the pair $(R\{t_d\}, (f))$ is henselian. By 4.2 applied to $(\operatorname{Spec} R\{t_d\}, \operatorname{Spec} R\{t_d\}/(f))$, one finds a reduced noetherian local ring $\widetilde{A}$ finite over $R\{t_d\}$ and étale outside $V(f)$ such that $A = R[[t_d]] \otimes_{R\{t_d\}} \widetilde{A}$ (in other words, A is the t_d-adic completion of $\widetilde{A}$). Writing $\operatorname{Spec} R\{t_d\}$ as a filtering projective limit of étale neighborhoods U_α of the origin in $\operatorname{Spec} R[t_d]$, one can descend $\operatorname{Spec} \widetilde{A}$ to such a U_α, yielding the desired maps g and h in (4.1.1). Let Z be a finite family of closed subschemes of X. We want to descend it to a suitable $\widetilde{X} = \operatorname{Spec} \widetilde{A}$. Suppose first that Z is defined by a single primary ideal. If $\dim Z = d$, Z is an irreducible component of X, and therefore is the completion of an irreducible component of $\widetilde{X}$ (by Popescu, the completion of an integral excellent henselian local scheme is integral). If $\dim Z < d$, by Weierstrass again, one can change coordinates to simultaneously ensure that X is étale outside $V(f)$ with f the above Weierstrass polynomial, and Z is finite over $\operatorname{Spec} R$ (and *a fortiori* finite over $\widetilde{A}$. Then Z descends to $\widetilde{X}$ (if $\widehat{B}$ is the J-adic completion of a local noetherian ring, any quotient of $\widehat{B}$ which is finite over B descends to B, as an Artin-Rees argument shows, [ILO]). The case of a finite family (Z_i) is treated similarly.

4.4. *Sketch of proof of 4.1 (b) ([Ga2], proof of Th. 2 (2)).*

We consider the H-equivariant morphism $u : C[[t_1, \cdots, t_{d-1}]] \to B$ of 3.5. By a change of coordinates, we may assume that u is étale on the open set of invertibiliy of an element $f \notin (\mathbf{m}_C, t_1, \cdots, t_{d-2})$. One can make f an H-invariant Weierstrass polynomial $f \in R[t_{d-1}]$, where $R = C[[t_1, \cdots, t_{d-2}]]$. Then, applying 4.2 as above to the henselian pair $(R\{t_{d-1}\}, R\{t_{d-1}\}/(f))$, one sees that B descends H-equivariantly to a finite local $R\{t_{d-1}\}$-algebra $\widetilde{B}$, endowed with an action of H compatible with that on R. Now, in the diagram

$$A \to A_1 = B^H \leftarrow A' = \widetilde{B}^H,$$

A_1 is local, complete, normal and finite over A of degree prime to ℓ, $A' \to A_1$ induces an isomorphism $\widehat{A'} \xrightarrow{\sim} A_1$, and A' is finite over $R^H\{t_{d-1}\} = C^H[[t_1, \cdots, t_{d-2}]]\{t_{d-1}\}$. The same passing to the limit argument then shows the existence of a diagram (4.1.2). The algebraization of the families Z of closed subschemes is done as in 4.3 : one has first to dispose of the case where the pull-back T of Z to $\operatorname{Spec} B$ consists of an irreducible component of the special fiber (or a subscheme defined by a power of the ideal defining such a component), and then, when the dimension of the special fiber is $< d-1$, choose coordinates making both f a Weierstrass polynomial as above and T finite over $\operatorname{Spec} R$, and one concludes as in 4.3.

5. Reduction to the equivariant log regular case

5.0. Let X be a noetherian scheme, endowed with an action (on the right) of a finite group G. Assume that the action is *admissible.* Recall [SGA 1 V 1] that this means that a quotient $p : X \to Y = X/G$ of X by G exists as a scheme (in the sense that for any scheme Z, p induces a bijection $\mathrm{Hom}(Y, Z) \xrightarrow{\sim} \mathrm{Hom}(X, Z)^G$), or equivalently that X is a union of G-stable affine open subschemes (or any orbit Gx is contained in an affine open subscheme). Moreover, the morphism p is integral, and $\mathcal{O}_Y \xrightarrow{\sim} (p_*\mathcal{O}_X)^G$. The action induced by G on any G-stable closed (resp. open) subscheme is again admissible.

One can ask whether p is finite (which implies that Y is noetherian (by Eakin's theorem, [Ma, p. 263])). Here are two cases where the answer is yes :

(a) If X is of finite type over a noetherian scheme S and G acts by S-automorphisms, then Y is of finite type over S and p is finite [SGA 1 V 1.5].

(b) If the order of G is invertible on X, then p is finite (3.4.1).

For $x \in X$ we denote by $G_d(x) = \{g \in G, gx = x\}$ the *decomposition group* at x, and $G_i(x) = \{x \in G_d(x), ga = a \ \forall a \in k(x)\}$ the *inertia group* at x. The decomposition group $G_d(x)$ acts on the geometric points $\overline{x}$ over x, and the stabilizer $G_{\overline{x}}$ of $\overline{x}$ is $G_i(x)$. We say that G acts *freely at x* if $G_i(x) = \{1\}$, and acts *freely* if it acts freely at each point x (equivalently, that for any scheme T, it acts freely on the set $X(T)$). We say that G acts *generically freely* if it acts freely at each maximal point of X. Suppose p is finite. Let $\overline{x}$ be a geometric point of X with image x. Consider the conditions :

(i) G acts freely at x (i. e. $G_{\overline{x}} = \{1\}$) ;

(ii) in a neighborhood of $p(\overline{x})$, the action of G makes X an étale Galois cover of Y of group G [SGA 1 V 2.3].

(iii) p is étale in a neighborhood of $\overline{x}$, i. e. the map induced by p on the strict localizations $X_{(\overline{x})} \to Y_{(p(\overline{x}))}$ is an isomorphism.

Then we have (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii). In particular, (i) is an open condition. G acts generically freely if and only if there is a dense open subset V of Y such that $p^{-1}(V)$ is an étale Galois cover of V of group G. In general, (iii) does not imply (i). However, if G acts *faithfully*, X is connected and p is étale, then G acts freely [SGA 1 V 2.4].

The main ingredient in this section is the following slightly weaker form of the nodal curve theorem of de Jong [dJ2, 2.4] :

Theorem 5.1. *Let $f : X \to Y$ be a proper morphism of integral excellent schemes, with generic fiber X_η of dimension 1. Let Z be a proper closed subset of X. Then there exists a finite group G and a commutative (non necessarily cartesian) square of G-schemes, where f' is projective :*

(5.1.1)
$$\begin{tikzcd} X' \arrow[r, "a"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "b"] & Y \end{tikzcd} ,$$

together with

- *a divisor D in X'*

• *a G-stable closed and nowhere dense subset $T' \subset Y'$*
satisfying the following properties :
- *G acts trivially on X, Y, faithfully on X', Y', and freely on $Y' - T'$*
- *a, b are alterations and $X'/G \to X$, $Y'/G \to Y$ are generically radicial*
- *f' is a nodal curve, smooth outside T'*
- *D is étale over Y' and contained in the smooth locus of f'*
- *$Z' := a^{-1}(Z)$ is contained in $D \cup f'^{-1}(T')$.*

Here by an *alteration* we mean a proper, surjective and generically finite morphism. A *nodal curve* is a proper and flat morphism, with geometric fibers of dimension 1, having at most ordinary quadratic singularities.

5.2. We now start the proof of 1.1 by induction on the dimension of X as explained in 3.1. By 2.8 we may assume X complete, local, of dimension d, and residue characteristic $p > 0$ (stronger results are available in characteristic zero !). We may even assume X normal. By 4.1 and the fact that uniformization data for (X', Z') will induce uniformization data for $(X, Z = X \times_{X'} Z')$, we may replace X by an X' appearing in (4.1.1) or (4.1.2), so that changing notations, and leaving out the assumption that X is complete, we may assume that we have a morphism $f : X \to Y$, with X and Y excellent, integral, Y being normal, integral, of dimension $d - 1$ and the generic fiber of f a curve. In addition, we have a nowhere dense closed subset Z of X, and a closed point x in Z. We want to show that, locally around x for the ℓ'-topology, we can uniformize (X, Z). We may assume X and Y affine, so that, compactifying f and changing notations, we may assume Y affine, normal, and f proper. From now on we forget about the point x and look for a (global) covering family $(X_i \to X)$ for the ℓ'-topology which uniformizes (X, Z). Replacing X by some blow-up and Z by its inverse image, we may assume that Z is a Cartier divisor in X.

We apply 5.1 to f and $Z \subset X$. As $X'/G \to X$ is covering for the ℓ'-topology, we may assume that $Y = Y'/G$, $X = X'/G$. Fix an ℓ-Sylow H of G. Consider the factorization

$$\begin{tikzcd} X' \arrow[r, "a_1"] \arrow[d, "f'"] & X'/H \arrow[r, "a_2"] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "b_1"] & Y'/H \arrow[r, "b_2"] & Y \end{tikzcd} .$$

As a_2 is covering for the ℓ'-topology, we may replace X by X'/H and Z by its inverse image in X'/H, Y by Y'/H, and then G by H, so that we may assume that G *is an ℓ-group.*

Apply the inductive assumption to $(Y, T := T'/G)$. We get a family (Y_i, T_i) uniformizing (Y, T) $((Y_i \to Y)$ covering for the ℓ'-topology, Y_i regular connected, T_i the support of a strict ncd). Take "normalized pull-backs" by b, i. e. let Y'_i be the normalization of a component of $Y' \times_Y Y_i$,. Replace Y by Y_i, Y' by Y'_i, G by the decomposition group G_i of Y'_i, and take the pull-back of the other data by $Y_i \to Y$, $Y'_i \to Y'$. Working separately over each Y_i, and changing notations, we may assume that in the diagram (5.1.1), $Y = Y'/G$ is regular connected, $T = T'/G$ is a strict ncd, and $X = X'/G$.

At this point, log geometry enters. Before continuing, let us review a few basic definitions and facts concerning log regularity and Kummer covers.

5.3. A *fine* (resp. *fs*, i. e. *fine and saturated*) *log scheme* [K1] is a scheme endowed with a log structure (for the étale topology) admitting étale locally charts modeled on a fine (i. e. finitely generated and integral) (resp. fs) monoid. The sheaf of monoids of X is usually denoted by M_X, the structural morphism $M_X \to \mathcal{O}_X$ by α, and we often write $\overline{M}$ for $M/\mathcal{O}^*$. All log schemes considered in these notes will be assumed locally noetherian.

Let X be an fs log scheme. If $\overline{x}$ is a geometric point of X, we often denote by $I_{\overline{x}}$ the ideal of the strict localization $\mathcal{O}_{X,\overline{x}}$ generated by $\alpha(M_{X,\overline{x}} - \mathcal{O}^*_{X,\overline{x}})$. One says that X is *log regular* at $\overline{x}$ if the subscheme $C_{\overline{x}}(X)$ of $X_{(\overline{x})}$ defined by $I_{\overline{x}}$ is regular and of codimension $\mathrm{rk}\,\overline{M}^{gp}_{X,\overline{x}}$. One says that X is *log regular* if it is log regular at each geometric point. This notion is due to Kato [K3] for Zariski log schemes. The variant for étale log schemes was treated by Niziol [N]. Here are some basic properties (*loc. cit.*) :

5.4. (a) If X is log regular at $\overline{x}$, then X is log regular in an étale neighborhood of $\overline{x}$.

(b) Suppose X log regular, let U be the open subset of triviality of its log structure, denote by $j : U \to X$ the inclusion. Then

$$M_X = \mathcal{O}_X \cap j_*\mathcal{O}^*_U.$$

We shall say that a pair (X, Z) consisting of a scheme X and a closed subset Z is a *log regular pair* if for the log structure on X defined by $M_X = \mathcal{O}_X \cap j_*\mathcal{O}^*_U$, where $j : U = X - Z \to X$ is the open complement of Z, X is log regular and Z is the complement of the open subset of triviality of its log structure.

(c) Assume X log regular. Let $X^{(i)}$ be the subset of X consisting of points x such that at a geometric point $\overline{x}$ over x, $\mathrm{rk}\,\overline{M}^{gp}_{X,\overline{x}} = i$. Then $X^{(i)}$ is locally closed, and underlies a regular subscheme of X, of codimension i, whose trace on $X_{(\overline{x})}$, for $x \in X^{(i)}$, is $C_{\overline{x}}(X)$. We call $X^{(i)}$ the *stratum* of codimension i defined by $\mathrm{rk}\,M^{gp}$. Here are two examples :

(i) If X is a regular noetherian scheme, with log structure defined by a strict ncd D, X is log regular, and $X^{(i)}$ consists of the points through which pass exactly i components of D.

(ii) If X is a toric variety over a field k, with torus T, equipped with its canonical log structure, X is log regular, and $X^{(i)}$ is the union of smooth orbits of T of codimension i.

(d) Assume X log regular at $\overline{x}$, let $P = \overline{M}_{X,\overline{x}}$, $k = k(\overline{x})$. Note that P is a sharp (i. e. $P^* = \{0\}$) fs monoid. Let $\widehat{X}_{\overline{x}}$ be the completion at the closed point of the strict localization of X at $\overline{x}$. Then X admits a chart modeled on P at $\overline{x}$, and such a chart gives rise to isomorphisms

(i)

$$\widehat{X}_{\overline{x}} \simeq \mathrm{Spec} k[[P]][[t_1, \cdots, t_n]]$$

if $\mathcal{O}_{X,x}$ is of equal characteristic,

(ii)

$$\widehat{X}_{\overline{x}} \simeq \mathrm{Spec} C(k)[[P]][[t_1, \cdots, t_n]]/(f)$$

if $\mathcal{O}_{X,x}$ is of mixed characteristic $(0, p)$, where $C(k)$ is a Cohen ring for k, and f is congruent to p modulo the ideal generated by $P - \{0\}$ and the t_i's.

(e) (Vidal [V]) Assume X log regular. Then X is regular at a geometric point $\overline{x}$ if and only if $\overline{M}_{X,\overline{x}} \simeq \mathbb{N}^r$. In particular, if X is regular, the open subset of triviality of the log structure is the complement of an ncd.

(f) Let Y be a log regular log scheme, and $f : X \to Y$ a log smooth map. Then X is log regular.

(g) (Gabber) Let (Y, T) be a log regular pair. Let $f : X \to Y$ be a nodal curve (5.1), smooth outside T. Let D be a divisor in X contained in the smooth locus of f and étale over Y. Then the pair $(X, f^{-1}(T) \cup D)$ is log regular and f is log smooth.

This follows from the local structure of f : by [SGA 7 XV 1.3.2] (see also [dJ1, 2.23]), if $\overline{x}$ is a geometric point of the non smoothness locus of X, with image $\overline{y}$ in Y, we have

$$\mathcal{O}_{X,\overline{x}} \simeq \mathcal{O}_{Y,\overline{y}}\{u, v\}/(uv - h),$$

where h is an element of $\mathcal{O}_{Y,\overline{y}}$ invertible on $Y - T$, hence belonging to $M_{Y,\overline{y}}$ (5.4 (b)). One can then extend a local chart c of Y at $\overline{y}$ modeled on $P = \overline{M}_{Y,\overline{y}}$ to a local chart d of X at $\overline{x}$ modeled on the monoid Q defined as the amalgamated sum of $\mathbb{N}^2$ and $\mathbb{Z} \times P$ along the maps $\mathbb{N} \to \mathbb{N}^2$, $1 \mapsto (1, 1)$ and $\mathbb{N} \to \mathbb{Z} \times P$, $1 \to (1, a)$ for the element a of P defined by $h = \varepsilon a$, ε a unit : d sends $\mathbb{N}^2$ to $\mathcal{O}_{X,\overline{x}}$ by $(1, 0) \mapsto u$, $(0, 1) \mapsto v$, and on $\mathbb{Z}$ is given by $1 \mapsto \varepsilon$.

5.5. We will need the notion of *Kummer étale cover*, whose definition we recall.

(a) A homomorphism $h : P \to Q$ of integral monoids is called *Kummer* if h is injective and if for any $q \in Q$ there exists $n \geq 1$ and $p \in P$ such that $nq = h(p)$,

(b) A morphism $f : X \to Y$ of fs log schemes is called *Kummer* if for all geometric point $\overline{x}$ of X with image $\overline{y} = f(\overline{x})$ in Y, the induced morphism $\overline{M}_{\overline{y}} \to \overline{M}_{\overline{x}}$ is Kummer.

(c) A morphism $f : X \to Y$ of fs log schemes is called *Kummer étale* if it is Kummer and log étale.

(d) A morphism $f : X \to Y$ of fs log schemes is called a *finite Kummer étale cover* if f is Kummer étale, and finite as a morphism of schemes. The *Kummer étale topology* on (the category of Kummer étale fs log schemes over) an fs log scheme S, is generated by surjective families of Kummer étale maps.

(e) Let Y be an fs log scheme and G a finite group. A *Kummer étale cover* of Y *of group* G is an fs log scheme X above Y, endowed with an action of G by Y-automorphisms making $p : X \to Y$ into a G-torsor for the Kummer étale topology, or equivalently such that the canonical map

$$G_Y \times_Y X \to X \times_Y X, \ (g, a) \mapsto (a, ag) \tag{5.5.1}$$

is an isomorphism (the fiber product on the right hand side being taken in the category of fs log schemes).

As the Kummer étale topology is weaker than the canonical topology (an fs log scheme Z defines a sheaf on Y for the Kummer étale topology), p induces a bijection $\mathrm{Hom}(Y, Z) \xrightarrow{\sim} \mathrm{Hom}(X, Z)^G$ (where homomorphisms are taken in the category of log schemes), in other words, Y is a geometric quotient of X (as a log scheme) : the natural maps $\mathcal{O}_Y \to (p_*\mathcal{O}_X)^G$ and $M_Y \to (p_*M_X)^G$ are isomorphisms. In particular, the underlying scheme of Y is the quotient by G of the underlying scheme of X.

Kummer étale covers will be further discussed in the next section. A basic fact about them is the following theorem of Fujiwara-Kato, a particular case of which will play a crucial role in the reduction in this section.

Theorem 5.6 ([F-K, 3.1]. *Let X be a log regular fs log scheme, and let U the open subset of triviality of its log structure. Then the restriction functor from the category of Kummer étale covers of X to the category of (finite) classical étale covers of U induces an equivalence with the subcategory of those étale covers $V \to U$ which are tamely ramified along $X - U$, i. e. such that if Z is the normalization of X in V, at all points $x \in X - U$ with $\dim \mathcal{O}_{X,x} = 1$, the restriction of Z to $\operatorname{Spec} \mathcal{O}_{X,x}$ is tamely ramified.*

5.7. Let us now return to the situation considered in 5.2 : we have a diagram (5.1.1), with $X = X'/G$, $Y = Y'/G$, $T = T'/G$, G a finite ℓ-group. Here Y' is normal integral, Y is regular, connected, and T is a strict ncd. As G acts freely on $Y' - T'$, the restriction of Y' over $Y - T$ is an étale Galois cover of group G. As Y' is normal, integral, and $Y'/G = Y$, Y' is the normalization of Y in $Y' - T'$. It then follows from 5.6 that Y' is the underlying scheme of the unique log scheme Kummer étale of group G over Y extending the étale cover $Y' - T' \to Y - T$. In particular, the pair (Y', T') is log regular. From 5.4 (g) we deduce that the pair $(X', f'^{-1}(T'))$ is log regular and f' is log smooth. Moreover, as the divisor D is étale over Y' and contained in the smooth locus of f', the pair $(X', f'^{-1}(T') \cup D)$ is also log regular, and with this new log structure on X', f' is also log smooth (with open subset of triviality contained in, not equal to, the inverse image of that of Y'). Finally, the inverse image Z' of Z in X' is a subdivisor of $f'^{-1}(T') \cup D$.

If the pair $(X = X'/G, (f'^{-1}(T') \cup D)/G)$ was log regular, then Kato-Niziol's resolution of singularities of log regular pairs would easily finish the proof. However, the quotient of a log regular scheme S by a finite group G of order invertible on S is not in general a log regular scheme, as the example of trivial log structures already shows. This issue is tackled in the next section.

6. Very tame actions

6.1. Let X be a noetherian scheme, endowed with an action (on the right) of a finite group G. We say that G acts *tamely* at a point x if the order of $G_i(x)$ is prime to char $k(x)$, and acts *tamely* if it acts tamely at each point. This notion is closely related to that of *Kummer étale cover*. The purpose of Gabber's theory of *very tame actions* is to make this relation more precise and exhibit conditions (stronger than tameness) ensuring that the quotient of a log regular scheme by a finite group action is a log regular scheme.

6.2. A *standard Kummer étale cover* of an fs log scheme Y is the pull-back by a strict map $Y \to \operatorname{Spec} \mathbb{Z}[P]$ of a morphism of log schemes of the form $\operatorname{Spec} \mathbb{Z}[h] : \operatorname{Spec} \mathbb{Z}[Q] \to \operatorname{Spec} \mathbb{Z}[P]$, where $h : P \to Q$ is a Kummer homomorphism of fs monoids such that the cokernel of h^{gp} is killed by an integer n invertible on Y.

Let $f : X \to Y$ be a standard Kummer étale cover. Choose a Kummer homomorphism $h : P \to Q$ such that the square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & \operatorname{Spec}\mathbb{Z}[Q] \arrow[d] \\ Y \arrow[r] & \operatorname{Spec}\mathbb{Z}[P] \end{tikzcd} ,$$

is cartesian, where the horizontal morphisms are strict. Applying the Cartier dual $D_{\mathbb{Z}}$ functor (on $\operatorname{Spec}\mathbb{Z}$) to the exact sequence

$$0 \to P^{gp} \to Q^{gp} \to \Gamma \to 0,$$

we get an exact sequence of diagonalizable groups (on $\operatorname{Spec}\mathbb{Z}$) :

$$0 \to \Delta \to T_Q \to T_P \to 0,$$

with T_Q and T_P split tori, and $\Delta = D_{\mathbb{Z}}(\Gamma)$ finite, becoming étale over Y (a product $\mu_{n_1} \times \cdots \times \mu_{n_r}$, if $n_1|\cdots|n_r$ are the invariants of $P^{gp} \subset Q^{qp}$, with $n = n_1 \cdots n_r$ invertible on Y). The torus T_P (resp. T_Q) acts on $Z_P = \operatorname{Spec}\mathbb{Z}[P]$ (resp. $Z_Q = \operatorname{Spec}\mathbb{Z}[Q]$), and the morphism $\operatorname{Spec}\mathbb{Z}[h]$ is equivariant with respect to $T_Q \to T_P$. Recall that this action is described in the following way : if S is a scheme, an S-valued point g of T_P, written as $g = \sum_{p \in P^{gp}} g_p p$, with $g_p \in \Gamma(S, \mathcal{O}_S^*)$, acts on the set of S-valued points of Z_P by sending a point $a = \sum_{p \in P} a_p p$, with $a_p \in \Gamma(S, \mathcal{O}_S)$, to the point $ga = \sum g_p a_p p$. In other words, the action of T_P on Z_P is given by the tautological family of *characters* $\chi_p : T_P \to \mathbb{G}_m$ indexed by P^{gp} defined by the pairing $T_P \otimes P^{gp} \to \mathbb{G}_m$. As the action of T_P on Z_P extends its action on T_P, T_P acts on the sheaves $\mathcal{O}_{Z_P}$, $(j_P)_*\mathcal{O}^*_{T_P}$ on Z_P, where $j_P : T_P \to Z_P$ is the open immersion, and on the sheaf of monoids $M_{Z_P} = \mathcal{O}_{Z_P} \cap (j_P)_*\mathcal{O}^*_{T_P}$ in a way compatible with the inclusion $M_{Z_P} \subset \mathcal{O}_{Z_P}$, i. e. it acts on the canonical log structure of Z_P. Similarly, T_Q acts on Z_Q by automorphisms of log schemes. The group Δ, via its injection into T_Q, acts on Z_Q by Z_P-automorphisms (of log schemes), and the resulting morphism

(6.2.1) $$\Delta \times_{\operatorname{Spec}\mathbb{Z}} Z_Q \to Z_Q \times_{Z_P} Z_Q \ , \ (g, a) \mapsto (a, ag)$$

(where the fiber product on the right hand side is taken in the category of fs log schemes) is an isomorphism. By pull-back to Y, one obtains an action of Δ_Y on X by Y-automorphisms giving rise to similar isomorphisms

(6.2.2) $$\Delta_Y \times_Y X \xrightarrow{\sim} X \times_Y X,$$

(6.2.3) $$\delta_{\overline{x}} \times X_{(\overline{x})} \xrightarrow{\sim} X_{(\overline{x})} \times_{Y_{(\overline{y})}} X_{(\overline{x})},$$

where $\delta_{\overline{x}} \subset (\Delta_Y)_{\overline{y}}$ is the stabilizer of a geometric point $\overline{x}$ of X and $\overline{y}$ the image of $\overline{x}$ in Y. The group $\delta_{\overline{x}}$ acts on $M_{\overline{x}}$: the image $q_{\overline{x}}$ of $q \in Q$ in $M_{\overline{x}}$ is sent by $g \in \delta_{\overline{x}}$ to $\chi_q(g)_{\overline{x}} q_{\overline{x}}$. It follows that $\delta_{\overline{x}}$ *acts trivially* on $\overline{M}_{\overline{x}}$. Moreover, $\delta_{\overline{x}}$ acts trivially on $C_{\overline{x}}(X) \subset X_{(\overline{x})}$ (5.3). Indeed, for $g \in \delta_{\overline{x}}$, we have $\chi_q(g)(\overline{x}) = 1$ if $q(\overline{x}) \neq 0$, so if $a \in \mathcal{O}_{X,\overline{x}}$ is the image of $\sum a_q q$ with $a_q \in \mathcal{O}_{Y,\overline{y}}$, the image ga of $\sum \chi_q(g) a_q q$ is congruent to a modulo the ideal $I_{\overline{x}}$ of $\mathcal{O}_{\overline{x}}$ (5.3).

6.3. The importance of standard Kummer étale covers comes from the fact that if $f : X \to Y$ is a Kummer étale morphism, then, if $\overline{x}$ is a geometric point of X and $\overline{y}$ its image in Y, the morphism induced by f on the corresponding strict localizations

$X_{(\overline{x})} \to Y_{(\overline{y})}$ is a standard Kummer étale cover, or equivalently, there exists an étale neighborhood U (resp. V) of $\overline{x}$ (resp. $\overline{y}$) such that f restricts to a standard Kummer étale cover $U \to V$. In particular, the *Kummer étale topology* on (the category of Kummer étale fs log schemes over) an fs log scheme S, is generated by standard Kummer étale covers and surjective families of étale maps.

6.4. Let $f : X \to Y$ be a Kummer étale cover of Y of group G, and let $x \in X$, $y = f(x)$, $\overline{y}$ a geometric point of Y above y, $\overline{x}$ a geometric point of $X_{\overline{y}}$ above x. Then the inertia group $G_{\overline{x}} = G_i(x) = G_d(\overline{x})$ acts on the strict localization $X_{(\overline{x})}$ of X at $\overline{x}$, by $Y_{(\overline{y})}$-automorphisms, making $X_{(\overline{x})}$ a $G_{\overline{x}}$-torsor on $Y_{(\overline{y})}$ for the Kummer étale topology. Let $f_x : X_{(\overline{x})} \to Y_{(\overline{y})}$ denote the induced morphism. We know that f_x is a standard Kummer étale cover : one can choose a chart of f modeled on a Kummer morphism $h : P \to Q$ making f_x into a $\delta_{\overline{x}}$-torsor over $Y_{(\overline{y})}$, for some finite étale diagonalizable group $\delta_{\overline{x}}$ over $Y_{(\overline{y})}$ (6.2.3). These two torsors are related by a unique isomorphism

$$G_{\overline{x}} \xrightarrow{\sim} \delta_{\overline{x}}. \tag{6.4.1}$$

In particular, the inertia group $G_{\overline{x}}$ is *abelian* and G acts tamely at x. Moreover, it follows from the observations at the end of 6.2 that *$G_{\overline{x}}$ acts trivially on $\overline{M}_{\overline{x}}$ and $C_{\overline{x}}(X)$.*

6.5. Let X be an fs log scheme endowed with an action of a finite group G.

(a) We will say that G acts *in a Kummer way*, or that the action is *Kummer-like*, if it makes X into a Kummer étale cover of an fs log scheme Y of group G. The log scheme Y and the map $p : X \to Y$ is then determined by (X, G) up to unique isomorphism (5.5 (e)).

The next definitions are due to Gabber.

(b) Let $\overline{x}$ be a geometric point of X, and $G_{\overline{x}}$ the corresponding inertia group. One says that G acts *very tamely* at $\overline{x}$ if the following conditions are satisfied :

(i) $G_{\overline{x}}$ is of order prime to $\mathrm{char}(k(\overline{x}))$ (i. e. G acts tamely at $\overline{x}$) and acts trivially on $\overline{M}_{\overline{x}}$.

(ii) $G_{\overline{x}}$ acts trivially on the stratum $C_{\overline{x}}(X)$ (5.3).

(c) Given an integer $n \geq 1$, a finite group G, and an fs monoid Q, a pairing

$$\chi : G^{ab} \otimes Q^{gp} \to \mu := \mu_n(\mathbb{C}),\ g \otimes q \mapsto \chi_q(g),$$

defines an action of G on the log scheme $\mathrm{Spec}\,\Lambda[Q]$, where $\Lambda = \mathbb{Z}[\mu][1/n]$: the action on the underlying scheme and on the sheaf of monoids is characterized by $g.q = \chi_q(g)q$ for $q \in Q$. If X is an fs log scheme endowed with an action of G, by an *equivariant chart of X modeled on Q and χ*, one means a G-equivariant map of log schemes

$$X \to \mathrm{Spec}\Lambda[Q].$$

Proposition 6.6. *Let a finite group G act on an fs log scheme X, and let $\overline{x}$ be a geometric point of X.*

(a) If condition (b) (i) of 6.5 is satisfied at $\overline{x}$, étale locally at $\overline{x}$, X admits a $G_{\overline{x}}$-equivariant chart modeled on $\overline{M}_{\overline{x}}$ for some pairing χ relative to $\mu = \mu_n(\mathbb{C})$, for n the exponent of $|G_{\overline{x}}|$.

(b) If moreover $G_{\overline{x}}$ acts trivially on the stratum $C_{\overline{x}}(X)$, i. e. G acts very tamely at $\overline{x}$, then étale locally around $\overline{x}$ and its image in Y, the quotient $Y = X/G_{\overline{x}}$ exists as a log scheme and the projection $X \to Y$ makes X into a strict closed sub log scheme of a standard Kummer étale cover of Y.

(c) If X is log regular, G acts admissibly, generically freely, and very tamely at each geometric point, then G acts on X in a Kummer way, and Y is log regular.

For (a), the choice of a splitting $s : \overline{M}^{gp}_{\overline{x}} \to M^{gp}_{\overline{x}}$ of the exact sequence

$$(*) \qquad 0 \to \mathcal{O}^*_{X,\overline{x}} \to M^{gp}_{\overline{x}} \to \overline{M}^{gp}_{\overline{x}} \to 0$$

(as a sequence of abelian groups) lifts the cohomology class of $(*)$ to a 1-cocycle $z \in Z^1(G_{\overline{x}}, Hom(\overline{M}^{gp}_{\overline{x}}, \mathcal{O}^*_{\overline{x}}))$ ($g(sa) = z(g)(a)sa$, for $a \in \overline{M}^{gp}_{\overline{x}}$), which describes the action of $G_{\overline{x}}$ on $M^{gp}_{\overline{x}}$. The image of z in $Z^1(G_{\overline{x}}, Hom(\overline{M}^{gp}_{\overline{x}}, k(\overline{x})^*))$ $(= H^1(G_{\overline{x}}, Hom(\overline{M}^{gp}_{\overline{x}}, k(\overline{x})^*)))$ is a pairing χ ; $z(g)$ is the unique lifting in $Hom(\overline{M}^{gp}_{\overline{x}}, \mathcal{O}^*_{\overline{x}})$ of its image in $Hom(\overline{M}^{gp}_{\overline{x}}, k(\overline{x})^*)$. The restriction of s to $Q = \overline{M}_{\overline{x}}$ gives an equivariant chart modeled on Q and χ. Note that the map $H^1(G_{\overline{x}}, Hom(\overline{M}^{gp}_{\overline{x}}, \mathcal{O}^*_{\overline{x}})) \to H^1(G_{\overline{x}}, Hom(\overline{M}^{gp}_{\overline{x}}, k(\overline{x})^*))$ is an isomorphism, since $(1+\mathbf{m}_{\overline{x}})^{\times}$ is n-divisible. For (b), one replaces X by its strict localization at $\overline{x}$ and G by $G_{\overline{x}}$. Then the quotient $Y = X/G$ exists, is strictly local (with same residue field) and $X \to Y$ is finite. One defines $P \subset Q$ as the intersection with Q of the subgroup P' of Q^{gp} consisting of those q for which $\chi_q(g) = 1$ for all $g \in G$ ($[Q^{gp} : P']$ is finite and prime to $\mathrm{char}(k(x))$). The equivariant chart $X \to \operatorname{Spec} \Lambda[Q]$ gives a morphism $Y \to \operatorname{Spec} \Lambda[P]$, defining a log structure on Y for which $(f_* M_X)^G = M_Y$ ($f : X \to Y$ the projection), and a chart of f modeled on the Kummer map $P \to Q$. Finally, as $\mathcal{O}_{X,\overline{x}}/I_{\overline{x}}$ is invariant under G, $\mathcal{O}_{Y,\overline{y}}$ (for $\overline{y}$ the image of $\overline{x}$) surjects to $\mathcal{O}_{X,\overline{x}}/I_{\overline{x}}$, hence $k(\overline{x})[Q] = k(\overline{y})[Q]$ surjects to $\mathcal{O}_{X,\overline{x}}/\mathbf{m}_{\overline{y}}\mathcal{O}_{X,\overline{x}}$, and the resulting map $X \to Y \times_{\operatorname{Spec} \Lambda[P]} \operatorname{Spec} \Lambda[Q]$ is a closed immersion. Under the assumptions of (c), because $C_{\overline{x}}(X)$ is regular, and projects isomorphically to $C_{\overline{y}}(Y) = C_{\overline{x}}(X)/G_{\overline{x}}$, and $\operatorname{rk} \overline{M}^{gp}_{\overline{x}} = \operatorname{rk} \overline{M}^{gp}_{\overline{y}}$, Y is log regular at $\overline{x}$, and the above closed immersion is an isomorphism, so $X_{(\overline{x})}$ is a $G_{\overline{x}}$-Kummer cover of $Y_{(\overline{y})}$. As G acts generically freely, the restriction of X over $Y_{(\overline{y})}$ is the extension from $G_{\overline{x}}$ to G of this torsor. (One could also use the local structure 5.4 (d), which would give - in the mixed characteristic case, for example - a $G_{\overline{x}}$-equivariant isomorphism $\widehat{\mathcal{O}}_{X,\overline{x}} \xrightarrow{\sim} R[[Q]][[t_1, \cdots, t_r]]/(f)$, with f and $R[[t_1, \cdots, t_r]]$ invariant, f congruent to p mod. $(Q - \{0\}, t_1, \cdots, t_r)$, and $\widehat{\mathcal{O}}_{Y,\overline{y}} \xrightarrow{\sim} R[[P]][[t_1, \cdots, t_r]]/(f)$.)

Corollary 6.7. *(a) If G acts very tamely on X at $\overline{x}$, then the inertia group $G_{\overline{x}}$ is abelian.*

(b) If G acts very tamely on X at $\overline{x}$, then G acts very tamely in an étale neighborhood of $\overline{x}$.

Proposition 6.8. *Let G act on the fs log scheme X in a Kummer way, and let Y be the quotient. Suppose X is log smooth over an fs base S, over which G acts trivially, and $X \to S$ is equivariant. Then Y is log smooth over S.*

This is a particular case of descent by exact log flat or log smooth maps [K2]. Here is a sketch of proof, due to Nakayama and Tsuji. By the technique of toric stacks (or log products) ([Ol], [KS 4.3.3]), which expresses log smoothness in terms of classical smoothness, one is reduced to proving the following result :

Proposition 6.9. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & Y' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Y \end{array} ,$$

be a cartesian square of noetherian schemes, where f underlies a G-Kummer étale cover of fs log schemes and g is of finite type. Then, g is flat (resp. smooth) if and only if g' is flat (resp. smooth).

It suffices to prove that if g' is flat, g is flat. We may assume that f is deduced by base change from $\operatorname{Spec}\Lambda[Q] \to \operatorname{Spec}\Lambda[P]$, for $\Lambda = \mathbb{Z}[\mu_n(\mathbb{C}), 1/n]$ and a Kummer homomorphism $h : P \to Q$ with Q^{gp}/P^{gp} killed by n. Then by exactness of h, P acts on the complement of Q in P, hence $\Lambda[P]$ is a direct summand of $\Lambda[Q]$ as a $\Lambda[P]$-module. One can then apply the following elementary result :

Lemma 6.10 (Tsuji). *Consider a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & Y' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Y \end{array} ,$$

where f is affine and $\mathcal{O}_Y$ is a direct factor of $f_*\mathcal{O}_{Y'}$ as an $\mathcal{O}_Y$-module. Then, g is flat if and only if g' is flat.

7. Canonical resolutions

7.1. Let X be a noetherian, quasi-excellent scheme. By a *resolution tower* of X one means a sequence of morphisms

$$X^{(n)} \xrightarrow{f_n} \cdots \xrightarrow{f_1} X^{(0)} = X , \tag{7.1.1}$$

where, for $1 \leq i \leq n$, f_i is a blow-up with non singular center disjoint from $(X^{(i-1)})_{\mathrm{reg}}$, and $X^{(n)}$ is regular. If Z is a nowhere dense closed subset of X, an *embedded resolution tower* of (X, Z) (or resolution tower for short) is a resolution tower (7.1.1) such that if one defines inductively $Z^{(0)} = Z$, $Z^{(i)} = f_i^{-1}(Z^{(i-1)})$, then f_i is a blow-up with non singular center contained in the union of $Z^{(i-1)}$ and the singular locus of $X^{(i-1)}$, and the pair $(X^{(n)}, Z^{(n)}_{\mathrm{red}})$ consists of a regular scheme and a ncd in it.

Note that if $X^{(\cdot)} \to X$ is a resolution tower of X (resp. of (X, Z)), its pull-back by any smooth map $X' \to X$ is a resolution tower of X' (resp. (X', Z')), where Z' is the pull-back of Z. Note also that if X is regular and Z is a ncd in X, then de Jong's canonical sequence of blow-ups [dJ1, 7.2] (see 8.3) makes Z a sncd.

7.2. Hironaka [H] proved the existence of embedded resolution towers for reduced schemes separated and of finite type over a field of characteristic zero (or, more generally, over a quasi-excellent local ring of characteristic zero). Temkin [T] showed how to deduce embedded resolution for reduced, noetherian, quasi-excellent schemes X of characteristic zero. Namely, he proves, for any closed nowhere dense subset Z in X, the existence of a modification $f : X' \to X$ with X' regular and $f^{-1}(Z)$ the support of a strict ncd (though he doesn't construct embedded resolution *towers*).

On the other hand, for reduced schemes separated and of finite type over a field k of characteristic zero, Bierstone-Milmann [B-M] showed the existence of *canonical* resolution towers (and embedded resolution towers), "canonical" meaning that these towers are compatible with pull-back by smooth maps. In other words, a suitable category $\mathcal{T}_k$ of resolution towers, fibered by $X^{(\cdot)} \mapsto X^{(0)}$ over the category $\mathcal{S}_k$ of reduced schemes separated and of finite type over k having *smooth* k-maps as morphisms has a *cartesian section* (*). This implies that, if G is a smooth group scheme over k, and X is a G-scheme (separated and of finite type), then X admits a G-equivariant resolution tower.

7.3. We will see that this procedure yields canonical resolutions for *toric varieties* and *log regular schemes*. A slight technical complication occurs here, though, as log blow-ups of fs log schemes involve normalizations, which do not appear in the resolution towers. This forces to work with non necessarily normal toric varieties and fine log schemes instead of fs log schemes. The following definitions are due to Gabber. See [ILO] for details.

(a) Let R be a noetherian, regular, integral ring and T a split torus over R, with character group Γ. A *toric scheme* (or *variety*) over R of torus T is an integral scheme X, separated and of finite over R, endowed with an open embedding $j : T \to X$ and an equivariant action of T (for its action on itself by translations), admitting a covering family by affine open T-stable subsets X_α. If X is affine, then $X = \operatorname{Spec} R[P]$, for a unique fine submonoid P of Γ such that $P^{gp} = \Gamma$, and thus X inherits a canonical *Zariski* fine log structure. If X is a toric scheme over R of torus T, with a covering by a family (X_α) as above, the log structures of the X_α's glue and define a (Zariski) fine log structure on X called *canonical*.

(b) Let X be a *fine* log scheme and X^{sat} its associated fs log scheme. One says that X is *log regular* at a geometric point $\overline{x}$ if X^{sat} is log regular at any geometric point above $\overline{x}$ (5.3). One says that X is *log regular* if X is log regular at every point.

For example, a toric scheme X over R, with torus T, is log regular, and T is the open subset of triviality of its log structure (however, the map $M_X \to \mathcal{O}_X \cap j_*\mathcal{O}_T^*$ is not an isomorphism in general, as the case where $T = \operatorname{Spec} \mathbb{C}[t, t^{-1}] \subset X = \operatorname{Spec} \mathbb{C}[t^2, t^3]$ already shows).

If $X \to Y$ is a log smooth map between fine log schemes and Y is log regular, then X is log regular.

7.4. Let $X = \operatorname{Spec} k[P]$ be an *affine toric variety* over a field k of characteristic zero, where P is a fine monoid with P^{gp} torsionfree, with associated torus $T = \operatorname{Spec} k[P^{gp}]$. One sees inductively that all the floors $X^{(i)}$ of the canonical resolution tower of X are again toric varieties, and the centers of blow-ups defining $X^{(i+1)}$ are finite disjoint unions

(*) Gabber warns that this statement lacks adequate references.

of smooth closures of orbits of T. Moreover, if $X_i = \mathrm{Spec}\, k[P_i]$ $(1 \le i \le N)$ is a finite family of such toric varieties, and $g_i : S \to X_i$ a family of smooth maps, with S in $\mathcal{S}_k$ (7.2), then the pull-backs of the towers $X_i^{(\cdot)}$ by the g_i's are isomorphic (as towers of schemes). If $S = \mathrm{Spec}\, k[Q]$ is also a toric variety and the g_i's are defined by homomorphisms of monoids $P_i \to Q$, the pull-backs of the towers are isomorphic as towers of toric varieties.

7.5. The blow-ups appearing in the resolution tower of $X = \mathrm{Spec}\, k[P]$ can be defined in a purely combinatorial way. To explain this we need to recall a few definitions and constructions.

A *fine fan*, in the sense of Kato [K1], is a topological space F endowed with a sheaf of monoids M_F, which is locally of the form $\mathrm{Spec}\, P$, for a fine monoid P ($\mathrm{Spec}\, P$ is the set of prime ideals of P, equipped with the Zariski topology, and the sheaf of monoids whose set of sections over an open subset where $f \in P$ is invertible is $\overline{P[1/f]} = P[1/f]/(P[1/f]^*)$. The fan F is called *torsionfree* if the P^{gp}'s are torsionfree. For example, if X is a toric scheme over R, as in 7.3, with its canonical (Zariski) fine log structure, then the set $F(X)$ of points $x \in X$ such that the ideal I_x of $\mathcal{O}_{X,x}$ generated by $\alpha(M_x - \mathcal{O}_x^*)$ is $\mathbf{m}_x$, equipped with the topology induced by the Zariki topology of X and the sheaf of monoids induced by $\overline{M}_X$, is a fine, torsionfree fan, called the *fan associated* with X (for $X = \mathrm{Spec}\, R[P]$, $F = \mathrm{Spec}\, P$). One has a canonical morphism of monoidal spaces (spaces endowed with a sheaf of monoids) $c : X \to F$ inducing an isomorphism $c^{-1}M_F \xrightarrow{\sim} \overline{M_X}$.

An *ideal* in a fine fan F is a sheaf of ideals of M_F, which, on open subsets where X is of the form $\mathrm{Spec}\, P$, is generated by an ideal of P. If I is an ideal of the fine fan F, there is a fine fan $F' = Bl_I(F)$, called the *blow-up* of I, together with a morphism of fans (i. e. of monoidal spaces) $f : F' \to F$, such that $f^{-1}I$ is *invertible* (i. e. locally generated by one element), and which is universal for this property. When $F = \mathrm{Spec}\, P$, and I is defined by an ideal (still denoted I) of P, F' is the union of the $\mathrm{Spec}\, P_a$, for $a \in I$, where P_a is the submonoid of P^{gp} generated by P and $b - a$, for $b \in I$. Let X be a fine Zariski log scheme, and $c : X \to F$ a *chart* of X on a fine fan F, i. e. a morphism of monoidal spaces inducing an isomorphism $c^{-1}M_F \xrightarrow{\sim} \overline{M_X}$ as above. Let I be an ideal in F, J the ideal in M_X generated by $c^{-1}I$, F' the blow-up of I in F, X' the *log blow-up* of J (when X is log regular, this is the *usual* blow-up of the ideal of $\mathcal{O}_X$ generated by J, [N, 4.3]). Then one has a diagram

$$\begin{tikzcd} X' \arrow[r, "c'"] \arrow[d, "f"] & F' \arrow[d, "g"] \\ X \arrow[r, "c"] & F \end{tikzcd} ,$$

where g (resp. f) is the blow-up (resp. log blow-up) of I (resp. J), the diagram of underlying monoidal spaces commutes, and c' is a chart of X' on F'. Moreover, the fine log scheme X', together with the morphism of log schemes f and the morphism of monoidal spaces c', is universal in the obvious sense. One says that X' is the *pull-back of* F' *by* c, and one writes

$$X' = X \times_F F'.$$

Suppose now that X is a toric variety over the field k, with torus T and associated fan

$c : X \to F$. Let X' be the blow-up of X centered at the closure of a T-orbit, corresponding to an ideal I in F. Then $X' = X \times_F F'$, where $F' = Bl_I(F)$.

Coming back to the situation in 7.4, we see that the resolution tower of $X = \operatorname{Spec} k[P]$ is deduced by pull-back from a tower of blow-ups of fans

$$F^{(n)} \xrightarrow{g_n} \cdots \xrightarrow{g_1} F^{(0)} = F \ ,$$

where F is the fan associated with X, and for $1 \leq i \leq n$,

$$X^{(i)} = X^{(i-1)} \times_{F^{(i-1)}} F^{(i)}.$$

This in turn allows to construct T-equivariant resolution towers of toric schemes of the form $X = \operatorname{Spec} \mathbb{Z}[P]$, with P as above, and $T = \operatorname{Spec} \mathbb{Z}[P^{gp}]$: define $X^{(i+1)}$ inductively as $X^{(i)} \times_{F^{(i)}} F^{(i+1)}$. If X_i $(1 \leq i \leq N)$ is a finite family of toric varieties $X_i = \operatorname{Spec} \mathbb{Z}[P_i]$, and $g_i : S \to X_i$ a family of smooth maps, with S a scheme of finite type over $\mathbb{Z}$, then the pull-backs of the towers $X_i^{(\cdot)}$ by the g_i's are isomorphic (as towers of schemes), and when $S = \operatorname{Spec} \mathbb{Z}[Q]$ and the g_i's are defined by homomorphisms $P_i \to Q$, the pull-backs are isomorphic as towers of toric schemes.

7.6. Let X be a fine noetherian log regular log scheme (7.3). Using local charts and 7.5 one constructs a resolution tower $X^{(\cdot)}$, where all the $X^{(i)}$'s are log regular log schemes and the maps f_i's log blow-ups. The top $X^{(n)}$ of the tower is a log regular fs log scheme, which is regular (as the stalks of $\overline{M}$ are of the form $\mathbb{N}^r$ (5.4 (e))), and for which the complement of the open subset of triviality of its log structure is an ncd (*loc. cit*). One can show (Gabber) that the underlying tower of schemes depends only on the underlying scheme of X (see [ILO]), at least if the maximal points of the strata of the natural stratification by the rank of M^{gp}, cf. 5.4 (c), are of characteristic zero ; one hopes that this restriction can be removed. It is called the *canonical resolution tower* of X. The composite map $X^{(n)} \to X$ is called the *canonical resolution* of X.

If a finite group G acts on X, using the functoriality again, one shows that G acts naturally on the floors of the tower, by automorphisms of log schemes, and the f_i's are G-equivariant.

8. Making actions very tame, end of proof

The main ingredient needed to complete the proof of 1.1 is the following theorem of Gabber (see [S] for a result of a similar nature, but in a relative situation) :

Theorem 8.1. *Let a finite group G act tamely and generically freely on a noetherian, separated, log regular fs log scheme (X, Z). Let T be the complement of the largest stable open subset of X over which G acts freely. Then there exists a projective equivariant modification $f : X' \to X$ such that if $Z' = f^{-1}(Z \cup T)$, then the pair (X', Z') is log regular and the action of G on X' is very tame.*

8.2. **Step 1** : *We may assume X regular and $Z = \sum Z_i$ a strict ncd.*
Use equivariant resolution of log regular schemes (7.6).

8.3. **Step 2** : *We may assume moreover that Z is G-strict.*

Recall that G-strictness means that Z is G-stable and that, for any irreducible component D of Z and any $g \in G$, if gD is distinct from D, then $D \cap gD = \emptyset$.

Use de Jong's canonical blow-up [dJ1, 7.2] :

$$\tilde{X} = X^{(1)} \to X^{(2)} \to \cdots \to X^{(d-1)} \to X^{(d)} = X$$

(here X is assumed to have pure dimension d, Z is decomposed as the disjoint union of locally closed subsets $Z^{(i)}$ (of codimension i) where the number of branches passing through a point is exactly i, and the maps are defined inductively by blowing-up $Z^{(d)}$ in X, then the closure of $Z^{(d-1)}$ in $X^{(d-1)}$, $\cdots$, the closure of $Z^{(i)}$ in $X^{(i)}$). If $f : \tilde{X} \to X$ is the composition, $\tilde{Z} = f^{-1}(Z)_{\mathrm{red}}$ is a G-strict ncd. The point is that the components of $\tilde{Z}$ are the closures of the inverse images by f of the components of the $Z^{(i)}$'s.

Note that G-strictness implies that, at each geometric point $\overline{x}$ of X, the inertia $G_{\overline{x}}$ *acts trivially on $\overline{M}_{\overline{x}}$*. Indeed, if D_i, for $1 \leq i \leq r$, are the branches of Z passing through $\overline{x}$, then $\overline{M}_{\overline{x}} = \oplus_{1 \leq i \leq r} \mathbb{N} e_i$, with e_i corresponding to D_i, and by G-strictness $gD_i = D_i$ for all i.

8.4. **Step 3** : *We may assume moreover that the inertia groups are abelian and that G acts freely on $X - Z$.*

This requires several lemmas. The next one seems to be well known, but a reference is apparently lacking.

Lemma 8.4.1. *Let G be a finite group acting on a regular scheme X and of order invertible on X. Then the fixed points scheme X^G is regular.*

Here the fixed point scheme X^G is characterized by $X^G(S) = X(S)^G$ for all schemes S.

We may assume X local, $X = \operatorname{Spec} A$, so $X^G = \operatorname{Spec} A_G$, $A_G = A/I$, I the ideal generated by $ga - a$, $a \in \mathbf{m}$, $g \in G$ ($\mathbf{m}$ the maximal ideal of A). We may furthermore assume that G acts trivially on the residue field k of A, otherwise X^G is empty. As $(X^G)\hat{} = \operatorname{Spec}(\widehat{A}/I\widehat{A}) = (\widehat{X})^G$, we may assume X complete, local. We prove the result by *linearizing* the action of G.

Assume first that A is of equal characteristic p. Choose a basis $(t_i)_{1 \leq i \leq r}$ of $T = \mathbf{m}/\mathbf{m}^2$ and elements $x_i \in \mathbf{m}$ lifting t_i, thus forming a regular sequence of parameters in A. Choose a field of representives, still denoted k, in A, and let

$$\varphi : k[[T]] \to A$$

be the homomorphism sending t_i to x_i. The averaged homomorphism $f : k[[T]] \to A$ sending $t = (t_i)$ to

$$y = (1/|G|) \sum_{g \in G} g \varphi g^{-1} t$$

is G-equivariant and y is congruent to x mod $\mathbf{m}^2$, hence a regular system of parameters. Then f induces an isomorphism $k[[T]]_G = k[[T_G]] \xrightarrow{\sim} A_G$, where T_G is the coinvariant space of G on T.

Assume now that A is of mixed characteristic $(0, p)$. Let C be a Cohen ring for k and choose $C \to A$ lifting $C \to k$. As G is of order prime to p, there exists a unique $C[G]$-module V, free of finite type over C, lifting $T = \mathbf{m}/\mathbf{m}^2$. Let (t_i) be a basis of T, (v_i) a basis of V lifting (t_i), and, as above, elements $x_i \in \mathbf{m}$ lifting t_i, thus forming a regular sequence of parameters in A. Extend $C \to A$ to

$$\varphi : C[[V]] \to A$$

by sending v_i to x_i. The averaged homomorphism $f : C[[V]] \to A$ sending $v = (v_i)$ to

$$y = (1/|G|) \sum_{g \in G} g\varphi g^{-1} v$$

is G-equivariant and y is congruent to x mod $\mathbf{m}^2$, hence a regular system of parameters. The homomorphism f is surjective and describes X as a regular divisor in $X' = \operatorname{Spec} C[[V]]$, defined by an equation $F = 0$, where F is congruent to p modulo the ideal generated by the v_i's. As F cuts a regular parameter in $X'^G = \operatorname{Spec} C[[V_G]]$, $X^G = X \times_{X'} X'^G$ is regular.

8.4.2. It follows from 8.4.1 that for (X, Z) as in 8.3, i. e. X regular acted on tamely by G, Z a G-strict ncd, if H is a subgroup of G, then the fixed points scheme X^H is regular (if $X^H \neq \emptyset$, at a geometric point $\overline{x}$ of X^H, $H \subset G_{\overline{x}}$), hence the blow-up $B = Bl_{X^H}(X)$ of X along X^H is regular. The normalizer $N = N_G(H)$ of H in G stabilizes X^H, hence acts on B, and the map $f : B \to X$ is equivariant with respect to $N \to G$. Moreover, $f^{-1}(X^H)$ is a regular divisor in B, and if D is a component of Z, as D is H-stable, $D \times_X X^H = D^H$ is regular, and the proper transform $\tilde{D} = Bl_{D^H}(D)$ is a regular divisor crossing $f^{-1}(X^H)$ transversally. It follows that the reduced total transform $f^{-1}(Z)_{\mathrm{red}}$ is an N-strict ncd in B.

Lemma 8.4.3. *Let $\overline{x}$ be a geometric point of X at which the inertia group $G_{\overline{x}}$ is not abelian, and let H be the commutator subgroup $(G_{\overline{x}}, G_{\overline{x}})$. Then $G_{\overline{x}} = N_{G_{\overline{x}}}(H)$ acts on $B = Bl_{X^H}(X)$, and at each geometric point $\overline{y}$ of B above $\overline{x}$, the inertia group $(G_{\overline{x}})_{\overline{y}}$ is strictly smaller than $G_{\overline{x}}$.*

The point $\overline{y}$ corresponds to a line L in the fiber at $\overline{x}$ of the normal bundle $T_{\overline{x}}/T^H_{\overline{x}}$ of X^H in X. If $G_{\overline{x}}$ fixes $\overline{y}$, then $G_{\overline{x}}$ acts on L by a character, hence H acts trivially on L, which is a contradiction, as $(T_{\overline{x}}/T^H_{\overline{x}})^H = 0$. (Note that this shows in particular that when $H \neq \{1\}$, X^H is of codimension ≥ 2 at $\overline{x}$.)

Lemma 8.4.4. *Let $f : B \to X$ be the join of the blow-ups of the subschemes X^H for all subgroups H of G, $H \neq \{1\}$, i. e. the blow-up of the product of the ideals defining the X^H's. Then :*

(a) G acts on B, and f is G-equivariant,

(b) B is regular and

$$f^{-1}(Z \cup (\cup_H (X^H)))_{\mathrm{red}}$$

is a G-strict ncd,

(c) G acts freely on $B - E$, where $E = \cup_H f^{-1}(X^H)$,

(d) if at some geometric point $\overline{x}$ of X, $G_{\overline{x}}$ is not abelian, then at any geometric point $\overline{y}$ of B above $\overline{x}$, $G_{\overline{y}}$ is strictly smaller than $G_{\overline{x}}$.

Proof : (a) follows from the formula

$$gX^H = X^{gHg^{-1}},$$

which implies that the union of the X^H's for H in a conjugacy class of subgroups of G is G-stable. Assertion (b) will follow from 8.4.1 via the analysis of the respective positions of the X^H's and the components of Z. Observe that by 8.4.1 all finite intersections $\cap_{i \in I} X^{H_i} \cap \cap_{j \in J} Z_j$ (I a finite subset of the set of subgroups of G, J a finite subset of the set of components of Z) are regular, as for any closed subscheme W of X, scheme-theoretically

$$\cap_{i \in I} W^{H_i} = W^H,$$

where H is the subgroup generated by the H_i's. This suggests the following definitions (Gabber). Let $F = (F_i)_{i \in I}$ be a finite family of closed subschemes of X.

One says that F is in *weakly good position* if all finite intersections of members of F are regular.

One says that F is in *good position* if at each geometric point $\overline{x}$ of X the finite intersections of members of F cut out on the cotangent space $T^*_{\overline{x}} = \mathbf{m}_{\overline{x}}/\mathbf{m}^2_{\overline{x}}$ to X at $\overline{x}$ a family of linear subspaces for which there exists a basis $(e_i)_{1 \leq i \leq r}$ of $T^*_{\overline{x}}$ such that each such intersection is defined by equations forming a part of the set of the (e_i)'s. This is equivalent to saying that the filtrations of the cotangent space cut out by the F_i's are *compatible* in the sense of M. Saito [MS]. For a family consisting of two elements, good position and weakly good position are equivalent. If F is in good position, F is in weakly good position.

Let $\mathcal{H}$ denote the family of subgroups of G, and $\mathcal{Z}$ the family of components of Z. One checks the following :

(i) The family $\mathcal{Z}$ is in good position.

(ii) The family $(X^H)_{H \in \mathcal{H}}$ is in weakly good position. It is not in good position in general, as the case of G the dihedral group D_3 acting in the standard way on the plane already shows.

(iii) For any H in $\mathcal{H}$, the family $(X^H, \mathcal{Z})$ is in good position.

(iv) Let $Y_1 \subset \cdots \subset Y_n$ is a nested sequence of closed regular subschemes of X and $D = (D_i)_{i \in I}$ a family of regular closed subschemes of X in good position. Assume that for each $i \in I$, $(D_i, (Y_j))$ is in good position, then the family $((D_i), (Y_j))$ is in good position. This is a particular case of general properties of compatible filtrations. It applies, for example, to the family $\mathcal{Z}$ and any family (X^{H_i}), for a decreasing sequence $H_1 \supset \cdots \supset H_n$ of subgroups of G.

(v) For every point $x \in B$, there is a nested sequence $H_1 \supset \cdots \supset H_n$ of subgroups of G such that if B' is the blow-up of the increasing family $X^{H_1} \subset \cdots \subset X^{H_n}$, the canonical morphism $B \to B'$ is a local isomorphism at x.

(vi) Let F be a family in good position such that the intersection of any two members of F is a disjoint union of members of F, and let $p : X' \to X$ be the join of the blow-ups of the members of F (blow-up of the ideal which is the product of the ideals of the members of

F). Then X' is regular, and $p^{-1}(\cup_{S\in F}S)_{\mathrm{red}}$ is a strict ncd in X'. (This is a generalization of the situation occuring in de Jong's blow-up (8.3).)

Assertion (b) of 8.4.4 follows from (i)-(vi). Assertion (c) is clear, and (d) follows from 8.4.3.

As the non abelian inertias strictly decrease, and the union of the X^H's is the complement of the largest open subset of X on which G acts freely, repeating the construction a finite number of times will complete step 3.

8.5. **Step 4** : *Etale locally around each geometric point $\overline{x}$ of X, up to enlarging Z, we may assume that the action of G on (X, Z) is very tame.*

Consider the stratum $S = C_{\overline{x}}(X)$. This is the trace on $X_{(\overline{x})}$ of $Z^{(i)}$, with $i = \mathrm{rk}\,\overline{M}^{gp}_{\overline{x}}$, in the notation of 8.3 (cf. 5.4 (c)). By G-strictness, it is $G_{\overline{x}}$-stable, but the inertia $G_{\overline{x}}$ does not necessarily act trivially on it, in other words, G does not necessarily act very tamely on (X, Z) at $\overline{x}$. However, as $G_{\overline{x}}$ is abelian, its action on the cotangent space $T^*_{X,\overline{x}} = \mathbf{m}_{\overline{x}}/\mathbf{m}^2_{\overline{x}}$ at $\overline{x}$ decomposes (non uniquely) $T^*_{X,\overline{x}}$ as a sum of $G_{\overline{x}}$-stable lines,

$$T^*_{X,\overline{x}} = \oplus_{1\le i\le n} kt_i,$$

where $k = k(\overline{x})$, with

$$T^*_{S,\overline{x}} = \oplus_{1\le i\le r} kt_i$$

for some $r \le n$. The action of $G_{\overline{x}}$ is given by characters χ_i. By the linearization process used in the proof of 8.4.1, one can lift the t_i's to parameters y_i's of the *completion* of X such that $gy_i = \chi_i(g)y_i$ for $g \in G_{\overline{x}}$. By approximation and algebraization, one finds similar parameters (z_i) in the *strict localization* $X_{(\overline{x})}$ for which $gz_i = \chi_i(g)z_i$. With these parameters, $\prod_{i>r} z_i = 0$ is an equation of Z at $\overline{x}$, and $z_{r+1} = \cdots = z_n = 0$ is a system of equations of S. Let D be the divisor in $X_{(\overline{x})}$ with equation $\prod_{1\le i\le r} z_i = 0$. Then G acts very tamely on $(X_{(\overline{x})}, D \cup Z_{(\overline{x})})$ at $\overline{x}$, as the new stratum $C_{\overline{x}}$ is reduced to $\{\overline{x}\}$, and we have an equivariant chart modeled on $\overline{M}_{\overline{x}} = \mathbb{N}^n$ and the characters χ_i.

8.6. **Step 5** : *Gluing local modifications, end of proof of 8.1.*

In an étale neighborhood U of $\overline{x}$, choose a $G_{\overline{x}}$-stable divisor D such that, in U, $D \cup Z$ is a $G_{\overline{x}}$-strict ncd and the action of $G_{\overline{x}}$ on $(U, H = D \cup Z)$ is very tame (for brevity we write Z for the trace of Z on U). The quotient $(V = U/G_{\overline{x}}, H/G_{\overline{x}})$ is log regular, and étale over X/G at the image $\overline{y}$ of $\overline{x}$. Moreover, the underlying scheme of V, which has toric singularities, and $Z/G_{\overline{x}} \subset H/G_{\overline{x}}$ depend only on (X, Z) restricted to U. Let $c : W \to V$ be the canonical resolution of V (7.6). The map c is log étale, W is regular and log regular, and the reduced subscheme of $c^{-1}(H/G_{\overline{x}}) = c^{-1}(D/G_{\overline{x}}) \cup c^{-1}(Z/G_{\overline{x}})$ is a strict ncd. Consider the normalized pull-back of U/V by c :

$$\begin{tikzcd} U \arrow[d, "p"] & \widetilde{U} \arrow[l, "d"'] \arrow[d, "q"] \\ V & W \arrow[l, "c"] \end{tikzcd} .$$

As p is a Kummer étale cover of group $G_{\overline{x}}$, the same is true of q. Hence $(\widetilde{U}, \widetilde{D} \cup \widetilde{Z})$, where $\widetilde{D} = d^{-1}(D)$, $\widetilde{Z} = d^{-1}(Z)$, is log regular. Note that the pair $(W, c^{-1}(Z/G_{\overline{x}}))$ depends only on (U, Z). Moreover, as $G_{\overline{x}}$ acts freely on $U - Z$, q restricts to an étale cover of group $G_{\overline{x}}$ of $W - c^{-1}(Z/G_{\overline{x}})$, so that $\widetilde{U}$ is the normalization of W in this cover. Therefore the pair $(\widetilde{U}, \widetilde{Z})$ is log regular, and with its (very tame) action of G through $G_{\overline{x}}$, depends only on (U, Z). These local constructions glue and yield the desired global G-equivariant modification of (X, Z).

8.7. *End of proof of 1.1.*

We return to the situation obtained at the end of 5.7. Let us summarize what we have achieved. We may assume that the given pair (X, Z) in 1.1 is of the form $(X = X'/G, Z = Z'/G)$, for X' underlying a log regular scheme, endowed with an action of a finite ℓ-group G, and Z' the support of an equivariant subdivisor of the complement D' of the locus of triviality of X' ($D' = f'^{-1}(T') \cup D$ in the notation of 5.7). Moreover, G acts freely on $X' - D'$. Now apply 8.1 to (X', D'). We get a commutative diagram

$$\begin{tikzcd}
(X'', D'') \arrow[r] \arrow[d, "p"] & (X''/G, D''/G) \arrow[d, "q"] \\
(X', D') \arrow[r] & (X, D) = (X'/G, D'/G)
\end{tikzcd},$$

where p is a G-equivariant modification, the horizontal maps are the canonical projections, (X'', D'') is log regular, with $X'' - D''$ contained in $p^{-1}(X' - D')$, and the action of G on (X'', D'') *very tame*. By 6.6 (c), $(X''/G, D''/G)$ is a log regular pair. Applying Kato-Niziol's desingularization ([K3], [N]) to $(X''/G, D''/G)$, we find a log blow-up $b : \widetilde{X} \to X''/G$, with $\widetilde{X}$ regular, and a strict ncd $\widetilde{D}$, such that $\widetilde{X} - \widetilde{D}$ is contained in $b^{-1}(X''/G - D''/G)$. Then $\widetilde{Z} := (qb)^{-1}(Z)$ is the support of a subdivisor of $\widetilde{D}$, hence the support of a strict ncd, too. This last modification qb uniformizes (X, Z) as required in 1.1.

The proofs of 1.3 and 1.4 are independent of 1.1. They rely on the following reinforcement of 8.1 :

Theorem 8.1' (Gabber). *In the situation of 8.1, assume that we are given a log regular pair (S, W) endowed with a trivial action of G, and a G-equivariant log smooth map $X \to S$. Then there exists a modification f satisfying the properties of 8.1 and such that moreover the pair (X', Z') is log smooth over (S, W) as well as its quotient $(X'/G, Z'/G)$.*

Here, by a pair (Y, E) being log smooth over (S, W) we mean that Y is log smooth over S (and E (resp. W) is the complement of the open subset of triviality of the log structure).

8.8. The log smoothness of (X', Z') over (S, W) is proven with the help of the following lemma on the local equivariant structure of log smooth maps :

Lemma 8.8.1 (Gabber). *Let $f : X \to S$ be an equivariant log smooth map between fine log schemes endowed with an action of a finite group G. Let $\overline{x}$ be a geometric point of X, with image $\overline{y}$ in S. Assume that G is the inertia group at $\overline{x}$ and is of order invertible in S. Assume furthermore that G acts trivially on $\overline{M}_{\overline{x}}$ and $\overline{M}_{\overline{y}}$ and we are given an equivariant chart $a : S \to \operatorname{Spec} \Lambda[P]$ at $\overline{y}$, modeled on some pairing $\chi : G^{ab} \otimes P^{gp} \to \mu = \mu_n(\mathbb{C})$ in the* *sense of 6.5 (c), where $\Lambda = \mathbb{Z}[1/n, \mu]$, with n the exponent of G. Then, up to replacing X by an étale equivariant neighborhood of $\overline{x}$, there is an equivariant chart $b : X \to \operatorname{Spec} \Lambda[Q]$ extending a, such that $P^{gp} \to Q^{gp}$ is injective, the torsion of its cokernel being invertible in X, and the resulting map $b' : X \to X' = S \times_{\operatorname{Spec} \Lambda[P]} \operatorname{Spec} \Lambda[Q]$ smooth. Moreover, up to shrinking X, b' lifts to an equivariant étale map $c : X \to X' \times_{\operatorname{Spec} \Lambda} \operatorname{Spec} \operatorname{Sym}_\Lambda(V)$, where V is a finitely generated projective Λ-module equipped with a G-action.*

To prove that (X', Z') is log smooth over (S, W) one has to check that log smoothness is preserved at each step of the construction. This is trivial for steps 1 and 2, as the resolutions used in them are sequences of log blow-ups. Note the following variant of 8.4.1 :

Lemma 8.8.2. *Let S be a fine log scheme and $f : X \to S$ a log smooth map. Assume f is equivariant with respect to an action of a finite group G of order invertible on S, G acting trivially on S. Let X^G denote the (fine) log scheme of fixed points of G. Then X^G is log smooth over S.*

It suffices to check the *formal* log smoothness of X^G over S. For this one uses averaging on G to lift invariant points to an exact thickening of order 1.

However, the formation of the fine log scheme of fixed points does not commute in general with taking the underlying scheme. Therefore the preservation of log smoothness at steps 3 and 4 does not follow from 8.8.2. Additional arguments, using 8.8.1, are needed to complete the proof of 8.1'.

8.9. *Sketch of proof of 1.3.*
By de Jong's theorem [dJ1, 4.1], up to a radicial extension of k, we may assume that we have a G-alteration $p_1 : X_1 \to X$, with G an ℓ-group, X_1 smooth and quasi-projective over k, $Z_1 = p_1^{-1}(Z)$ the support of a strict ncd, and $X_1/G \to X$ generically finite of degree prime to ℓ. By 8.1' we can find a G-equivariant modification $f : X_2 \to X_1$ and a G-strict equivariant divisor E_2 on X_2, containing $Z_2 = f^{-1}(Z_1)$ as a subdivisor, such that the pair (X_2, E_2) is log smooth over k (with the trivial log structure), as well as the quotient $(X_3, E_3) = (X_2, E_2)/G$. Finally, there exists a log modification $X' \to X_3$, such that X' is log smooth and smooth over k, and the (reduced) inverse image E' of E_3 a sncd. The (reduced) inverse image Z' of Z in X' is then a sub sncd of E'. As $X_3 \to X$ is an ℓ'-alteration, so is $X' \to X$.

8.10. *Sketch of proof of 1.4.*
The proof is similar to that of 1.3. One applies de Jong's theorem [dJ1, 6.5]. One first reduces to the case where we have a G-alteration $p_1 : X_1 \to X$ of group G an ℓ-group (with X_1 quasi-projective over S and $X_1/G \to X$ generically finite of degree prime to ℓ), and a sncd E_1 in X_1 containing the inverse image of Z as a subdivisor, together with a G-alteration S_1 of S, such that the pair (X_1, E_1) is semistable over S_1 (with its standard log structure), and in particular log smooth. Then one proceeds as before, using 8.1' applied to the composition $X_1 \to S_1 \to S_1/G$.

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