Tunneling effect for Witten Laplacians and Kramers operators (After Helffer-Klein-Nier, Helffer-Nier and Hérau-Hitrik-Sjöstrand)

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Main goals

We are interested in the exponentially small eigenvalues of the Dirichlet realization of the semiclassical Witten Laplacian on 0-forms

$$\Delta^{(0)}_{f,h} = -h^2\Delta + |\nabla f(x)|^2 - h\Delta f(x)\ .$$

Our aim is to describe the recent results which have been recently obtained in three cases :

- The case of $\mathbb{R}^n$ (Bovier-Gayrard-Klein [BoGayKl], Helffer-Klein-Nier [HKN]),
- The case of a compact riemannian manifold ([HKN]),
- The case of a bounded set Ω with regular boundary (HelfferNier[HelNi2] (Dirichlet realization[1]).

[1] D. Le Peutrec, a student of F. Nier, is considering the "twisted" Neumann case.

We would also like to discuss the case of the Kramers-Fokker-Planck type operators. If V is a C^∞ potential, we can consider the maximal accretive realization of

$$K(h) = y \cdot h\nabla_x - \nabla_x V(x) \cdot h\nabla_y + \frac{1}{2}(-h\nabla_y + y) \cdot (h\nabla_y + y) \; .$$

with h be a small parameter.

We will try to show, that like in the case of the Witten Laplacians, the existence of some Witten complex can play an important role in the analysis of the spectrum in a small neighborhood of 0.

Main assumptions for the Witten Laplacian case

Assumption 1
The function f is assumed to be a C^∞- function on $\overline{\Omega}$ and a Morse function on Ω.

In the case when $\Omega = \mathbb{R}^n$,
Assumption 2

$$\liminf_{|x|\to+\infty} |\nabla f(x)|^2 > 0 \,,$$

and

$$|D_x^\alpha f| \leq C_\alpha(|\nabla f|^2 + 1) \,,$$

for $|\alpha| = 2$.

In the case with boundary,
Assumption 3
The function f has no critical points at the boundary and the function $f_{/\partial\Omega}$ is a Morse function on $\partial\Omega$.

Initiated by E. Witten

It is known (see Simon, Witten, Helffer-Sjostrand and more recently Chang-Liu, Helffer-Nier2) that the Witten Laplacians on functions $\Delta_{f,h}^{(0)}$ admits exactly m_0 eigenvalues in some interval $[0, Ch^{\frac{6}{5}}]$ for $h > 0$ small enough, where m_0 is the number of local minima in Ω.

This is easy to guess by considering, near each of the local minima $U_j^{(0)}$, the function $\chi_j(x) \exp -\frac{f}{h}$, where χ_j is a suitable cut-off function localizing near $U_j^{(0)}$ as suitable quasimode. This shows that these eigenvalues are actually exponentially small as $h \to 0$.

Note we consider the Dirichlet problem. So Assumption 3 implies that the eigenfunctions corresponding to low lying eigenvalues are localized far from the boundary.

Witten Laplacians on p-forms

Moreover this can be extended (see Simon, Witten, Helffer-Sjostrand, Chang-Liu) to Laplacians on p-forms, $p > 1$.
These Laplacians are obtained by deforming the De Rham complex d :

$$d_{f,h} = \exp -\frac{f}{h}(hd) \exp \frac{f}{h} = hd + df\wedge \;,$$

and by associating the Laplacian :

$$\Delta_{f,h} = (d_{f,h} + d_{f,h}^*)^2 \;,$$

where $d_{f,h}^*$ is its L^2-adjoint. By restriction to the p-forms we get $\Delta_{f,h}^{(p)}$.

Morse Inequalities

In the compact case, this was the main point of the semi-classical proof suggested by Witten of the Morse inequalities.

Each of the W-Laplacians is essentially selfadjoint and an analysis based on the harmonic approximation shows that the dimension of the eigenspace corresponding to $]0, h^{\frac{6}{5}}]$ is, for h small enough, equal to m_p the number of critical points of index p (the index at a critical point U being defined as the number of negative eigenvalues of the Hessian of f at U.

Note that the dimension of the kernel of $\Delta_{h,f}^{(p)}$ is equal to the Betti number b_p, so this gives the "weak Morse Inequalities" :

$$b_p \leq m_p \;, \quad \text{for all } p \in \{0, \dots, n\} \;.$$

Questions in the case with boundary

In the case with boundary, two natural questions appear :

What is the interesting selfadjoint realization to consider (in order for example to show a Morse inequality) ?

How do we define the notion of critical point and of index for a point at the boundary?

We mainly concentrate here on the analysis of the Witten Laplacians on 0-forms and 1-forms.

Our aim is to get the optimal accuracy asymptotic formulas for the m_0 first eigenvalues of the Dirichlet realization of $\Delta^{(0)}_{f,h}$.

Previously via a probabilistic approach : Freidlin-Wentcel [FrWe], Holley-Kusuoka-Strook[HolKusStr], Miclo [Mic], Kolokoltsov [Kol], Bovier-Eckhoff-Gayrard-Klein [BEGK] and Bovier-Gayrard-Klein [BoGayKl], but the proof of optimal accuracy (except may be for the case of dimension 1) has been obtained in [HKN] and [HelNi2].

The Witten Laplacian is associated to the Dirichlet form

$$C_0^\infty(\Omega) \ni u \mapsto \int_\Omega |(h\nabla + \nabla f)u(x)|^2 \; dx \, .$$

The probabilists look equivalently at :

$$C_0^\infty(\Omega) \ni v \mapsto h^2 \int_\Omega |\nabla v(x)|^2 \; e^{-2f(x)/h} \, dx \, .$$

The case of $\mathbb{R}^n$

In the case of $\mathbb{R}^n$ and under assumptions 1 and 2 (together with a generic assumption), one gets :

Theorem (Bovier-Eckhoff-Gayrard-Helffer-Klein-Nier) The first eigenvalues $\lambda_k(h)$, $k \in \{2, \dots, m_0\}$, of $\Delta_{f,h}^{(0)}$ have the following expansions :

$$\lambda_k(h) = \frac{h}{\pi} |\widehat{\lambda}_1(U_{j(k)}^{(1)})| \sqrt{\frac{\left|\det(\mathrm{Hess} f(U_k^{(0)}))\right|}{\left|\det(\mathrm{Hess} f(U_{j(k)}^{(1)}))\right|}} \times \exp -\frac{2}{h}\left(f(U_{j(k)}^{(1)}) - f(U_k^{(0)})\right) \times (1 + r_1(h)) \,,$$

with $r_1(h) = o(1)$.
Here $\left(U_k^{(0)}\right)_k$ denote the suitably ordered sequence local minima of f, $U_{j(k)}^{(1)}$ is a “saddle point ” attached in a specific way to the $U_k^{(0)}$ and $\widehat{\lambda}_1(U_{j(k)}^{(1)})$ is the negative eigenvalue of $\mathrm{Hess} f(U_{j(k)}^{(1)})$.

Actually, the estimate

$$r_1(h) = \mathcal{O}(h^{\frac{1}{2}}|\log h|) \ ,$$

is obtained in [BoGayKl] (under weaker assumptions on f) and a complete asymptotics

$$r_1(h) \sim \sum_{j\geq 1} r_{1j} h^j \ ,$$

is obtained in [HKN].

Here we have left out the case $k=1$, which leads to a specific assumption (see Assumption 2) in the case of $\mathbb{R}^n$ for f at ∞. This implies that $\Delta^{(0)}_{f,h}$ is essentially selfadjoint and that the bottom of the essential spectrum is bounded below by some $\epsilon_0 > 0$ (independently of $h \in]0, h_0]$, h_0 small enough). If the function $\exp -\frac{f}{h}$ is in L^2, then

$$\lambda_1(h) = 0 \ .$$

The approach given in [HKN] intensively uses, together with the techniques of [HelSj4],

- the Witten Laplacian is associated to a cohomology complex
- $\exp -\frac{f(x)}{h}$ is in the kernel of the Witten Laplacian on 0-forms

This permits to construct – and this is specific of the case of $\Delta_{f,h}^{(0)}$– very efficiently quasimodes. The restriction of $d_{f,h}$ to p-forms is denoted by $d_{f,h}^{(p)}$. We note that we have the relation

$$d_{f,h}^{(0)}\Delta_{f,h}^{(0)} = \Delta_{f,h}^{(1)}d_{f,h}^{(1)}\ .$$

Witten complex, Reduced Witten complex

It is more convenient to consider the singular values of the restricted differential $d_{f,h} : F^{(0)} \to F^{(1)}$. The space $F^{(\ell)}$ is the m_ℓ-dimensional spectral subspace of $\Delta_{f,h}^{(\ell)}$, $\ell \in \{0, 1\}$,

$$F^{(\ell)} = \mathrm{Ran}\, 1_{I(h)}(\Delta_{f,h}^{(\ell)}) \;,$$

with $I(h) = [0, h^{\frac{6}{5}}]$ and the property

$$1_{I(h)}(\Delta_{f,h}^{(1)}) d_{f,h} = d_{f,h} 1_{I(h)}(\Delta_{f,h}^{(0)}) \;.$$

We will analyze :

$$\beta_{f,h}^{(\ell)} := (d_{f,h}^{(\ell)})_{/F^{(\ell)}} \;.$$

We will mainly concentrate on the case $\ell = 0$.

Singular values

In order to exploit all the information which can be extracted from well chosen quasimodes, working with singular values of $\beta^{(0)}_{f,h}$ happens to be more efficient than considering their squares, the eigenvalues of $\Delta^{(0)}_{f,h}$. The main point is probably that the errors appear "multiplicatively" when computing the matrix of $\beta^{(0)}_{f,h}$ in approximate well localized "almost" orthogonal basis of $F^{(0)}$ and $F^{(1)}$.
By this we mean :

$$\lambda = \lambda^{app}(1 + error) \ ,$$

instead of additively

$$\lambda = \lambda^{app} + error \ ,$$

as for example in [HelSj4]. Here λ_{app} will be explicitly obtained from the WKB analysis.

The main result in the case with boundary

In the case with boundary, the function $\exp -\frac{f}{h}$ does not satisfy the Dirichlet condition, so the smallest eigenvalue can not be 0.

For this case, a starting reference is [FrWe]. It says (in particular) that, if f has a unique non degenerate local minimum x_{min}, then the lowest eigenvalue $\lambda_1(h)$ of the Dirichlet realization $\Delta_{f,h}^{(0)}$ in Ω satisfies :

$$\lim_{h\to 0} -h\,\log \lambda_1(h) = 2 \inf_{x\in\partial\Omega} (f(x) - f(x_{min}))\ .$$

Other results are given in the case of many local minima but they are again limited to the determination of logarithmic equivalents.

It is shown in [HelNi2] that, under a suitable generic assumption "Assgeneric", one can

- label the m_0 local minima
- introduce an injective map j from the set of the local minima into the set of the m_1 (generalized) saddle points of the Morse functions in $\overline{\Omega}$ of index 1.

We recall that ∇f does not vanish at the boundary. Our problem leads us to define a point of index 1 at the boundary as a point U which is a local minimum of $f_{/\partial\Omega}$ and for which the external normal derivative of f is strictly positive.

At a generalized critical point U with index 1, we can associate the Hessians $\mathrm{Hess} f(U)$, if $U \in \Omega$, or $(\mathrm{Hess} f\big|_{\partial\Omega})(U)$, if $U \in \partial\Omega$. When $U \in \Omega$, $\widehat{\lambda}_1(U)$ denotes the negative eigenvalue of $\mathrm{Hess} f(U)$.

Theorem (Helffer-Nier)
Under Assumptions 1, 3 and "Assgeneric", there exists h_0 such that, for $h \in (0, h_0]$, the spectrum in $[0, h^{\frac{3}{2}})$ of the Dirichlet realization of $\Delta_{f,h}^{(0)}$ in Ω, consists of m_0 eigenvalues $\lambda_1(h) < \ldots < \lambda_{m_0}(h)$ of multiplicity 1, which are exponentially small and admit the following asymptotic expansions :

$$\lambda_k(h) = \frac{h}{\pi} |\widehat{\lambda}_1(U_{j(k)}^{(1)})| \sqrt{\frac{\left|\det(\mathrm{Hess} f(U_k^{(0)}))\right|}{\left|\det(\mathrm{Hess} f(U_{j(k)}^{(1)}))\right|}} \left(1 + h c_k^1(h)\right) \times$$
$$\times \exp -\frac{2}{h} \left(f(U_{j(k)}^{(1)}) - f(U_k^{(0)}) \right) , \quad \text{if } U_{j(k)}^{(1)} \in \Omega ,$$

and

$$\lambda_k(h) = \frac{2h^{1/2} |\nabla f(U_{j(k)}^{(1)})|}{\pi^{1/2}} \sqrt{\frac{\left|\det(\mathrm{Hess} f(U_k^{(0)}))\right|}{\left|\det(\mathrm{Hess} f\big|_{\partial\Omega}(U_{j(k)}^{(1)}))\right|}} \left(1 + h c_k^1(h)\right) \times$$
$$\times \exp -\frac{2}{h} \left(f(U_{j(k)}^{(1)}) - f(U_k^{(0)}) \right) , \quad \text{if } U_{j(k)}^{(1)} \in \partial\Omega .$$

Here $c_k^1(h)$ admits a complete expansion :

$$c_k^1(h) \sim \sum_{m=0}^{\infty} h^m c_{k,m} \, .$$

This theorem extends to the case with boundary the previous results of [BoGayKl] and [HKN] (see also the books [FrWe] and [Kol] and references therein).

About the proof in the case with boundary

As in [HelSj4], the proof is deeply connected with the analysis of the small eigenvalues of a suitable realization (which is **not** the Dirichlet realization) of the Laplacian on the 1-forms. In order to understand the strategy, three main points have to be explained.

First point: define the Witten complex and the associate Laplacian.

The case of a compact manifold was treated in the foundational paper of Witten.
The case of $\mathbb{R}^n$ requires some care (See [Jo] or [HelNi1]).

The case with boundary creates specific new problems.

Our starting problem being the analysis of the Dirichlet realization of the Witten Laplacian, we were led to find the right realization of the Witten Laplacian on 1-forms in the case with boundary in order to extend the commutation relation

$$\Delta^{(1)}_{f,h} d^{(0)}_{f,h} = d^{(0)}_{f,h} \Delta^{(0)}_{f,h} .$$

in a suitable "strong" sense (at the level of the selfadjoint realizations).

Towards the boundary conditions

The answer was present in the literature [CL] in connection with the analysis of the relative cohomology.
Let us explain how we can find the right condition by looking at the eigenfunctions.

If u is eigenfunction of the Dirichlet realization of $\Delta^{(0)}_{f,h}$, then by commutation relation, $d^{(0)}_{f,h}u$ (which can not vanish) should be an eigenfunction in the domain of the realization of $\Delta^{(1)}_{f,h}$. But $d^{(0)}_{f,h}u$ does not satisfy the Dirichlet condition in all its components, but only in its tangential components.

This is the natural condition in the definition of the variational domain to take for the quadratic form $\omega \mapsto ||d^{(1)}_{f,h}\omega||^2 + ||d^{(0)}_{f,h}{}^*\omega||^2$.

The selfadjoint realization $\Delta_{f,h}^{(1)\,DT}$ obtained as the Friedrichs extension associated to the quadratic form gives the right answer.

Observing also that $d_{f,h}^{(0),*}(d_{f,h}^{(0)}u) = \lambda u$ (with $\lambda \neq 0$), we get the second natural (Neumann type)-boundary condition saying that a one form ω in the domain of the operator $\Delta_{f,h}^{(1)\,DT}$ should satisfy

$$d_{f,h}^{(0),*}\omega_{/\partial\Omega} = 0\ .$$

Second point :“rough” localization of the spectrum of this Laplacian on 1-forms.

The analysis was performed in [CL], in the spirit of Witten's idea, extending the so called Harmonic approximation. But these authors, because they were interested in the Morse theory, used the possibility to add symplifying assumptions on f and the metric near the boundary. We emphasize that [HelNi1] and [HKN] treat the general case.

Third point : construction of WKB solutions for the critical points.

This was one in [HelSj4] for the case without boundary, as an extension of previous constructions of [HelSj1].

The new point is the construction of WKB near critical points of the restriction of the Morse function at the boundary is done in [HelNi1] for 1-forms.

The analysis of a point of index 1 at the boundary of a WKB solution is done in [HelNi1]. Let us explain the main lines of the construction.

The construction is done locally around a local minimum U_0 with $\partial_n f(U_0) > 0$. The function Φ is a local solution of the eikonal equation

$$|\nabla\Phi|^2 = |\nabla f|^2 \ ,$$

which also satisfies

$$\Phi = f \text{ on } \partial\Omega$$

and

$$\partial_n \Phi = -\partial_n f \text{ on } \partial\Omega$$

and we normalize f so that $f(U_0) = f(0) = 0$.
We first consider a local solution u_0^{wkb} near the point $x = 0$ of

$$e^{\frac{\Phi}{h}} \Delta_{f,h}^{(0)} u_0^{wkb} = \mathcal{O}(h^\infty) \ ,$$

with u_0^{wkb} in the form

$$u_0^{wkb} = a(x,h) e^{-\frac{\Phi}{h}} \ ,$$

$$a(x,h) \sim \sum_j a_j(x) h^j \ ,$$

and the condition at the boundary

$$a(x,h) e^{-\frac{\Phi}{h}} = e^{-\frac{f}{h}} \quad \text{on } \partial\Omega \ ,$$

which leads to the condition

$$a(x,h)\big|_{\partial\Omega} = 1 \ .$$

In order to verify locally the boundary condition for our future u_1^{wkb}, we substract $e^{-\frac{f}{h}}$ and still obtain

$$e^{\frac{\Phi}{h}} \, \Delta_f^{(0)} (u_0^{wkb} - e^{-\frac{f}{h}}) = \mathcal{O}(h^\infty) \ .$$

We now define the WKB solution u_1^{wkb} by considering :

$$u_1^{wkb} := d_{f,h} u_0^{wkb} = d_{f,h} (u_0^{wkb} - e^{-\frac{f}{h}}) \ .$$

The 1-form $u_1^{wkb} = d_{f,h} u_0^{wkb}$ satisfies locally the Dirichlet tangential condition on the boundary and u_1^{wkb} gives a good approximation for a ground state of a suitable realization of $\Delta_{f,h}^{(1)}$ in a neighborhood of this boundary critical point.

Semi-classical analysis for Kramers operators

Here we follow a recent work of Hérau-Nier-Sjöstrand [HerHiSj] in the continuation of Hérau-Nier, Helffer-Nier, Hérau-Sjöstrand-Stolk ...

In its simplest version, the Kramers-Fokker-Planck operator is the following operator

$$K(h) = y\cdot h\nabla_x - \nabla_x V(x)\cdot h\nabla_y + \frac{1}{2}(-h\nabla_y + y)\cdot(h\nabla_y + y)\,. \tag{1}$$

It has been observed by [TT-NK] that this operator has the same structure as the Witten Laplacian on $\mathbb{R}^{2n}$. This has been exploited in a quite recent preprint of [HerHiSj] and this is this point that we will briefly describe.

Starting form the function

$$\Phi(x,y) = \frac{y^2}{2} + V(x) \ , \tag{2}$$

we can introduce as before the Witten complex $d_{\Phi,h}$ attached to Φ by

$$d_{\Phi,h} = hd + d\Phi\wedge \ ,$$

and we would like to recover the Kramers operator as a "Witten Laplacian".

The case of V quadratic leads to explicit computations (as can be seen in the book by Risken).

This can of course not be done in a standard way because $K(h)$ is not selfadjoint !!

But ....

The operator $K(h)$ can indeed be written as

$$K(h) = d^{*,A}_{\Phi,h} \cdot d_{\Phi,h} = \Delta^{(0)}_{\Phi,h,A} \;,$$

where $d^{*,A}_{\Phi,h}$ is defined on the 1-forms by

$$\langle d^{*,A}_{\Phi,h}\omega \mid u\rangle = \langle A\omega \mid d_{\Phi,h}u\rangle \;,$$

with A the $(2n)\times(2n)$ matrix defined by

$$A := \frac{1}{2}\begin{pmatrix} 0 & I_n \\ -I_n & I_n \end{pmatrix}$$

The lost of selfadjointness is due to to the fact that A is not symmetric.

Like in the case where $A = I_{2n}$ (Standard Witten Laplacian), one can get

$$d_{\Phi,h}\, \Delta^{(0)}_{\Phi,h,A} = \Delta^{(1)}_{\Phi,h,A}\, d_{\Phi,h} \;.$$

Asssumption[2] on V :

V is a Morse function without critical point at ∞ and has precisely three critical points, two local minima $U_{\pm 1}$, and a "saddle point" U_0 of index one.

Theorem (Hérau-Hitrik-Sjöstrand)
Under the previous assumption, there exist $C > 0$ and h_0 such that, $\forall h \in]0, h_0]$, $K(h)$ has in the disk $D(0, \frac{h}{C})$ exactly two eigenvalues $\mu_1 = 0$ and $\mu_2(h)$. Here μ_2 is real and of the form

$$\mu_2(h) = h \left(a_1(h) \exp -2\frac{S_1}{h} + a_{-1}(h) \exp -2\frac{S_{-1}}{h} \right) , \tag{3}$$

where, for $j = \pm 1$,

$$a_j(h) \sim \sum_{\ell \geq 0} a_{j,\ell} h^\ell ,$$

$$a_{j,0} > 0 \text{ and } S_j = V(U_0) - V(U_j) .$$

[2] roughly

Note that the fact that μ_2 is real is a consequence of the property that $K(h)$ is a real operator and of the localization, inside $D(\frac{h}{C})$, of its spectrum.

It would be nice to extend these results in the same spirit as for the results for the Witten Laplacian. As in the case of the Witten Laplacian, a new difficulty (solved in [HKN] for the Witten Laplacian) arises when there are more than two minima. One can still control the largest exponentially small eigenvalue but one could become blocked for the smallest non zero eigenvalue (problem of resonant wells).

It would be interesting to have an expression of $a_{j,0}$. According to M. Hitrick, this should not be a problem and is quite analog to what is done for the usal Witten Laplacian.

As we have seen the Witten Laplacian corresponds to $A = I_{2n}$. So $\mu_2^A(h)$ as a function of the matrix A is always of the same order (assuming for example that $S_1 < S_{-1}$.)

This is to compare with the comparison estimates between $\mu_2^{I_{2n}}(h)$ and $\mu_2^A(h)$ obtained via their interpretation of the rate of return to the equilibrium given in the book of Helffer-Nier.

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