Introduction to semi-classical analysis for the Schrödinger operators

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The aim is to present the basic mathematical techniques in semi-classical analysis involving the theory of h-pseudodifferential operators and to illustrate how they permit to solve natural questions about spectral distribution and localization of eigenfunctions. Although semi-classical methods can be applied to many problems, we choose to remain quite close in this short presentation to the initial goals of the theory, that is the verification of the correspondence principle for the Schrödinger operator. More details are given in [20].

From classical mechanics to quantum mechanics

The initial goal of semi-classical mechanics is to explore the correspondence principle, due to Bohr in 1923 [4], which states that one should recover as the Planck constant h tends to zero the classical mechanics from the quantum mechanics. So we start with a very short presentation of these two theories.

Classical mechanics

We start (we present the Hamiltonian formalism) from a C^∞ function on $\mathbb{R}^{2n}$: $(x,\xi) \mapsto p(x,\xi)$ which will permit to describe the motion of the system in consideration and is called the Hamiltonian. The variable x corresponds in the simplest case to the position and ξ to the impulsion of one particle. The evolution is then described, starting of a given point (y,η), by the so called Hamiltonian equations

$$\begin{aligned} dx_j/dt &= (\partial p/\partial \xi_j)(x(t),\xi(t))\ ,\ \text{for}\ \ j = 1,\cdots,n\ ; \\ d\xi_j/dt &= -(\partial p/\partial x_j)(x(t),\xi(t))\ ,\ \text{for}\ j = 1,\cdots,n\ . \end{aligned} \tag{1}$$

The classical trajectories are then defined as the integral curves of a vector field defined on $\mathbb{R}^{2n}$ called the hamiltonian vector field associated with p and defined by $H_p = ((\partial p/\partial \xi), -(\partial p/\partial x))$. All these definitions are more generally relevant in the framework of symplectic geometry on a symplectic manifold M, but we choose for simplicity to explain the theory on $\mathbb{R}^{2n}$, which can be seen the cotangent vector bundle $T^*\mathbb{R}^n$, and is the "local" model of the general situation. This space is equipped naturally with a symplectic structure defined by giving at each point a non degenerate 2-form, which is here $\sigma \ := \sum_j d\xi_j \wedge dx_j$. This 2- form permits to associate canonically to a 1-form on $T^*\mathbb{R}^n_x$ a vector field on $T^*\mathbb{R}^n_x$. In this correspondence, if p is a function on $T^*\mathbb{R}^n_x$, H_p is associated with the differential dp.

In this talk, we keep in mind as guiding example the example of the Hamiltonian $p(x,\xi) = \xi^2 + V(x)$, also called the Schrödinger Hamiltonian and more specifically the case of the harmonic oscillator where $V(x) = \sum_{j=1}^n \mu_j x_j^2 \quad (\text{with } \mu_j > 0)$, which is the natural approximation of a potential near its minimum, when non degenerate.

In the framework of the classical mechanics the main questions could be :

- Are the trajectories bounded ?
- Are there periodic trajectories ?
- Is one trajectory dense in its energy surface ?
- Is the energy surface compact ?

The solution of these questions could be very difficult. Let us just mention the trivial fact that, if $p^{-1}(\lambda)$ is compact for some λ, then the conservation of energy law

$$p(x(t), y(t)) = p(y, \eta)\ . \tag{2}$$

leads to the property that the trajectories starting of one point (y, η) remain in the set $\{p^{(-1)}(p(y, \eta))$ in $\mathbb{R}^{2n}$ and are hence bounded. This is in particular the case for the harmonic oscillator.

Quantum mechanics

The quantum theory is born around 1920. It is structurally related to the classical mechanics in a way that we shall describe very briefly. In quantum mechanics, our basic object will be a (possibly non-bounded) selfadjoint operator defined on a dense subspace of an Hilbert space $\mathcal{H}$. In order to simplify, we shall always take $\mathcal{H} = L^2(\mathbb{R}^n)$.

This operator can be associated with p by different techniques called quantizations. We choose here to present a procedure called the Weyl-quantization procedure (which is already present in 1928), which under suitable assumptions on p and its derivatives will be defined for $u \in \mathcal{S}(\mathbb{R}^n)$ by

$$\begin{aligned} p^w(x, hD_x, h)u(x) = \\ (2\pi h)^{-n} \iint \exp\left(\frac{i}{h}(x-y)\cdot\xi\right)\, p\left(\frac{x+y}{2}, \xi, h\right)\, u(y)\, dy\, d\xi\, . \end{aligned} \tag{3}$$

The operator $p^w(x, hD_x, h)$ is called an h-pseudodifferential operator of Weyl-symbol p. One can also write $\mathrm{Op}_{\mathrm{h}}^{\mathrm{w}}(p)$ in order to emphasize that it is the operator associated to p by the Weyl quantization. Here h is a parameter which plays the role of the Planck constant.
Of course, one has to give a sense to these integrals and this is the object of the theory of the oscillatory integrals. If $p = 1$, we observe that the associated operator is nothing else, by Plancherel's formula, than the identity :

$$u(x) = (2\pi h)^{-n} \cdot \int\int \exp(\frac{i}{h}(x-y)\cdot\xi)\ u(y)\, dy\, d\xi\ .$$

A way to rewrite any h-differential operator $\sum_{|\alpha|\leq m} a_\alpha(x)(hD_x)^\alpha$ as an h-pseudodifferential operator is to apply it to the Plancherel identity. In particular, we observe that if $p(x,\xi) = \xi^2 + V(x)$, then the h-Weyl quantization associated with p is the Schrödinger operator : $-h^2\Delta + V$. Other interesting examples appear naturally in solid state physics. Let us for example mention the Harper's operator H (see [26]) whose symbol is $(x,\xi) \mapsto \cos\xi + \cos x$. and which can also be written, for $u \in L^2(\mathbb{R}^n)$, by

$$(Hu)(x) = \frac{1}{2}(u(x+h) + u(x-h)) + \cos x \; u(x)\,.$$

We shall later recall how to relate the properties of p and the properties of the associated operator. More precisely, we shall describe under which conditions on p the operator $p^w(x, hD_x; h)$ is semi-bounded, symmetric, essentially selfadjoint, compact, with compact resolvent, trace class, Hilbert-Schmidt (See [Rob] for an extensive presentation.)

But before to look later at a more general situation, let us consider the case of the Schrödinger operator : $S_h = -h^2\Delta + V(x)$. If V is -say continuous- bounded from below, S_h, which is a priori defined on $\mathcal{S}(\mathbb{R}^n)$ as a differential operator, admits a unique selfadjoint extension on $L^2(\mathbb{R}^n)$.
We are first interested in the nature of the spectrum. If V tends to $+\infty$ as $|x| \to \infty$, one can show that S_h, more precisely its selfadjoint realization, has compact resolvent and its spectrum consists of a sequence of eigenvalues tending to ∞. We are next interested in the asymptotic behavior of these eigenvalues.

In the case of the harmonic operator, corresponding to

$$V(x) = \sum_{j=1}^{n} \mu_j x_j^2 \ (\text{with } \ \mu_j > 0)\,,$$

the criterion of compact resolvent is satisfied and the spectrum is described as the set of the

$$\lambda_\alpha(h) = \sum_{j=1}^{n} \sqrt{\mu_j}(2\alpha_j + 1)h\,,$$

for $\alpha \in \mathbb{N}^n$.

We have also in this case a complete description of the normalized associated eigenfunctions which are constructed recursively starting from the first eigenfunction corresponding to $\lambda_0(h) = \sum_j \sqrt{\mu_j}\, h$:

$$\phi_0(x;h) = (\prod_{j=1}^{n} \mu_j^{\frac{1}{8}})(\frac{2}{\pi})^{\frac{n}{2}} \cdot h^{-\frac{n}{4}} \cdot \exp(-\sum_j \sqrt{\mu_j} x_j^2/h)\,. \qquad (4)$$

The eigenfunction ϕ_0 is strictly positive and decays exponentially. Moreover, (and here we enter in the semi-classical world), the local decay in a fixed closed set avoiding $\{0\}$ (which is measured by its L^2 norm) is exponentially small as $h \to 0$. In particular, this says that the eigenfunction lives asymptotically in the set $V(x) \leq \lambda(h)$ which has to be understood as the projection by the map $(x, \xi) \mapsto x$ of the energy surface which is classically attached to the eigenvalue $\lambda(h)$, that is $p(x, \xi) = \lambda(h)$. This is a typical semi-classical statement which will be true in full generality.

From quantum mechanics to classical mechanics : semi-classical mechanics

Before to describe the mathematical tools involved in the exploration of the correspondence principle, let us describe a few results which are typical in the semi-classical context. They concern **Weyl's asymptotics** and the localization of the eigenfunctions.

We start with the case of the Schrödinger operator S_h, but we emphasize however that the h-pseudodifferential techniques are not limited to this situation.

We assume that V is a C^∞ function on $\mathbb{R}^n$ which is semi-bounded and satisfies $\inf V < \underline{\lim}_{|x|\to\infty} V(x)$. The Weyl Theorem gives that the essential spectrum is contained in

$$[\, \underline{\lim}_{|x|\to\infty} V(x)\,,\, +\infty\,[\,.$$

It is also clear that the spectrum is contained in $[\inf V, +\infty[$. In the interval $I = [\inf V, \underline{\lim}_{|x|\to\infty} V(x)[$, the spectrum is discrete, that is has only isolated eigenvalues with finite multiplicity. For any E in I, it is consequently interesting to look at the counting function of the eigenvalues contained in $[\inf V, E]$

$$N_h(E) = \sharp\{\lambda_j(h)\,;\ \lambda_j(h) \leq E\}\,. \tag{5}$$

The main semi-classical result is then

Theorem : Weyl's asymptotics

With the previous assumptions, we have :

$$\lim_{h\to 0} h^n N_h(E) = (2\pi)^{-n} \int_{V(x)\leq E} (E - V(x))^{\frac{n}{2}}\, dx \,.$$

The main term in the expansion of $N_h(E)$, which will be denoted by $W_h(E) := (2\pi h)^{-n} \int_{V(x)\leq E}(E - V(x))^{\frac{n}{2}}\, dx$, is called the Weyl term. It has an analog for the analysis of the counting function for Laplacians on compact manifolds (see [41] and references therein), but let us emphasize that here E is fixed and that one looks at the asymptotics as $h \to 0$. In the other case h is fixed and one looks[1] at the asymptotics as $E \to +\infty$.

[1] Note that on a compact manifold and for the Laplacian, the formula $N_h(E) = N_1(\frac{E}{h^2})$ permits easily to go from one point of view to the other.

Although this formula is rather old (first as folk theorem), many efforts have been done by mathematicians for analyzing, first when E is not a critical value of V, the remainder (see [Rob], [Iv]) $N_h(E) - W_h(E)$, whose behavior is again related to classical analysis. If $h^{n+1}\,(N_h(E) - W_h(E))$ can be shown to be bounded, it appears to be $o(1)$ if the measure of the periodic points for the flow is of measure 0 ([35], [Iv]).
Beyond the analysis of the counting function, one is also interested (for example in questions concerning the groundstate energy of an atom with a large number of particles N satisfying the Pauli exclusion principle (see in [RaSi])) in other quantities like the Riesz means, which are defined, for a given $s \geq 0$, by

$$N_h^s(E) = \sum_j (E - \lambda_j)_+^s \,.$$

The case $s = 0$ corresponds to the counting function. It is then natural to ask for the asymptotic behavior as $h \to 0$ of these functions.
We have for example the following result (See Helffer-Robert [24], Ivrii-Sigal [30], Ivrii [Iv]), when E is not a critical value of V in the interval I,

$$N_h^s(E) = (2\pi h)^{-n} \left(\int_{p_E(x,\xi)\leq 0} (-p_E(x,\xi))^s \, dx \cdot d\xi \right) + \mathcal{O}(h^{-n+\inf(1+s,2)}) \,,$$

with $p_E(x,\xi) = \xi^2 + V(x) - E$.

Localized version (Karadzhov)

$$\sum_j (E-\lambda_j(h))_+^s \psi_j(x;h)^2 = (2\pi h)^{-n} \left(\int_{p_E(x,\xi)\leq 0} (-p_E(x,\xi))^s d\xi \right) + \mathcal{O}(h^{-n+\inf(1+s,2)})\,,$$

uniformy for $x \in K$ where K is compact in $\{V(x) < E\}$.

$$\sum_j (E-\lambda_j(h))_+^s \psi_j(x;h)^2 = (2\pi h)^{-n} L_{s,n} (E-V(x))_+^{(s+n)/2} + \mathcal{O}(h^{-n+\inf(1+s,2)})\,.$$

Applications of this formula in Signal theory : B. Helffer and M. Laleg-Kirati [19] .

Uncertainty principle and Weyl term

The Weyl term can be heuristically understood in the following way. According to the uncertainty principle, a “quantum” particule should occupy at least a volume of order h^n in the phase space with the measure $dx\, d\xi$ (proportional to $(\sum_{j=1}^n d\xi_j \wedge dx_j)^n$). This guess is a consequence of the inequality

$$\frac{h}{2}||u||^2 \leq \left(\int_{\mathbb{R}} (x - x_0)^2 |u|^2 \, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |(\frac{h}{i}\frac{d}{dx} - \ \xi_0)u|^2 dx\right)^{\frac{1}{2}} \ ,$$

expressing the non commutation of $(\frac{h}{i}\frac{d}{dx} - \xi_0)$ and of $(x - x_0)$.

When $||u|| = 1$, and, when x_0 (mean position) and ξ_0 (mean impulsion) are defined by

$$x_0 := \int_{\mathbb{R}} x|u|^2 \, dx$$

and

$$\xi_0 := \frac{h}{i} \int_{\mathbb{R}} u'(x) \cdot \bar{u} \, dx \, ,$$

this inequality is expressing the impossibility for a quantum particle to have a simultaneous small localization in position and impulsion. Consequently the maximal number of "quantum" particles which can live in the the region $p_E(x,\xi) \leq 0$ is approximately (up to some universal multiplicative constant) the volume of this region divided by $(2\pi h)^n$.

Localization of the eigenfunctions

The localization property was already observed on the specific case of the harmonic oscillator. But this was a consequence of an explicit description of the eigenfunctions. This is quite important to have a good description of the decay of the eigenfunctions (as $h \to 0$) outside the classically permitted region without to have to know an explicit formula.

Various approaches can be used.

The first one fits very well in the case of the Schrödinger operator (more generally to h-pseudodifferential operators with symbols admitting holomorphic extensions in the ξ variable) and gives exponential decay. This is based on the so-called Agmon estimates (see Agmon [Ag], Helffer-Sjöstrand [25] or Simon [37]). This is the starting point of the analysis of the tunneling (see [Hel], [DiSj] and [Mar]).

The second one is an elementary application of the h-pseudodifferential formalism which will be described later and leads for example to the following statement.

Proposition: localization of the eigenfunctions

Let E in I and let $(\lambda(h_j), \phi_{(h_j)}(x))$ a sequence in $I \times L^2(\mathbb{R}^n)$ where $\lambda(h_j) \to E$ and $h_j \to 0$ as $j \to \infty$, $x \mapsto \phi_{(h_j)}(x)$ is an L^2-normalized eigenfunction associated with $\lambda(h_j)$ with norm 1. Let Ω be a relatively compact set in $\mathbb{R}^n$ such that

$$V^{-1}(]-\infty, E]) \cap \bar{\Omega} = \emptyset\,.$$

Then,

$$||\phi_{(h_j)}||_{L^2(\Omega)} = \mathcal{O}(h_j^{+\infty})\,.$$

Short introduction to the h-pseudodifferential calculus

Basic calculus : the class S^0

We shall mainly discuss the most simple called the S^0 calculus. Let us simply say here that the S^0 calculus is sufficient once we have suitably (micro)-localized the problem (for example by the functional calculus).
This class of symbols p is simply defined by

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leq C_{\alpha,\beta}\ ,$$

for all $(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n$.
The symbols can possibly be h dependent. With this symbol, one can associate an h-pseudodifferential operator by (3). This operator is a continuous operator on $\mathcal{S}(\mathbb{R}^n)$ but can also be defined by duality on $\mathcal{S}'(\mathbb{R}^n)$.
The first basic analytical result is the Calderon-Vaillancourt (See for example [Ho]) theorem establishing the L^2 continuity.

The second important property is the existence of a calculus. If a is in S^0 and b is in S^0 then the composition $a^w(x, hD_x) \circ b^w(x, hD_x)$ of the two operators is a pseudodifferential operator associated with an h-dependent symbol c in S^0:

$$a^w(x, hD_x) \circ b^w(x, hD_x) = c^w(x, hD_x; h) .$$

We immediately meet symbols admitting expansions in powers of h, called regular symbols, i.e. admitting expansions of the type

$$a(x, \xi; h) \sim \sum_j a_j(x, \xi) h^j \ , \ b(x, \xi; h) \sim \sum_j b_j(x, \xi) h^j \ .$$

In this case c has a similar expansion :

$$c(x,\xi;h) \sim \left[\exp\left(\frac{ih}{2}(D_x\cdot D_\eta - D_y\cdot D_\xi)\right)\left(a(x,\xi;h)\cdot b(y,\eta;h)\right)\right]_{x=y\,;\,\xi=\eta}\,.$$

The symbol a_0 is called the principal symbol. At the level of principal symbols, the rule is that

$$c_0 = a_0 \cdot b_0 \ .$$

Another important property is the correspondence between commutator of two operators and Poisson brackets. The principal symbol of the commutator $\frac{1}{h}(a^w \circ b^w - b^w \circ a^w)$ is $\frac{1}{i}\{a_0, b_0\}$, where $\{f, g\}$ is the Poisson bracket of f and g :

$$\{f, g\}(x, \xi) = H_f g = \sum_j \left(\partial_{\xi_j} f \cdot \partial_{x_j} g \ - \ \partial_{x_j} f \cdot \partial_{\xi_j} g\right) \ .$$

About global classes

The class S^0 is far to be sufficient for analyzing the global spectral problem and we refer to [3], [Ho] or [Rob] for an extensive presentation of the theory and for the discussion of other quantizations. Our initial operators (think of the harmonic oscillator) have usually not this property. We are consequently obliged to construct more general classes including these examples and permitting to realize this localization. Similar considerations should be done if one start of a problem on a compact manifold. Once such class is introduced, one of the main points to analyze is the existence of a quasi-inverse for a suitably defined elliptic operator. The more general classes were introduced by Beals-Feffermann (see also the most general Hörmander calculus [Ho]), but it is sometimes better to have for a specific problem an adapted class of pseudo-differential operators.

Elliptic theory

Once one has a pseudo-differential calculus, the main point is to have a class of invertible operators, such that the inverse is also in the class. This is what we call an elliptic theory and the typical statement is :

Theorem: construction of the inverse

Let P be an h-pseudodifferential operator associated to a symbol p in S^0. We assume that it is elliptic in the sense that $\frac{1}{p}$ belongs to S^{reg}. Then there exists an h-pseudodifferential operator Q with symbol in S^{reg} such that

$$Q \cdot P = I + R \quad ; \quad P \cdot Q = I + S \,.$$

The remainders R and S are operators with symbols in $\mathcal{O}(h^\infty)$.

The proof is rather easy, once the formalism of composition and the notion of principal symbol have been understood. One can indeed start from the operator Q_0 of symbol $\frac{1}{p}$ and observe that

$$Q_0 P = I + R_1$$

with

$$R_1 \in \mathcal{O}(h^\infty)$$

. The operator

$$(I + R_1)^{-1} Q_0 \sim \left(\sum_{j \geq 0} (-1)^j R_1^j \right) Q_0$$

gives essentially the solution.

Essential selfadjointness and semiboundedness

We now sketch two applications of this calculus in spectral theory. We shall usually consider in our applications an h-pseudodifferential operator P whose Weyl symbol p is regular

$$(H0) \quad p(x,\xi;h) \sim h^j p_j(x,\xi)\,.$$

(We refer to [Rob, Ho, DiSj] for a more precise formulation). Moreover we assume that

$$(H1) \quad (x,\xi) \mapsto p(x,\xi;h) \in \mathbb{R}\,.$$

This implies, as can be immediately seen from (3), that p^w is symmetric (= formally selfadjoint) :

$$\langle p^w\, u\,,\ v\rangle_{L^2} = \langle u\,,\ p^w\, v\rangle_{L^2}\,,\ \forall u,v \in \mathcal{S}(\mathbb{R}^n)\,.$$

The third assumption is that the principal symbol is bounded from below (and there is no restriction to assume that it is positive)

$$(H2) \quad p_0(x,\xi) \geq 0 \,.$$

This assumption implies that the operator itself is bounded from below. This result belongs to the family of the so called "Garding's inequality" theorems. More precisely, the assumption gives the existence of a constant C such that for any $u \in \mathcal{S}(\mathbb{R}^n)$

$$\langle Pu, u\rangle_{L^2\times L^2} \geq -C\ h||u||^2 \,.$$

Under suitable assumptions (mainly the previous ones), the main result is that P is, for h small enough, essentially self-adjoint. This means that the operator which was initially defined on $\mathcal{S}(\mathbb{R}^n)$ by the pseudodifferential operator with symbol p admits a unique selfadjoint extension.

The functional calculus

It is well known by the spectral theorem for a selfadjoint operator P that a functional calculus exists for Borel functions. What is important here is to find a class of functions (actually essentially C_0^∞) such that $f(P)$ is a nice pseudodifferential operator in the same class as P with simple rules of computation for the principal symbol.

We are starting from the general formula (see [DiSj])

$$f(P) = -\pi^{-1} \lim_{\epsilon \to 0^+} \iint_{|\mathrm{Im}\, z| \geq \epsilon} \frac{\partial \tilde{f}}{\partial \bar{z}}(x, y)\, (z - P)^{-1} dx\, dy$$

which is true for any selfadjoint operator and any f in $C_0^\infty(\mathbb{R})$.

Here $(x, y) \mapsto \tilde{f}(x, y)$ is a compactly supported, almost analytic extension of f in $\mathbb{C}$. This means that $\tilde{f} = f$ on $\mathbb{R}$ and that for any $N \in \mathbb{N}$ there exists a constant C_N such that $|\frac{\partial \tilde{f}(z)}{\partial \bar{z}}| \leq C_N |\mathrm{Im}\, z|^N$.
The main result due to Helffer-Robert [22] is that, for P an h-regular pseudodifferential operator satisfying (H0)-(H3) and f in $C_0^\infty(\mathbb{R})$, then $f(P)$ is a pseudodifferential operator whose Weyl's symbol $p_f(x, \xi; h)$ admits a formal expansion in powers of h

$$p_f(x, \xi; h) \sim h^j p_{f,j}(x, \xi) \ ,$$

with

$$\begin{aligned} p_{f,0} &= f(p_0) \\ p_{f,1} &= p_1 \cdot f'(p_0) \\ p_{f,j} &= \sum_{k=1}^{2j-1} (-1)^k (k!)^{-1} d_{j,k} f^{(k)}(p_0) \ \ \forall j \geq 2 \ , \end{aligned}$$

where the $d_{j,k}$ are universal polynomial functions of the $\partial_x^\alpha \partial_\xi^\beta p_\ell$ with $|\alpha| + |\beta| + \ell \leq j$.

The main point in the proof is that we can construct a parametrix (= approximate inverse) for $(P - z)^{-1}$ for $\mathrm{Im}\, z \neq 0$ with a nice control as $\mathrm{Im}\, z \to 0$. The constants controling the estimates on the symbols are exploding as $\mathrm{Im}\, z \to 0$ but the choice of the almost analytic extension of f absorbs any negative power of $|\mathrm{Im}\, z|$.
As a consequence, we get that if in some interval I

$$(H4) \quad p_0^{-1}(I + [-\epsilon_0, \epsilon_0]) \text{ is compact },$$

for some $\epsilon_0 > 0$, then the spectrum is, for h small enough, discrete in I.

In particular, we get that, if $p(x,\xi) \to +\infty$ as $|x| + |\xi| \to +\infty$, then the spectrum of P_h is discrete (P_h has compact resolvent).
Under Assumption (H4), we get more precisely

Theorem: Trace formula

Let P be an h-regular pseudodifferential operator satisfying (H0)-(H4), with $I = [E_1, E_2]$, then for any g in $C_0^\infty(]E_1, E_2[)$, we have:

$$\mathrm{Tr}\,[g(P(h))] = h^{-n} \sum_{j \geq 0} h^j \; T_j(g) + \mathcal{O}(h^\infty) \text{ as } \; h \to 0 \,,$$

where $g \mapsto T_j(g)$ are distributions in $\mathcal{D}'(]E_1, E_2[)$.
In particular we have

$$\begin{aligned} T_0(g) &= (2\pi)^{-n} \iint g(p_0(x,\xi))\, dx\, d\xi \,, \\ T_1(g) &= (2\pi)^{-n} \iint g'(p_0(x,\xi))\, p_1(x,\xi)\, dx\, d\xi \,. \end{aligned}$$

This theorem is just obtained by integration of the preceding one, because in the good cases the trace of a trace-class pseudodifferential operator $\mathrm{Op}^w(a)$ is given by the integral of the symbol a over $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$. According to (3), the distribution kernel is given by the oscillatory integral :

$$K(x,y;h) = (2\pi h)^{-n} \int_{\mathbb{R}^n} \exp(\frac{i}{h}(x-y)\cdot\xi)\ \ a(\frac{x+y}{2},\xi;h)d\xi\ , \quad (6)$$

and the trace of $\mathrm{Op}^w(a)$ is the integral over $\mathbb{R}^n$ of the restriction to the diagonal of : $K(x,x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} \ a(x,\xi;h)d\xi$.
Of course, one could think of using the theorem with g, characteristic function of an interval, in order to get for example, the behavior of the counting function attached to this interval. This is of course not directly possible and this will be only obtained through Tauberian theorems and at the price of additional errors.

Another interest is that for suitable f (possibly h-dependent) the operator $f(P)$ could have better properties that the initial operator. This idea will for example applied for the theorem concerning the clustering. It appears in particular very powerful in dimension 1 where we can in some interval of energy find a function $t \mapsto f(t;h)$ admitting an expansion in powers of h such that $f(P;h)$ has the spectrum of the harmonic oscillator. This is a way to get the Bohr-Sommerfeld conditions (See Helffer-Robert [23], in connexion with Maslov [Mas] or Voros [39]) :

$$f(\lambda_n(h)\,;\,h) \sim (2n+1)h\;,$$

modulo $\mathcal{O}(h^\infty)$.

We have tried in this short survey to present some of the techniques which were the starting techniques for the development of the "mathematical" semi-classical analysis. Of course this is very limited and semi-classical methods go far beyond the verification of the correspondence principle. One can refer to semi-classical analysis for many other problems where the same analysis (with a small parameter h) is relevant but where h is no more the Planck constant. This could be a flux (Harper's equation) or the inverse of a flux, the inverse of a mass (Born-Oppenheimer's approximation), of an energy or of a number of particle. We have not developed this point of view here.

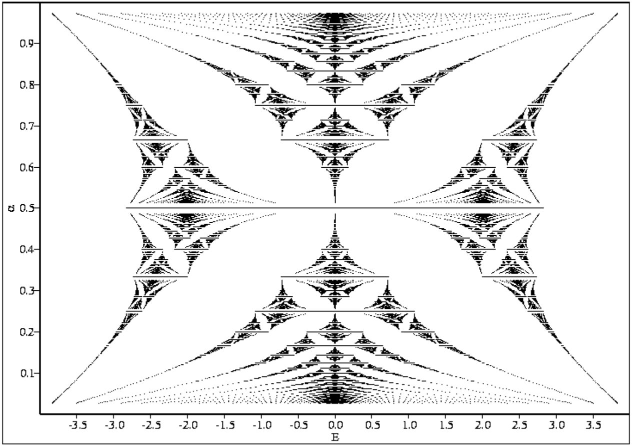

Figure: Hofstadter's butterfly: spectrum of $\cos hD_x + \cos x$, $h/2\pi$ rational

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