Spectral theory and applications. An elementary introductory course. Bucarest Version 2010

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Abstract

We intend to present in this course the basic tools in spectral analysis and to illustrate the theory by presenting examples coming from the Schrödinger operator theory and from various branches of physics : statistical mechanics, superconductivity, fluid mechanics. We also give some introduction to non self-adjoint operators theory with emphasis on the role of the pseudo-spectrum. Other examples are treated in the 2008-2009 version.

Contents

1 Introduction

Our starting point could be the theory of Hermitian matrices, that is of the matrices satisfying : $A^\star = A$. If we look for eigenvectors and corresponding eigenvalues of A, that is for pairs (u, λ) with $u \in \mathbb{C}^k$, $u \neq 0$ and $\lambda \in \mathbb{C}$ such that $Au = \lambda u$, we know that the eigenvalues are real and that one can find an orthonormal basis of eigenvectors associated with real eigenvalues.

In order to extend this theory to the case of spaces with infinite dimension (that is replacing the space $\mathbb{C}^m$ by a general Hilbert space $\mathcal{H}$), the first attempt consists in developing the theory of compact selfadjoint operators. But it is far to cover all the interesting cases that are present in Quantum Mechanics. So our aim is to present a general theory but it is perhaps good to start by looking at specific operators and to ask naive questions about the existence of pairs (u, λ) with u in some suitable domain, $u \neq 0$ and $\lambda \in \mathbb{C}$ such that $Au = \lambda u$. We shall discover in particular that the answer at these questions may depend strongly on the choice of the domain and on the precise definition of the operator.

1.1 The free Laplacian

The Laplacian $-\Delta$ has no eigenfunctions in L^2, but it has for any $\lambda \in \mathbb{R}^+$ an eigenfunction in $\mathcal{S}'(\mathbb{R}^m)$ (actually in L^∞) and for any $\lambda \in \mathbb{C}$ an eigenfunction in $\mathcal{D}'(\mathbb{R}^m)$. So what is the right way to extend the theory of Hermitian matrices on $\mathbb{C}^k$?
On the other hand, it is easy to produce (take for simplicity $m = 1$) approximate eigenfunctions in the form $u_n(x) = \frac{1}{\sqrt{n}} \exp ix\cdot\xi \; \chi(\frac{x-n^2}{n})$, where χ is a compactly supported function of L^2-norm equal to 1.

1.2 The harmonic oscillator

As we shall see the harmonic oscillator

$$H = -d^2/dx^2 + x^2$$

plays a central role in the theory of quantum mechanics. When looking for eigenfunctions in $\mathcal{S}(\mathbb{R})$, we obtain that there is a sequence of eigenvalues λ_n $(n \in \mathbb{N})$

$$\lambda_n = (2n-1) \,.$$

In particular the fundamental level (in other words the lowest eigenvalue) is

$$\lambda_1 = 1$$

and the splitting between the two first eigenvalues is 2.
The first eigenfunction is given by

$$\phi_1(x) = c_1 \, \exp -\frac{x^2}{2} \tag{1.2.1}$$

and the other eigenfunctions are obtained by applying the creation operator

$$L^+ = -d/dx + x \ . \tag{1.2.2}$$

We observe indeed that

$$H = L^+ \cdot L^- \ + \ 1 \ , \tag{1.2.3}$$

where

$$L^- = d/dx + x \ , \tag{1.2.4}$$

and has the property

$$L^- \phi_1 = 0. \tag{1.2.5}$$

Note that if $u \in L^2$ is a distributional solution of $L^+ u = 0$, then $u = 0$. Note also that if $u \in L^2$ is a distributional solution of $L^- u = 0$, then $u = \mu \phi_1$ for some $\mu \in \mathbb{R}$.
The n^{th}-eigenfunction is then given by

$$\phi_n = c_n (L^+)^{n-1} \phi_1 \ .$$

This can be shown by recursion using the identity

$$L^+ (H + 2) = H L^+ \ . \tag{1.2.6}$$

It is easy to see that $\phi_n(x) = P_n(x) \exp -\frac{x^2}{2}$ where $P_n(x)$ is a polynomial of order $n-1$. One can also show that the ϕ_n are mutually orthogonal. The proof of this point is identical to the finite dimensional case, if we observe the following identity (expressing that H is symmetric) :

$$< Hu, v >_{L^2} = < u, Hv >_{L^2} \ , \ \forall u \in \mathcal{S}(\mathbb{R}), \forall v \in \mathcal{S}(\mathbb{R}) \ , \tag{1.2.7}$$

which is obtained through an integration by parts.
Then it is a standard exercise to show that the family $(\phi_n)_{n \in \mathbb{N}}$ is total and that we have obtained an orthonormal hilbertian basis of L^2, which in some sense permits to diagonalize the operator H.
Another way to understand the completeness is to show that starting of an eigenfunction u in $\mathcal{S}'(\mathbb{R})$ associated with $\lambda \in \mathbb{R}$ solution (in the sense of distribution) of

$$Hu = \lambda u \ ,$$

then there exists $k \in \mathbb{N}$ and $c_k \neq 0$ such that $(L^-)^k u = c_k \phi_1$ and that the corresponding λ is equal to $(2k+1)$.
For this proof, we admit that any eigenfunction can be shown to be in $\mathcal{S}(\mathbb{R})$ and use the identity

$$L^- (H - 2) = H L^- \ , \tag{1.2.8}$$

and the inequality

$$< Hu, u > \geq 0 \ , \ \forall u \in \mathcal{S}(\mathbb{R}) \ . \tag{1.2.9}$$

This property is called "positivity" of the operator.
Actually one can show by various ways that

$$< Hu, u >\geq ||u||^2 \; , \; \forall u \in \mathcal{S}(\mathbb{R}) \; . \tag{1.2.10}$$

One way is to first establish the Heisenberg Principle[1] :

$$||u||^2_{L^2(\mathbb{R})} \leq 2||xu||_{L^2}||u'||_{L^2} \; , \; \forall u \in \mathcal{S}(\mathbb{R}) \; . \tag{1.2.11}$$

The trick is to observe the identity

$$1 = \frac{d}{dx} \cdot x - x \cdot \frac{d}{dx} \; . \tag{1.2.12}$$

The inequality (1.2.10) is simply the consequence of the identity

$$< Hu, u >= ||u'||^2 + ||xu||^2 \; , \tag{1.2.13}$$

which is proved through an integration by parts, and of the application in (1.2.11) of Cauchy-Schwarz inequality.

Another way is to directly observe the identity

$$< Hu, u >= ||L^- u||^2 + ||u||^2 \; , \; \forall u \in \mathcal{S} \; . \tag{1.2.14}$$

1.3 The problem of the boundary

We mainly consider the operator $-\frac{d^2}{dx^2}$ and look at various problems that can be asked naively about the existence of eigenfunctions for the problem in $L^2(]0,1[)$.

1.3.1 Ill-posed problems

Look first at pairs $(u, \lambda) \in H^1(]0,1[) \times \mathbb{C}$ $(u \neq 0)$ such that

$$-du/dx = \lambda u \; , \; u(0) = 0 \, .$$

It is immediate to see that no such pairs exist. We will come back to this example later when analyzing non self-adjoint problems.
Look now at pairs $(u, \lambda) \in H^2(]0,1[) \times \mathbb{C}$ $(u \neq 0)$ such that

$$-d^2u/dx^2 = \lambda u \; .$$

We can find for any λ two linearly independent solutions.

[1] Here is a more "physical" version. If u is normalized by $||u||_{L^2(\mathbb{R})} = 1$, $|u|^2 \; dx$ defines a probability measure. One can define $\langle x \rangle = \int x|u|^2 dx$, mean value of the position and the variance $\sigma_x = \langle (x - \langle x \rangle)^2 \rangle$. Similarly, we can consider: $\langle D_x \rangle := \int (D_x u) \cdot \bar{u}(x) dx$ and $\sigma_{D_x} := ||(D_x - \langle D_x \rangle)u||^2$. Then (1.2.11) can be extended in the form :

$$\sigma_x \cdot \sigma_{D_x} \geq \frac{1}{4} \; .$$

1.3.2 The periodic problem

Here we consider pairs $(u, \lambda) \in H^{2,per}(]0,1[) \times \mathbb{C}$ $(u \neq 0)$ such that

$$-d^2u/dx^2 = \lambda u \ .$$

Here

$$H^{2,per}(]0,1[) = \{u \in H^2(]0,1[) \ , \ u(0) = u(1) \text{ and } u'(0) = u'(1)\} \ .$$

Here we recall that $H^2(]0,1[)$ is included in $C^1([0,1])$ by the Sobolev injection theorem. It is an easy exercise to show that the pairs are described by two families

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$$\lambda = 4\pi^2 n^2 \ , \ u_n = \mu \cos 2\pi n x \ , \quad \text{for } n \in \mathbb{N} \ , \ \mu \in \mathbb{R} \setminus 0 \ ,$$

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$$\lambda = 4\pi^2 n^2 \ , \ v_n = \mu \sin 2\pi n x \ , \quad \text{for } n \in \mathbb{N}^* , \ \mu \in \mathbb{R} \setminus 0 \ .$$

One observes that $\lambda = 0$ is the lowest eigenvalue and that its multiplicity is one. This means that the corresponding eigenspace is of dimension one (the other eigenspaces are of dimension 2). Moreover an eigenfunction in this subspace never vanishes in $]0,1[$. This is quite evident because $u_0 = \mu \neq 0$.
One observes also that one can find an orthonormal basis in $L^2(]0,1[)$ of eigenfunctions by normalizing the family $(\cos 2\pi n x \ (n \in \mathbb{N}), \sin 2\pi n x \ (n \in \mathbb{N}^*))$ or the family $\exp 2\pi i n x \ (n \in \mathbb{Z})$.
We are just recovering the L^2-theory of the Fourier series.

1.3.3 The Dirichlet problem

Here we consider pairs $(u, \lambda) \in H^{2,D}(]0,1[) \times \mathbb{C}$ $(u \neq 0)$ such that $-d^2u/dx^2 = \lambda u$.
Here

$$H^{2,D}(]0,1[) = \{u \in H^2(]0,1[) \ , \ u(0) = u(1) = 0\} \ .$$

It is again an easy exercise to show that the pairs are described by

$$\lambda = \pi^2 n^2 \ , \ v_n = \mu \sin \pi n x \ , \quad \text{for } n \in \mathbb{N}^* , \ \mu \in \mathbb{R} \setminus 0 \ .$$

One observes that $\lambda = \pi^2$ is the lowest eigenvalue, that its multiplicity is one (Here all the eigenspaces are one-dimensional) and that an eigenfunction in this subspace neither vanishes in $]0,1[$.

1.3.4 The Neumann problem

Here we consider pairs
$(u, \lambda) \in H^{2,N}(]0,1[) \times \mathbb{C}$ $(u \neq 0)$ such that

$$-d^2u/dx^2 = \lambda u \ .$$

Here

$$H^{2,N}(]0,1[) = \{u \in H^2(]0,1[)\ ,\ u'(0) = u'(1) = 0\}\ .$$

It is again an easy exercise to show that the pairs are described by

$$\lambda = \pi^2 n^2\ ,\ v_n = \mu \cos \pi n x\ , \quad \text{for } n \in \mathbb{N},\ \mu \in \mathbb{R} \setminus 0\ .$$

One observes that $\lambda = 0$ is the lowest eigenvalue, that its multiplicity is one (Here all the eigenspaces are one-dimensional) and that the corresponding eigenspace is of dimension one and that an eigenfunction in this subspace neither vanishes in $]0,1[$.

1.3.5 Conclusion

All these examples enter in the so called Sturm-Liouville theory. We have emphasized on one property which was always verified in each case: the eigenspace corresponding to the lowest eigenvalue is one dimensional and one can find a strictly positive (in $]0,1[$ or in $]-\infty,+\infty[$ in the case of the harmonic oscillator) corresponding eigenfunction. **We suggest to the reader to come back at this introduction after have read the course. He will surely realize that the theory has permitted to clarify many sometimes badly posed problems.**

2 Unbounded operators, adjoints, Selfadjoint operators

2.1 Unbounded operators

We consider an Hilbert space $\mathcal{H}$. The scalar product will be denoted by : $\langle u\,,\,v\rangle_{\mathcal{H}}$ or more simply by : $\langle u\,,\,v\rangle$ when no confusion is possible. We take the convention that the scalar product is antilinear with respect to the second argument.
A linear operator (or more simply an operator) T in $\mathcal{H}$ is a linear map $u \mapsto Tu$ defined on a subspace $\mathcal{H}_0$ of $\mathcal{H}$, denoted by $D(T)$ and which is called the domain of T. We shall also denote by $R(T)$ (or Im T or $Range(T)$) the range of $\mathcal{H}_0$ by T. We shall say that T is bounded if it is continuous from $D(T)$ (with the topology induced by the topology of $\mathcal{H}$) into $\mathcal{H}$. When $D(T) = \mathcal{H}$, we recover the notion of linear continuous operators on $\mathcal{H}$. We recall that with

$$||T||_{\mathcal{L}(\mathcal{H})} = \sup_{u \neq 0} \frac{||Tu||_{\mathcal{H}}}{||u||_{\mathcal{H}}}\ , \tag{2.1.1}$$

$\mathcal{L}(\mathcal{H})$ is a Banach space. When $D(T)$ is not equal to $\mathcal{H}$, we shall always assume that

$$D(T) \text{ is dense in } \mathcal{H}. \tag{2.1.2}$$

Note that, if T is bounded, then it admits a unique continuous extension to $\mathcal{H}$. In this case the generalized notion is not interesting.

We are mainly interested in extensions of this theory and would like to consider unbounded operators.
– When using this word, we mean more precisely "non necessarily bounded operators". –
The point is to find a natural notion replacing this notion of boundedness. This is the object of the next definition.

The operator is called **closed** if the graph $G(T)$ of T is closed in $\mathcal{H} \times \mathcal{H}$. We recall that

$$G(T) := \{(x, y) \in \mathcal{H} \times \mathcal{H}\ ,\ x \in D(T)\ ,\ y = Tx\}\ . \tag{2.1.3}$$

Equivalently, we can say

Definition 2.1.1. *(Closed operators).*
Let T be an operator on $\mathcal{H}$ with (dense) domain $D(T)$. We say that T is closed if the conditions

- $u_n \in D(T)$,
- $u_n \to u$ *in* $\mathcal{H}$,
- $Tu_n \to v$ *in* $\mathcal{H}$

imply

- $u \in D(T)$,
- $v = Tu$.

Example 2.1.2.

1. $T_0 = -\Delta$ *with* $D(T_0) = C_0^\infty(\mathbb{R}^m)$ *is not closed.*
   For this, it is enough to consider[2] some u in $H^2(\mathbb{R}^m)$ and not in $C_0^\infty(\mathbb{R}^n)$ and to consider a sequence $u_n \in C_0^\infty$ such that $u_n \to u$ in H^2. The sequence $(u_n, -\Delta u_n)$ is contained in $G(T_0)$ and converges in $L^2 \times L^2$ to $(u, -\Delta u)$ which does not belong to $G(T_0)$.

2. $T_1 = -\Delta$ with $D(T_1) = H^2(\mathbb{R}^m)$ is closed.
   We observe indeed that if $u_n \to u$ in L^2 and $(-\Delta u_n) \to v$ in L^2 then $-\Delta u = v \in L^2$. The last step is to observe that this implies that $u \in H^2(\mathbb{R}^m)$ (take the Fourier transform) and $(u, -\Delta u) \in G(T_1)$.

[2] We recall that the Sobolev space $H^s(\mathbb{R}^m)$ is defined as the space

$$H^s(\mathbb{R}^m) := \{u \in \mathcal{S}'(\mathbb{R}^m) \mid (1 + |\xi|^2)^{\frac{s}{2}} \hat{u} \in L^2(\mathbb{R}^m)\}\ .$$

Here $\mathcal{S}'$ is the set of tempered distributions. $H^s(\mathbb{R}^m)$ is equipped with the natural Hilbertian norm :

$$||u||^2_{H^s} := \int_{\mathbb{R}^m} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi\ .$$

By Hilbertian norm, we mean that the norm is associated to a scalar product.
When $s \in \mathbb{N}$, we can also describe H^s by

$$H^s(\mathbb{R}^m) := \{u \in L^2(\mathbb{R}^m) \mid D_x^\alpha u \in L^2\ ,\ \forall \alpha \text{ s. t. } |\alpha| \leq s\}\ .$$

The natural norm associated with the second definition is equivalent to the first one.

This example suggests another definition.

Definition 2.1.3.
The operator T is called **closable** *if the closure of the graph of T is a graph.*

We can then define the closure $\overline{T}$ of the operator by a limit procedure via its graph. We observe indeed that we can consider

$$D(\overline{T}) := \{x \in \mathcal{H} \mid \exists y \text{ s. t. } (x,y) \in \overline{G(T)}\}.$$

For any $x \in D(\overline{T})$, the assumption that $\overline{G(T)}$ is a graph says that y is unique. One can consequently define $\overline{T}$ by

$$\overline{T}x = y \ .$$

In a more explicit way, the domain of $\overline{T}$ is the set of the $x \in \mathcal{H}$ such that $x_n \to x \in \mathcal{H}$ and Tx_n is a Cauchy sequence, and for such x we define $\overline{T}x$ by

$$\overline{T}x = \lim_{n \to +\infty} Tx_n \ .$$

Example 2.1.4.
$T_0 = -\Delta$ *with* $D(T_0) = C_0^\infty$ *is closable and is closure is* T_1.

Let us prove it, as an exercise. Let $\overline{T_0}$ the closure of T_0. Let $u \in L^2$ such that there exists $u_n \in C_0^\infty$ such that $u_n \to u$ in L^2 and $-\Delta u_n \to v$ in L^2. We get by distribution theory that $u \in L^2$ satisfies $-\Delta u = v \in L^2$. By the ellipticity of the Laplacian (use the Fourier transform), we get that $u \in H^2$. We have consequently shown that $D(\overline{T_0}) \subset H^2$. But C_0^∞ is dense in H^2 and this gives the inverse inclusion : $H^2 \subset D(\overline{T_0})$. We have consequently,

$$H^2 = D(T_1) = D(\overline{T_0}) \ ,$$

and it is then easy to verify that $T_1 = \overline{T_0}$.

These examples lead to a more general question.

Realization of differential operators as unbounded operators. Let $\Omega \subset \mathbb{R}^n$ and let $P(x, D_x)$ be a partial differential operator with C^∞ coefficients in Ω. Then the operator P^Ω defined by

$$D(P^\Omega) = C_0^\infty(\Omega) \ , \ P^\Omega u = P(x, D_x)u, \forall u \in C_0^\infty(\Omega) \ ,$$

is closable. Here $\mathcal{H} = L^2(\Omega)$. We have indeed

$$\overline{G(P^\Omega)} \subset \tilde{G}_\Omega := \{(u,f) \in \mathcal{H} \times \mathcal{H} \mid P(x, D_x)u = f \text{ in } \mathcal{D}'(\Omega)\} \ .$$

The proof is then actually a simple exercise in distribution theory. This inclusion shows that $\overline{G(P^\Omega)}$ is a graph. Note that the corresponding operator is defined as P^Ω_{min} with domain

$$D(P^\Omega_{min}) = \left\{u \in L^2(\Omega) \mid \exists \text{ a sequence } u_n \in C_0^\infty(\Omega) \text{ s.t } \left\{ \begin{array}{l} u_n \to u \text{ in } L^2(\Omega) \\ P(x,D_x)u_n \text{ converges in } L^2(\Omega) \end{array} \right. \right\}.$$

The operator P^Ω_{min} is then defined for such u by

$$P^\Omega_{min} u = \lim_{n\to+\infty} P(x,D_x)u_n \; .$$

Using the theory of distributions, this gives :

$$P^\Omega_{min} u = P(x,D_x)u \; .$$

Note that there exists also a natural closed operator whose graph is $\tilde{G}_\Omega$ and extending P^Ω : this is the operator $\tilde{P}^\Omega$, with domain

$$\tilde{D}^\Omega := \{u \in L^2(\Omega) \;,\; P(x,D_x)u \in L^2(\Omega)\} \;,$$

and such that

$$\tilde{P}^\Omega u = P(x,D_x)u \;, \forall u \in \tilde{D}^\Omega \;,$$

where the last equality is in the distributional sense. Note that $\tilde{P}^\Omega$ is an extension of P^Ω_{min} in the sense that :

$$\tilde{P}^\Omega u = P^\Omega_{min} u \;,\; \forall u \in D(P^\Omega_{min}) \; .$$

Conclusion.
We have associated with a differential operator $P(x,D_x)$ in an open set Ω three natural operators. It would be important to know better the connection between these three operators.

Remark 2.1.5. *(Link between continuity and closeness).*
If $\mathcal{H}_0 = \mathcal{H}$, the closed graph Theorem says that a closed operator T is continuous.

2.2 Adjoints.

When we have an operator T in $\mathcal{L}(\mathcal{H})$, it is easy to define the Hilbertian adjoint $T^\star$ by the identity :

$$\langle T^\star u, v\rangle_\mathcal{H} = \langle u, Tv\rangle_\mathcal{H} \;, \forall u \in \mathcal{H}, \forall v \in \mathcal{H} \; . \tag{2.2.1}$$

The map $v \mapsto \langle u, Tv\rangle_\mathcal{H}$ defines a continuous antilinear map on $\mathcal{H}$ and can be expressed, using Riesz's Theorem, by the scalar product by an element which is called T^*u. The linearity and the continuity of T^* is then easily proved using (2.2.1).

Let us now give the definition of the adjoint of an unbounded operator.

Definition 2.2.1. *(Adjoint)*
If T is an unbounded operator on $\mathcal{H}$ whose domain $D(T)$ is dense in $\mathcal{H}$, we first define the domain of T^* by

$$\begin{aligned} D(T^*) \quad &= \{u \in \mathcal{H}, D(T) \ni v \mapsto \langle u, Tv \rangle, \\ &\text{can be extended as an antilinear continuous form on } \mathcal{H}\} \; . \end{aligned}$$

Using the Riesz Theorem, there exists $f \in \mathcal{H}$ such that

$$< f \,,\, v >=< u \,,\, Tv) \;,\; \forall u \in D(T^*), \forall v \in D(T) \;.$$

The uniqueness of f is a consequence of the density of $D(T)$ in $\mathcal{H}$ and we can then define T^*u by

$$T^* u = f \;.$$

Remark 2.2.2.
When $D(T) = \mathcal{H}$ and if T is bounded , then we recover as T^* the Hilbertian adjoint.

Example 2.2.3.

$$T_0^* = T_1 \;.$$

Let us treat in detail this example. We get

$$\begin{aligned} D(T_0^*) \quad &= \{u \in L^2 \mid \text{ the map } C_0^\infty \ni v \mapsto \langle u, -\Delta v \rangle, \\ &\text{can be extended as an antilinear continuous form on } L^2\} \; . \end{aligned}$$

We observe that

$$\langle u, -\Delta v \rangle_{L^2} = \int_{\mathbb{R}^m} u(\overline{-\Delta v}) dx = (-\Delta u)(\bar{v}) \;.$$

The last equality just means that we are considering the distribution $(-\Delta u)$ on the test function $\bar{v}$. The condition appearing in the definition is just that this distribution is in $L^2(\mathbb{R}^m)$. Coming back to the definition of $D(T_0^*)$, we get

$$D(T_0^*) = \{u \in L^2 \mid -\Delta u \in L^2\} \;.$$

But as already seen, this gives

$$D(T_0^*) = H^2 \;,\; T_0^* u = -\Delta u \;,\; \forall u \in H^2 \;.$$

Proposition 2.2.4. .
T^* is a closed operator.

Proof.
Let (v_n) be a sequence in $D(T^*)$ such that $v_n \to v$ in $\mathcal{H}$ and $T^* v_n \to w^*$ in $\mathcal{H}$ for some pair (v, w^*). We would like to show that (v, w^*) belongs to the graph of T^*.
For all $u \in D(T)$, we have :

$$\langle Tu, v \rangle = \lim_{n \to +\infty} \langle Tu, v_n \rangle = \lim_{n \to +\infty} \langle u, T^* v_n \rangle = \langle u, w^* \rangle \;. \tag{2.2.2}$$

Coming back to the definition of $D(T^*)$, we get from (2.2.2) that $v \in D(T^*)$ and $T^*v = w^*$. This means that (v, w^*) belongs to the graph of T^*.

Proposition 2.2.5. .
Let T be an operator in $\mathcal{H}$ with domain $D(T)$. Then the graph $G(T^*)$ of T^* can be characterized by

$$G(T^*) = \{V(\,\overline{G(T)}\,)\}^\perp \,, \tag{2.2.3}$$

where V is the unitary operator defined on $\mathcal{H} \times \mathcal{H}$ by

$$V\{u, v\} = \{v, -u\} \,. \tag{2.2.4}$$

Proof.
We just observe that for any $u \in D(T)$ and $(v, w^*) \in \mathcal{H} \times \mathcal{H}$ we have the identity

$$\langle V(u, Tu)\,,\,(v, w^*)\rangle_{\mathcal{H}\times\mathcal{H}} = \langle Tu, v\rangle_{\mathcal{H}} - \langle u, w^*\rangle_{\mathcal{H}} \,.$$

The right hand side vanishes for all $u \in D(T)$ iff $v \in D(T^*)$ and $w^* = T^*v$, that is if (v, w^*) belongs to $G(T^*)$. The left hand side vanishes for all $u \in D(T)$ iff (v, w^*) belongs to $V(G(T))^\perp$.
Standard Hilbertian analysis, using the continuity of V and $V^{-1} = -V$, then shows that

$$\{V(G(T))\}^\perp = \{\overline{V(G(T))}\}^\perp = \{V(\overline{G(T)})\}^\perp \,.$$

End of proof We have not analyzed till now under which condition the domain of the adjoint is dense in $\mathcal{H}$. This is one of the objects of the next theorem.

Theorem 2.2.6.
Let T be a closable operator. Then we have

1. *$D(T^*)$ is dense in $\mathcal{H}$,*
2. *$T^{**} := (T^*)^* = \overline{T}$, where we have denoted by $\overline{T}$ the operator whose graph is $\overline{G(T)}$.*

Proof.
For the first point, let us assume that $D(T^*)$ is not dense in $\mathcal{H}$. Then there exists $w \neq 0$ such that w is orthogonal to $\overline{D(T^*)}$.
We consequently get that for any $v \in D(T^*)$, we have

$$\langle (0, w)\;,\;(T^*v, -v)\rangle_{\mathcal{H}\times\mathcal{H}} = 0 \,.$$

This shows that $(0, w)$ is orthogonal to $V(G(T^*))$.
But the previous proposition gives :

$$V(\overline{G(T)}) = G(T^*)^\perp \,.$$

We now apply V to this identity and get, using $V^2 = -I$,

$$V\left(G(T^*)^\perp\right) = \overline{G(T)} \,.$$

But, for any closed subspace $\mathcal{M} \subset \mathcal{H} \times \mathcal{H}$, we have

$$V(\mathcal{M}^\perp) = [V(\mathcal{M})]^\perp ,$$

as a consequence of the identity

$$\langle V(u,v) \, , \, (x,y) \rangle_{\mathcal{H}\times\mathcal{H}} = \langle (u,v) \, , \, V(x,y) \rangle_{\mathcal{H}\times\mathcal{H}} .$$

We finally obtain that $(0,w)$ belongs to the closure of the graph of T, that is the graph of $\overline{T}$ because T is closable, and consequently that $w = 0$. This gives the contradiction.
For the second point, we first observe that, $D(T^*)$ being dense in $\mathcal{H}$, we can of course define $(T^*)^*$. Using again the proposition and the closeness of T^*, we obtain $G(T^{**}) = \overline{G(T)}$ and $T^{**} = \overline{T}$.
This means more explicitly that

$$D(T^{**}) = D(\overline{T}) \, , \, T^{**}u = \overline{T}u \, , \, \forall u \in D(\overline{T}) .$$

End of the proof.

2.3 Symmetric and selfadjoint operators.

Definition 2.3.1. *(symmetric operators).*
We shall say that $T : \mathcal{H}_0 \mapsto \mathcal{H}$ is **symmetric** *if it satisfies*

$$\langle Tu, v\rangle_{\mathcal{H}} = \langle u, Tv\rangle_{\mathcal{H}} \, , \forall u, v \, \in \mathcal{H}_0 .$$

Example 2.3.2. .
$T = -\Delta$ with $D(T) = C_0^\infty(\mathbb{R}^m)$.

If T is symmetric it is easy to see that

$$D(T) \subset D(T^*) \tag{2.3.1}$$

and that

$$Tu = T^*u \, , \, \forall u \in D(T) . \tag{2.3.2}$$

The two conditions (2.3.1) and (2.3.2) express the property that $(T^*, D(T^*))$ is an extension of $(T, D(T))$.

Exercise 2.3.3.
Show that a symmetric operator is closable.

Hint :
Show that, if u_n is a sequence in $D(T)$ such that, for some $\ell \in \mathcal{H}$, we have $u_n \to 0$ and $Tu_n \to \ell$, then $\ell = 0$.

For a symmetric operator, we have consequently two natural closed extensions:

- The **minimal** one denoted by T_{min} (or previously $\overline{T}$), which is obtained by taking the operator whose graph is the closure of the graph of T,
- The **maximal** one denoted by T_{max} the adjoint of T.

If T^{sa} is a selfadjoint extension of T, then T^{sa} is automatically an extension of T_{min} and admits[3] as an extension T_{max}.

Definition 2.3.4.
We shall say that T is **selfadjoint** *if $T^* = T$, i. e.*

$$D(T) = D(T^*) \ , \quad and \ \ Tu = T^*u \ , \quad \forall u \in D(T) \ .$$

Starting of a symmetric operator, it is a natural question to ask for the existence and the uniqueness of a selfadjoint extension. We shall see later that a natural way is to prove the equality between T_{min} and T_{max}.

Exercise 2.3.5. Analysis of differential operators.
Give simple criteria in the case of operators with constant coefficients for obtaining symmetric operators on $C_0^\infty(\mathbb{R}^m)$. In particular, verify that the operator $D_{x_j} = \frac{1}{i}\partial_{x_j}$ is symmetric.

Proposition 2.3.6.
A selfadjoint operator is closed.

This is immediate because T^* is closed.

Proposition 2.3.7.
Let T be a selfadjoint operator which is invertible. Then T^{-1} is also selfadjoint.

By invertible, we mean here that T admits an inverse T^{-1} from $R(T)$ into $D(T)$. Let us first show that $R(T)$ is dense in $\mathcal{H}$. Let $w \in \mathcal{H}$ such that $< Tu, w >_{\mathcal{H}} = 0$, $\forall u \in D(T)$.
Coming back to the definition of T^*, this implies in particular that $w \in D(T^*)$ and $T^*w = 0$. But T is selfadjoint and injective and this implies that $w = 0$.
We consequently know that $D(T^{-1})$ is dense in $\mathcal{H}$.
Coming back to the analysis of the corresponding graphs it is now easy to show the second assertion by coming back to the corresponding graphs and by using Proposition 2.2.5.

Remark 2.3.8.
If T is selfadjoint $T + \lambda I$ is selfadjoint for any real λ.

[3] Use Proposition 2.2.5.

3 Representation theorems

We assume that the reader knows about this material but recall it for completeness[4].

3.1 Riesz's Theorem.

Theorem 3.1.1. *(Riesz's Theorem)*
Let $u \mapsto F(u)$ a linear continuous form on $\mathcal{H}$. Then there exists a unique $w \in \mathcal{H}$ such that

$$F(u) = \langle u, w \rangle_{\mathcal{H}} \ , \ \forall u \in \mathcal{H} \ . \tag{3.1.1}$$

There is a similar version with antilinear maps :

Theorem 3.1.2.
Let $u \mapsto F(u)$ a antilinear continuous form on $\mathcal{H}$. Then there exists a unique $w \in \mathcal{H}$ such that

$$F(u) =< w, u >_{\mathcal{H}} \ , \ \forall u \in \mathcal{H} \ . \tag{3.1.2}$$

3.2 Lax-Milgram's situation.

Let us now consider a continuous sesquilinear form a defined on $V \times V$:

$$(u, v) \mapsto a(u, v) \ .$$

We recall that, because of the sesquilinearity, the continuity can be expressed by the existence of C such that

$$|a(u, v)| \leq C \ ||u||_V \ \cdot \ ||v||_V \ , \ \forall u, v \in V \ . \tag{3.2.1}$$

It is immediate to associate, using the Riesz Theorem, a linear map $A \in \mathcal{L}(V)$ such that

$$a(u, v) =< Au, v >_V \ . \tag{3.2.2}$$

Definition 3.2.1. *(V-ellipticity)*
We shall say that a is V-elliptic, if there exists $\alpha > 0$, such that

$$|a(u, u)| \geq \alpha \, ||u||_V^2 \ , \ \forall u \in V \ . \tag{3.2.3}$$

Theorem 3.2.2. *(Lax-Milgram's Theorem)*
Let a be a continuous sesquilinear form on $V \times V$. If a is V-elliptic, then A is an isomorphism from V onto V.

The proof is in three steps.

Step 1 :A is injective.
We get indeed from (3.2.3)

$$|\langle Au, u \rangle_V| \geq \alpha ||u||_V^2 \ , \ \forall u \in V \ . \tag{3.2.4}$$

[4]Here we follow, almost verbatim, the book of D. Huet [Hu].

Using Cauchy-Schwarz in the left hand side, we first get

$$||Au||_V \cdot ||u||_V \geq \alpha ||u||_V^2 \ , \ \forall u \in V \ ,$$

and consequently

$$||Au||_V \geq \alpha ||u||_V \ , \ \forall u \in V \ . \tag{3.2.5}$$

This gives clearly the injectivity but actually more.

Step 2 : $A(V)$ is dense in V.
Let us consider $u \in V$ such that $< Av, u >_V = O \ , \ \forall v \in V$. In particular, we can take $v = u$. This gives $a(u,u) = 0$ and $u = 0$ using (3.2.3).

Step 3 :$R(A) := A(V)$ is closed in V.
Let v_n a Cauchy sequence in $A(V)$ and u_n the sequence such that $Au_n = v_n$. But using (3.2.5), we get that u_n is a Cauchy sequence which is consequently convergent to some $u \in V$. But the sequence Au_n tends to Au by continuity and this shows that $v_n \to v = Au$ and $v \in R(A)$.

Step 4 : A^{-1} is continuous.
The three previous steps show that A is bijective. The continuity of A^{-1} is a consequence of (3.2.5) or of the Banach Theorem.

Remark 3.2.3. .
Let us suppose for simplicity that V is a real Hilbert space. Using the isomorphism $\mathcal{I}$ between V and V' given by the Riesz Theorem, one gets also a natural operator $\mathcal{A}$ from V onto V' such that

$$a(u,v) = (\mathcal{A}u)(v) \ , \ \forall v \in V \ . \tag{3.2.6}$$

We have

$$\mathcal{A} = \mathcal{I} \circ A \ .$$

3.3 An alternative point of view: $V, \mathcal{H}, V'$.

We now consider two Hilbert spaces V and $\mathcal{H}$ such that

$$V \subset \mathcal{H} \ . \tag{3.3.1}$$

By this notation of inclusion, we mean also that the injection of V into $\mathcal{H}$ is continuous or equivalently that there exists a constant $C > 0$ such that, for any $u \in V$, we have

$$||u||_{\mathcal{H}} \leq C \, ||u||_V \ .$$

We also assume that

$$V \text{ is dense in } \mathcal{H}. \tag{3.3.2}$$

In this case, there exists a natural injection from $\mathcal{H}$ into the space V' which is defined as the space of continuous linear forms on V. We observe indeed that if $h \in \mathcal{H}$ then $V \ni u \mapsto < u, h >_{\mathcal{H}}$ is continuous on V. So there exists $\ell_h \in V'$ such that

$$\ell_h(u) = < u, h >_{\mathcal{H}} \ , \forall u \in V \ .$$

The injectivity is a consequence of the density of V in $\mathcal{H}$.
We can also associate to the sesquilinear form a an unbounded operator S on $\mathcal{H}$ in the following way.
We first define $D(S)$ by

$$D(S) = \{u \in V \mid v \mapsto a(u,v) \text{ is continuous on } V \text{ for the topology induced by } \mathcal{H}\}. \tag{3.3.3}$$

Using again the Riesz Theorem and assumption (3.3.2), this defines Su in $\mathcal{H}$ by

$$a(u,v) = < Su, v >_{\mathcal{H}} \ , \ \forall v \in V \ . \tag{3.3.4}$$

Theorem 3.2.2 is completed by

Theorem 3.3.1. .
Under the same assumptions, S is bijective from $D(S)$ onto $\mathcal{H}$ and $S^{-1} \in \mathcal{L}(\mathcal{H})$. Moreover $D(S)$ is dense in $\mathcal{H}$.

Proof.
We first show that S is injective. This is a consequence of

$$\alpha ||u||_{\mathcal{H}}^2 \leq C\, \alpha ||u||_V^2 \leq C |a(u,u)| = C \mid < Su, u >_{\mathcal{H}} \mid \leq C\, ||Su||_{\mathcal{H}} \cdot ||u||_{\mathcal{H}} \ , \ \forall u \in D(S) \ ,$$

which leads to

$$\alpha ||u||_{\mathcal{H}} \leq C\, ||Su||_{\mathcal{H}} \ , \ \forall u \in D(S) \ . \tag{3.3.5}$$

We get directly the surjectivity in the following way. If $h \in \mathcal{H}$ and if $w \in V$ is chosen such that

$$\langle h, v \rangle_{\mathcal{H}} = \langle w, v \rangle_V \ , \forall v \in V \ ,$$

(which follows from Riesz's Theorem), we can take $u = A^{-1} w$ in V, which is a solution of

$$a(u,v) = < w, v >_V \ .$$

We then show that $u \in D(S)$, using the identity

$$a(u,v) = \langle h, v \rangle_{\mathcal{H}} \ , \ \forall v \in V \ ,$$

and get simultaneously

$$Su = h \ .$$

The continuity of S^{-1} is a consequence of (3.3.5).
Let us show the last statement of the theorem, i.e. the density of $D(S)$ in $\mathcal{H}$.
Let $h \in \mathcal{H}$ s. t.

$$\langle u, h \rangle_{\mathcal{H}} = 0 \ , \ \forall u \in D(S) \ .$$

By the surjectivity of S, there exists $v \in D(S)$ s. t. :

$$Sv = h \ .$$

We get

$$< Sv, u >_{\mathcal{H}} = 0 \ , \ \forall u \in D(S)$$

Taking $u = v$ and using the $V-ellipticity$, we get that $v = 0$ and consequently $h = 0$.

The hermitian case.

We now consider an hermitian sesquilinear form, that is satisfying

$$a(u, v) = \overline{a(v, u)} \ , \ \forall u, v \in V \ . \tag{3.3.6}$$

This property is transmitted to S in the following way

Theorem 3.3.2.
If a is hermitian and V-elliptic, we have

1. *S is closed;*
2. *$S = S^*$;*
3. *$D(S)$ is dense in V.*

Proof of 2.
We first observe that the assumption of Hermiticity gives

$$< Su, v >_{\mathcal{H}} = < u, Sv >_{\mathcal{H}} \ , \ \forall u \in D(S) \ , \ \forall v \in D(S) \ . \tag{3.3.7}$$

In other words S is symmetric. This means in particular that

$$D(S) \subset D(S^*) \ . \tag{3.3.8}$$

Let $v \in D(S^*)$. Using the surjectivity of S, there exists $v_0 \in D(S)$ such that

$$Sv_0 = S^* v \ .$$

For all $u \in D(S)$, we get that

$$< Su, v_0 >_{\mathcal{H}} = < u, Sv_0 >_{\mathcal{H}} = < u, S^* v >_{\mathcal{H}} = < Su, v >_{\mathcal{H}} \ .$$

Using again the surjectivity of S, we get $v = v_0 \in D(S)$. This shows that $D(S) = D(S^*)$ and $Sv = S^* v$, $\forall v \in D(S)$.

Proof of 1.
S is closed because S^* is closed and $S = S^*$.

Proof of 3.
Let $h \in V$ such that

$$< u, h >_V = 0 \ , \ \forall u \in D(S) \ .$$

Let $f \in V$ such that $Af = h$ (A is an isomorphism from V onto V).
We then have

$$0 = < u, h >_V = < u, Af >_V = \overline{< Af, u >_V} = \overline{a(f, u)} = a(u, f) = < Su, f >_{\mathcal{H}} \ .$$

Using the surjectivity, we get $f = 0$ and consequently $h = 0$.

Remark 3.3.3.
Theorem 3.3.2 gives us a rather easy way to construct selfadjoint operators. This will be combined with some completion argument in the next Section to get the Friedrichs extension.

4 Semi-bounded operators and Friedrichs extension.

4.1 Definition

Definition 4.1.1.
Let T_0 be a symmetric unbounded operator of domain $D(T_0)$. We say that T_0 is semibounded (from below) if there exists a constant C such that

$$< T_0 u, u >_{\mathcal{H}} \geq -C \, ||u||^2_{\mathcal{H}} \ , \ \forall u \in D(T_0) \ . \tag{4.1.1}$$

Example 4.1.2. *(The Schrödinger operator).*
We consider on $\mathbb{R}^m$ the operator

$$P_V(x, D_x) := -\Delta + V(x) \ , \tag{4.1.2}$$

where $V(x)$ is a continuous function on $\mathbb{R}^m$ (called the potential) such that there exists C s.t. :

$$V(x) \geq -C \ , \ \forall x \in \mathbb{R}^m \ . \tag{4.1.3}$$

Then the operator T_0 defined by

$$D(T_0) = C_0^\infty(\mathbb{R}^m) \text{ and } T_0 u = P_V(x, D_x) u \ , \ \forall u \in D(T_0) \ ,$$

is a symmetric, semibounded operator.
We have indeed, with $\mathcal{H} = L^2(\mathbb{R}^m)$,

$$\begin{array}{ll} < P(x, D_x) u, u >_{\mathcal{H}} & = \int_{\mathbb{R}^m} (-\Delta u + V u) \cdot \bar{u} \, dx \\ & = \int_{\mathbb{R}^m} |\nabla u(x)|^2 dx + \int_{\mathbb{R}^m} V(x) |u(x)|^2 \, dx \\ & \geq -C \, ||u||^2_{\mathcal{H}} \ . \end{array} \tag{4.1.4}$$

Exercise 4.1.3.
Let us consider on $L^2(\mathbb{R}^2; \mathbb{C}^2)$ the operator $\sum_{j=1}^2 \alpha_j D_{x_j}$ with domain $\mathcal{S}(\mathbb{R}^2; \mathbb{C}^2)$. Here the α_j are 2×2 Hermitian matrices such that :

$$\alpha_j \cdot \alpha_k + \alpha_k \cdot \alpha_j = 2 \delta_{jk} \ .$$

Show that this operator symmetric but not semi-bounded. This operator is called the Dirac operator. Its domain is $H^1(\mathbb{R}^2 \, , \mathbb{C}^2)$ and its square is the Laplacian :

$$(\sum_{j=1}^2 \alpha_j D_{x_j})^2 = (-\Delta) \otimes I_{\mathbb{C}^2} \ .$$

4.2 Analysis of the Coulomb case.

There are two important inequalities which are useful when considering the Coulomb case which plays an important role in atomic Physics. By Coulomb case, we mean the analysis on $\mathbb{R}^3$ of the operator

$$S_Z := -\Delta - \frac{Z}{r} \ ,$$

or of the Klein-Gordon operator

$$K_Z := \sqrt{-\Delta + 1} - \frac{Z}{r} \ .$$

The operator $\sqrt{-\Delta+1}$ can easily defined on $\mathcal{S}(\mathbb{R}^3)$, using the Fourier transform $\mathcal{F}$, by

$$\mathcal{F}(\sqrt{-\Delta+1}\, u)(p) = \sqrt{p^2+1}(\mathcal{F}u)(p) \ .$$

The first one is the Hardy Inequality (which can be found for example in the book of Kato ([Ka], p. 305-307)) :

$$\int_{\mathbb{R}^3} |x|^{-2}|u(x)|^2\, dx \leq 4 \int_{\mathbb{R}^3} |p|^2 |\hat{u}(p)|^2\, dp \tag{4.2.1}$$

and the second one is due to Kato and says that

$$\int_{\mathbb{R}^3} |x|^{-1}|u(x)|^2\, dx \leq \frac{\pi}{2} \int_{\mathbb{R}^3} |p|\ |\hat{u}(p)|^2\, dp\,. \tag{4.2.2}$$

One proof of the Hardy inequality consists in writing that, for any $\gamma \in \mathbb{R}$ and any $u \in C_0^\infty(\mathbb{R}^3;\mathbb{R})$, we have :

$$\int_{\mathbb{R}^3} |\nabla u + \gamma \frac{x}{|x|^2} u|^2\, dx \geq 0 \ . \tag{4.2.3}$$

This leads to :

$$\int_{\mathbb{R}^3} |\nabla u|^2 + \gamma^2 \frac{1}{|x|^2}\, |u|^2\, dx \geq -2\gamma \int_{\mathbb{R}^3} \nabla u \cdot \frac{x}{|x|^2} u\, dx \ .$$

But an integration by part gives :

$$\begin{aligned} -2 \textstyle\int_{\mathbb{R}^3} \nabla u \cdot \frac{x}{|x|^2} u\, dx &= \textstyle\int_{\mathbb{R}^3} |u(x)|^2 (\mathrm{div}\, (\frac{x}{|x|^2}))\, dx \\ &= \textstyle\int_{\mathbb{R}^3} |u(x)|^2 (\frac{1}{|x|^2})\, dx \ . \end{aligned}$$

Optimizing over γ leads to $\gamma = \frac{1}{2}$ and gives then the result.

Remark 4.2.1. .
The same idea works for $N \geq 3$ but fails for $N = 2$. So a good exercise[5] *is to look for substitutes in this case by starting from the inequality :*

$$\int_{\mathbb{R}^3} |\nabla u - \gamma(x) \frac{x}{|x|^2} u|^2\, dx \geq 0 \ . \tag{4.2.4}$$

[5] We thank M.J. Esteban for explaining to us the trick.

The function $\gamma(x)$ can be assumed radial : $\gamma(x) = g(|x|)$ and the question is to find a differential inequality on g leading at a weaker Hardy's type inequality. One can for example try $g(r) = \ln(r)$.

For the proof of Kato's inequality, there is a another tricky nice estimate which, as far as we know, goes back to Hardy and Littlewood. In the case of the Coulomb potential, we can write, using the explicit computation for the Fourier-transform of $x \mapsto \frac{1}{|x|}$:

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \widehat{u}(p)\frac{1}{|p-p'|^2}\overline{\widehat{u}}(p')dp\cdot dp' &= \int_{\mathbb{R}^3\times\mathbb{R}^3} \widehat{u}(p)\frac{h(p)}{h(p')}\cdot\frac{h(p')}{h(p)}\frac{1}{|p-p'|^2}\overline{\widehat{u}}(p')dp\cdot dp' \\ &= \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{1}{|p-p'|}\widehat{u}(p)\frac{h(p)}{h(p')}\cdot\frac{h(p')}{h(p)}\frac{1}{|p-p'|}\overline{\widehat{u}}(p')dp\cdot dp'\,,\end{aligned}$$

where h is a strictly positive measurable function to be determined later.
We then use Cauchy-Schwarz in the last equality in order to get

$$\begin{aligned}&|\int\int \widehat{u}(p)\frac{1}{|p-p'|^2}\overline{\widehat{u}}(p')dp\cdot dp'| \\ &\quad\leq \left(\int|\widehat{u}(p)|^2|\frac{h(p)}{h(p')}|^2\frac{1}{|p-p'|^2}dp'\;dp\right)^{\frac{1}{2}}\times\left(\int|\widehat{u}(p')|^2|\frac{h(p')}{h(p)}|^2\frac{1}{|p-p'|^2}dp'\;dp\right)^{\frac{1}{2}} \\ &\quad= \int|\widehat{u}(p)|^2(\int|\frac{h(p)}{h(p')}|^2\frac{1}{|p-p'|^2}dp')\;dp \\ &\quad= \int h(p)^2|\widehat{u}(p)|^2(\int|\frac{1}{h(p')}|^2\frac{1}{|p-p'|^2}dp')\;dp\,.\end{aligned}$$

We now write $p' = \omega'|p|$ in the integral $\int|\frac{1}{h(p')}|^2\frac{1}{|p-p'|^2}dp'$. We then take

$$h(p) = |p|\,.$$

The integral becomes

$$\int|\frac{1}{h(p')}|^2\frac{1}{|p-p'|^2}dp' = |p|^{-1}\int\frac{1}{|\omega'|^2|\omega-\omega'|^2}d\omega'\,,$$

with $p = \omega|p|$.
This is clearly a convergent integral. Moreover, observing the invariance by rotation, one can show that the integral is independent of $\omega \in \mathbb{S}^2$. Hence we can compute it with $\omega = (1,0,0)$.
We finally obtain the existence of an explicit constant C such that

$$|\int\int\widehat{u}(p)\frac{1}{|p-p'|^2}\overline{\widehat{u}}(p')\;dp.dp'| \leq C\int_{\mathbb{R}^3}|p|||\widehat{u}(p)|^2\;dp\,.$$

The optimization of the trick leads to $C = \frac{\pi}{2}$.

Let us now show how one can use these inequalities.

If we use (4.2.1), we get the semi-boundedness for any $Z > 0$ for the Schrödinger Coulomb operator (using Cauchy-Schwarz Inequality).

$$\int_{\mathbb{R}^3}\frac{1}{r}|u|^2\;dx \leq (\int\frac{1}{r^2}|u|^2dx)^{\frac{1}{2}}\cdot||u||\,.$$

But we can rewrite the Hardy Inequality in the form

$$\int_{\mathbb{R}^3} \frac{1}{r^2}|u|^2\, dx \leq 4 < -\Delta u, u >_{L^2(\mathbb{R}^3)} \ .$$

So we get, for any $\epsilon > 0$,

$$\int_{\mathbb{R}^3} \frac{1}{r}|u|^2\, dx \leq \epsilon < -\Delta u, u >_{L^2} + \frac{1}{\epsilon}||u||^2 \ . \tag{4.2.5}$$

This leads to :

$$< S_Z u, u >_{L^2} \geq (1 - \epsilon Z) < -\Delta u, u > - \frac{Z}{\epsilon}||u||^2 \ .$$

Taking $\epsilon = \frac{1}{Z}$, we have finally shown that

$$< S_Z u, u >_{L^2} \geq -Z^2||u||^2 \ . \tag{4.2.6}$$

Here we are not optimal[6]

but there is another way to see that the behavior with respect to Z is correct. We just observe some invariance of the model. Let us suppose that we have proved the inequality for $Z = 1$. In order to treat the general case, we make a change of variable $x = \rho y$. The operator S_Z becomes in the new coordinates :

$$\tilde{S}_Z = \rho^{-2}(-\Delta_y) - \frac{Z}{\rho y} \ .$$

Taking $\rho = Z^{-1}$, we obtain

$$\tilde{S}_Z = Z^2 \left(-\Delta_y - \frac{1}{y} \right) \ .$$

The other inequality (4.2.2) is with this respect much better and quite important for the analysis of the relativistic case. Let us see what we obtain in the case of Klein-Gordon using Kato's Inequality.
We have

$$< K_Z\, u \ , \ u >_{L^2} \geq (1 - Z\frac{\pi}{2}) < \sqrt{-\Delta + 1}\, u \ , \ u >_{L^2} \ .$$

Here the nature of the result is different. The proof gives only that K_Z is semibounded if $Z \leq \frac{2}{\pi}$. This is actually more than a technical problem !

[6]It can be proven (see any standard book in quantum mechanics) that the negative spectrum of this operator is discrete and is described by a sequence of eigenvalues tending to 0 : $-\frac{Z^2}{4n^2}$ with $n \in \mathbb{N}^*$. An eigenfunction related to the lowest eigenvalue $-\frac{1}{4}$ ($Z = 1$) is given by $x \mapsto \exp -\frac{1}{2}|x|$. To prove that, one can instead of using Hardy or Kato use the fact that

$$||\nabla u - \rho \frac{x}{|x|} u||^2 \geq 0 \,,$$

and then optimize over ρ.

4.3 Friedrichs's extension

Theorem 4.3.1.
A symmetric semibounded operator T_0 on $\mathcal{H}$ (with $D(T_0)$ dense in $\mathcal{H}$) admits a selfadjoint extension.

The extension constructed in the proof is the so-called Friedrichs extension. The proof can be seen as a variant of Lax-Milgram's Lemma. We can assume indeed by possibly replacing T_0 by $T_0 + \lambda_0 Id$ that T_0 satisfies

$$< T_0 u, u >_{\mathcal{H}} \geq ||u||^2_{\mathcal{H}} \ , \ \forall u \in D(T_0)\ . \tag{4.3.1}$$

Let us consider the associated form a priori defined on $D(T_0) \times D(T_0)$:

$$(u, v) \mapsto a_0(u, v) := \langle T_0 u \ , \ v\rangle_{\mathcal{H}}\ . \tag{4.3.2}$$

The inequality (4.3.1) says that

$$a_0(u, u) \geq ||u||^2_{\mathcal{H}} \ , \ \forall u \in D(T_0)\ . \tag{4.3.3}$$

We introduce V as the completion in $\mathcal{H}$ of $D(T_0)$ for the norm

$$u \mapsto p_0(u) = \sqrt{a_0(u, u)}\ .$$

More concretely $u \in \mathcal{H}$ belongs to V, if there exists $u_n \in D(T_0)$ such that $u_n \to u$ in $\mathcal{H}$ and u_n is a Cauchy sequence for the norm p_0.
As a natural norm for V, we get as a candidate :

$$||u||_V = \lim_{n \to +\infty} p_0(u_n)\ , \tag{4.3.4}$$

where u_n is a Cauchy sequence for p_0 tending to u in $\mathcal{H}$.
Let us show that the definition does not depend on the Cauchy sequence. This is the object of the

Lemma 4.3.2.
Let x_n a Cauchy sequence in $D(T_0)$ for p_0 such that $x_n \to 0$ in $\mathcal{H}$. Then $p_0(x_n) \to 0$.

Proof of the lemma.
First observe that $p_0(x_n)$ is a Cauchy sequence in $\mathbb{R}^+$ and consequently convergent in $\overline{\mathbb{R}^+}$.
Let us suppose by contradiction that

$$p_0(x_n) \to \alpha > 0\ . \tag{4.3.5}$$

We first observe that

$$a_0(x_n, x_m) = a_0(x_n, x_n) + a_0(x_n, x_m - x_n)\ , \tag{4.3.6}$$

and that a Cauchy-Schwarz inequality is satisfied :

$$|a_0(x_n, x_m - x_n)| \leq \sqrt{a_0(x_n, x_n)} \cdot \sqrt{a_0(x_m - x_n, x_m - x_n)}\ . \tag{4.3.7}$$

Using also that x_n is a Cauchy sequence for p_0 and (4.3.6), we obtain that

$$\forall \epsilon > 0, \exists N \text{ s. t. } \forall n \geq N, \forall m \geq N, |a_0(x_n, x_m) - \alpha^2| \leq \epsilon \,. \tag{4.3.8}$$

We take $\epsilon = \frac{\alpha^2}{2}$ and consider the corresponding N given by (4.3.8). Coming back to the definition of a_0 we obtain,

$$|a_0(x_n, x_m)| = |< T_0 x_n, x_m >| \geq \frac{1}{2}\alpha^2 \;,\; \forall n \geq N, \forall m \geq N \,. \tag{4.3.9}$$

But as $m \to +\infty$, the left hand side in (4.3.9) tends to 0 because by assumption $x_m \to 0$ and this gives a contradiction.
##.
We now observe that

$$||u||_V \geq ||u||_{\mathcal{H}} \;, \tag{4.3.10}$$

as a consequence of (4.3.3) and (4.3.4).
This means that the injection of V in $\mathcal{H}$ is continuous. Note also that V, which contains $D(T_0)$, is dense in $\mathcal{H}$, by density of $D(T_0)$ in $\mathcal{H}$. Moreover, we get a natural scalar product on V by extension of a_0 :

$$< u, v >_V := \lim_{n \to +\infty} a_0(u_n, v_n) \;, \tag{4.3.11}$$

where u_n and v_n are Cauchy sequences for p_0 tending respectively to u and v in $\mathcal{H}$.
By the second version of the Lax-Milgram Theorem (in the context $V, \mathcal{H}, V'$) applied with

$$a(u, v) := \;\; < u, v >_V \;,$$

we get an unbounded selfadjoint operator S on $\mathcal{H}$ extending T_0 whose domain $D(S)$ satisfies $D(S) \subset V$.

Remark 4.3.3. *(Friedrichs extension starting from a sesquilinear form)*
One can also start directly from a semi-bounded sesquilinear form a_0 defined on a dense subspace of $\mathcal{H}$.
As we shall see below, this is actually the right way to proceed for the Neumann realization of the Laplacian, where we consider on $C^\infty(\overline{\Omega})$ the sequilinear form

$$(u, v) \mapsto \langle \nabla u \,,\, \nabla v \rangle \,.$$

4.4 Applications

The reader should have some minimal knowledge of Sobolev Spaces and traces of distributions for reading this subsection (See for example Brézis [Br]).

Application 1: The Dirichlet realization.
Let Ω be an open set in $\mathbb{R}^m$ such that $\overline{\Omega}$ is compact and let T_0 be the unbounded operator defined by

$$D(T_0) = C_0^\infty(\Omega) \ , \ T_0 = -\Delta \ .$$

The involved Hilbert space is $\mathcal{H} = L^2(\Omega)$. It is clear that T_0 is symmetric and positive[7] (hence semi-bounded). Following the previous general construction, we prefer to consider : $\tilde{T}_0 := T_0 + Id$.
It is easy to see[8] that V is the closure in $H^1(\Omega)$ of $C_0^\infty(\Omega)$. This means, at least if Ω is regular, the space $H_0^1(\Omega)$. The domain of S is then described as

$$D(S) := \{u \in H_0^1(\Omega) \mid -\Delta u \in L^2(\Omega)\}.$$

S is then the operator $(-\Delta + 1)$ acting in the sense of distributions.
When Ω is regular, a standard regularity theorem (see [Lio2], [LiMa]) permits to show that

$$D(S) = H^2(\Omega) \cap H_0^1(\Omega) \ . \tag{4.4.1}$$

So we have shown the following theorem

Theorem 4.4.1.
The operator T_1 defined by

$$D(T_1) = H^2(\Omega) \cap H_0^1(\Omega) \ , \ T_1 = -\Delta \ ,$$

is selfadjoint and called the Dirichlet realization of $-\Delta$ in Ω.

We have just to observe that $T_1 = S - 1$ and to use Remark 2.3.8.
Note that T_1 is a selfadjoint extension of T_0.
Note that by the technique developed in Subsection **??**, we have also constructed another selfadjoint extension of T_0. So we have constructed, when Ω is relatively compact, two different selfadjoint realizations of T_0. We say in this case that T_0 is not essentially selfadjoint.

Application 2: The harmonic oscillator.
We can start from

$$H_0 = -\Delta + |x|^2 + 1 \ ,$$

with domain

$$D(H_0) = C_0^\infty(\mathbb{R}^m) \ .$$

[7]We shall in fact see later that it is strictly positive.

[8]We recall that there are two ways for describing $H_0^1(\Omega)$. In the first definition we just take the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.
In the second definition, we describe $\mathcal{H}_0^1(\Omega)$ as the subspace in $H^1(\Omega)$ of the distributions whose trace is zero at the boundary. This supposes that the boundary $\Gamma = \partial\Omega$ is regular. In this case, there exists a unique application γ_0 continuous from $H^1(\Omega)$ onto $H^{\frac{1}{2}}(\Gamma)$ extending the map $C^\infty(\bar{\Omega}) \ni u \mapsto u_\Gamma$. It is a standard result (cf Brézis [Br] or Lions-Magenes [LiMa]) that $\mathcal{H}_0^1(\Omega) = H_0^1(\Omega)$, when the boundary is regular.

Following the scheme of the construction of the Friedrichs extension, we first get that

$$V = B^1(\mathbb{R}^m) := \{u \in H^1(\mathbb{R}^m) \mid x_j u \in L^2(\mathbb{R}^m)\ ,\ \forall j \in [1, \cdots, m]\}\ .$$

One can indeed first show that $V \subset B^1(\mathbb{R}^m)$ and then get the equality by proving the property that $C_0^\infty(\mathbb{R}^m)$ is dense in $B^1(\mathbb{R}^m)$. One can then determine the domain of S as

$$D(S) = \{u \in B^1(\mathbb{R}^m) \mid (-\Delta + |x|^2 + 1)u \in L^2(\mathbb{R}^m)\}\ .$$

By a regularity theorem (differential quotients method [LiMa]), one can show that

$$D(S) = B^2(\mathbb{R}^m) := \{u \in H^2(\mathbb{R}^m) \mid x^\alpha u \in L^2(\mathbb{R}^m)\ ,\ \forall \alpha \text{ s. t. } |\alpha| \leq 2\}.$$

Application 3 : Schrödinger operator with Coulomb potential
We consequently start from

$$D(T_0) = C_0^\infty(\mathbb{R}^3)\,,\ T_0 = -\Delta - \frac{1}{r}\ . \tag{4.4.2}$$

We have seen that T_0 is semibounded and replacing T_0 by T_0+2, the assumptions of the proof of Friedrich's extension theorem are satisfied. We now claim that

$$V = H^1(\mathbb{R}^3)\,.$$

Having in mind that $C_0^\infty(\mathbb{R}^3)$ is dense in $H^1(\mathbb{R}^3)$, we have just to verify that the norm p_0 and the norm $||\cdot||_{H^1(\mathbb{R}^3)}$ are equivalent on $C_0^\infty(\mathbb{R}^3)$. This results immediately of (4.2.5).
With a little more effort, one gets that $D(S) = H^2(\mathbb{R}^3)$.

Application 4 : Neumann problem.
Let Ω be a bounded domain with regular boundary in $\mathbb{R}^m$. Take $\mathcal{H} = L^2(\Omega)$. Let us consider the sesquilinear form :

$$a_0(u, v) = \int_\Omega \langle \nabla u\ ,\ \nabla v\rangle_{\mathbb{C}^m} dx + \int_\Omega u\bar{v} dx\ ,$$

on $C^\infty(\overline{\Omega})$.
Using Remark 4.3.3 and the density of $C^\infty(\overline{\Omega})$ in $H^1(\Omega)$, we can extend the sesquilinear form to $V = H^1(\Omega)$ According to the definition of the domain of S, we observe that, for $u \in D(S)$, then it should exist some $f \in L^2(\Omega)$ such that, for all $v \in H^1(\Omega)$:

$$a(u, v) = \int_\Omega f(x)\overline{v(x)} dx\ . \tag{4.4.3}$$

Then, one can find first show (by taking in (4.4.3) $v \in C_0^\infty(\Omega)$) that, in the sense of distribution,

$$-\Delta u + u = f\ , \tag{4.4.4}$$

and consequently that :

$$D(S) \subset W(\Omega) := \{u \in H^1(\Omega) \mid -\Delta u \in L^2(\Omega)\}\ . \tag{4.4.5}$$

But this is not enough for characterizing the domain.
We refer to [Lio1], [Lio2], [LiMa] or better [DaLi], Vol. 4 (p. 1222-1225)) for a detailed explanation. We first recall the Green-Riemann Formula :

$$\int_\Omega \langle \nabla u \mid \nabla v\rangle = \int_\Omega (-\Delta u)\cdot \bar{v} dx + \int_{\partial\Omega} (\partial u/\partial\nu)\bar{v}\ d\mu_{\partial\Omega}\ , \tag{4.4.6}$$

where $d\mu_{\partial\Omega}$ is the induced measure on the boundary, which is clearly true for $u \in H^2(\Omega)$ (or for $u \in C^1(\overline{\Omega})$) and $v \in H^1(\Omega)$. We unfortunately do not know that $W(\Omega) \subset H^2(\Omega)$ and the inclusion in $H^1(\Omega)$ is not sufficient for defining the normal trace. But this formula can be extended in the following way.

We first observe that, for $v \in H^1_0(\Omega)$ and $u \in W$, we have :

$$\int_\Omega \langle \nabla u \mid \nabla v\rangle = \int_\Omega (-\Delta u)\cdot \bar{v} dx\ . \tag{4.4.7}$$

This shows that the expression

$$\Phi_u(v) := \int_\Omega \langle \nabla u \mid \nabla v\rangle - \int_\Omega (-\Delta u)\cdot \bar{v} dx\ ,$$

which is well defined for $u \in W$ and $v \in H^1(\Omega)$ depends only of the restriction of v to $\partial\Omega$.

If $v_0 \in C^\infty(\partial\Omega)$, we can then extend v_0 inside Ω as a function $v = Rv_0$ in $C^\infty(\overline{\Omega})$ and the distribution [9] is defined as the map $v_0 \mapsto \Phi_u(Rv_0)$.

One observes also that, when $u \in C^1(\overline{\Omega})$ or $u \in H^2(\Omega)$, the Green-Riemann formula shows that :

$$\Phi_u(v_0) = \int_{\partial\Omega} (\partial u/\partial\nu)\bar{v}_0\ d\mu_{\partial\Omega}\ .$$

So we have found a natural way to extend the notion of trace of the normal derivative for $u \in W$ and we write :

$$\gamma_1 u = \Phi_u\ .$$

We then conclude (using (4.4.4) and again (4.4.3) this time in full generality) that :

$$D(S) = \{u \in W(\Omega) \mid \gamma_1 u = 0\}\ ,$$

and that

$$S = -\Delta + 1\ .$$

The operator S is called the Neumann realization of the Laplacian in $L^2(\Omega)$.

[9] By chosing a more specific R continuous from $H^{\frac{1}{2}}(\partial\Omega)$ into $H^1(\Omega)$, we get that Φ_u can be extended as a linear form on $H^{\frac{1}{2}}(\partial\Omega)$.

Remark 4.4.2.
Another "standard" regularity theorem shows that

$$D(S) = \{u \in H^2(\Omega) \mid \gamma_1 u = 0\} \,,$$

and the notion of normal trace $u \mapsto \gamma_1 u$ for $u \in H^2(\Omega)$ is more standard[10].

[10] The trace is in $H^{\frac{1}{2}}(\partial\Omega)$.

5 Compact operators : general properties and examples.

5.1 Definition and properties.

We just recall here very briefly the basic properties of compact operators and their spectral theory. We will emphasize on examples. We refer to the book by H. Brézis [Br] (Chap. VI).
Let us recall that an operator T from a Banach E into a Banach F is compact if the range of the unit ball in E by T is relatively compact in F. We denote by $\mathcal{K}(E,F)$ the space of compact operators which is a closed subspace in $\mathcal{L}(E,F)$.
There is an alternative equivalent definition in the case when E and F are Hilbert spaces by using sequences. The operator is compact if and only if, for any sequence x_n which converges weakly in E, Tx_n is a strongly convergent sequence in F. Here we recall that a sequence is said to be weakly convergent in $\mathcal{H}$, if, for any $y \in \mathcal{H}$, the sequence $< x_n, y >_{\mathcal{H}}$ is convergent. Such a sequence is bounded (Banach-Steinhaus's Theorem) and we recall that, in this case, there exists a unique $x \in \mathcal{H}$, such that $< x_n, y >_{\mathcal{H}} \to < x, y >_{\mathcal{H}}$ for all $y \in \mathcal{H}$. In this case, we write : $x_n \rightharpoonup y$.

Let us recall that when one composes a continuous operator and a compact operator (in any order) one gets a compact operator. This could be one way to prove the compactness.
Another efficient way for proving compactness of an operator T is to show that it is the limit (for the norm convergence) of a sequence of continuous operators with finite rank (that is whose range is a finite dimensional space). We observe indeed that a continuous operator with finite rank is clearly a compact operator (in a finite dimensional space the closed bounded sets are compact).

5.2 Examples

Continuous kernels .
The first example of this type is the operator T_K associated to the continuous kernel K on $[0,1] \times [0,1]$.
By this we mean that the operator T_K is defined by

$$E \ni u \mapsto (T_K u)(x) = \int_0^1 K(x,y)u(y)dy \, . \tag{5.2.1}$$

Here E could be the Banach space $C^0([0,1])$ (with the Sup norm) or $L^2(]0,1[)$.

Proposition 5.2.1.
If the kernel K is continuous, then the operator T_K is compact from E into E.

There are two standard proofs for this proposition. The first one is based on Ascoli's Theorem giving a criterion relating equicontinuity of a subset of continuous functions on a compact and relatively compact sets in $C^0([0,1])$.

The other one is based on the Stone-Weierstrass Theorem permitting to recover the operator as the limit of a sequence of finite rank operators T_{K_n} associated to kernels K_n of the form $K_n(x,y) = \sum_{j=1}^{j_n} f_{j,n}(x) g_{j,n}(y)$.

Let us study three other examples.
The first example comes from statistical mechanics, the second one from the spectral theory for the Dirichlet realization of the Laplacian and the third one from Quantum Mechanics.

The transfer operator. The transfer operator is the operator associated with the kernel

$$K_t(x,y) = \exp -\frac{V(x)}{2} \exp -t|x-y|^2 \exp -\frac{V(y)}{2} \;, \tag{5.2.2}$$

where $t > 0$, and V is a $C^\infty(\mathbb{R})$ function such that

$$\int_{\mathbb{R}} \exp -V(x) dx < +\infty \;.$$

The L^2- boundedness of operators with integral kernel is proven very often through the

Lemma 5.2.2. *Schur's Lemma*
Let $\mathbf{K}$ *an operator associated with an integral kernel* K*, that is a function* $(x,y) \mapsto K(x,y)$ *on* $\mathbb{R}^m \times \mathbb{R}^m$*, satisfying*

$$\begin{array}{l} M_1 := \sup_{x\in\mathbb{R}^m} \int_{\mathbb{R}^m} |K(x,y)| \, dy < +\infty \;, \\ M_2 := \sup_{y\in\mathbb{R}^m} \int_{\mathbb{R}^m} |K(x,y)| \, dx < +\infty \;. \end{array} \tag{5.2.3}$$

Then $\mathbf{K}$*, initially defined for* $u \in C_0^\infty(\mathbb{R}^m)$ *by*

$$(Ku)(x) = \int_{\mathbb{R}^m} K(x,y) u(y) dy \;,$$

can be extended as a continuous linear operator in $\mathcal{L}(L^2(\mathbb{R}^m))$ *(still denoted by* $\mathbf{K}$ *or* T_K*), whose norm satisfies*

$$||\mathbf{K}|| \leq \sqrt{M_1 M_2} \;. \tag{5.2.4}$$

Proof:
By the Cauchy-Schwarz inequality, we have

$$|Ku(x)|^2 \leq \int |K(x,y)||u(y)|^2 dy \int |K(x,y)| dy \leq M_1 \int |K(x,y)||u(y)|^2 dy \;.$$

Integrating with respect to x and using Fubini's Theorem, we then obtain the result.

In our case, the operator T_K is actually an Hilbert-Schmidt operator, i.e. an operator whose integral kernel is in $L^2(\mathbb{R}^m \times \mathbb{R}^m)$ (with $m = 1$). One can indeed prove, using Cauchy-Schwarz's inequality, show that :

$$|Ku(x)|^2 \leq \int |u(y)|^2 dy \int |K(x,y)|^2 dy ,$$

and one obtains :

$$||T_K|| \leq ||K||_{L^2(\mathbb{R}^m \times \mathbb{R}^m)} . \tag{5.2.5}$$

It is then easy to show that T_K is a compact operator. Its kernel K is indeed the limit in L^2 of a sequence K_n such that T_{K_n} is of finite rank. If ϕ_j is an orthonormal basis in $L^2(\mathbb{R}^m)$, one first shows that the basis $\phi_k \otimes \overline{\phi_\ell}$ is an orthonormal basis in $L^2(\mathbb{R}^m \times \mathbb{R}^m)$. Here we have by definition :

$$(\phi_k \otimes \overline{\phi_\ell})(x,y) := \phi_k(x)\overline{\phi_\ell}(y) .$$

We then obtain

$$K(x,y) = \sum_{k,\ell} c_{k,\ell}\phi_k(x)\overline{\phi_\ell}(y) .$$

We now introduce

$$K_n(x,y) := \sum_{k+\ell \leq n} c_{k,\ell}\phi_k(x)\overline{\phi_\ell}(y) .$$

It is then easy to see that T_{K_n} is of finite rank because its range is included in the linear space generated by the ϕ_k's ($k = 1, \cdots, n$). Moreover, we have

$$\lim_{n \to +\infty} ||K - K_n||_{L^2(\mathbb{R}^m \times \mathbb{R}^m)} = 0 .$$

Coming back to the corresponding operators, we get

$$\lim_{n \to +\infty} ||T_K - T_{K_n}||_{\mathcal{L}(L^2(\mathbb{R}^m))} = 0 .$$

The inverse of the Dirichlet operator .

We come back to the operator S, which was introduced in the study of the Dirichlet realization. One can show the following

Proposition 5.2.3.
The operator S^{-1} is compact.

Proof.
The operator S^{-1} can indeed be considered as the composition of a continuous operator from L^2 into $V = H_0^1(\Omega)$ and of the continuous injection of V into $L^2(\Omega)$. If Ω is relatively compact, we know (cf [Br]) that we have compact injection from $H^1(\Omega)$ into $L^2(\Omega)$. Hence the injection of V into L^2 is compact and S^{-1} is compact. For the continuity result, we observe that, for all $u \in D(S)$:

$$||Su||_{\mathcal{H}}||u||_{\mathcal{H}} \geq \langle Su \mid u \rangle = a(u,u) \geq \alpha ||u||_V^2 \geq \alpha ||u||_V ||u||_{\mathcal{H}} .$$

This gives, for all $u \in D(S)$, the inequality :

$$||Su||_{\mathcal{H}} \geq \alpha ||u||_V \ . \tag{5.2.6}$$

Using the surjectivity of S, we get :

$$||S^{-1}||_{\mathcal{L}(\mathcal{H},V)} \leq \frac{1}{\alpha} \ . \tag{5.2.7}$$

Note that in our example $\alpha = 1$ but that this part of the proof is completely general.

The inverse of the harmonic oscillator.

The analysis is analogous. We have seen that the Sobolev space $H_0^1(\mathbb{R})$ has to be replaced by the space

$$B^1(\mathbb{R}) := \{u \in L^2(\mathbb{R}) \ , \ xu \in L^2 \text{ and } du/dx \in L^2\} \ .$$

We can then prove, using a standard precompactness criterion, that $B^1(\mathbb{R})$ has compact injection in $L^2(\mathbb{R})$. One has in particular to use the inequality :

$$\int_{|x|\geq R} |u(x)|^2 dx \leq \frac{1}{R^2} ||u||^2_{B^1(\mathbb{R})} \ . \tag{5.2.8}$$

It is very important to realize that the space $H^1(\mathbb{R})$ is not compactly injected in L^2. To understand this point, it is enough to consider the sequence $u_n = \chi(x-n)$ where χ is a function in $C_0^\infty(\mathbb{R})$ with norm in L^2 equal to 1. It is a bounded sequence in H^1, which converges weakly in H^1 to 0 and is not convergent in $L^2(\mathbb{R})$.

5.3 Adjoints and compact operators

We recall[11] that the adjoint of a bounded operator in the Hilbertian case is bounded. When E and F are different Hilbert spaces , the Hilbertian adjoint is defined through the identity :

$$< Tx, y >_F = < x, T^* y >_E \ , \ \forall x \in E \, , \, \forall y \in F \ . \tag{5.3.1}$$

Example 5.3.1.
Let Ω be an open set in $\mathbb{R}^m$ and Π_Ω the operator of restriction to Ω: $L^2(\mathbb{R}^m) \ni u \mapsto u_{/\Omega}$. Then $\Pi_\Omega^\star$ is the operator of extension by 0.

Exercise 5.3.2.
Let γ_0 be the trace operator on $x_m = 0$ defined from $H^1(\mathbb{R}^m_+)$ onto $H^{\frac{1}{2}}(\mathbb{R}^{m-1})$. Determine the adjoint.

[11] We are mainly following Brézis's exposition [Br].

In an Hilbert space, we have

$$M^{\perp} = \overline{M}^{\perp} \ .$$

In particular, if M is a closed subspace, we have already used the property

$$(M^{\perp})^{\perp} = M \ . \tag{5.3.2}$$

In the case of bounded operators ($T \in \mathcal{L}(E, F)$), one gets easily the properties

$$N(T^*) = R(T)^{\perp} \ , \tag{5.3.3}$$

and

$$\overline{R(T)} = (N(T^*))^{\perp} \ . \tag{5.3.4}$$

Let us also recall the proposition

Proposition 5.3.3.
The adjoint of a compact operator is compact.

5.4 Precompactness

We assume that the reader is aware of the basic results concerning compact sets in metric spaces. We in particular recall that in a complete metric space E, an efficient way to show that a subset M is relatively compact is to show, that for any $\epsilon > 0$, one can recover M by a finite family of balls of radius ϵ in E.

The second standard point is to remember Ascoli's Theorem, giving a criterion for a bounded subset in $C^0(K)$ (K compact in $\mathbb{R}^m$) to be relatively compact in term of uniform equicontinuity. Ascoli's Theorem gives in particular :

- the compact injection of $C^1(K)$ into $C^0(K)$
- the compact injection of $H^m(\Omega)$ into $C^0(K)$, for $m > \frac{n}{2}$ and with $K = \overline{\Omega}$.

Let us recall finally a general proposition permitting to show that a subset in L^2 is relatively compact.

Proposition 5.4.1. *Let $A \subset L^2(\mathbb{R}^m)$. Let us assume that :*

1. *A is bounded in $L^2(\mathbb{R}^m)$, that is there exists $M > 0$ such that :*

$$||u||_{L^2} \leq M \ , \ \forall u \in A \ .$$

2. *The expression $\epsilon(u, R) := \int_{|x| \geq R} |u(x)|^2 dx$ tends to zero as $R \to +\infty$ uniformly with respect to $u \in A$.*
3. *For $h \in \mathbb{R}^m$, let τ_h defined on L^2 by : $(\tau_h u)(x) = u(x - h)$. Then the expression $\delta(u, h) := ||\tau_h u - u||_{L^2}$ tends to zero as $h \to 0$, uniformly with respect to A.*

Then A is relatively compact in L^2.

This proposition can be applied for showing :

- the compact injection of $H_0^1(\Omega)$ in $L^2(\Omega)$ when Ω is regular and bounded,
- the compact injection of $H^1(\Omega)$ in $L^2(\Omega)$ when Ω is regular and bounded,
- the compact injection of $B^1(\mathbb{R}^m)$ in $L^2(\mathbb{R}^m)$.

6 Spectral theory for bounded operators.

6.1 Fredholm's alternative

Let us first recall Riesz's Theorem.

Theorem 6.1.1.
Let E be a normed linear space such that $\overline{B_E}$ is compact then E is finite dimensional.

Let us now describe Fredholm's alternative.

Theorem 6.1.2.
Let $T \in \mathcal{K}(E)$. Then
(i) $N(I-T)$ is finite dimensional.
(ii) $R(I-T)$ is closed (of finite codimension).
(iii) $R(I-T) = E$ if and only if $N(I-T) = \{0\}$.

We shall only use this theorem in the Hilbertian framework, so $E = \mathcal{H}$, and we shall prove it for simplicity under the additional assumption that $T = T^*$.
Proof.
We divide the proof in successive steps.
Step 1.
(i) is a consequence of Riesz's Theorem.
Step 2.
Let us show that $R(I-T)$ is closed.
Let y_n a sequence in $R(I-T)$ such that $y_n \to y$ in $\mathcal{H}$. We would like to show that $y \in R(I-T)$.
Let x_n in $N(I-T)^\perp$ such that $y_n = x_n - Tx_n$.

Step 2a. Let us first show the weaker property that the sequence x_n is bounded.
Let us indeed suppose that there exists a subsequence x_{n_j} such that $||x_{n_j}|| \to +\infty$. Considering $u_{n_j} = x_{n_j}/||x_{n_j}||$, we observe that

$$(*) \quad u_{n_j} - Tu_{n_j} \to 0 \ .$$

The sequence being bounded, we observe that (after possibly extracting a subsequence) one can consider that the sequence u_{n_j} is weakly convergent. This implies that Tu_{n_j} is convergent (T is compact). Using now $(*)$, we get the convergence of u_{n_j} to u :

$$u_{n_j} \to u \ , \ Tu = u \ , \ ||u|| = 1 \ .$$

But $u \in N(I-T)^\perp$, hence we get $u = 0$ and a contradiction.

Step 2b. We have consequently obtained that the sequence x_n is bounded. One can consequently extract a subsequence x_{n_j} that weakly converges to x_∞ in $\mathcal{H}$. Using the compactness of T, we get Tx_{n_j} converges strongly to Tx_∞.

Hence the sequence x_{n_j} tends strongly to $y + Tx_\infty$.
We have finally

$$y + Tx_\infty = x_\infty \ ,$$

and consequently proved that $y = x_\infty - Tx_\infty$.

Step 3.
If $N(I-T) = \{0\}$, then $N(I-T^*) = 0$ (here we use for simplification our additional assumption) and $R(I-T)$ being closed, we get

$$R(I-T) = N(I-T^*)^\perp = \mathcal{H} \ .$$

The converse is also immediate as $T = T^*$.

Step 4.
We have

$$R(I-T)^\perp = N(I-T^*) = N(I-T) \ .$$

This shows, according to (i) that $R(I-T)$ is of finite codimension (second statement of (ii)).

This ends the proof of Fredholm's alternative in the particular case that T is selfadjoint.

Remark 6.1.3.
Under the same asumptions, it is possible to show that

$$dim\ N(I-T) = \ dim\ N(I-T^*) \ .$$

6.2 Resolvent set for bounded operators

In this subsection, E could be a Banach on $\mathbb{R}$ or $\mathbb{C}$, but we will essentially need the Hilbertian case in the applications treated here.

Definition 6.2.1. *(Resolvent set)*
For $T \in \mathcal{L}(E)$, the resolvent set is defined by

$$\varrho(T) = \{\lambda \in \mathbb{C} \ ; \ (T - \lambda I) \textit{ is bijective from } E \textit{ on } E\}. \tag{6.2.1}$$

Note that in this case $(T-\lambda I)^{-1}$ is continuous (Banach's Theorem). It is easy to see that $\varrho(T)$ is an open set in $\mathbb{C}$. If $\lambda_0 \in \varrho(T)$, we oberve that

$$(T-\lambda) = (T-\lambda_0)\left(Id + (\lambda - \lambda_0)(T-\lambda_0)^{-1}\right)$$

Hence $(T-\lambda)$ is invertible if $|\lambda - \lambda_0| < ||(T-\lambda_0)^{-1}||^{-1}$. We also get the following identity for all $\lambda,\ \lambda_0 \in \varrho(T)$:

$$(T-\lambda)^{-1} - (T-\lambda_0)^{-1} = (\lambda - \lambda_0)(T-\lambda)^{-1}(T-\lambda_0)^{-1} \ . \tag{6.2.2}$$

Definition 6.2.2. *(Spectrum)*
The spectrum of T, $\sigma(T)$, is the complementary set of $\varrho(T)$ in $\mathbb{C}$.

Note that $\sigma(T)$ is a closed set in $\mathbb{C}$. This is typically the case when T is a compact injective operator in a Banach space of infinite dimension.

We say that λ is an eigenvalue if $N(T-\lambda I) \neq 0$. $N(T-\lambda I)$ is called the eigenspace associated with λ.

Definition 6.2.3. *(Point spectrum)*
The point spectrum $\sigma_p(T)$ of T is defined as the set of the eigenvalues of T.

Example 6.2.4. *(Basic example)*
Let $\mathcal{H} = L^2(]0,1[)$ and $f \in C^0([0,1])$. Let T_f be the operator of multiplication by f. Then one has :

$$\sigma(T_f) = \; Im\; f =: \{\lambda \in \mathbb{C} \mid \exists x \in [0,1] \; with \; f(x) = \lambda\} \; .$$

$$\sigma_p(T_f) = Sta(f) =: \{\lambda \in \mathbb{C} \mid meas(f^{-1}(\lambda)) > 0\} \; .$$

For the first assertion, it is first immediate to see that if $\lambda \notin \; Im\; f$, then $T_{(f-\lambda)^{-1}}$ is a continuous inverse of $T_f - \lambda$. On the other side, if $\lambda = f(x_0)$ for some $x_0 \in]0,1[$ then we have $(T_f - \lambda) u_n \to 0$ and $||u_n|| = 1$ for $u_n = \frac{1}{\sqrt{n}}\chi(\frac{x-x_0}{n})$, where χ is a C_0^∞ function such that $||\chi|| = 1$. This shows that $f(]0,1[) \subset \sigma(T_f)$ and we can conclude by considering the closure.

Note that the point spectrum is not necessarily closed. Note also that one can have a strict inclusion of the point spectrum in the spectrum as can be observed in the following example :

Example 6.2.5.
Let us consider $E = \ell^2(\mathbb{N})$ and let T be the shift operator defined by :

$$(Tu)_0 = 0 \; , \; (Tu)_n = u_{n-1} \; , \; n > 0 \; ,$$

where $u = (u_0, \cdots, u_n, \cdots) \in \ell^2(\mathbb{N})$. Then it is easy to see that T est injective (so 0 is not an eigenvalue) and is not surjective (so 0 is in the spectrum of T).

As another interesting example, one can consider :

Example 6.2.6.
Let $E = \ell^2(\mathbb{Z}, \mathbb{C})$ and let T be the operator defined by :

$$(Tu)_n = \frac{1}{2}\left(u_{n-1} + u_{n+1}\right) \; , \; n \in \mathbb{Z} \; ,$$

for $u \in \ell^2(\mathbb{Z})$. Then it is easy to see (by using expansion in Fourier series [12]) that T has no eigenvalues and its spectrum is $[-1,+1]$.

[12] Using the isomorphism between $\ell^2(\mathbb{Z};\mathbb{C})$ and $L^2(S^1\,;\,\mathbb{C})$, which associates to the sequence $(u_n)_{n\in\mathbb{Z}}$ the fonction $\sum_{n\in\mathbb{Z}} u_n \exp in\theta$, one has to analyze the operator of multiplication by $\cos\theta$:

$$f \mapsto \mathcal{T} f = \cos\theta f \; .$$

It is then analyzed as in example 6.2.4. One concludes that the spectrum of $\mathcal{T}$ is $[-1,+1]$.

Exercise 6.2.7.
Let $\alpha \in [0,1]$. Let p and q integers which are mutually prime. Analyze the spectrum Σ_α of the operator H_α defined on $\ell^2(\mathbb{Z})$ by

$$(H_\alpha u)_n = \frac{1}{2}\,(u_{n-1} + u_{n+1}) + \cos 2\pi(\frac{p}{q}n + \alpha)\; u_n \;,\; n \in \mathbb{Z}\;.$$

In order to make the analysis easier, one can admit (particular case of the so-called Floquet theory), that one has $\Sigma = \cap_{\theta\in[0,1]}\Sigma_\theta$, where Σ_θ is the spectrum of H_α reduced to the space of the u's in ℓ^∞ such that $u_{n+q} = \exp 2i\pi\theta\; u_n$, for $n \in \mathbb{Z}$.
This operator plays an important role in Solid State Physics and is called the Harper's operator.
We now replace the rational $\frac{p}{q}$ by an irrational number β. So we consider the operator $H_{\beta,\alpha} := \frac{1}{2}(\tau_{+1} + \tau_{-1}) + \cos 2\pi(\beta\cdot\; + \alpha)$ on $\ell^2(\mathbb{Z},\mathbb{C})$, where τ_k $(k \in \mathbb{Z})$ is the operator defined on $\ell^2(\mathbb{Z},\mathbb{C})$ by $(\tau_k u)_n = u_{n-k}$.
Show that, if $\beta \notin \mathbb{Q}$, then the spectrum of $H_{\beta,\alpha}$ is independent of α. For this, one can first prove that $H_{\beta,\alpha}$ is unitary equivalent with $H_{\beta,\alpha+k\beta}$ for any $k \in \mathbb{Z}$. Secondly, one can use the density of the set $\{\alpha + \beta\mathbb{Z} + \mathbb{Z}\}$ in $\mathbb{R}$. Finally, one can use the inequality

$$||H_{\beta,\alpha} - H_{\beta,\alpha'}|| \leq 2\pi|\alpha - \alpha'|\;.$$

Proposition 6.2.8.
The spectrum $\sigma(T)$ is a compact set included in the ball $\overline{B}(0, ||T||)$.

This proposition is immediate if we observe that $(I - \frac{T}{\lambda})$ is invertible if $|\lambda| > ||T||$.

6.3 Spectral theory for compact operators

In the case of a compact operator, one has a more precise description of the spectrum.

Theorem 6.3.1.
Let $T \in \mathcal{K}(E)$ where E is an infinite dimensional Banach space. Then

1. $$0 \in \sigma(T)\;.$$

2. $$\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}\;.$$

3. *We are in one (and only one) of the following cases*

   - *either* $\sigma(T) = \{0\}$,
   - *either* $\sigma(T) \setminus \{0\}$ *is finite,*
   - *or* $\sigma(T)\setminus\{0\}$ *can be described as a sequence of distincts points tending to* 0.

4. Each $\lambda \in \sigma_p(T) \setminus \{0\}$ is isolated and $\dim N(T - \lambda I) < +\infty$.

Proof.
a) If $0 \notin \sigma(T)$, then T admits a continuous inverse T^{-1} and the identity, which can be considered as :

$$I = T \circ T^{-1} \ ,$$

is a compact operator, as the composition of the compact operator T and the continuous operator T^{-1}. Using Riesz's Theorem we get a contradiction in the case E is supposed of infinite dimension.
b) The fact that, if $\lambda \neq 0$ and $\lambda \in \sigma(T)$, then λ is an eigenvalue, comes directly from the Fredholm's alternative applied to $(I - \frac{T}{\lambda})$.
c) The last step comes essentially from the following lemma :

Lemma 6.3.2.
Let $(\lambda_n)_{n\geq 1}$ a sequence of distincts points $\lambda_n \to \lambda$ and $\lambda_n \in \sigma(T) \setminus \{0\}$, for all $n > 0$.
Then $\lambda = 0$.

Proof.
We just give the proof in the Hilbertian case. For all $n > 0$, let e_n be a normalized eigenfunction such that $(T - \lambda_n)e_n = 0$ and let E_n be the vectorial space spanned by $\{e_1, e_2, \cdots, e_n\}$. Let us show that we have a strict inclusion of E_n in E_{n+1}.
We prove this point by recursion. Let us assume the result up to order $n-1$ and let us show it at order n. If $E_{n+1} = E_n$, then $e_{n+1} \in E_n$ and we can write

$$e_{n+1} = \sum_{j=1}^{n} \alpha_j e_j \ .$$

Let us apply T to this relation. Using the property that e_{n+1} is an eigenfunction, we obtain

$$\lambda_{n+1} \left(\sum_{j=1}^{n} \alpha_j e_j \right) = \sum_{j=1}^{n} \alpha_j \lambda_j e_j \ .$$

Using the recursion assumption, $\{e_1, \cdots, e_n\}$ is a basis of E_n and the λ_j being distincts, we obtain $\alpha_j = 0$ for all $j = 1, \cdots, n$ and a contradiction with $||e_{n+1}|| = 1$.
So we can find a sequence u_n such that $u_n \in E_n \cap E_{n-1}^{\perp}$ and $||u_n|| = 1$. T being compact, one would extract a convergent subsequence (still denoted by Tu_n) from the sequence (Tu_n) and, if $\lambda_n \to \lambda \neq 0$, one would also have the convergence of this subsequence $(\frac{1}{\lambda_n} Tu_n)$ and consequently the Cauchy property.
Let us show that this leads to a contradiction. We remark that

$$(T - \lambda_n)E_n \subset E_{n-1} \ .$$

Let $n > m \geq 2$. We have

$$\begin{aligned} ||\frac{Tu_n}{\lambda_n} - \frac{Tu_m}{\lambda_m}||^2 &= ||\frac{(T-\lambda_n)u_n}{\lambda_n} - \frac{(T-\lambda_m)u_m}{\lambda_m} + u_n - u_m||^2 \\ &= ||\frac{(T-\lambda_n)u_n}{\lambda_n} - \frac{(T-\lambda_m)u_m}{\lambda_m} - u_m||^2 + ||u_n||^2 \\ &\geq ||u_n||^2 = 1 \; . \end{aligned}$$

We can consequently not extract a Cauchy subsequence from the sequence $\frac{1}{\lambda_n} Tu_n$. This is in contradiction with the assumption $\lambda \neq 0$.
This ends the proof of the lemma and of the theorem.

We shall now consider the Hilbertian case and see which new properties can be obtained by using the additional assumption that T is selfadjoint.

6.4 Spectrum of selfadjoint operators.

As $T = T^*$, the spectrum is real. If $\text{Im}\,\lambda \neq 0$, one immediately verifies that :

$$|\text{ Im }\lambda\,|\;||u||^2 \leq |\text{ Im }\langle (T-\lambda)u, u\rangle \leq ||(T-\lambda)u|| \cdot ||u|| \; .$$

This shows immediately that the map $(T - \lambda)$ is injective and with close range. But the orthogonal of the range of $(T - \lambda)$ is the kernel of $(T - \bar{\lambda})$ which is reduced to 0. So $(T - \lambda)$ is bijective.
This was actually, a consequence of the Lax-Milgram theorem (in the simple case when $V = H$), once we have observed the inequality

$$|\langle (T-\lambda)u, u\rangle| \geq |\text{ Im }\lambda\,|\;||u||^2 \; .$$

Using again Lax-Milgram's theorem, we can show

Theorem 6.4.1.
Let $T \in \mathcal{L}(\mathcal{H})$ be a selfadjoint operator. Then the spectrum of T is contained in $[m, M]$ with $m = \inf \langle Tu, u\rangle/||u||^2$ and $M = \sup \langle Tu, u\rangle/||u||^2$. Moreover m and M belong to the spectrum of T.

Proof:
We have already mentioned that the spectrum is real. Now if $\lambda > M$, we can apply the Lax-Milgram to the sequilinear-form $(u, v) \mapsto \lambda\langle u, v\rangle - \langle Tu, v\rangle$.
Let us now show that $M \in \sigma(T)$.

We observe that, by Cauchy-Schwarz applied to the scalar product $(u, v) \mapsto M\langle u, v\rangle - \langle Tu, v\rangle$, we have :

$$|\langle Mu - Tu, v\rangle| \leq \langle Mu - Tu, u\rangle^{\frac{1}{2}}\;\langle Mv - Tv, v\rangle^{\frac{1}{2}} \; .$$

In particular, we get :

$$||Mu - Tu||_{\mathcal{H}} \leq ||M - T||^{\frac{1}{2}}_{\mathcal{L}(\mathcal{H})} \langle Mu - Tu, u\rangle^{\frac{1}{2}} \; . \tag{6.4.1}$$

Let $(u_n)_{n\in\mathbb{Z}}$ be a sequence such that $||u_n|| = 1$ and $\langle Tu_n, u_n\rangle \to M$ as $n \to +\infty$. By (6.4.1), we get that $(T - M)u_n$ tends to 0 as $n \to +\infty$. This implies that $M \in \sigma(T)$. If not, we would get that $u_n = (M - T)^{-1}((M - T)u_n)$ tends to 0 in contradiction with $||u_n|| = 1$.

This theorem admits the following important corollary

Corollary 6.4.2.
Let $T \in \mathcal{L}(\mathcal{H})$ be a selfadjoint operator such that $\sigma(T) = \{0\}$. Then $T = 0$.

We first indeed show that $m = M = 0$ and consequently that $< Tu, u >_{\mathcal{H}} = 0$ for all $u \in \mathcal{H}$. But $< Tu, v >$ can be written as a linear combination of terms of the type $< Tw, w >_{\mathcal{H}}$ and this gives the result by taking $v = Tu$.

Another connected property which is useful is the

Proposition 6.4.3.
If T is positive and selfadjoint then $||T|| = M$.

The proof is quite similar. We observe (Cauchy-Schwarz for $(u, v) \mapsto < Tu\, ,\, v >$) that :

$$|\langle Tu, v\rangle| \leq \langle Tu, u\rangle^{\frac{1}{2}}\ \langle Tv, v\rangle^{\frac{1}{2}}\ .$$

This implies, using Riesz's Theorem :

$$||Tu|| \leq ||T||^{\frac{1}{2}}(\langle Tu, u\rangle)^{\frac{1}{2}}\ .$$

Coming back to the definition of $||T||$, we obtain :

$$||T|| \leq ||T||^{\frac{1}{2}} M^{\frac{1}{2}}\ ,$$

and the inequality :

$$||T|| \leq M\ .$$

But it is immediate that :

$$\langle Tu, u\rangle \leq ||T||\ ||u||^2\ .$$

This gives the converse inequality and the proposition.

6.5 Spectral theory for compact selfadjoint operators

We have a very precise description of the selfadjoint compact operators.

Theorem 6.5.1.
Let $\mathcal{H}$ be a separable Hilbert[13] space and T a compact selfadjoint operator. Then $\mathcal{H}$ admits an Hilbertian basis consisting of eigenfunctions of T.

[13]that is having a countable dense set.

Proof.
Let $(\lambda_n)_{n\geq 1}$ be the sequence of disjoint eigenvalues of T, except 0. Their existence comes from Theorem 6.3.1 and we also observe that the eigenvalues are real.
Let us define $\lambda_0 = 0$.
We define $E_0 = N(T)$ and $E_n = N(T - \lambda_n I)$; We know (from Riesz's Theorem) that

$$0 < \text{ dim } E_n < +\infty \ .$$

Let us show that $\mathcal{H}$ is the Hilbertian sum of the $(E_n)_{n\geq 0}$.

(i) The spaces (E_n) are mutually orthogonal. If $u \in E_m$ and $v \in E_n$ with $m \neq n$, we have

$$< Tu, v >_{\mathcal{H}} = \lambda_m < u, v >_{\mathcal{H}} = < u, Tv >_{\mathcal{H}} = \lambda_n < u, v >_{\mathcal{H}} \ ,$$

and consequently

$$< u, v >_{\mathcal{H}} = 0 \ .$$

(ii) Let F be the linear space spanned by the $(E_n)_{n\geq 0}$. Let us verify that F is dense in $\mathcal{H}$. It is clear that $TF \subset F$ and, using the selfadjoint character of T, we have also $TF^\perp \subset F^\perp$. The operator $\tilde{T}$, obtained by restriction of T to $F^\perp$, is a compact selfadjoint operator. But one shows easily that $\sigma(\tilde{T}) = \{0\}$ and consequently $\tilde{T} = 0$. But $F^\perp \subset N(T) \subset F$ and hence $F^\perp = \{0\}$. F is consequently dense in $\mathcal{H}$.
(iii) To end the proof, one chooses in each E_n an Hilbertian basis. Taking the union of these bases, one obtains an Hilbertian basis of $\mathcal{H}$ effectively formed with eigenfunctions of T.

Remark 6.5.2.
If T is a compact selfadjoint operator, we can write any u in the form

$$u = \sum_{n=0}^{+\infty} u_n \ , \quad \textit{with } u_n \in E_n \ .$$

This permits to write

$$Tu = \sum_{n=1}^{+\infty} \lambda_n u_n \ .$$

If, for $k \in \mathbb{N}^*$, we define T_k by

$$T_k u = \sum_{n=1}^{k} \lambda_n u_n \ ,$$

we easily see that T_k is of finite rank and that

$$||T - T_k|| \leq \sup_{n \geq k+1} |\lambda_n| \ .$$

Hence the operator T appears as the limit in $\mathcal{L}(\mathcal{H})$ of the sequence T_k as $k \to +\infty$.

7 Examples.

We go back to our previous examples and analyze their properties.

7.1 The transfer operator.

7.1.1 Compactness

The transfer operator (which was introduced in (5.2.2)) is compact and admits consequently a sequence of eigenvalues λ_n tending to 0 as $n \to +\infty$. Let us show the

Lemma 7.1.1.
The transfer operator is positive and injective.

Proof of the lemma.
Let $u \in L^2(\mathbb{R})$. We can write[14], with $\phi(x) = \exp -\frac{V(x)}{2} u(x)$

$$< T_K u, u >_{\mathcal{H}} = \int_{\mathbb{R}^2} \exp -t|x-y|^2 \phi(x)\overline{\phi(y)} dx dy \; .$$

Using the properties of the Fourier transform and of the convolution, we deduce

$$< T_K u, u >_{\mathcal{H}} = c_t \int_{\mathbb{R}} \exp -\frac{|\xi|^2}{4t} |\widehat{\phi}(\xi)|^2 d\xi \; ,$$

where $c_t > 0$ is a normalization constant.

The spectrum is consequently the union of a sequence of positive eigenvalues and of its limit 0. T_K can be diagonalized in an orthonormal basis of eigenfunctions associated with strictly positive eigenvalues. We emphasize that 0 is in the spectrum but not an eigenvalue.
Theorem 6.4.1 says also that $||T_K||$ is the largest eigenvalue and is isolated. A natural question is then to discuss the multiplicity of each eigenvalue, i. e. the dimension of each associated eigenspace. We shall see later (Krein-Rutman's Theorem) that the largest eigenvalue is of multiplicity 1.

7.1.2 About the physical origin of the problem.

Our initial problem was to find a rather general approach for the estimate of the decay of correlations attached to "gaussian like" measures of the type

$$\exp -\Phi(X)\, dX \tag{7.1.1}$$

on $\mathbb{R}^n$ with Φ. One proof of this estimate (in the case when Φ has a particular structure) is based on the analysis of the transfer matrix method, originally

[14]If we make only the weak assumption that $\exp -V \in L^1(\mathbb{R})$, it is better to start for the proof of the positivity by considering u's in $C_0^\infty(\mathbb{R})$ and then to treat the general case by using the density of C_0^∞ in L^2.

due to Kramers-Wannier, that we have already seen for the study of the Ising model. We present here briefly the technique for our toy model. We shall only consider the case when $d = 1$ and treat the periodic case, that is the case when $\{1, \cdots, n\}$ is a representation of $\Lambda^{per} = \mathbb{Z}/n\mathbb{Z}$.

We consider the particular potential Φ

$$\Phi^{(n)}(X) \equiv \Phi(X) \equiv \frac{1}{h}\left(\sum_{j=1}^{n} V(x_j) + \frac{|x_j - x_{j+1}|^2}{4}\right) \tag{7.1.2}$$

where we take the convention that $x_{n+1} = x_1$ and where h is possibly a semiclassical parameter which is sometimes chosen equal to one if we are not interested in the "semiclassical" aspects. More generally, we could more generally consider examples of the form:

$$\Phi_h(X) = \frac{1}{h}\left(\sum_{j=1}^{n} \left(V(x_j) + I(x_j, x_{j+1})\right)\right) \tag{7.1.3}$$

where I is a symmetric "interaction" potential on $\mathbb{R} \times \mathbb{R}$. Let us mention however that the example (7.1.2) appears naturally in quantum field theory when the so called "lattice approximation" is introduced. For this special class of potentials, we shall demonstrate that the informations given by the transfer operator method are complementary to the other approachs we have explained before. We shall present the "dictionary" between the properties of the measure $h^{-\frac{n}{2}} \cdot \exp -\Phi_h(X)\, dX$ on $\mathbb{R}^n$ and the spectral properties of the transfer operator $\mathbf{K_V}$ (which is also called Kac operator for some particular models introduced by M. Kac) whose integral kernel is real and given on $\mathbb{R} \times \mathbb{R}$ by

$$K_V(x,y) = h^{-\frac{1}{2}} \exp -\frac{V(x)}{2h} \;\cdot\; \exp -\frac{|x-y|^2}{4h} \;\cdot\; \exp -\frac{V(y)}{2h}\ . \tag{7.1.4}$$

By integral kernel (or distribution kernel), we mean[15] a distribution in $\mathcal{D}'(\mathbb{R}^2)$ such that the operator is defined from $C_0^\infty(\mathbb{R})$ into $\mathcal{D}'(\mathbb{R})$ by the formula

$$\int_{\mathbb{R}} (\mathbf{K_V} u)(x) v(x)\, dx = \int_{\mathbb{R}\times\mathbb{R}} K_V(x,y)\, u(x)\, v(y)\, dx\, dy\ ,\ \forall u\ ,\ v \in C_0^\infty(\mathbb{R})\ . \tag{7.1.5}$$

This dictionary permits to obtain interesting connections between estimates for the quotient μ_2/μ_1 of the two first largest eigenvalues of the transfer operator and corresponding estimates controlling the speed of convergence of thermodynamic quantities. In particular this speed of convergence is exponentially rapid as $n \to +\infty$.

We know that when the operator $\mathbf{K}$ is compact, symmetric and injective, then there exists a decreasing (in modulus) sequence μ_j of eigenvalues tending

[15] Here we shall always consider much more regular kernels which are in particular continuous. So the notation $\int$ can be interpreted in the usual sense. In general, this means that the distribution kernel K_V is applied to the test function $(x,y) \mapsto u(x)\, v(y)$.

to 0 and a corresponding sequence of eigenfunctions u_j which can be normalized in order to get an orthonormal basis of $L^2(\mathbb{R})$. Moreover, the operator becomes the limit in norm of the family of operators $\mathbf{K}^{(N)}$ whose corresponding kernel are defined by

$$K^{(N)}(x,y) = \sum_{j=1}^{N} \mu_j u_j(x)\, u_j(y) \; . \tag{7.1.6}$$

We recall indeed that

$$||\mathbf{K} - \mathbf{K}^{(N)}|| \leq \sup_{j>N} |\mu_j| \; . \tag{7.1.7}$$

The symmetric operators[16] are called trace class if we have in addition the property that

$$||\mathbf{K}||_{tr} := \sum_j |\mu_j| < +\infty \tag{7.1.8}$$

In this case, we get that

$$||\mathbf{K} - \mathbf{K}^{(N)}||_{tr} \leq \sum_{j>N} |\mu_j| \; . \tag{7.1.9}$$

For a trace class symmetric operator, we can in particular define the trace as

$$\mathrm{Tr}\;\mathbf{K} = \sum_j \mu_j \; , \tag{7.1.10}$$

and this operation is continuous on the space of the trace class operators

$$|\;\mathrm{Tr}\;\mathbf{K}\;| \leq ||\mathbf{K}||_{tr} \; . \tag{7.1.11}$$

This of course extends the usual notion of trace for matrices. We can actually compute directly the trace of a trace-class operator in the following way. We first observe that

$$\mathrm{Tr}\;\mathbf{K} = \lim_{N\to+\infty} \mathrm{Tr}\;\mathbf{K}^{(N)} \; . \tag{7.1.12}$$

Then we observe that

$$\mathrm{Tr}\;\mathbf{K}^{(N)} = \sum_{j=1}^{N} \mu_j = \int_{\mathbb{R}} K^{(N)}(x,x)dx \; . \tag{7.1.13}$$

We consequently obtain that

$$\mathrm{Tr}\;\mathbf{K} = \lim_{N\to+\infty} \int_{\mathbb{R}} K^{(N)}(x,x)dx \; . \tag{7.1.14}$$

[16] When $\mathbf{K}$ is not symmetric, trace class operators can still be defined by considering $\sqrt{\mathbf{K}^*\mathbf{K}}$. Note that when K is trace class, one can compute the trace by considering any orthonormal basis e_i : $\mathrm{Tr}\;\mathbf{K} = \sum_i \langle \mathbf{K}e_i, e_i\rangle$.

If we observe that $x \mapsto \sum_j |\mu_j||u_j(x)|^2$ is in $L^1(\mathbb{R})$, then it is natural[17] to hope (but this is not trivial!) that $x \mapsto K(x,x)$ is in L^1 and that

$$\text{Tr }\mathbf{K} = \int_{\mathbb{R}} K(x,x)\,dx\;. \tag{7.1.15}$$

Note that it is only when $\mathbf{K}$ is positive that the finiteness of the right hand side in (7.1.15) will imply the trace class property.
In the case the operator is Hilbert-Schmidt, that is with a kernel K in L^2, then the operator is also compact and we have the identity

$$\sum_j \mu_j^2 = \int_{\mathbb{R}\times\mathbb{R}} |K(x,y)|^2\; dx\,dy\;. \tag{7.1.16}$$

This gives an easy criterion for verifying the compactness of the operator. Let us first look at the thermodynamic limit. This means that we are interested in the limit $\lim_{n\to+\infty} \frac{1}{n}\ln\left(\int_{\mathbb{R}^n} \exp -\Phi(X)\; dX\right)$. We start from the decomposition:

$$\exp -\Phi(X) = K_V(x_1,x_2)\cdot K_V(x_2,x_3)\cdots K_V(x_{n-1},x_n)\cdot K_V(x_n,x_1) \tag{7.1.17}$$

and we observe that

$$\int_{\mathbb{R}^n} \exp -\Phi(X)\; dX = \int_{\mathbb{R}} K_{V,n}(y,y)\; dy \tag{7.1.18}$$

where $K_{V,n}(x,y)$ is the distribution kernel of $(\mathbf{K_V})^n$. Our assumption on V permits to see that $(K_V)^n$ is trace class[18] and we rewrite (7.1.18) in the form:

$$\int_{\mathbb{R}^n} \exp -\Phi(X)\; dX = \text{ Tr }\left[(\mathbf{K_V})^n\right] = \sum_j \mu_j^n\;, \tag{7.1.19}$$

where the μ_j are introduced in (7.1.24).
We note also for future use that

$$K_{V,n}(x,y) = \sum_j \mu_j^n u_j(x)\,u_j(y)\;. \tag{7.1.20}$$

[17] Let us sketch a proof of (7.1.15) in our particular case. $\mathbf{K}$ being positive and with an explicit kernel : $\exp -\frac{V(x)}{2}\;\exp -\mathcal{J}|x-y|^2\;\exp -\frac{V(y)}{2}$, one can find an Hilbert-Schmidt operator L satisfying $L^\star L = K$. The kernel of L is given, for a suitable θ, by $L(x,y) = c_\theta\;\exp -\theta|x-y|^2 \exp -\frac{V(y)}{2}$. We note indeed that $c_\theta \exp -\theta|x-y|^2$ is the distribution kernel of $\exp t_\theta\Delta$ for a suitable $t_\theta > 0$.
Then we observe that L is Hilbert-Schmidt and that $||L||^2_{H.S} = \sum_j \mu_j = \text{ Tr } K$. Using the previously mentioned formula for the Hilbert-Schmidt norm and the property that

$$K(x,x) = \int L^\star(x,z)\,L(z,x)\,dz = \int L^\star(z,x)\,L(z,x)\,dz\;,$$

one obtains (7.1.15).

[18] See for example [Ro].

In particular we get for the thermodynamic limit:

$$\lim_{n \to \infty} \frac{\ln \int_{\mathbb{R}^n} \exp -\Phi(X)\, dX}{n} = \ln \mu_1 \ . \tag{7.1.21}$$

Moreover the speed of the convergence is easily estimated by:

$$- \ln | \frac{\ln \int_{\mathbb{R}^n} \exp -\Phi(X)\, dX}{n} - \ln \mu_1 | = -n \ln(\frac{\mu_2}{\mu_1}) - \ln k_2 + \ln n + \mathcal{O}(\exp -\delta_2 n) \ , \tag{7.1.22}$$

where k_2 is the multiplicity of μ_2.

7.1.3 Krein-Rutman's Theorem.

We observe now that the kernel $(x, y) \mapsto K_V(x, y)$ satisfies the condition

$$K_V(x, y) > 0 \ , \ \forall x, y \in \mathbb{R} \ . \tag{7.1.23}$$

In particular it satisfies the assumptions of the extended Perron-Frobenius Theorem also called Krein-Rutman's Theorem and our positive operator $\mathbf{K_V}$ admits consequently a largest eigenvalue μ_1 equal to $||K_V||$ which is simple and corresponds to a unique strictly positive normalized eigenfunction which we denote by u_1. Let μ_j the sequence of eigenvalues that we order as a decreasing sequence tending to 0:

$$0 \le \mu_{j+1} \le \mu_j \le .... \le \mu_2 < \mu_1. \tag{7.1.24}$$

We shall denote by u_j a corresponding orthonormal basis of eigenfunctions with

$$K_V u_j = \mu_j u_j \ , \ ||u_j|| = 1 \ . \tag{7.1.25}$$

Let us present the statements:

Definition 7.1.2.
Let A be a bounded positive operator on a Hilbert space $\mathcal{H} = L^2(X, d\nu)$ where (X, ν) is a measured space. Then we say that A has a strictly positive kernel if, for each choice of a non negative function $\theta \in \mathcal{H}$ ($||\theta|| \neq 0$), we have

$$0 < A\theta$$

almost everywhere.

It is immediate to see that the transfer operator satisfies this condition. The theorem generalizing the Perron-Frobenius Theorem is then the following:

Theorem 7.1.3.
Let A be a bounded positive compact symmetric operator on $\mathcal{H}$ having a strictly positive kernel and let[19] *$||A|| = \lambda$ be the largest eigenvalue of A. Then λ has multiplicity 1 and the corresponding eigenfunction u_λ can be chosen to be a strictly positive function.*

[19] See Proposition 6.4.3

Proof:
Since A maps real functions into real functions, we may assume that u_λ is real. We now prove that

$$\langle Au_\lambda, u_\lambda\rangle \leq \langle A|u_\lambda|, |u_\lambda|\rangle.$$

This is an immediate consequence of the strict positivity of the kernel. We just write:

$$u_\lambda = u_\lambda^+ - u_\lambda^-$$

and

$$|u_\lambda| = u_\lambda^+ + u_\lambda^-$$

and the above inequality is then a consequence of

$$\langle Au_\lambda^+, u_\lambda^-\rangle \geq 0 \ ,$$

and

$$\langle u_\lambda^+, Au_\lambda^-\rangle \geq 0 \ .$$

We then obtain

$$\lambda ||u_\lambda||^2 = \langle Au_\lambda, u_\lambda\rangle \leq \langle A|u_\lambda|, |u_\lambda|\rangle \leq ||A|| \, ||u_\lambda||^2 = \lambda ||u_\lambda||^2 \ .$$

This implies

$$\langle Au_\lambda, u_\lambda\rangle = \langle A|u_\lambda|, |u_\lambda|\rangle \ .$$

This equality means

$$\langle u_\lambda^+, Au_\lambda^-\rangle + \langle u_\lambda^-, Au_\lambda^+\rangle = 0 \ .$$

We then get a contradiction unless $u_\lambda^+ = 0$ or $u_\lambda^- = 0$. We can then assume $u_\lambda \geq 0$ and the assumption gives again

$$0 < \langle \theta, Au_\lambda\rangle = \lambda \langle \theta, u_\lambda\rangle \ ,$$

for any positive θ. This gives

$$u_\lambda \geq 0 \quad a.e \ .$$

But

$$u_\lambda = \lambda^{-1} A u_\lambda$$

and this gives

$$u_\lambda > 0 \quad a.e \ .$$

Finally if there are two linearly independent eigenfunctions v_λ and u_λ corresponding to λ, we would obtain the same property for v_λ by considering as new Hilbert space the orthogonal of u_λ in $\mathcal{H}$. But it is impossible to have two orthogonal vectors which are strictly positive.
q.e.d.

Remark 7.1.4.
In the case $t = 0$, we keep the positivity but lose the injectivity ! The spectrum is easy to determine. We are indeed dealing with the orthonormal projector associated to the function $x \mapsto \exp -\frac{V(x)}{2}$. The real number 1 is a simple eigenvalue and 0 is an eigenvalue whose corresponding eigenspace is infinite dimensional.

7.2 The Dirichlet realization, a model of operator with compact resolvent.

We can apply Theorem 6.5.1 to the operator $(-\Delta_D + Id)^{-1}$. We have seen that this operator is compact and it is clearly injective (by construction). It was also seen as a selfadjoint and positive. Moreover, the norm of this operator is less or equal to 1.

There exists consequently a sequence of distinct eigenvalues μ_n tending to 0 (with $0 < \mu_n \leq 1$) and a corresponding orthonormal basis of eigenfunctions such that $(\Delta_D + I)^{-1}$ is diagonalized. If $\phi_{n,j}$ $(j = 1, \ldots, k_n)$ is a corresponding basis of eigenfunctions associated with μ_n, that is, if

$$(-\Delta_D + I)^{-1}\phi_{n,j} = \mu_n \phi_{n,j} \ ,$$

we first observe that $\phi_{n,j} \in D(-\Delta_D + I)$; hence $\phi_{n,j} \in H_0^1(\Omega) \cap H^2(\Omega)$ (if Ω is relatively compact with regular boundary) and

$$-\Delta_D \phi_{n,j} = (\frac{1}{\mu_n} - 1)\phi_{n,j} \ .$$

The function $\phi_{n,j}$ is consequently an eigenfunction of $-\Delta_D$ associated with the eigenvalue $\lambda_n = (\frac{1}{\mu_n} - 1)$.

Let us show, as one easily guesses, that this basis $\phi_{n,j}$ permits effectively the diagonalization of $-\Delta_D$.

Let us indeed consider $u = \sum_{n,j} u_{n,j}\phi_{n,j}$ in the domain of $-\Delta_D$. Let us consider the scalar product $< -\Delta_D u, \phi_{m,\ell} >_{\mathcal{H}}$. Using the selfadjoint character of $-\Delta_D$, we get

$$< -\Delta_D u, \phi_{m,\ell} >_{\mathcal{H}} = < u, -\Delta_D \phi_{m,\ell} >_{\mathcal{H}} = \lambda_m u_{m,\ell} \ .$$

Observing, that $D(\Delta_D) = R(S^{-1})$, one obtains that the domain of $-\Delta_D$ is characterized by

$$D(-\Delta_D) = \{u \in L^2 \mid \sum_{n,j} \lambda_n^2 |u_{n,j}|^2 < +\infty\} \ .$$

Here we have used the property that for any N, we have the identity :

$$\sum_{n \leq N} u_{n,j}\phi_{n,j} = S(\sum_{n \leq N} \lambda_n u_{n,j}\phi_{n,j}) \ .$$

We have consequently given a meaning to the following diagonalization formula

$$-\Delta_D = \sum_n \lambda_n \Pi_{E_n} \ , \tag{7.2.1}$$

where Π_{E_n} is the orthogonal projector on the eigenspace E_n associated with the eigenvalue λ_n.

Let us remark that it results from the property that the sequence μ_n tends to 0 the property that λ_n tend to $+\infty$.

Let us also prove the

Lemma 7.2.1.
The lowest eigenvalue of the Dirichlet realization of the Laplacian in a relatively compact domain Ω is strictly positive :

$$\lambda_1 > 0 \,. \tag{7.2.2}$$

Proof.
We know that $\lambda_1 \geq 0$; the Dirichlet realization of the Laplacian is indeed positive. If $\lambda_1 = 0$, a corresponding normalized eigenfunction ϕ_1 would satisfy

$$< -\Delta\phi_1, \phi_1 >= 0 \,,$$

and consequently

$$\nabla\phi_1 = 0 \,, \quad \text{dans } \Omega \,.$$

This leads first $\phi_1 = \text{ Cste }$ in each connected component of Ω but because the trace of ϕ_1 on $\partial\Omega$ vanishes ($\phi_1 \in H_0^1(\Omega)$) implies that $\phi_1 = 0$. So we get a contradiction.

Finally it results from standard regularity theorems (See [Br], [LiMa]) that the eigenfunctions belong (if Ω is regular) to $C^\infty(\overline{\Omega})$.

We now show the

Proposition 7.2.2.
The lowest eigenvalue is simple and one can choose the first eigenfunction to be strictly positive.

The natural idea is to apply Krein-Rutman's Theorem to $(-\Delta_D + 1)^{-1}$. One has to show that this operator is positivity improving. This is indeed the case if the domain is connected **but the proof of this property will not be given**. We will only show that $(-\Delta_D + I)^{-1}$ is positivity preserving and observe that this implies (following the proof of Krein-Rutman's theorem) that if Ω is an eigenfunction then $|\Omega|$ is an eigenfunction. Then the proof of Proposition 7.2.2 will be completed by using the properties of superharmonic functions.

Lemma 7.2.3.
The operator $(-\Delta_D + I)^{-1}$ is positivity preserving.

The proof is a consequence of the Maximum principle. It is enough to show that

$$-\Delta u + u = f \,,\; \gamma_0 u = 0 \;\text{ and }\; f \geq 0 \;\text{ a.e}\,,$$

implies that $u \geq 0$ almost everywhere.
We introduce $u^+ = \max(u, 0)$ and $u_- = -\inf(u, 0)$. A standard proposition (see below[20] or the book of Lieb-Loss) shows that u^+ and u^- belong to $H_0^1(\Omega)$. Multiplying by u^-, we obtain

$$\int \nabla u^+ \cdot \nabla u^- dx - ||\nabla u^-||^2 - ||u^-||^2 = \int_\Omega f u^- \, dx \geq 0 \,,$$

[20] Recall that $|u| = u^+ + u^-$ and $u = u^+ - u^-$.

which implies, using the corollary of the next proposition $u^- = 0$.

Proposition 7.2.4.
Suppose that $f \in L^1_{\rm loc}(\mathbb{R}^n)$ with $\nabla f \in L^1_{\rm loc}(\mathbb{R}^n)$. Then also $\nabla |f| \in L^1_{\rm loc}(\mathbb{R}^n)$ and with the notation

$$\operatorname{sign} z = \begin{cases} \frac{\overline{z}}{|z|}, & z \neq 0 \\ 0, & z = 0, \end{cases} \tag{7.2.3}$$

we have

$$\nabla |f|(x) = \operatorname{Re}\{\operatorname{sign}(f(x)) \nabla f(x)\} \;\; \textit{almost everywhere}. \tag{7.2.4}$$

In particular,

$$\big|\nabla |f|\big| \leq |\nabla f|\,,$$

almost everywhere.

Corollary 7.2.5.
Under the assumptions of Proposition 7.2.4, we have if f is real

$$\nabla f^+(x) = \frac{1}{2}(\operatorname{sign}(f(x)) + 1)\nabla f(x)\} \;\; \textit{almost everywhere}, \tag{7.2.5}$$

and

$$\nabla f^-(x) = \frac{1}{2}(\operatorname{sign}(f(x)) - 1)\nabla f(x)\} \;\; \textit{almost everywhere}. \tag{7.2.6}$$

Hence

$$\nabla f^+(x) \cdot \nabla f^-(x) = \frac{1}{4}(\operatorname{sign}(f(x))^2 - 1)|\nabla f(x)|^2 \leq 0 \;\; \textit{almost everywhere}. \tag{7.2.7}$$

Proof of Proposition 7.2.4.
Suppose first that $u \in C^\infty(\mathbb{R}^n)$ and define $|z|_\epsilon = \sqrt{|z|^2 + \epsilon^2} - \epsilon$, for $z \in \mathbb{C}$ and $\epsilon > 0$. We observe that

$$0 \leq |z|_\epsilon \leq |z[\text{ and } \lim_{\epsilon \to 0} |z|_\epsilon = |z|\ .$$

Then $|u|_\epsilon \in C^\infty(\mathbb{R}^n)$ and

$$\nabla |u|_\epsilon = \frac{\operatorname{Re}\,(\overline{u}\,\nabla u)}{\sqrt{|u|^2 + \epsilon^2}}. \tag{7.2.8}$$

Let now f be as in the proposition and define f_δ as the convolution

$$f_\delta = f * \rho_\delta$$

with ρ_δ being a standard approximation of the unity for convolution. Explicitly, we take a $\rho \in C_0^\infty(\mathbb{R}^n)$ with

$$\rho \geq 0, \qquad \int_{\mathbb{R}^n} \rho(x)\, dx = 1,$$

and define $\rho_\delta(x) := \delta^{-n}\rho(x/\delta)$, for $x \in \mathbb{R}^n$ and $\delta > 0$.
Then $f_\delta \to f$, $|f_\delta| \to |f|$ and $\nabla f_\delta \to \nabla f$ in $L^1_{\rm loc}(\mathbb{R}^n)$ as $\delta \to 0$.

Take a test function $\phi \in C_0^\infty(\mathbb{R}^n)$. We may extract a subsequence $\{\delta_k\}_{k\in\mathbb{N}}$ (with $\delta_k \to 0$ for $k \to \infty$) such that $f_{\delta_k}(x) \to f(x)$ for almost every $x \in \operatorname{supp}\phi$. We restrict our attention to this subsequence. For simplicity of notation we omit the k from the notation and write $\lim_{\delta\to 0}$ instead of $\lim_{k\to\infty}$.

We now calculate, using dominated convergence and (7.2.8),

$$\begin{aligned}
\int (\nabla\phi)|f|\, dx &= \lim_{\epsilon\to 0} \int (\nabla\phi)|f|_\epsilon\, dx \\
&= \lim_{\epsilon\to 0}\lim_{\delta\to 0} \int (\nabla\phi)|f_\delta|_\epsilon\, dx \\
&= -\lim_{\epsilon\to 0}\lim_{\delta\to 0} \int \phi \frac{\operatorname{Re}\,(\overline{f_\delta}\nabla f_\delta)}{\sqrt{|f_\delta|^2+\epsilon^2}}\, dx
\end{aligned}$$

Using the pointwise convergence of $f_\delta(x)$ and $\|\nabla f_\delta - \nabla f\|_{L^1(\operatorname{supp}\phi)} \to 0$, we can take the limit $\delta \to 0$ and get

$$\int (\nabla\phi)|f|\, dx = -\lim_{\epsilon\to 0} \int \phi \frac{\operatorname{Re}\,(\overline{f}\nabla f)}{\sqrt{|f|^2+\epsilon^2}}\, dx. \tag{7.2.9}$$

Now, $\phi\nabla f \in L^1(\mathbb{R}^n)$ and $\frac{\overline{f(x)}}{\sqrt{|f|^2+\epsilon^2}} \to \operatorname{sign} f(x)$ as $\epsilon \to 0$, so we get (7.2.4) from (7.2.9) by dominated convergence. ∎

We can now look directly (which is a consequence of this positivity improving property, but we have not given the proof!) at the property that the first eigenfunction does not vanish. The eigenfunction is positive, belongs to $C^\infty(\overline{\Omega})$ by a regularity theorem and satisfies

$$-\Delta u = \lambda u \geq 0\,.$$

Hence u is superharmonic and (see Lieb-Loss) satisfies the mean value property : For all $y \in \Omega$, for all $R > 0$ such that $B(y,R) \in \Omega$, then

$$u(y) \geq \frac{1}{\operatorname{vol}(B(y,R))} \int_{B(y,R)} u(z)\, dz\,.$$

Moreover, we know that $\inf u = 0$. Applying this mean value property with y (if any) such that $u(y) = 0$, we obtain that $u = 0$ in $B(y,R)$. Using in addition a connectedness argument, we obtain that in a connected open set u is either identically 0 or strictly positive.

Let us come back to what appears in the proof of Krein-Rutman's theorem. u_λ^+ and u_λ^- are either 0 or strictly positive eigenfunctions and we have also $\langle u_\lambda^+ \,,\, u_\lambda^- \rangle = 0$. Hence u_λ^+ or u_λ^- should vanish. We can then show the simplicity of the first eigenvalue as in the proof of Krein-Rutman's theorem. Hence we have finally completed the proof of Proposition 7.2.2.

7.3 Extension to operators with compact resolvent

What we have done for the analysis of the Dirichlet realization is indeed quite general. It can be applied to selfadjoint operators, which are bounded from below and with compact resolvent.
We show that in this case, there exists an infinite sequence (if the Hilbert space is infinite dimensional) of real eigenvalues λ_n tending to $+\infty$ such that the corresponding eigenspaces are mutually orthogonal, of finite dimension and such that their corresponding Hilbertian sum is equal to $\mathcal{H}$.
Typically, one can apply the method to the Neumann realization of the Laplacian in a relatively compact domain Ω or to the harmonic oscillator in $\mathbb{R}^m$.

7.4 Operators with compact resolvent : the Schrödinger operator in an unbounded domain.

We just recall some criteria of compactness for the resolvent of the Schrödinger operator $P = -\Delta + V$ in $\mathbb{R}^m$ in connection with the precompactness criterion.
In the case of the Schrödinger equation on $\mathbb{R}^m$ and if V is C^∞ and bounded from below, the domain of the selfadjoint extension is always contained in

$$Q(P) := H^1_V(\mathbb{R}^m) = \{u \in H^1(\mathbb{R}^m) | (V+C)^{\frac{1}{2}} u \in L^2(\mathbb{R}^m)\} \,.$$

$Q(P)$ is usually called the form domain of the form

$$u \mapsto \int_{\mathbb{R}^m} |\nabla u|^2 dx + \int_{\mathbb{R}^m} V(x)|u(x)|^2 \, dx \,.$$

It is then easy to see that, if V tends to ∞, then the injection of H^1_V in L^2 is compact (using a criterion of precompactness). We then obtain, observing that $(P+1)^{-1}$ is continuous from L^2 into H^1_V, that the resolvent $(P+\lambda)^{-1}$ is compact for $\lambda \notin \sigma(P)$.
In the case of a compact manifold M and if we consider the Laplace-Beltrami operator on M, then the compactness of the resolvent is obtain without additional assumption on V. The domain of the operator is $H^2(M)$ and we have compact injection from $H^2(M)$ into $L^2(M)$.
The condition that $V \to \infty$ as $|x| \to \infty$ is not a necessary condition. We can indeed replace it by the weaker sufficient condition

Proposition 7.4.1.
Let us assume that the injection of $H^1_V(\mathbb{R}^m)$ into $L^2(\mathbb{R}^m)$ is compact then P has compact resolvent.

More concretely, the way to verify this criterion is to show the existence of a continuous function $x \mapsto \rho(x)$ tending to ∞ as $|x| \to +\infty$ such that

$$H^1_V(\mathbb{R}^m) \subset L^2_\rho(\mathbb{R}^m) \ . \tag{7.4.1}$$

Of course, the preceding case corresponds to $\rho = V$, but, as typical example of this strategy, we shall show in exercise 7.4.3 that the Schrödinger operator on $\mathbb{R}^2$, $-\Delta + x^2 \cdot y^2 + 1$, has compact inverse.
On the other hand, the criterion that $V \to +\infty$ as $|x| \to +\infty$ is not not too far from optimality.
We can indeed prove

Lemma 7.4.2.
Suppose that $V \geq 0$ and that there exists $r > 0$ and a sequence σ_n such that $|\sigma_n| \to +\infty$ and such that

$$\sup_{n \in \mathbb{N}} \sup_{x \in B(\sigma_n, r)} V(x) < +\infty \ . \tag{7.4.2}$$

Then $-\Delta + V + 1$ is not with compact inverse.

Proof.
Let us consider the sequence

$$\phi_n(x) = \psi(x - \sigma_n) \ . \tag{7.4.3}$$

Here ψ is a compactly supported function of L^2 norm 1 and with support in $B(0, r)$.
We observe that the ϕ_n are an orthogonal sequence (after possibly extracting a subsequence for obtaining that the supports of ϕ_n and $\phi_{n'}$ are disjoint for $n \neq n'$) which satisfies for some constant C

$$||\phi_n||^2_{H^2} + ||V\phi_n||^2_{L^2} \leq C \ . \tag{7.4.4}$$

In particular, there exists C such that :

$$||(-\Delta + V + 1)\phi_n||_{L^2} \leq C \ , \ \forall n \in \mathbb{N} \ .$$

But we can not extract from this sequence a strongly convergent sequence in L^2, because ϕ_n is weakly convergent to 0 and $||\phi_n|| = 1$. So the operator $(-\Delta + V + I)^{-1}$ can not be a compact operator.

Exercise 7.4.3.
Show that the unbounded operator on $L^2(\mathbb{R}^2)$

$$P := -\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + x^2 y^2 \ ,$$

has compact resolvent.

Hint.
One can introduce

$$X_1 = \frac{1}{i}\partial_x \ , \ X_2 = \frac{1}{i}\partial_y \ , \ X_3 = xy \ ,$$

and show, for $j = 1, 2$ and for a suitable constant C, the following inequality :

$$||(x^2 + y^2 + 1)^{-\frac{1}{4}}[X_j, X_3]u||^2 \leq C \left(< Pu \, , \, u >_{L^2(\mathbb{R}^2)} + ||u||^2\right) \ ,$$

for all $u \in C_0^\infty(\mathbb{R}^2)$.
One can also observe that

$$i[X_1, X_3] = y \ , \ i[X_2, X_3] = x \ ,$$

and then

$$\begin{aligned} &||(x^2 + y^2 + 1)^{-\frac{1}{4}}[X_1, X_3]u||^2 + ||(x^2 + y^2 + 1)^{-\frac{1}{4}}[X_2, X_3]u||^2 + ||u||^2 \\ &\quad \geq ||(x^2 + y^2 + 1)^{\frac{1}{4}}u||^2 \ . \end{aligned}$$

For the control of $||(x^2 + y^2 + 1)^{-\frac{1}{4}}[X_1, X_3]u||^2$, one can remark that

$$||(x^2 + y^2 + 1)^{-\frac{1}{4}}[X_1, X_3]u||^2 = \langle \frac{-iy}{(x^2 + y^2 + 1)^{\frac{1}{4}}} u \mid (X_1 \, X_3 - X_3 \, X_1)u \rangle \ ,$$

perform an integration by parts, and control a commutator.

7.5 The Schrödinger operator with magnetic field

We can consider on $\mathbb{R}^m$ the so-called Schrödinger operator with magnetic field :

$$P_{A,V} := -\Delta_A + V \ , \tag{7.5.1}$$

where

$$-\Delta_A := \sum_{j=1}^{m} (\frac{1}{i}\partial_{x_j} - A_j(x))^2 \ . \tag{7.5.2}$$

Here $x \mapsto \vec{A} = (A_1(x), \cdots, A_n(x))$ is a vector field on $\mathbb{R}^m$ called the "magnetic potential" and V is called the electric potential. It is easy to see that, when V is semi-bounded, the operator is symmetric and semi-bounded on $C_0^\infty(\mathbb{R}^m)$. We can therefore consider the Friedrichs extension and analyze the property of this selfadjoint extension.
A general question arises if one can get operator of the type $P_{A,V}$ which are with compact resolvent if $V = 0$. This is the problem which is called the problem of the magnetic bottle.
The "heuristical" idea is that the module of the magnetic field can play in some sense the role of the electric potential if it does not oscillate too rapidly ($m \geq 2$).

For defining the magnetic field it is probably easier to consider the magnetic potential as a one-form

$$\sigma_A = \sum_{j=1}^{n} A_j(x) dx_j \ .$$

The magnetic field is then defined as the two form

$$\omega_B = d\sigma_A = \sum_{j<k} (\partial_{x_j} A_k - \partial_{x_k} A_j) dx_j \wedge dx_k \ .$$

The case when $m = 2$ is particularly simple. In this case,

$$\omega_B = B dx_1 \wedge dx_2 \ ,$$

and we can identify ω_B with the function[21] $x \mapsto B(x) = \mathrm{curl}\ (\vec{A})(x)$.
The proof is particularly simple in the case when $B(x)$ has a constant sign (say $B(x) \geq 0$). In this case, we immediately have the inequality

$$\int_{\mathbb{R}^2} B(x)|u(x)|^2 dx \leq < -\Delta_A u, u >_{L^2} \ . \tag{7.5.3}$$

We observe indeed the following identities between operators

$$B(x) = \frac{1}{i}[X_1, X_2] \ , \ -\Delta_A = X_1^2 + X_2^2 \ . \tag{7.5.4}$$

Here

$$X_1 = \frac{1}{i}\partial_{x_1} - A_1(x) \ , \ X_2 = \frac{1}{i}\partial_{x_2} - A_2(x) \ .$$

Note also that :

$$\langle -\Delta_A u \ , \ u \rangle = ||X_1 u||^2 + ||X_2 u||^2 \ ,$$

for all $u \in C_0^\infty(\mathbb{R}^2)$.
To obtain (7.5.3) is then easy through an integration by parts. One can also use, introducing $Z = X_1 + iX_2$, the positivity of Z^*Z or ZZ^*.
We then easily obtain as in the previous example that the operator is with compact resolvent if $B(x) \to +\infty$.
As a simple example, one can think of

$$\vec{A} = (-x_1^2 x_2, +x_1 x_2^2) \ ,$$

which gives

$$B(x) = x_1^2 + x_2^2 \ .$$

Note that the case $m = 2$ is rather particular and it is more difficult to treat $m > 2$. We have indeed to introduce partition of unity.

[21] Here, for french readers, curl denotes the rotational (in french "rotationnel") rot .

7.6 Laplace Beltrami operators on a Riemannian compact manifold

If M is a compact riemannian manifold, it is well known that in this case one can canonically define a measure $d\mu_M$ on M and consequently the Hilbertian space $L^2(M)$. We have also a canonical definition of the gradient. At each point x of M, we have indeed a scalar product on T_xM giving an isomorphism between T_xM and T_x^*M. Using this family of isomorphisms we have a natural identification between the C^∞-vector fields on M and the 1-forms on M. In this identification, the vector field $\mathrm{grad}\, u$ associated to a C^∞ function on M corresponds to the 1-form du.
Considering on $C^\infty(M) \times C^\infty(M)$ the sesquilinear form

$$(u,v) \mapsto a_0(u,v) := \int_M < \mathrm{grad}\, u(x)\ ,\ \mathrm{grad}\, v(x) >_{T_xM}\ d\mu_M\ .$$

There is a natural differential operator $-\Delta_M$ called the Laplace-Beltrami operator on M such that

$$a_0(u,v) =< -\Delta_M u\ ,\ v >_{L^2(M)}\ .$$

In this context, it is not diffficult to define the Friedrichs extension and to get a selfadjoint extension of $-\Delta_M$ as a selfadjoint operator on $L^2(M)$. The domain is easily characterized as being $H^2(M)$, the Sobolev space naturally associated to $L^2(M)$ and one can show that the injection of $H^2(M)$ into $L^2(M)$ is compact because M is compact. The selfadjoint extension of $-\Delta_M$ has compact resolvent and the general theory can be applied to this example.

The case on the circle $\mathbb{S}^1$

The simplest model is the operator $-d^2/d\theta^2$ on the circle of radius one whose spectrum is $\{n^2\ ,\ n \in \mathbb{N}\}$. For $n > 0$ the multiplicity is 2. An orthonormal basis is given by the functions $\theta \mapsto (2\pi)^{-\frac{1}{2}} \exp in\theta$ for $n \in \mathbb{Z}$. Here the form domain of the operator is $H^{1,per}(S^1)$ and the domain of the operator is $H^{2,per}(S^1)$. These spaces have two descriptions. One is to describe these operators as $H^{1,per} := \{u \in H^1(]0,2\pi[\ \mid u(0) = u(2\pi)\}$ and $H^{2,per} := \{u \in H^2(]0,2\pi[\ \mid u(0) = u(2\pi)\ ,\ u'(0) = u'(2\pi)\}$.
The other way is to consider the Fourier coefficients of u.
The Fourier coefficients of $u \in H^{k,per}$ are in $\mathbf{h}^k$. Here

$$\mathbf{h}^k := \{u_n \in \ell^2(\mathbb{Z}) \mid n^k u_n \in \ell^2(\mathbb{Z})\}.$$

It is then easy to prove the compact injection from $H^{1,per}$ in $L^2(S^1)$ or equivalently from $\mathbf{h}^1$ into ℓ^2.
More generally, elliptic symmetric positive operators of order $m > 0$ admit a selfadjoint extension with compact resolvent. We refer to the book by Berger-Gauduchon-Mazet [BGM] for this central subject in Riemannian geometry.

The Laplacian on $\mathbb{S}^2$
One can also consider the Laplacian on $\mathbb{S}^2$. We describe as usual $\mathbb{S}^2$ by the spherical coordinates, with

$$x = \cos\phi \sin\theta, y = \sin\phi\sin\theta, z = \cos\theta\,, \text{ with } \phi \in [-\pi, \pi[\,,\, \theta \in]0, \pi[\,, \quad (7.6.1)$$

and we add the two poles "North" and "South", corresponding to the two points $(0, 0, 1)$ and $(0, 0, -1)$.
We are looking for eigenfunctions of the Fiedrichs extension of

$$\mathbf{L}^2 = -\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\,\sin\theta\,\frac{\partial}{\partial\theta} \quad (7.6.2)$$

in $L^2(\sin\theta d\theta\, d\phi)$, satisfying

$$\mathbf{L}^2 Y_{\ell m} = \ell(\ell+1)Y_{\ell m}\,. \quad (7.6.3)$$

The standard spherical harmonics, corresponding to $\ell \geq 0$ and for an integer $m \in \{-\ell, \dots, \ell\}$, are defined by

$$Y_{\ell m}(\theta, \phi) = c_{\ell,m} \exp im\phi \frac{1}{\sin^m\theta}(-\frac{1}{\sin\theta}\frac{d}{d\theta})^{\ell-m}\sin^{2\ell}\theta\,, \quad (7.6.4)$$

where $c_{\ell,m}$ is an explicit normalization constant.
For future extensions, we prefer to take this as a definition for $m \geq 0$ and then to observe that

$$Y_{\ell,-m} = \hat{c}_{\ell,m}\overline{Y_{\ell,m}}\,. \quad (7.6.5)$$

For $\ell = 0$, we get $m = 0$ and the constant. For $\ell = 1$, we obtain, for $m = 1$, the function $(\theta, \phi) \mapsto \sin\theta \exp i\phi$ and for $m = -1$, the function $\sin\theta\exp -i\phi$ and for $m = 0$ the function $\cos\theta$, which shows that the multiplicity is 3 for the eigenvalue 2.

To show the completeness it is enough to show that, for given $m \geq 0$, the orthogonal family (indexed by $\ell \in \{m + \mathbb{N}\}$) of functions $\theta \mapsto \psi_{\ell,m}(\theta) := \frac{1}{\sin^m\theta}(-\frac{1}{\sin\theta}\frac{d}{d\theta})^{\ell-m}\sin^{2\ell}\theta$ span all $L^2(]0, \pi[, \sin\theta d\theta)$.
For this, we consider $\chi \in C_0^\infty(]0, \pi[)$ and assume that

$$\int_0^\pi \chi(\theta)\psi_{\ell,m}(\theta)\sin\theta d\theta = 0\,,\, \forall \ell \in \{m + \mathbb{N}\}\,.$$

We would like to deduce that this implies $\chi = 0$. After a change of variable $t = \cos\theta$ and an integration by parts, we obtain that this problem is equivalent to the problem to show that, if

$$\int_{-1}^1 \psi(t)((1-t^2)^\ell)^{(\ell-m)}(t)dt = 0\,,\, \forall\ell \in \{m + \mathbb{N}\}\,,$$

then $\psi = 0$.
Observing that the space spanned by the functions $(1-t^2)^{-m}((1-t^2)^\ell)^{(\ell-m)}$ (which are actually polynomials of exact order ℓ) is the space of all polynomials we can conclude the completeness.

8 Selfadjoint unbounded operators and spectral theory.

8.1 Introduction

We assume that $\mathcal{H}$ is an Hilbert space. Once we have a selfadjoint operator we can apply the basic spectral decomposition, which we shall now describe without to give complete proofs. Before to explain the general case, let us come back to the spectral theorem for compact operators T or operators with compact resolvent. This will permit us to introduce a new vocabulary.
We have seen that one can obtain a decomposition of $\mathcal{H}$ in the form

$$\mathcal{H} = \oplus_{k\in\mathbb{N}} V_k \ , \tag{8.1.1}$$

such that

$$Tu_k = \lambda_k u_k \ , \ \text{ if } u_k \in V_k \ . \tag{8.1.2}$$

Hence we have decomposed $\mathcal{H}$ into a direct sum of orthogonal subspaces V_k in which the selfadjoint operator T is reduced to multiplication by λ_k.
If P_k denotes the orthogonal projection operator onto V_k, we can write

$$I = \sum_k P_k \ , \tag{8.1.3}$$

(the limit is in the strong-convergence sense) and

$$Tu = \sum_k \lambda_k P_k u \ , \ \forall u \in D(T) \ . \tag{8.1.4}$$

Here we recall the definition :

Definition 8.1.1.
An operator $P \in \mathcal{L}(\mathcal{H})$ is called an orthogonal projection if $P = P^*$ and $P^2 = P$.

If we assume that T is semibounded (with compact resolvent[22]), we can introduce for any $\lambda \in \mathbb{R}$

$$G_\lambda = \oplus_{\lambda_k \leq \lambda} V_k \ , \tag{8.1.5}$$

and E_λ is the orthogonal projection onto G_λ :

$$E_\lambda = \sum_{\lambda_k \leq \lambda} P_k \ . \tag{8.1.6}$$

It is easy to see that the function $\lambda \mapsto E_\lambda$ has values in $\mathcal{L}(\mathcal{H})$ and satisfy the following properties :

- $E_\lambda = E_\lambda^*$;
- $E_\lambda \cdot E_\mu = E_{\inf(\lambda,\mu)}$;

[22]Note that if $(T - \lambda_0)^{-1}$ is compact for some $\lambda_0 \in \rho(T)$, then it is true for any $\lambda \in \rho(T)$.

- for all λ, $E_{\lambda+0} = E_\lambda$;
- $\lim_{\lambda\to-\infty} E_\lambda = 0$, $\lim_{\lambda\to+\infty} E_\lambda = Id$.
- $E_\lambda \geq 0$

All the limits above are in the sense of the strong convergence.
We also observe that

$$E_{\lambda_k} - E_{\lambda_k - 0} = P_k \ .$$

Then in the sense of vectorvalued distributions, we have

$$dE_\lambda = \sum_k \delta_{\lambda_k} \otimes P_k \ , \tag{8.1.7}$$

where δ_{λ_k} is the Dirac measure at the point λ_k. Hence, in the sense of Stieltjes integrals (this will be explained in more detail below), one can write

$$x = \int_{-\infty}^{+\infty} dE_\lambda(x) \ ,$$

and

$$T = \int_{-\infty}^{+\infty} \lambda dE_\lambda \ .$$

This is in this form that we shall generalize the previous formulas to the case of any selfadjoint operator T.

Functional calculus for operators with compact resolvent :
If f is a continuous (or piecewise continuous function) one can also define $f(T)$ as

$$f(T) = \sum_k f(\lambda_k) \cdot P_k \ ,$$

as an unbounded operator whose domain is

$$D(f(T)) = \{x \in \mathcal{H} \mid \sum_k |f(\lambda_k)|^2 ||x_k||^2 < +\infty\} \ ,$$

where $x_k = P_k x$.

We can also write $f(T)$ in the form :

$$< f(T)x, y >_{\mathcal{H}} = \int_{\mathbb{R}} f(\lambda) \ d < E_\lambda x, y > \ ,$$

where the domain of $f(T)$ is described as

$$D(f(T)) = \{x \in \mathcal{H} \mid \int_{\mathbb{R}} |f(\lambda)|^2 \ d < E_\lambda x, x >_{\mathcal{H}} < +\infty\} \ .$$

Remark 8.1.2.
There are, for semibounded operators with compact resolvent, two possible conventions for the notation of the eigenvalues. The first one is to classify them into an increasing sequence

$$\mu_j \leq \mu_{j+1}$$

counting each eigenvalue according to its multiplicity. The second one is to describe them as a strictly increasing sequence λ_k with λ_k eigenvalue of multiplicity m_k.

We now present a list of properties which are easy to verify in this particular case and which will be still true in the general case.

1. If f and g coincide on $\sigma(T)$, then $f(T) = g(T)$. For any $(x, y) \in \mathcal{H} \times \mathcal{H}$, the support of the measure $d\langle E_\lambda x, y\rangle$ is contained[23] in $\sigma(T)$.

2. If f and g are functions on $\mathbb{R}$,

$$f(T)g(T) = (f \cdot g)(T)\ .$$

In particular, if $(T - z)$ is invertible, the inverse is given by $f(T)$ where f is a continuous function such that $f(\lambda) = (\lambda - z)^{-1}$, $\forall \lambda \in \sigma(T)$.

3. If f is bounded, then $f(T)$ is bounded and we have

$$||f(T)|| \leq \sup_{\lambda \in \sigma(T)} |f(\lambda)|\ . \tag{8.1.8}$$

4. The function f may be complex. Note that, in this case, we get

$$f(T)^\star = \overline{f}(T)\ . \tag{8.1.9}$$

An interesting case is, for $z \in \mathbb{C} \setminus \mathbb{R}$, the function $\lambda \mapsto (\lambda - z)^{-1}$. Then we get from (8.1.8)

$$||(T - z)^{-1}||_{\mathcal{L}(\mathcal{H})} \leq |\ \mathrm{Im}\ z|^{-1}\ . \tag{8.1.10}$$

5. More generally, this works also for $z \in \mathbb{R} \setminus \sigma(T)$. We then obtain in this case the spectral theorem

$$||(T - z)^{-1}||_{\mathcal{L}(\mathcal{H})} \leq d(z, \sigma(T))^{-1}\ . \tag{8.1.11}$$

6. If $f \in C_0^\infty(\mathbb{R})$, we have :

$$f(T) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{|Imz| > \epsilon} (\frac{\partial \tilde{f}}{\partial \bar{z}})(T - z)^{-1} dx.dy\ . \tag{8.1.12}$$

[23] At the moment, we have not defined $\sigma(T)$ when T is unbounded. Think in the case of operators with compact resolvent of the set of the eigenvalues !

Here $\tilde{f}$ is defined by :

$$\tilde{f}(x,y) = (f(x) + iyf'(x))\chi(y) \ ,$$

where $\chi(y)$ is equal to 1 in a neighborhood of 0 and with compact support. This formula can be proven using the Green-Formula (first prove it with T replaced by the scalar λ) or using that :

$$\partial_{\bar{z}} \frac{1}{z-\lambda} = \pi \delta_{(\lambda,0)} \ ,$$

where $\delta_{(\lambda,0)}$ is the Dirac measure at $(\lambda, 0) \in \mathbb{R}^2$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. One should observe that $\tilde{f}$ is not holomorphic but "almost" holomorphic in the sense that :

$$\partial \tilde{f} / \partial \bar{z} = \mathcal{O}(y) \ ,$$

as $y \to 0$.

Exercise 8.1.3.
Show that, one can also find, for any $N \geq 1$, $\tilde{f} = \tilde{f}^N$ such that in addition

$$\partial \tilde{f} / \partial \bar{z} = \mathcal{O}(y^N) \ ,$$

as $y \to 0$.

8.2 Spectrum.

We now come back to the notion of spectrum that we have only met for bounded operators.

Definition 8.2.1.
The resolvent set of a closed operator T is the set of the λ in $\mathbb{C}$ such that the range of $(T-\lambda)$ is equal to $\mathcal{H}$ and such that $(T-\lambda)$ admits a continuous operator denoted by $R(\lambda)$ whose range is included in $D(T)$ such that :

$$R(\lambda)(T-\lambda) = I_{D(T)} \ ,$$

and

$$(T-\lambda)R(\lambda) = I_{\mathcal{H}} \ .$$

As in the bounded case, we observe that the resolvent set is open. Note also that the continuity of $R(\lambda)$ is actually a consequence of the property that the graph of $R(\lambda)$ is closed (using that T is closed) and that $R(\lambda)$ is defined on $\mathcal{H}$.

Definition 8.2.2.
The spectrum of a closed operator T is defined as the complementary set in $\mathbb{C}$ of the resolvent set.

It is then rather easy to show that the spectrum $\sigma(T)$ is closed in $\mathbb{C}$.
The proof of the fact that the spectrum is contained in $\mathbb{R}$ if T is selfadjoint is very close to the bounded case.
The spectrum $\sigma(T)$ is not empty if T is selfadjoint.
The proof is by contradiction. If T has empty spectrum T^{-1} is a bounded selfadjoint operator with spectrum equal to $\{0\}$. We observe indeed that, for $\lambda \neq 0$, the inverse of $T^{-1} - \lambda$ is given, if $\lambda^{-1} \in \rho(T)$, by $\lambda^{-1}T(T - \lambda^{-1})^{-1}$. Hence T^{-1} should be the 0 operator[24], which contradicts $T \circ T^{-1} = I$. This is no longer true in the non selfadjoint case. At the end of the chapter, we give some example appearing naturally in various questions in Fluid Mechanics.

8.3 Spectral family and resolution of the identity.

Definition 8.3.1.
A family of orthogonal projectors $E(\lambda)$ (or E_λ), $-\infty < \lambda < \infty$ in an Hilbert space $\mathcal{H}$ is called a resolution of the identity (or spectral family) if it satisfies the following conditions :

- $$E(\lambda)E(\mu) = E(\min(\lambda, \mu)) \,, \tag{8.3.1}$$

- $$E(-\infty) = 0 \,,\; E(+\infty) = I \,, \tag{8.3.2}$$

  where $E(\pm\infty)$ is defined [25] by

  $$E(\pm\infty)x = \lim_{\lambda \to \pm\infty} E(\lambda)x \tag{8.3.3}$$

  for all x in $\mathcal{H}$,

- $$E(\lambda + 0) = E(\lambda) \,, \tag{8.3.4}$$

  where $E(\lambda + 0)$ is defined by

  $$E(\lambda + 0)x = \lim_{\mu \to \lambda \,,\, \mu > \lambda} E(\mu)x \,. \tag{8.3.5}$$

Remark 8.3.2.
We have shown an example of such a family in the previous subsection.

Proposition 8.3.3.
Let $E(\lambda)$ be a resolution of identity (=spectral family); then for all $x, y \in \mathcal{H}$, the function

$$\lambda \mapsto < E(\lambda)x, y > \tag{8.3.6}$$

[24] Here we refer to our analysis of the spectrum of a bounded self-adjoint operator.
[25] (8.3.1) gives the existence of the limit (cf also Lemma 8.3.4). The limit in (8.3.3) is taken in $\mathcal{H}$. We observe indeed that $\lambda \mapsto \langle E(\lambda)x, x\rangle = ||E(\lambda)x||^2$ is monotonically increasing.

is a function of bounded variation whose total variation[26] *satisfies*

$$V(x,y) \leq ||x|| \cdot ||y|| \ , \ \forall x, y \in \mathcal{H} \ . \tag{8.3.7}$$

Proof.
Let $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. We first get from the assumption (8.3.1) that

$$E_{]\alpha,\beta]} = E_\beta - E_\alpha$$

is an orthogonal projection. From the Cauchy-Schwarz inequality we have

$$\begin{array}{ll} \sum_{j=2}^n | < E_{]\lambda_{j-1},\lambda_j]} x, y > | & = \sum_{j=2}^n | < E_{]\lambda_{j-1},\lambda_j]} x, E_{]\lambda_{j-1},\lambda_j]} y > | \\ & \leq \sum_{j=2}^n ||E_{]\lambda_{j-1},\lambda_j]} x|| \ ||E_{]\lambda_{j-1},\lambda_j]} y]|| \\ & \leq \left(\sum_{j=2}^n ||E_{]\lambda_{j-1},\lambda_j]} x||^2 \right)^{\frac{1}{2}} \ \left(\sum_{j=2}^n ||E_{]\lambda_{j-1},\lambda_j]} y||^2 \right)^{\frac{1}{2}} \\ & = \left(||E_{]\lambda_1,\lambda_n]} x||^2 \right)^{\frac{1}{2}} \ \left(||E_{]\lambda_1,\lambda_n]} y||^2 \right)^{\frac{1}{2}} \ . \end{array}$$

But for $m > n$, we get

$$||x||^2 \geq ||E_{]\lambda_n,\lambda_m]} x||^2 = \sum_{i=n}^{m-1} ||E_{]\lambda_i,\lambda_{i+1}]} x||^2 \ . \tag{8.3.8}$$

We finally obtain that, for any finite sequence $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, we have

$$\sum_{j=2}^n | < E_{]\lambda_{j-1},\lambda_j]} x, y > | \leq ||x|| \cdot ||y|| \ .$$

This shows the bounded variation and the estimate of the total variation defined as

$$V(x,y) := \sup_{\lambda_1,\cdots,\lambda_n} \sum_{j=2}^n | < E_{]\lambda_{j-1},\lambda_j]} x, y > | \ . \tag{8.3.9}$$

Hence we have shown that, for all x and y in $\mathcal{H}$, the function $\lambda \mapsto \langle E(\lambda)x, y\rangle$ is with bounded variation and we can then show the existence of $E(\lambda + 0)$ and $E(\lambda - 0)$. This is the object of

Lemma 8.3.4.
If $E(\lambda)$ is a family of projectors satisfying (8.3.1) and (8.3.2), then, for all $\lambda \in \mathbb{R}$, the operators

$$E_{\lambda+0} = \lim_{\mu\to\lambda \ \mu>\lambda} E(\mu) \ , \ E_{\lambda-0} = \lim_{\mu\to\lambda \ \mu<\lambda} E(\mu) \ , \tag{8.3.10}$$

are well defined when considering the limit for the strong convergence topology.

[26] See the definition in (8.3.9)

Proof.
Let us show the existence of the left limit. From (8.3.8), we get that, for any $\epsilon > 0$, there exists $\lambda_0 < \lambda$ such that, $\forall \lambda'$, $\forall \lambda'' \in [\lambda_0, \lambda[$, such that $\lambda' < \lambda''$

$$||E_{]\lambda',\lambda'']}x||^2 \leq \epsilon \,.$$

It is then easy to show that $E_{\lambda-\frac{1}{n}}\, x$ is a Cauchy sequence converging to a limit and that this limit does not depend on the choice of the sequence tending to λ. The proof of the existence of the limit from the right is the same. This ends the proof of the lemma.

It is then classical (Stieltjes integrals) that one can define for any continuous complex valued function $\lambda \mapsto f(\lambda)$ the integrals $\int_a^b f(\lambda) d\langle E(\lambda)x, y\rangle$ as a limit[27] of Riemann sums.

Proposition 8.3.5.
Let f be a continuous function on $\mathbb{R}$ with complex values and let $x \in \mathcal{H}$. Then it is possible to define for $\alpha < \beta$, the integral

$$\int_\alpha^\beta f(\lambda)\, dE_\lambda x$$

as the strong limit in $\mathcal{H}$ of the Riemann sum :

$$\sum_j f(\lambda_j')(E_{\lambda_{j+1}} - E_{\lambda_j})x \,, \tag{8.3.11}$$

where

$$\alpha = \lambda_1 < \lambda_2 < \cdots < \lambda_n = \beta \,,$$

and

$$\lambda_j' \in]\lambda_j, \lambda_{j+1}] \,,$$

when $\max_j |\lambda_{j+1} - \lambda_j| \to 0$.

The proof is easy using the uniform continuity of f. Note also that the notation could be misleading. May be $\int_{]\alpha,\beta]} f(\lambda) dE_\lambda x$ is less ambiguous.

We now arrive like in the standard theory to the generalized integral.

Definition 8.3.6.
For any given $x \in \mathcal{H}$ and any continuous function f on $\mathbb{R}$, the integral :

$$\int_{-\infty}^{+\infty} f(\lambda) dE_\lambda x$$

is defined as the strong limit in $\mathcal{H}$, if it exists of $\int_\alpha^\beta f(\lambda) dE_\lambda x$ when $\alpha \to -\infty$ and $\beta \to +\infty$.

[27] The best is to first consider the case when $x = y$ and then use a depolarisation formula, in the same way that, when we have an Hilbertian norm, we can recover the scalar product from the norm.

Remark 8.3.7.
The theory works more generally for any borelian function (cf Reed-Simon, Vol. 1 [RS-I]). This can be important, because we are in particular interested in the case when $f(t) = 1_{]-\infty,\lambda]}(t)$.
One possibility for the reader who wants to understand how this can be made is to look at Rudin's book [Ru1], which gives the following theorem (Theorem 8.14, p. 173)

Theorem 8.3.8. .

1. *If μ is a complex Borel measure on $\mathbb{R}$ and if*

$$(\star) \quad f(x) = \mu(]-\infty, x]) \ , \ \forall x \in \mathbb{R} \ ,$$

then f is a normalized function with bounded variation (NBV). By NBV we mean, with bounded variation, but also continuous from the right and such that $\lim_{x\to -\infty} f(x) = 0$.

2. *Conversely, to every $f \in NBV$, there corresponds a unique complex Borel measure μ such that $(\star)$ is satisfied.*

Theorem 8.3.9.
For x given in $\mathcal{H}$ and if f is a complex valued continuous function on $\mathbb{R}$, the following conditions are equivalent

-
$$\int_{-\infty}^{+\infty} f(\lambda) dE_\lambda x \ \textit{exists} \ ; \tag{8.3.12}$$

-
$$\int_{-\infty}^{+\infty} |f(\lambda)|^2 d||E_\lambda x||^2 < +\infty; \tag{8.3.13}$$

-
$$y \mapsto \int_{-\infty}^{+\infty} f(\lambda) d(< E_\lambda y, x >_{\mathcal{H}}) \tag{8.3.14}$$

is a continuous linear form.

Hint for the proof.

a)
(8.3.12) implies (8.3.14) essentially by using repeatedly the Banach-Steinhaus Theorem (also called Uniform Boundedness Theorem) and the definition of the integral.

b)
Let us prove that (8.3.14) implies (8.3.13).

Let F be the linear form appearing in (8.3.14). If we introduce

$$y = \int_\alpha^\beta \overline{f(\lambda)} dE_\lambda x \,,$$

then we first observe (coming back to the Riemann integrals) that

$$y = E_{]\alpha,\beta]} y \,.$$

It is then not too difficult to show that

$$\begin{array}{ll} \overline{F(y)} & = \int_{-\infty}^{+\infty} \overline{f(\lambda)} d < E_\lambda x, y > \\ & = \int_{-\infty}^{+\infty} \overline{f(\lambda)} d < E_\lambda x, E_{]\alpha,\beta]} y > \\ & \int_{-\infty}^{+\infty} \overline{f(\lambda)} d < E_{]\alpha,\beta]} E_\lambda x, y > \\ & = \int_\alpha^\beta \overline{f(\lambda)} d < E_\lambda x, y > \\ & = ||y||^2 \,. \end{array}$$

Using (8.3.14), we get $||y||^2 \leq ||F|| \cdot ||y||$ and consequently

$$||y|| \leq ||F|| \,. \tag{8.3.15}$$

Here we observe that the r.h.s. is independent of α and β.

On the other hand, coming back to Riemann sums, we get

$$||y||^2 = \int_\alpha^\beta |f(\lambda)|^2 d||E_\lambda x||^2 \,.$$

We finally obtain

$$\int_\alpha^\beta |f(\lambda)|^2 d||E_\lambda x||^2 \leq ||F||^2 \,. \tag{8.3.16}$$

Hence, taking the limits $\alpha \to -\infty$ and $\beta \to +\infty$, we obtain (8.3.13).

c)
For the last implication, it is enough to observe that, for $\alpha' < \alpha < \beta < \beta'$, we have

$$|| \int_{\alpha'}^{\beta'} f(\lambda) dE_\lambda x - \int_\alpha^\beta f(\lambda) dE_\lambda x||^2 = \int_{\alpha'}^{\alpha} |f(\lambda)|^2 d||E_\lambda x||^2 + \int_\beta^{\beta'} |f(\lambda)|^2 d||E_\lambda x||^2 \,.$$

Theorem 8.3.10.
Let $\lambda \mapsto f(\lambda)$ be a real-valued continuous function. Let

$$D_f := \{x \in \mathcal{H}, \int_{-\infty}^{+\infty} |f(\lambda)|^2 \, d\langle E(\lambda)x, x\rangle \ < \ \infty \ , \ \}.$$

Then D_f is dense in $\mathcal{H}$ and we can define T_f whose domain is defined by

$$D(T_f) = D_f \ ,$$

and

$$\langle T_f x, y\rangle \;=\; \int_{-\infty}^{+\infty} f(\lambda) d\langle E(\lambda)x, y\rangle$$

for all x in $D(T_f)$ and y in $\mathcal{H}$.
The operator T_f is selfadjoint.
Finally, we have

$$T_f E_\lambda \textit{ is an extension of } E_\lambda T_f \;. \tag{8.3.17}$$

Proof of the theorem.
From property (8.3.2), we obtain that, for any y in $\mathcal{H}$, there exists a sequence (α_n, β_n) such that $E_{]\alpha_n,\beta_n]} y \to y$ as $n \to +\infty$.
But $E_{]\alpha,\beta]} y$ belongs to D_f, for any α, β, and this shows the density of D_f in $\mathcal{H}$.
We now observe that f being real and E_λ being symmetric the symmetry is clear. The selfadjointness is proven by using Theorem 8.3.9.

We observe that, for $f_0 = 1$, we have $T_{f_0} = I$ and for $f_1(\lambda) = \lambda$, we obtain a selfadjoint operator $T_{f_1} := T$.
In this case we say that

$$T = \int_{-\infty}^{+\infty} \lambda \; dE(\lambda)$$

is a spectral decomposition of T and we shall note that

$$||Tx||^2 = \int_{-\infty}^{+\infty} \lambda^2 d\,\langle E(\lambda)x, x\rangle \;=\; \int_{-\infty}^{+\infty} \lambda^2 d\,||E(\lambda)x||^2 \;,$$

for $x \in \; D(T)$.
More generally

$$||T_f x||^2 = \int_{-\infty}^{+\infty} |f(\lambda)|^2 d((E(\lambda)x, x)) = \int_{-\infty}^{+\infty} |f(\lambda)|^2 d(||E(\lambda)x||^2)$$

for $x \in D(T_f)$.

Conclusion. We have consequently seen in this subsection how one can associate to a spectral family of projectors a selfadjoint operator. We have seen in the introduction that the converse was true for a compact operator or an operator with compact resolvent. It remains to prove that this is true in the general case.

8.4 The spectral decomposition Theorem.

The spectral decomposition Theorem makes explicit that the preceding situation is actually the general one.

Theorem 8.4.1.
Any selfadjoint operator T in an Hilbert space $\mathcal{H}$ admits a spectral decomposition such that

$$\begin{array}{l} < Tx, y >= \int_{\mathbb{R}} \lambda \, d < E_\lambda x, y >_{\mathcal{H}} \, , \\ Tx = \int_{\mathbb{R}} \lambda d(E_\lambda x) \, . \end{array} \tag{8.4.1}$$

"Proof".
We shall only give the main points of the proof. We refer to [Hu], [Le-Br] or [DaLi] for detailed proofs or to [RS-I] for another proof which we describe now shortly. Another interesting proof is based on Formula 8.1.12 and presented in the book of Davies [Da].

Step 1.
It is rather natural to imagine that it is essentially enough to treat the case when T is a bounded selfadjoint operator (or at least a normal bounded operator, that is satisfying $T^*T = TT^*$). If A is indeed a general semibounded selfadjoint operator, one can come back to the bounded case by considering $(A+\lambda_0)^{-1}$, with λ_0 real, which is bounded and selfadjoint. In the general case[28], one can consider $(A+i)^{-1}$.

Step 2.
We analyze first the spectrum of $P(T)$ where P is a polynomial.

Lemma 8.4.2. .
If P is a polynomial, then

$$\sigma(P(T)) = \{P(\lambda) \mid \lambda \in \sigma(T)\} \tag{8.4.2}$$

Proof.
We start from the identity $P(x) - P(\lambda) = (x-\lambda)Q_\lambda(x)$ and from the corresponding identity between bounded operators $P(T) - P(\lambda) = (T-\lambda)Q_\lambda(T)$. This permits to construct the inverse of $(T-\lambda)$ if one knows the inverse of $P(T) - P(\lambda)$.

Conversely, we observe that, if $z \in \mathbb{C}$ and if $\lambda_j(z)$ are the roots of $\lambda \mapsto (P(\lambda) - z)$, then we can write :

$$(P(T) - z) = c \prod_j (T - \lambda_j(z)) \, .$$

This permits to construct the inverse of $(P(T) - z)$ if one has the inverses of $(T - \lambda_j(z))$ (for all j).

Lemma 8.4.3.
Let T be a bounded self-adjoint operator. Then

$$||P(T)|| = \sup_{\lambda \in \sigma(T)} |P(\lambda)| \, . \tag{8.4.3}$$

[28]Here we recall that an example of operator which is not semibounded is given in Exercise 4.1.3

We first observe that

$$||P(T)||^2 = ||P(T)^* P(T)||$$

This is the consequence of the general property for bounded linear operators that :

$$||A^* A|| = ||A||^2 \;.$$

We recall that the proof is obtained by observing first that :

$$\begin{array}{ll} ||A^* A|| & = \sup_{||x|| \leq 1, ||y|| \leq 1} |\langle A^* A x, y\rangle| \\ & = \sup_{x,y \; ||x|| \leq 1, ||y|| \leq 1} |\langle Ax, Ay\rangle \\ & \leq ||A||^2 \;, \end{array}$$

and secondly that :

$$||A||^2 = \sup_{||x|| \leq 1} \langle Ax, Ax\rangle = \sup_{||x|| \leq 1} \langle A^* A x, x\rangle \leq ||A^* A|| \;.$$

We then observe that :

$$\begin{array}{lll} ||P(T)||^2 & = ||(\bar{P}P)(T)|| & \\ & = \sup_{\mu \in \sigma(\bar{P}P)(T)} |\mu| & \text{(using Theorem 6.4.1)} \\ & = \sup_{\lambda \in \sigma(T)} |(\bar{P}P)(\lambda)| & \text{(using Lemma 8.4.2)} \\ & = \left(\sup_{\lambda \in \sigma(T)} |P(\lambda)|^2\right) & . \end{array}$$

Step 3.
We have defined a map Φ from the set of polynomials into $\mathcal{L}(\mathcal{H})$ defined by

$$P \mapsto \Phi(P) = P(T) \;, \tag{8.4.4}$$

which is continuous

$$||\Phi(P)||_{\mathcal{L}(\mathcal{H})} = \sup_{\lambda \in \sigma(T)} |P(\lambda)| \;. \tag{8.4.5}$$

The set $\sigma(T)$ is a compact in $\mathbb{R}$ and using the Stone-Weierstrass theorem (which states the density of the polynomials in $C^0(\sigma(T))$), the map Φ can be uniquely extended to $C^0(\sigma(T))$. We still denote by Φ this extension. The properties of Φ are described in the following theorem

Theorem 8.4.4.
Let T be as selfadjoint continuous operator on $\mathcal{H}$. Then there exists a unique map Φ from $C^0(\sigma(T))$ into $\mathcal{L}(\mathcal{H})$ with the following properties :

1.

$$\begin{array}{ll} \Phi(f+g) = \Phi(f) + \Phi(g) \;, & \Phi(\lambda f) = \lambda \Phi(f) \;; \\ \Phi(1) = Id \;, & \Phi(\bar{f}) = \Phi(f)^* \;; \\ \Phi(fg) = \Phi(f) \circ \Phi(g) \;. & \end{array}$$

2.

$$||\Phi(f)|| = \sup_{\lambda \in \sigma(T)} |f(\lambda)| \;.$$

3. *If f is defined by $f(\lambda) = \lambda$, then $\Phi(f) = T$.*

4. $$\sigma(\Phi(f)) = \{f(\lambda) \mid \lambda \in \sigma(T)\} \ .$$

5. *If ψ satisfies $T\psi = \lambda\psi$, then $\Phi(f)\psi = f(\lambda)\psi$.*

6. *If $f \geq 0$, then $\Phi(f) \geq 0$.*

All these properties are obtained by showing first the properties for polynomials P and then extending the properties by continuity to continuous functions. For the last item, note that :

$$\Phi(f) = \Phi(\sqrt{f}) \cdot \Phi(\sqrt{f}) = \Phi(\sqrt{f})^* \cdot \Phi(\sqrt{f}) \ .$$

Step 4.
We are now ready to introduce the measures. Let $\psi \in \mathcal{H}$. Then

$$f \mapsto < \psi, f(T)\psi >_{\mathcal{H}} = < \psi, \Phi(f)\psi >_{\mathcal{H}}$$

is a positive linear functional on $C^0(\sigma(T))$. By measure theory (Riesz Theorem) (cf Rudin [Ru1]), there exists a unique measure μ_ψ on $\sigma(T)$, such that

$$< \psi, f(T)\psi >_{\mathcal{H}} = \int_{\sigma(T)} f(\lambda) d\mu_\psi(\lambda) \ . \tag{8.4.6}$$

This measure is called the spectral measure associated with the vector $\psi \in \mathcal{H}$. This measure is a Borel measure. This means that we can extend the map Φ and (8.4.6) to Borelian functions.
Using the standard Hilbert calculus (that is the link between sesquilinear form and quadratic forms) we can also construct for any x and y in $\mathcal{H}$ a complex measure $d\mu_{x,y}$ such that

$$< x, \Phi(f)y >_{\mathcal{H}} = \int_{\sigma(T)} f(\lambda) d\mu_{x,y}(\lambda) \ . \tag{8.4.7}$$

Using the Riesz representation Theorem (Theorem 3.1.1), this gives as, when f is bounded, an operator $f(T)$. If $f = 1_{]-\infty,\mu]}$, we recover the operator $E_\mu = f(T)$ which permits to construct indeed the spectral family announced in Theorem 8.4.1.

Remark 8.4.5.
Modulo some care concerning the domains of the operator, the properties mentioned at the first subsection of this section for operators with compact resolvent are preserved in the case of an unbounded selfadjoint operator.

8.5 Applications of the spectral theorem:

One of the first applications of the spectral theorem (Property 2.) is the following property :

Proposition 8.5.1.

$$d(\lambda, \sigma(T))||x|| \leq ||(T-\lambda)x|| \,, \tag{8.5.1}$$

for all x in $D(T)$.

This proposition is frequently used in the following context. Except very special cases like the harmonic oscillator, it is usually difficult to get explicitely the values of the eigenvalues of an operator. One consequently tries to localize these eigenvalues by using approximations. Let us suppose for example that one has found λ_0 and y in $D(L)$ such that

$$||(T-\lambda_0)y|| \leq \epsilon \tag{8.5.2}$$

and $||y|| = 1$ then we deduce the existence of λ in the spectrum of L such that $|\lambda - \lambda_0| \leq \epsilon$.
Standard examples are the case of hermitian matrices or the case of the anharmonic oscillator $T := -h^2 \frac{d^2}{dx^2} + x^2 + x^4$. In the second case the first eigenfunction of the harmonic oscillator $-h^2 \frac{d^2}{dx^2} + x^2$ can be used as approximate eigenfunction y in (8.5.2) with $\lambda_0 = h$. We then find (8.5.2) with $\epsilon = \mathcal{O}(h^2)$.
Another application is, using the property that the spectrum is real, the following inequality

$$|\operatorname{Im} \lambda| \; ||x|| \leq ||(T-\lambda)x|| \tag{8.5.3}$$

and this gives an upper bound on the norm of $(T-\lambda)^{-1}$ in $\mathcal{L}(\mathcal{H})$ by $1/|\operatorname{Im} \lambda|$.
One can also consider the operator $T_\epsilon = -d^2/dx^2 + x^2 + \epsilon x^4$. One can show that near each eigenvalue of the harmonic oscillator $(2n+1)$, then there exists, when $\epsilon > 0$ is small enough, an eigenvalue $\lambda_n(\epsilon)$ of T_ϵ.
Another good example to analyze is the construction of a sequence of approximate eigenfunctions considered in Subsection 1.1. From the construction of u_n such that, with $T = -\Delta$,

$$||(T-\xi^2)u_n||_{L^2(\mathbb{R}^m)} = \mathcal{O}(\frac{1}{n}) \,,$$

one obtains that

$$d(\sigma(T), \xi^2) \leq \frac{C}{n} \,,\ \forall n \in \mathbb{N} \,.$$

As $n \to +\infty$, we get $\xi^2 \in \sigma(T)$.
It is then easy to show that

$$\sigma(P) = [0, +\infty[.$$

It is enough to prove indeed, using the Fourier transform, that, for any $b > 0$, $(-\Delta + b)$ has an inverse $(-\Delta + b)^{-1}$ sending L^2 onto H^2.

Here we have followed in a particular case the proof of the following general theorem

Theorem 8.5.2.
Let T be a selfadjoint operator. Then $\lambda \in \sigma(P)$ if and only if there exists a sequence $(u_n)_{n\in\mathbb{N}}$, $u_n \in D(T)$ such that $||u_n|| = 1$ and $||(T-\lambda)u_n|| \to 0$ as $n \to +\infty$.

Exercise 8.5.3.
Show the "only if", after reading of the proof of Proposition 8.5.4.

Note also the following characterization.

Proposition 8.5.4.

$$\sigma(T) = \{\lambda \in \mathbb{R},\ s.t.\ \forall \epsilon > 0, E(]\lambda-\epsilon, \lambda+\epsilon[) \neq 0\} \tag{8.5.4}$$

The proof uses in one direction the explicit construction of $(T-\lambda)^{-1}$ through Proposition 8.5.1. If λ and $\epsilon_0 > 0$ are such $E_{]\lambda-\epsilon_0,\lambda+\epsilon_0[} = 0$. Then there exists a continuous function f on $\mathbb{R}$, such that $f(t) = (t-\lambda)^{-1}$ on the support of the measure dE_λ. This permits to construct the inverse and to show that λ is in the resolvent set of T.
Conversely, let λ in the set defined by the r.h.s of (8.5.4) later denoted by $\tilde{\sigma}(T)$. For any $n \in \mathbb{N}^*$, let us take x_n such that $||x_n|| = 1$ and $E(]\lambda - \frac{1}{n}, \lambda + \frac{1}{n}[)x_n = x_n$. Using property 2. of Theorem 8.4.4, with the function $t \mapsto f_n(t) = (t-\lambda)1_{]\lambda-\frac{1}{n},\lambda+\frac{1}{n}[}(t)$, we get :

$$||(T-\lambda)x_n|| = ||(T-\lambda)E(]\lambda - \frac{1}{n}, \lambda + \frac{1}{n}[)x_n|| \leq \frac{1}{n}||x_n|| = \frac{1}{n}\ .$$

Applying Proposition 8.5.1, we obtain :

$$\tilde{\sigma}(T) \subset \sigma(T)\ .$$

8.6 Examples of functions of a selfadjoint operator

We shall for example meet usually in spectral theory the functions

1. f is the characteristic function of $]-\infty, \lambda]$, $Y_{]-\infty,\lambda]}$; $\Phi(f) = f(T)$ is then $\Phi(f) = E(\lambda)$.

2. f is the characteristic function of $]-\infty, \lambda[$, $Y_{]-\infty,\lambda[}$; $f(T)$ is then $\Phi(f) = E(\lambda - 0)$.

3. f is a compactly supported continuous function. $f(T)$ will be an operator whose spectrum is localized in the support of f.

4. $f_t(\lambda) = \exp(it\lambda)$ with t real.
   $f_t(T)$ is then a solution of the functional equation

$$\begin{aligned} (\partial_t - iT)(f(t,T)) &= 0 \\ f(0,T) &= Id \end{aligned}$$

   We note here that, for all real t, $f_t(T) = \exp(itT)$ is a bounded unitary operator .

5. $g_t(\lambda) = \exp(-t\lambda)$ with t real positive. $g_t(T)$ is then a solution of the functional equation

$$\begin{aligned}(\partial_t + T)(g(t,T)) &= 0 \,,\ \text{for } t \geq 0\\ g(0,T) &= Id \,.\end{aligned}$$

We have discussed in the introduction the case of an operator with compact resolvent. The other case to understand for the beginner is of course the case of the free Laplacian $-\Delta$ on $\mathbb{R}^n$. Using the Fourier transform $\mathcal{F}$, we get as unbounded operator the operator of multiplication by ξ^2. It is not difficult to define directly the functional calculus which simply becomes for a borelian function ϕ :

$$\phi(-\Delta) = \mathcal{F}^{-1}\phi(\xi^2)\mathcal{F} \,. \tag{8.6.1}$$

One possibility is to start from $(-\Delta+1)^{-1}$, for which this formula is true and to then follow what was our construction of the functional calculus. Another possibility is to use the Formula (8.1.12) and to use that (8.6.1) is satisfied for $(-\Delta+z)^{-1}$, with $z \in \mathbb{C} \setminus \mathbb{R}$.

The spectral family is then defined by

$$< E(\lambda)f, g >_{L^2(\mathbb{R}^m)} = \int_{\xi^2 \leq \lambda} \hat{f}(\xi) \cdot \bar{\hat{g}}(\xi)\, d\xi \,.$$

8.7 Spectrum and spectral measures

Another interest of the spectral theorem is to permit the study of the different properties of the spectrum according to the nature of the spectral measure and this leads to the definition of the continuous spectrum and of the pure point spectrum. Let us briefly discuss (without proof) these notions.
Starting of a selfadjoint operator T, one defines $\mathcal{H}_{pp}$ (pure point subspace) as the set defined as

$$\mathcal{H}_{pp} = \{\psi \in \mathcal{H} \mid f \mapsto < f(T)\psi, \psi >_{\mathcal{H}} \text{ is a pure point measure } \} \tag{8.7.1}$$

We recall that a measure on X is pure point if

$$\mu(X) = \sum_{x \in X} \mu(\{x\}) \,. \tag{8.7.2}$$

One can verify that H_{pp} is a closed subspace of $\mathcal{H}$ and that the corresponding orthogonal projection $\Pi_{\mathcal{H}_{pp}}$ satisfies

$$\Pi_{\mathcal{H}_{pp}} D(T) \subset D(T) \,.$$

In this case $T_{/\mathcal{H}_{pp}}$ is naturally defined as unbounded operator on $\mathcal{H}_{pp}$ and one defines the pure point spectrum of T by

$$\sigma_{pp}(T) = \sigma(T_{/\mathcal{H}_{pp}}) \,. \tag{8.7.3}$$

We can similarly define $\mathcal{H}_c$ (continuous subspace) as the set defined as

$$\mathcal{H}_c = \{\psi \in \mathcal{H} \mid f \mapsto < f(T)\psi, \psi >_{\mathcal{H}} \text{ is a continuous measure } \} \tag{8.7.4}$$

We recall that a measure on X is continuous if

$$\mu(\{x\})) = 0 \; , \; \forall x \in X \; . \tag{8.7.5}$$

One can verify that H_c is a closed subspace of $\mathcal{H}$ and that the corresponding orthogonal projection $\Pi_{\mathcal{H}_c}$ satisfies

$$\Pi_{\mathcal{H}_c} D(T) \subset D(T) \; .$$

Moreover, it can be shown (See [RS-I]), that

$$\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_c \; . \tag{8.7.6}$$

In this case $T_{/\mathcal{H}_c}$ is naturally defined as unbounded operator on $\mathcal{H}_c$ and one defines the continuous spectrum of T by

$$\sigma_c(T) = \sigma(T_{/\mathcal{H}_c}) \; . \tag{8.7.7}$$

Example 8.7.1.
The spectrum of $-\Delta$ is continuous.

We observe indeed that

$$\lim_{\epsilon \to 0 \; , \; \epsilon > 0} \int_{||\xi|^2 - \lambda| \leq \epsilon} |\hat{f}(\xi)|^2 d\xi = 0 \; , \; \forall f \in L^2(\mathbb{R}^m) \; .$$

One can still refine this discussion by using the natural decomposition of the measure given by the Radon-Nikodym Theorem. This leads to the notion of absolutely continuous spectrum and of singularly continuous spectrum.

9 Non-self adjoint operators and ϵ-pseudospectrum

When the operators are not selfadjoint, one should think that the spectrum is not the right object because it becomes very unstable by perturbation. It has been realized in the recent years that a family of sets (parametrized by $\epsilon > 0$) in $\mathbb{C}$ called the ϵ-pseudospectrum is the right object for getting this stability.

9.1 Main definitions and properties

Here we follow Chapter 4 in the book by L.N. Trefethen and M. Embree [TrEm].

Definition 9.1.1.
If A is a closed operator with dense domain $D(A)$ in an Hilbert space $\mathcal{H}$, the ϵ-pseudospectrum $\sigma_\epsilon(A)$ of A is defined by

$$\sigma_\epsilon(A) := \{z \in \mathbb{C} \mid ||(zI - A)^{-1}|| > \frac{1}{\epsilon}\}.$$

Remark 9.1.2.
In one part of the literature, $>$ is replaced by $\geq$ in the above definition. We have chosen the definition which leads to the simplest equivalent definitions.
We will be interested in this notion in the limit $\epsilon \to 0$.

We take the convention that $||(zI - A)^{-1}|| = +\infty$ if $z \in \sigma(A)$, where $\sigma(A)$ denotes the spectrum of A, so it is clear that we always have :

$$\sigma(A) \subset \sigma_\epsilon(A) .$$

When A is selfadjoint (or more generally normal), $\sigma_\epsilon(A)$ is, by the Spectral Theorem, given by,

$$\sigma_\epsilon(A) = \{z \in \mathbb{C} \mid d(z, \sigma(A)) < \epsilon\} .$$

So this is only in the case of non self-adjoint operators that this new concept (first appearing in numerical analysis, see Trefethen [Tr1, Tr2]) becomes interesting. Although formulated in a rather abstract way, the following (weak version of) a result by Roch-Silbermann [RoSi] explains rather well to what corresponds the ϵ-pseudospectrum :

Proposition 9.1.3.

$$\sigma_\epsilon(A) = \bigcup_{\{\delta A \in \mathcal{L}(\mathcal{H}) \ s.\ t.\ ||\delta A||_{\mathcal{L}(\mathcal{H})} < \epsilon\}} \sigma(A + \delta A) .$$

In other words, z is in the ϵ-pseudospectrum of A if z is in the spectrum of some perturbation $A + \delta A$ of A with $||\delta A|| < \epsilon$. This is indeed a natural notion thinking of the fact that the models we are analyzing are only approximations of the real problem and of the fact that the numerical analysis of the model goes through the analysis of explicitly computable approximated problems. Numerical examples are treated in [Tr2].

Proof
Let us first show the easy part of this characterization of the ϵ-pseudospectrum. If $||(z-A)^{-1}|| \leq \frac{1}{\epsilon}$, it is clear that for any δA such that $||\delta A|| < \epsilon$, $(A+\delta A - z)$ is invertible. Its inverse is obtined by observing that

$$(z-A)^{-1}(z-A-\delta A) = I - (z-A)^{-1}\delta A\,.$$

But the left hand side is invertible because

$$||(z-A)^{-1}\delta A|| \leq ||(z-A)^{-1}||\;||\delta A|| < 1\;.$$

The inverse is consequently given by

$$(z-A-\delta A)^{-1} = \left(\sum_j ((z-A)^{-1}\delta A)^j\right)(z-A)^{-1}\,.$$

The converse is not very difficult. If $||(z-A)^{-1}|| > \frac{1}{\epsilon}$, by definition of the norm, there exists $u \in \mathcal{H}$ such that $||u|| = 1$ and

$$||(z-A)^{-1}u|| = \mu > \frac{1}{\epsilon}\,.$$

Let $v = (z-A)^{-1}u$. Let (δA) the linear bounded operator such that

$$(\delta A)x = \mu^{-2} u \,\langle v\,|x\rangle\,,\;\forall x \in \mathcal{H}\,.$$

It is clear that v is an eigenfunction of $A+\delta A$ associated to z and that $||\delta A|| = \frac{1}{\mu} < \epsilon$.
Hence we have found a perturbation $A+\delta A$ of A such that $z \in \sigma(A+\delta A)$ and $||\delta A|| < \epsilon$.
Another presentation for defining the ϵ-pseudospectrum is to say that $z \in \sigma_\epsilon(A)$ if and only if :
either $z \in \sigma(A)$ or if there exists an ϵ-pseudoeigenfunction that is an $u \in D(A)$ such that $||u|| = 1$ and $||(z-A)u|| < \epsilon$.

Theorem 9.1.4 (ϵ-Pseudospectrum of the adjoint)**.**
For any closed densely defined operator A and any $\epsilon > 0$, we have

$$\sigma_\epsilon(A^*) = \overline{\sigma_\epsilon(A)}\,,$$

where for a subset Σ in $\mathbb{C}$ we denote by $\overline{\Sigma}$ the set

$$\overline{\Sigma} = \{z \in \mathbb{C}\,|\,\bar{z} \in \Sigma\}\,.$$

Proof
This is immediate using that, if $z \notin \sigma(A)$,

$$\|(z-A)^{-1}\| = \|(\bar{z}-A^*)^{-1}\|\,.$$

9.2 ϵ-Pseudospectrum : complete analysis of the differentiation operator.

We consider the operator A defined on $L^2(]0,1[)$ by

$$D(A) = \{u \in H^1(]0,1[)\,,\, u(1) = 0\}\,,$$

and

$$Au = u'\,,\, \forall u \in D(A)\,.$$

This is clearly a closed operator with dense domain.
The adjoint of A is defined on $L^2(]0,1[)$ by

$$D(A^*) = \{u \in H^1(]0,1[)\,,\, u(0) = 0\}\,,$$

and

$$Au = -u'\,,\, \forall u \in D(A^*)\,.$$

Lemma 9.2.1.
$\sigma(A) = \emptyset$ and A has compact resolvent.

First we can observe that $(A - z)$ is injective on $D(A)$ for any $z \in \mathbb{C}$. Then one easily verifies that for any $z \in \mathbb{C}$, the inverse is given by

$$[(z - A)^{-1} f](x) = \int_x^1 \exp z(x - s)\ f(s)\, ds\ .$$

It is also clear that this operator is compact.

To analyze the ϵ-pseudospectrum is more interesting. For this we need to estimate

$$\psi(z) := \|(z - A)^{-1}\|\,.$$

The first remark is that ψ depends only on $\mathrm{Re}\, z$. For this we can observe that the map $u \mapsto \exp i\alpha x\, u$ is a unitary transform on $L^2(]0,1[)$, which maps $D(A)$ onto $D(A)$.
The main result is the following

Theorem 9.2.2.
The function ψ is a subharmonic function which depends only of $Re\ z$ and satisfies

$$\psi(z) \leq \frac{1}{Re\ z}\,,\ \ \textit{for}\ \ Re\ z > 0\,, \tag{9.2.1}$$

and

$$\psi(z) = -\frac{\exp - \ Re\ z}{2\ Re\ z} + \mathcal{O}(\frac{1}{|\ Re\ z|})\,,\ \textit{for}\ \ Re\ z < 0\,. \tag{9.2.2}$$

This implies for the ϵ-pseudospectrum of A :

Corollary 9.2.3.
For $\epsilon > 0$, the ϵ-pseudospectrum of A is an hyperplane of the form

$$\sigma_\epsilon(A) = \{z \in \mathbb{C} \,|\, Rez < c_\epsilon\} \tag{9.2.3}$$

with

$$c_\epsilon \sim \begin{cases} (\ln \epsilon) \;\; as\ \epsilon\,, \to 0 \\ \epsilon \;\; as\ \epsilon \to +\infty\,. \end{cases} \tag{9.2.4}$$

Rough estimate from below for Re $z < 0$
A first (non optimal) step is to think[29] "semi-classical". **Let us take z real** and let us consider

$$x \mapsto \phi_z(x) := |2z|^{\frac{1}{2}} \exp zx\,.$$

This function is not in Ker $(z - A)$ because it does not satisfies the boundary condition. But when $z \to -\infty$, the boundary condition at $x = 1$ is "almost satisfied". Actually ϕ_z lives very close to 0. Moreover, for $z < 0$,

$$\|\phi_z\|^2 = 1 - \exp 2z$$

Hence the norm tends to 1 as $z \to -\infty$.
Let us indeed consider for $\eta > 0$ a cut-off function χ_η such that $\chi_\eta = 1$ on $[0, 1-\eta]$ and $\chi\eta = 0$ on $[1 - \frac{\eta}{2}, 1]$ and let us introduce

$$\phi_{z,\eta}(x) = \chi_\eta \phi_z\ .$$

We now observe that $\phi_{z,\eta} \in D(A)$ and that

$$(z - A)\phi_{z,\eta}(x) = -\chi'_\eta \phi_z\,.$$

The L^2 norm of the right hand side is exponentially small like $\exp(1-\eta)z$.
This shows that, for any $\eta > 0$ there exists $z_\eta < 0$ such that

$$\frac{1}{C_\eta} \exp -(1-\eta)z \le \psi(z)\,, \;\; \text{for } z < z_\eta\,. \tag{9.2.5}$$

This is not as good as in the statement of the theorem but this suggests a rather general point of view. We will complete later the analysis of the behavior of ψ as Re $z \mapsto -\infty$.

Rough estimate from above for Re $z > 0$
Here we will try to estimate $\psi(z)$ from above by using an a priori estimate for $(A - z)$ (with z real).
For $u \in D(A)$, we have

$$\langle (A - z)u \,|\, u\rangle = -z||u||^2 + \int_0^1 u'(t)\bar{u}(t)dt\ .$$

[29] The semi-classical parameter is $h = \frac{1}{\mathrm{Re}\, z}$

But by an integration by parts, we observe that

$$\int_0^1 u'(t)\bar{u}(t)\, dt = -\int_0^1 u(t)\bar{u}'(t) dt - |u(0)|^2 \,.$$

Coming back to the previous equality and taking the real part, we obtain

$$-\ \mathrm{Re}\ \langle (A-z)u \mid u\rangle = z\|u\|^2 + \frac{1}{2}|u(0)|^2 \geq z\|u\|^2 \,.$$

Then we obtain

$$||(A-z)u|| \geq z||u||\,,\ \forall u \in D(A)\,, \tag{9.2.6}$$

which implies (9.2.1).

Control of the resolvent using Schur's Lemma
The operator $(A-z)^{-1}$ being an operator associated to an integral kernel, one can analyze what is given by Schur's lemma or by the Hilbert-Schmidt criterion. The kernel is defined by

$$K(x,s) = \begin{cases} 0 \text{ for } s < x \\ \exp z(x-s) \text{ for } x < s \end{cases}$$

According to Schur's Lemma, we have to consider $\sup_x \int K(x,s)ds$ and $\sup_s \int K(x,s)dx$. If we are interested with the Hilbert-Schmidt norm, we have to compute $\int\int K(x,s)^2 dxds$. All these computations can be done rather explicitely !
For $z \neq 0$, we have

$$\sup_x \int K(x,s)ds = \frac{1}{z}(1-\exp -z)\,,$$

and

$$\sup_s \int K(x,s)dx = \frac{1}{z}(1-\exp -z)$$

This gives

$$\psi(z) \leq \frac{1}{z}(1-\exp -z)\ . \tag{9.2.7}$$

This is actually an improved version of (9.2.1) for $z > 0$ and for $z < 0$, it is better to write it in the form

$$\psi(z) \leq \frac{-1}{z}(\exp -z - 1)\ , \tag{9.2.8}$$

and to compare it to the lower bound obtained in (9.2.5).

A more accurate estimate for $z < 0$
We can rewrite $(z-A)^{-1}$ in the form

$$(z-A)^{-1} = R_1 - R_2\,,$$

with

$$R_1 v(x) := \int_0^1 \exp z(x-s)\; v(s), ds$$

and

$$R_2 v(x) := \int_0^x \exp z(x-s)\; v(s), ds$$

Observing that R_2^* can be treated as for the proof of (9.2.7), we first obtain

$$||R_2|| \leq -\frac{1}{\text{Re } z}\,.$$

It remains to control $||R_1||$. This norm can be computed explicitly. We have indeed

$$||R_1 v|| = ||\exp zx||\;|\int_0^1 \exp -zs\; v(s)\, ds|$$

Hence we have just to compute the norm of the linear form

$$v \mapsto \int_0^1 \exp -zs\; v(s)\, ds$$

which is the L^2-norm of $s \mapsto \exp -zs$.
This gives

$$||R_1|| = ||\exp zx||\cdot||\exp -zx|| = -\frac{1}{2z}\sqrt{(1-e^{2z})(e^{-2z}-1)} = -\frac{1}{2z}e^{-z}(1-e^{2z})\,.$$

Combining the estimates of $||R_1||$ and $||R_2||$ leads to (9.2.2).

Remark 9.2.4.
One can discretize the preceding problem by considering, for $n \in \mathbb{N}^*$, the matrix $A_n = nA_1$ with $A_1 = I + J$ where J is the $n \times n$ matrix such that $J_{i,j} = \delta_{i+1,j}$. One can observe that the spectrum of A_n is $-n$. It is also interesting to analyze the pseudospectrum.

Remark 9.2.5.
There is a semi-classical version of the pseudospectrum for families A_h. One can then relate the ϵ appearing in the definition of the ϵ-pseudospectrum with the parameter h (which could typically be in $]0, h_0]$). For example, we can consider $\epsilon(h) = h^N$.

Exercise 9.2.6.
Analyze the pseudospectrum of $\frac{d}{d\theta} + \lambda g(\theta)$ on the circle.

9.3 Another example of non selfadjoint operator without spectrum

We consider the spectrum of the operator

$$A = \frac{d}{dx} + x^2 \,,$$

on the line.
We can take as domain $D(A)$ the space of the $u \in L^2(\mathbb{R})$ such that $Au \in L^2(\mathbb{R})$. We note that $C_0^\infty(\mathbb{R})$ is dense for the graph norm in $D(A)$. Hence A is the closed extension of the differential operator $\frac{d}{dx} + x^2$ with domain $C_0^\infty(\mathbb{R})$. We note that the operator is not selfadjoint. The adjoint is $-\frac{d}{dx} + x^2$.

The two following inequalities can be useful.

$$\text{Re } \langle Au \,|\, u\rangle \geq ||xu||^2 \geq 0 \,. \tag{9.3.1}$$

This inequality is first proved for $u \in C_0^\infty(\mathbb{R})$ and then extended to $u \in D(A)$ using the density of $C_0^\infty(\mathbb{R})$. A first consequence is that

$$||xu||^2 \leq ||Au||\,||u|| \leq \frac{1}{2}\left(||Au||^2 + ||u||^2\right) \,. \tag{9.3.2}$$

This implies that $D(A)$ (with graph norm) has continuous injection in the weighted space L^2_ρ with $\rho(x) = |x|$. Together with the fact that $D(A) \subset H^1_{loc}(\mathbb{R})$, this implies that $(A + I)$ is invertible and that the inverse is compact.

The second inequality is

$$||x^2 u||^2 + ||u'||^2 \leq C\left(||Au||^2 + ||u||^2\right) \,. \tag{9.3.3}$$

To prove it, we observe that

$$||Au||^2 = ||u'||^2 + ||x^2 u||^2 + 2 \text{ Re } \langle u' \,,\, x^2 u\rangle \,.$$

Using an integration by parts, we get

$$-2 \text{ Re } \langle u' \,,\, x^2 u\rangle = 2\langle xu \,,\, u\rangle \,.$$

Hence we get (using Cauchy-Schwarz)

$$||Au||^2 \geq ||u'||^2 + ||x^2 u||^2 - 2||xu||\,||u|| \,.$$

One can then use (9.3.2) to get the conclusion.
(9.3.3) permits to obtain that

$$D(A) = \{u \in H^1(\mathbb{R}) \,,\, x^2 u \in L^2(\mathbb{R})\} \,, \tag{9.3.4}$$

which is not obvious at all.

Proposition 9.3.1.
A has an inverse, and the inverse is compact. Moreover its spectrum is empty.

We consider on $\mathbb{R}$, the differential equation

$$u' + x^2 u = f \; . \tag{9.3.5}$$

For all $f \in L^2(\mathbb{R})$, let us show that there exists a unique solution in $L^2(\mathbb{R})$ de (9.3.5). An elementary calculus gives

$$u(x) = \exp -\frac{1}{3}x^3 \int_{-\infty}^{x} \exp \frac{1}{3}y^3 f(y)\, dy \; .$$

One has to work a little for showing that $u \in L^2$ (this is easier if f is compactly supported). If we denote by $\mathbf{K}$ the operator which associates to f the solution u, the distribution kernel of $\mathbf{K}$ is given by

$$K(x,y) = \begin{cases} 0 & \text{if } y \geq x \\ \exp \frac{1}{3}(y^3 - x^3) & \text{if } y < x \end{cases} \; .$$

We note that if there exists an eigenvalue of $\mathbf{K}$ $\lambda \neq 0$ and if u_λ is a corresponding eigenfunction, then u_λ satisfies

$$u'_\lambda + x^2 u_\lambda = \frac{1}{\lambda} u_\lambda \; . \tag{9.3.6}$$

From this we deduce that $\mathbf{K}$ has no non zero eigenvalue. One can indeed solve explicitly (9.3.6) :

$$u_\lambda(x) = C \exp -\frac{x^3}{3} \exp \frac{1}{\lambda} x \; .$$

It is then easy to see that no one can be in $L^2(\mathbb{R})$ when $C \neq 0$.
To show that $\mathbf{K}$ is compact, we can actually show that $\mathbf{K}$ is Hilbert-Schmidt. That is, we will show that $K(x,y)$ is in $L^2(\mathbb{R}^2)$. We have

$$\int_{y<x} \exp \frac{2}{3}(y^3 - x^3)\, dxdy = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{x} \exp \frac{2}{3}(y^3 - x^3)\, dy \right) dx \; .$$

Dividing the domain of integration in two parts, we first consider :

$$\int_{-1}^{+1} \left(\int_{-\infty}^{x} \exp \frac{2}{3}(y^3 - x^3)\, dy \right) dx \; ,$$

which is bounded from above by

$$2e \int_{-\infty}^{1} \exp \frac{2}{3} y^3 \, dy \; .$$

Then we look at

$$\int_{|x|>1} \left(\int_{-\infty}^{x} \exp \frac{2}{3}(y^3 - x^3)\, dy \right) dx \; .$$

Here we observe that

$$\exp \frac{2}{3}(y^3 - x^3) = \exp \frac{2}{3}(y-x)(y^2+yx+x^2)\ ,$$

and that

$$(y^2+yx+x^2) \geq \frac{1}{2}(y^2+x^2) \geq \frac{1}{2}x^2\ .$$

This leads, as $y \leq x$, to the upper bound la majoration

$$\exp \frac{2}{3}(y^3-x^3) \leq \exp \frac{1}{3}(y-x)x^2\ .$$

and to

$$\int_{|x|>1}\left(\int_{-\infty}^{x} \exp \frac{2}{3}(y^3-x^3)\, dy\right) dx \leq 3 \int_{|x|>1} \frac{1}{x^2} dx\ < +\infty.$$

This implies that the spectrum of $\mathbf{K}$ is 0.

Non self-adjoint effects.

We can also try to estimate the "solution" operator $\mathbf{K}_\lambda$ corresponding to the equation

$$u' + x^2 u = \lambda f\ . \tag{9.3.7}$$

It is easy, to see that we can reduce the computation to the case when λ réel. We have indeed

$$\mathbf{K}_\lambda = \exp -i\ \mathrm{Im}\ \lambda\, x\, \mathbf{K}_{\ \mathrm{Re}\ \lambda}\ \exp i\ \mathrm{Im}\ \lambda\, x\ .$$

For $\lambda < 0$, we easily find, observing that $u \in \mathbb{S}(\mathbb{R})$ and

$$\mathrm{Re}\ \langle u' + x^2 u - \lambda \,|\, u\rangle_{L^2} = \langle x^2 u - \lambda \,|\, u\rangle \geq -\lambda ||u||^2\ ,$$

the estimate

$$||\mathbf{K}_\lambda||_{\mathcal{L}(L^2)} \leq -\frac{1}{\lambda}\ .$$

The case $\lambda = 0$ has been treated before. Without to much effort, we get the same result for $|\ \mathrm{Re}\ \lambda| \leq 1$.
So we have to consider the case when $\lambda \geq 1$ and control the estimate as $\lambda \to +\infty$.
Proceeding as in the case $\lambda = 0$, we first obtain

$$K_\lambda(x,y) = \left\{ \begin{array}{ll} 0 & \text{si } y \geq x \\ \exp\left(\frac{1}{3}(y^3-x^3) - \lambda(y-x)\right) & \text{si } y < x \end{array} \right. . \tag{9.3.8}$$

Again, we see that $K_\lambda(x,y)$ is in $L^2(\mathbb{R}^2)$:

$$||K_\lambda||^2_{L^2(\mathbb{R}^2)} = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{x} \exp\left(\frac{2}{3}(y^3-x^3) - 2\lambda(y-x)\right) dy\right) dx\ .$$

The proof is similar to the case $\lambda = 0$. We cut the domain of integration.

$$\int_{-1-4\sqrt{\lambda}}^{-1} \left(\int_{-\infty}^{x} \exp\left(\frac{2}{3}(y^3 - x^3) - 2\lambda(y-x) \right) dy \right) dx \;,$$

is bounded after a new partition of the domain of integration by

$$4\sqrt{\lambda} e^{\frac{2}{3}(1+4\lambda^{\frac{1}{2}})^3} \int_{-\infty}^{-1-4\sqrt{\lambda}} \exp\left(\frac{2}{3} y^3 - 2\lambda y \right) dy \;\leq 3\sqrt{\lambda}\, e^{\frac{2}{3}(1+2\lambda^{\frac{1}{2}})^3} \;,$$

and by

$$\begin{aligned} &\textstyle\int_{-1-4\sqrt{\lambda}}^{-1} \left(\int_{-1-4\sqrt{\lambda}}^{x} \exp\left(\frac{2}{3}(y^3 - x^3) - 2\lambda(y-x) \right) dy \right) dx \\ &\quad \leq 4\sqrt{\lambda} \sup_{x \in]-1-4\sqrt{\lambda}, -1[} \left(\textstyle\int_{-1-4\sqrt{\lambda}}^{x} \exp -2\lambda(y-x)\, dy \right) \\ &\quad \leq 2\lambda^{-\frac{1}{2}} e^{8\lambda^{\frac{3}{2}}} \;. \end{aligned}$$

Then we consider

$$\int_{-1}^{+1} \left(\int_{-\infty}^{x} \exp\left(\frac{2}{3}(y^3 - x^3) - 2\lambda(y-x) \right) dy \right) dx \;,$$

which is bounded by

$$\begin{aligned} &2e^{\frac{2}{3}+2\lambda} \textstyle\int_{-\infty}^{-1-4\sqrt{\lambda}} \exp(\frac{2}{3} y^3 - 2\lambda y)\, dy \\ &\quad \leq 2e^{\frac{2}{3}+2\lambda} \\ &\leq C e^{\frac{2}{3}+2\lambda} \;. \end{aligned}$$

and by

$$\begin{aligned} &\textstyle\int_{-1}^{+1} \left(\int_{-1-4\sqrt{\lambda}}^{x} \exp\left(\frac{2}{3}(y^3 - x^3) - 2\lambda(y-x) \right) dy \right) dx \\ &\textstyle\int_{-1}^{+1} \left(\int_{-1-4\sqrt{\lambda}}^{x} \exp -2\lambda(y-x)\, dy \right) dx \\ &\leq \frac{1}{\lambda} \exp(4\lambda(1+2\sqrt{\lambda})) \end{aligned}$$

We look now at

$$\int_{1}^{1+4\sqrt{\lambda}} \left(\int_{-\infty}^{x} \exp\left(\frac{2}{3}(y^3 - x^3) - 2\lambda(y-x) \right) dy \right) dx \;,$$

which is similarly controlled by

$$4\sqrt{\lambda} \exp 2\lambda(1+4\sqrt{\lambda}) \int_{-\infty}^{-1-4\sqrt{\lambda}} \exp\left(\frac{2}{3} y^3 - 2\lambda y \right) dy$$

and by

$$\int_{1}^{1+4\sqrt{\lambda}} \left(\int_{-1-4\sqrt{\lambda}}^{x} \exp -2\lambda(y-x)\, dy \right) dx \;.$$

Finally, we have to estimate

$$\int_{|x|\geq 1+4\sqrt{\lambda}} \left(\int_{-\infty}^{x} \exp\left(\frac{2}{3}(y^3-x^3) - 2\lambda(y-x)\right) dy \right) dx \,,$$

Here we observe that

$$\exp\frac{2}{3}(y^3-x^3) = \exp\frac{2}{3}(y-x)(y^2+yx+x^2)\,,$$

and that

$$(y^2+yx+x^2) \geq \frac{1}{2}(y^2+x^2) \geq \frac{1}{2}x^2\,.$$

This leads to the upper bound

$$\exp\frac{2}{3}(y^3-x^3)\exp -2\lambda(y-x) \leq \exp\frac{1}{3}(y-x)(x^2-6\lambda)\,.$$

By simple integration, we get :

$$\begin{aligned} &\int_{|x|>1+4\sqrt{\lambda}} \left(\int_{-\infty}^{x} \exp\left(\tfrac{2}{3}(y^3-x^3) - 2\lambda(y-x)\right) dy \right) dx \\ &\quad \leq 3\int_{|x|>1+4\sqrt{\lambda}} \tfrac{1}{x^2-6\lambda} dx \; < +\infty\,. \end{aligned}$$

The last bound can be controlled independently of $\operatorname{Re}\lambda$. Hence, we have finally proved the existence of $C>0$ such that, for $\operatorname{Re}\lambda \geq 1$, we have

$$||\mathbf{K}_\lambda||_{HS} \leq C|\lambda|^C \exp C \operatorname{Re}\lambda^{\frac{3}{2}}\,.$$

Remark 9.3.2.
Using the Laplace integral method, one can get the asymptotics of $||\mathbf{K}_\lambda||_{HS}$.
Note that $||\mathbf{K}_\lambda||_{HS} \geq ||\mathbf{K}_\lambda||_{\mathcal{L}(L^2)}$.
One can also find a lower bound of $||\mathbf{K}_\lambda||_{\mathcal{L}(L^2)}$ *using quasimodes.*
Note that $\mathbf{K}_\lambda$ *is the resolvent of the unbounded operator* $-\frac{d}{dx}+x^2$. *This operator is* **not** *selfadjoint. It has compact resolvent and empty spectrum. Moreover, the norm of the resolvent depends only of* $\operatorname{Re}\lambda$.

Remark 9.3.3.
Using the Fourier transform, on can see that the operator is isospectral to the complex Airy operator :

$$D_x^2 + ix\,, \tag{9.3.9}$$

This will be analyzed in the next subsection.

9.4 The non selfadjoint harmonic oscillator

Other interesting example to analyze in the same spirit is the complex harmonic oscillator

$$H_1 := -\frac{d^2}{dx^2} + ix^2$$

(See Davies (1999), [Tr2], [Zw] and references therein). The spectrum can be seen as the spectrum of $-\frac{d^2}{dx^2}+x^2$ rotated by $\frac{\pi}{4}$ in the complex plane.

Let us sketch how we can guess the result.
If we make a dilation $x=\rho y$, the operator $-\frac{d^2}{dx^2}+ix^2$ becomes the operator $-\rho^{-2}\frac{d^2}{dy^2}+i\rho y^2$ which is unitary equivalent.
We can now consider the family of operator

$$\rho \mapsto H_\rho := -\rho^{-2}\frac{d^2}{dx^2}+i\rho^2x^2\,,$$

for ρ in a sector in $\mathbb{C}$.
It can be shown that for ρ in (an open neighborhood of) the sector $\arg\rho\in[-\frac{\pi}{8},0]$ the spectrum is discrete and independent of ρ (this is a mixture of Kato's theory and of the so called Combes-Thomas argument).

Taking $\rho=\exp -i\frac{\pi}{8}$, we get

$$H_{-\frac{\pi}{8}} = \exp i\frac{\pi}{4}\,(-\frac{d^2}{dx^2}+x^2)\,.$$

Hence :

$$\sigma(H_1)=\{\exp i\frac{\pi}{4}\,(2n+1)\,,n\in\mathbb{N}\}\,. \tag{9.4.1}$$

Let us give rigorously a part of an alternative proof. If we start from the basis of eigenfunctions of the standard harmonic oscillator u_n. It is easy to see that $u_n(\rho^{-1}\exp -i\frac{\pi}{8}\,x)$ is an eigenfunction of H_ρ in $L^2(\mathbb{R})$. Hence we have proven one inclusion in (9.4.1).
What is less clear is to show that the family $x\mapsto u_n(\rho^{-1}\exp -i\frac{\pi}{8}\,x)$ which is no more orthogonal generates a dense subspace in L^2. This is discussed in Davies paper.

Remark 9.4.1.
This example shows that when the operators are not selfadjoint many things can occur. In particular the fact that if $z\in\rho(T)$, then $||(T-z)^{-1}||$ is controlled by $\frac{1}{d(\lambda,\sigma(T)}$ becomes wrong.

9.5 The complex Airy operator in $\mathbb{R}$

This operator can be defined as the closed extension $\mathcal{A}$ of the differential operator on $C_0^\infty(\mathbb{R})$ $\mathcal{A}_0^+ := D_x^2+i\,x$. We observe that $\mathcal{A}=(\mathcal{A}_0^-)^*$ with $\mathcal{A}_0^- := D_x^2-i\,x$ and that its domain is

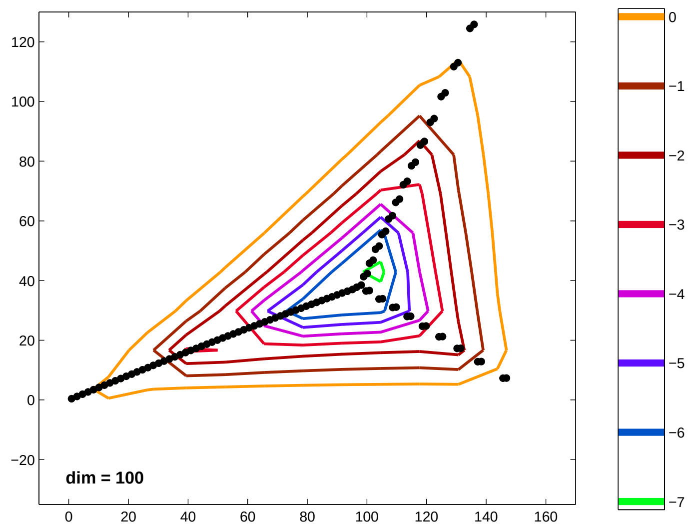
$$D(\mathcal{A})=\{u\in H^2(\mathbb{R})\,,\,x\,u\in L^2(\mathbb{R})\}\,.$$

In particular $\mathcal{A}$ has compact resolvent.

It is also easy to see that

$$\mathrm{Re}\,\langle \mathcal{A}u\,|\,u\rangle \geq 0\,. \tag{9.5.1}$$

Figure 1: Davies operator: pseudospectra



Hence $-\mathcal{A}$ is the generator of a semi-group S_t of contraction,

$$S_t = \exp -t\mathcal{A}\,. \tag{9.5.2}$$

Hence all the results of this theory can be applied.
In particular, we have, for $\mathrm{Re}\,\lambda < 0$

$$||(\mathcal{A}-\lambda)^{-1}|| \leq \frac{1}{|\,\mathrm{Re}\,\lambda|}\,. \tag{9.5.3}$$

One can also show that the operator is maximally accretive.

A very special property of this operator is that, for any $a \in \mathbb{R}$,

$$T_a\mathcal{A} = (\mathcal{A} - ia)T_a\,, \tag{9.5.4}$$

where T_a is the translation operator $(T_a u)(x) = u(x-a)$.
As immediate consequence, we obtain that the spectrum is empty and that the resolvent of $\mathcal{A}$, which is defined for any $\lambda \in \mathbb{C}$ satisfies

$$||(\mathcal{A}-\lambda)^{-1}|| = ||(\mathcal{A}-\;\mathrm{Re}\,\lambda)^{-1}||\,. \tag{9.5.5}$$

The most interesting property is the control of the resolvent for $\mathrm{Re}\,\lambda \geq 0$.

Proposition 9.5.1.
There exist two positive constants C_1 and C_2, such that

$$C_1\,|\;Re\,\lambda|^{-\frac{1}{4}}\exp\frac{4}{3}\;Re\,\lambda^{\frac{3}{2}} \leq ||(\mathcal{A}-\lambda)^{-1}|| \leq C_2\,|\;Re\,\lambda|^{-\frac{1}{4}}\exp\frac{4}{3}\;Re\,\lambda^{\frac{3}{2}}\,, \tag{9.5.6}$$

The proof of the (rather standard) upper bound is based on the direct analysis of the semi-group in the Fourier representation. We note indeed that

$$\mathcal{F}(D_x^2 + i\,x)\mathcal{F}^{-1} = \xi^2 + \frac{d}{d\xi}\,. \tag{9.5.7}$$

Then we have

$$\mathcal{F} S_t \mathcal{F}^{-1} v = \exp(-\xi^2 t + \xi t^2 - \frac{t^3}{3}) v(\xi - t)\,, \tag{9.5.8}$$

and this implies immediately

$$||S_t|| = \exp \max_{\xi} (-\xi^2 t + \xi t^2 - \frac{t^3}{3}) = \exp(-\frac{t^3}{12})\,. \tag{9.5.9}$$

Then one can get an estimate of the resolvent by using, for $\lambda \in \mathbb{C}$, the formula

$$(\mathcal{A} - \lambda)^{-1} = \int_0^{+\infty} \exp -t(\mathcal{A} - \lambda)\, dt\,. \tag{9.5.10}$$

For a closed accretive operator, (9.5.10) is standard when $\operatorname{Re}\lambda < 0$, but estimate (9.5.9) on S_t gives immediately an holomorphic extension of the right hand side to the whole space giving for $\lambda > 0$ the estimate

$$||(\mathcal{A} - \lambda)^{-1}|| \leq \int_0^{+\infty} \exp(\lambda t - \frac{t^3}{12})\, dt\,. \tag{9.5.11}$$

The asymptotic behavior as $\lambda \to +\infty$ of this integral is immediately obtained by using the Laplace method and the dilation $t = \lambda^{\frac{1}{2}} s$ in the integral.

In our case, everything can be computed explicitly. We observe that :

$$\int_0^{+\infty} \exp(-\xi^2 t + \xi t^2 - \frac{t^3}{3}) e^{\lambda t} v(\xi - t) dt = \int_{-\infty}^{\xi} \exp(\frac{s^3 - \xi^3}{3} + \lambda(\xi - s)s)\, v(s)\, ds\,,$$

which gives effectively the expression of $(\mathcal{A} - \lambda)^{-1}$ as an operator with integral kernel given in (9.3.8).

The proof of the lower bound is obtained by constructing quasimodes for the operator $(\mathcal{A} - \lambda)$ in its Fourier representation. We observe (assuming $\lambda > 0$), that

$$\xi \mapsto u(\xi; \lambda) := \exp\left(-\frac{\xi^3}{3} + \lambda\xi - \frac{2}{3}\lambda^{\frac{3}{2}}\right) \tag{9.5.12}$$

is a solution of

$$(\frac{d}{d\xi} + \xi^2 - \lambda) u(\xi; \lambda) = 0\,. \tag{9.5.13}$$

Multiplying $u(\cdot; \lambda)$ by a cut-off function χ_λ with support in $]-\sqrt{\lambda}, +\infty[$ and $\chi_\lambda = 1$ on $]-\sqrt{\lambda}+1, +\infty[$, we obtain a very good quasimode, concentrated as $\lambda \to +\infty$, around $\sqrt{\lambda}$, with an error term giving almost[30] the announced lower bound for the resolvent.

Of course this is a very special case of a result on the pseudo-spectra but this leads to an almost optimal result.

[30]One should indeed improve the cut-off for getting an optimal result

10 Essentially selfadjoint operators

10.1 Introduction

In most of the examples which were presented, the abstract operators are associated with differential operators. These differential operators are naturally defined on $C_0^\infty(\Omega)$ or $\mathcal{D}'(\Omega)$. Most of the time (for suitable potentials increasing slowly at ∞) they are also defined (when $\Omega = \mathbb{R}^m$) on $\mathcal{S}(\mathbb{R}^m)$ or $\mathcal{S}'(\mathbb{R}^m)$. It is important to understand how the abstract point of view can be related to the PDE point of view. The theory of the essential selfadjointness gives a clear understanding of the problem. The question is to decide if, starting from a symmetric operator T, whose domain $D(T) = \mathcal{H}_0$ is dense in $\mathcal{H}$, there exists a unique selfadjoint extension T^{ext} of T. We recall that it means that $D(T) \subset D(T^{ext})$ and $T^{ext}u = Tu$, $\forall u \in D(T)$. This leads to

Definition 10.1.1.
A symmetric operator T with domain $\mathcal{H}_0$ is called essentially selfadjoint if its closure is selfadjoint.

Proposition 10.1.2.
If T is essentially selfadjoint, then its selfadjoint extension[31] is unique.

Indeed suppose that S is a selfadjoint extension of T. Then S is closed and being an extension of T, is also an extension of its smallest extension $\overline{T}$. We recall from Theorem 2.2.6 that $\overline{T} = T^{**}$. Thus, $S = S^* \subset (T^{**})^* = T^{**}$, and so $S = T^{**}$.

Example 10.1.3.
Here we give a list of examples and counter-examples which will be analyzed later.

1. The differential operator $-\Delta$ with domain $\mathcal{H}_0 = C_0^\infty(\mathbb{R}^m)$ is essentially selfadjoint (see later).

2. The differential operators $-\Delta + |x|^2$ with domain $\mathcal{H}_0 = C_0^\infty(\mathbb{R}^m)$ or $H_1 = \mathcal{S}(\mathbb{R}^m)$ are essentially selfadjoint and admit consequently a unique selfadjoint extension. This extension is the same for the two operators. The domain can be explicitely described as

$$B^2(\mathbb{R}^m) = \{u \in L^2(\mathbb{R}^m) \mid x^\alpha D_x^\beta u \in L^2(\mathbb{R}^m) \, , \, \forall \alpha, \beta \in \mathbb{N}^m \text{ with } |\alpha| + |\beta| \leq 2\} \, .$$

3. The differential operator $-\Delta$ with domain $C_0^\infty(\Omega)$ (where Ω is an open bounded set with smooth boundary) is not essentially selfadjoint. There exists a lot of selfadjoint extensions related to the choice of a boundary problem. As we have seen before, we have already met two such extensions :

[31] Although, it will not help in this course, note that the converse is true. See for example the book [Ro], in which the essential selfadjointness is defined differently.

- the Dirichlet realization whose domain is the set $H_0^1(\Omega) \cap H^2(\Omega)$,
- the Neumann realization whose domain is the set : $\{u \in H^2(\Omega) \mid (\partial u/\partial \nu)_{/\partial\Omega} = 0\}$

4. The Laplace Beltrami operator on a compact manifold M with domain $C^\infty(M)$ is essentially selfadjoint on $L^2(M)$. The domain of the selfadjoint extension can be described as $H^2(M)$ (which can be described using local charts).

10.2 Basic criteria.

We now give some criteria in order to verify that an operator is essentially selfadjoint. As already mentioned, one can prove essential selfadjointness by proving that the minimal closed extension $T_{min} := \overline{T}$ coincides with T^*. One easily verifies (see Proposition 2.2.5 in conjonction with the definition of $\overline{T}$) that $T^* = \overline{T}^*$ and we recall that $T^{**} = \overline{T}$.
We now observe the

Proposition 10.2.1.
Any closed symmetric extension of T is a restriction of T^*.

Proof.
Let S is a closed symmetric extension of T.
We have indeed $T \subset S \subset S^*$ and, observing that $S^* \subset T^*$, we consequently get $S \subset T^*$.

In particular, if $\overline{T}$ is selfadjoint then $\overline{T}$ is the unique selfadjoint extension of T.

We can characterize the selfadjointness through the following general criterion

Theorem 10.2.2.
Let T be a closed symmetric operator. Then the following statements are equivalent :

1. *T is selfadjoint.*
2. *Ker $(T^\star \pm i) = \{0\}$;*
3. *Range $(T \pm i) = \mathcal{H}$.*

Proof.
1. implies 2.
This property was already observed (because $T = T^*$ and $\pm i \notin \sigma(T)$).
2. implies 3. .
We first observe that the property that ker $(T^\star + i) = \{0\}$ implies that $R(T - i)$ is dense in $\mathcal{H}$. Note that the converse is also true. For getting 3., it remains

to show that $R(T-i)$ is closed. But, for all ϕ in $D(T)$, we have (using the symmetry of T)

$$||(T-i)\phi||^2 = ||T\phi||^2 + ||\phi||^2 \; . \tag{10.2.1}$$

If ϕ_n is a sequence in $D(T)$ such that $(T+i)\phi_n$ converges to some ψ_∞, then the previous identity shows that ϕ_n is a Cauchy sequence, so there exists ϕ_∞ such that $\phi_n \to \phi_\infty$ in $\mathcal{H}$. But $T\phi_n = (T+i)\phi_n - i\phi_n$ is convergent and using that the graph is closed, we obtain that $\phi_\infty \in D(T)$ and $T\phi_\infty = \psi_\infty - i\phi_\infty$.

3. implies 1. .

Let $\phi \in D(T^*)$. Let $\eta \in D(T)$ such that $(T-i)\eta = (T^*-i)\phi$. T being symmetric, we have also $(T^*-i)(\eta-\phi) = 0$. But, if $(T+i)$ is surjective, then (T^*-i) is injective and we get $\phi = \eta$. This proves that $\phi \in D(T)$.

Remark 10.2.3.

Here we have used and proved during the proof of the assertion "2. implies 3." the following lemma

Lemma 10.2.4.

If T is closed and symmetric, then $R(T \pm i)$ is closed.

This theorem gives as a corollary a criterion for essential selfadjointness in the form

Corollary 10.2.5.

Let A with domain $D(A)$ be a symmetric operator. Then the following are equivalent

1. *A is essentially selfadjoint.*
2. *$Ker(A^* \pm i) = \{0\}$.*
3. *The two spaces $R(A \pm i)$ are dense in $\mathcal{H}$.*

We have indeed essentially to apply the previous theorem to $\overline{A}$, observing in addition that $\overline{A}$ is symmetric and using Lemma 10.2.4.

Let us here emphasize that in this case, to specify the operator A, it is not necessary to give the exact domain of A but a core for A that is a subspace D such that the closure of $A_{/D}$ is $\overline{A}$.

In the same spirit, we have in the semibounded case the following

Theorem 10.2.6.

Let T be a positive, symmetric operator. Then the following statements are equivalent :

1. *T is essentially selfadjoint.*
2. $$Ker\ (T^* + b) = \{0\} \text{ for some } b > 0\; .$$
3. *Range $(T+b)$ is dense for some $b > 0$.*

The proof essentially the same as for the previous corollary, if one observes that if T is positive then the following trivial estimate is a good substitute for (10.2.1) :

$$\langle (T+b)u, u\rangle \geq b||u||^2 \ .$$

Example 10.2.7. *(The free Laplacian)*
The operator $-\Delta$ with domain C_0^∞ is essentially selfadjoint. Its selfadjoint extension is $-\Delta$ with domain H^2.

10.3 The Kato -Rellich theorem

We would like to consider the case when $P = -\Delta + V$ when V is regular and tends to 0 as $|x| \to +\infty$. Here V can be considered as a perturbation of the Laplacian. One can then apply a general theorem due to Kato-Rellich.

Theorem 10.3.1.
Let A be a selfadjoint operator, B be a symmetric operator whose domain contains $D(A)$. Let us assume the existence of a and b such that $0 \leq a < 1$ and $b \geq 0$ such that

$$||Bu|| \leq a\,||Au|| + b\,||u|| \tag{10.3.1}$$

for all $u \in D(A)$. Then $A+B$ is selfadjoint on $D(A)$.
If A is essentially selfadjoint on[32] $D \subset D(A)$, then $A+B$ has the same property.

Proof.

Step 1.
We start from the following identity which only uses that $(A+B)$ with domain $D(A)$ is symmetric.

$$||(A+B\pm i\lambda)u||^2 = ||(A+B)u||^2 + \lambda^2||u||^2 \ , \ \forall u \in D(A) \ . \tag{10.3.2}$$

By the triangle inequality and the symmetry of $A+B$, we get for a real $\lambda > 0$, and for any $u \in D(A)$:

$$\begin{array}{ll} \sqrt{2}||(A+B-i\lambda)u|| & \geq ||(A+B)u|| + \lambda||u|| \\ & \geq ||Au|| - ||Bu|| + \lambda||u|| \\ & \geq (1-a)||Au|| + (\lambda-b)||u||. \end{array} \tag{10.3.3}$$

We now choose $\lambda > b$.

Step 2.
Let us show that $(A+B)$ with domain $D(A)$ is closed. If we start indeed from a pair (u_n, f_n) with $u_n \in D(A)$ and $f_n = (A+B)u_n$ such that $(u_n, f_n) \to (u, f)$ in $\mathcal{H}$. From (10.3.3), we get that Au_n is a Cauchy sequence in $\mathcal{H}$. A being closed, we get $u \in D(A)$ and the existence of g such that $Au_n \to g = Au$.
Now from (10.3.2) and (10.3.1), we get also that Bu_n is a Cauchy sequence and

[32] By this we mean that the closure of $A_{/D}$ is A.

there exists v such that $Bu_n \to v$ in $\mathcal{H}$.
We claim that $Bu = v$. We have indeed, for any $h \in D(A)$,

$$< v, h >_{\mathcal{H}} = \lim_{n \to +\infty} < Bu_n, h >_{\mathcal{H}} = \lim_{n \to +\infty} < u_n, Bh >_{\mathcal{H}} = < u, Bh >_{\mathcal{H}} = < Bu, h >_{\mathcal{H}} .$$

Using the density of $D(A)$, we get $v = Bu$. (We could have also used that B is closable).
We conclude by observing that $(A + B)u = f$ (with $f = g + v$) as expected.

Step 3.
In order to apply Theorem 10.2.2, we have to show that $(A+B\pm i\lambda)$ is surjective. The main element in the proof is the following

Lemma 10.3.2. .
For $\lambda > 0$ large enough, we have

$$||B(A \pm i\lambda)^{-1}|| < 1 . \tag{10.3.4}$$

Proof.
We observe that, for $u \in D(A)$,

$$||(A \pm i\lambda)u||^2 = ||Au||^2 + \lambda^2 ||u||^2 . \tag{10.3.5}$$

For $u \in D(A)$, we have, using two times (10.3.5) and then (10.3.1)

$$\begin{aligned} ||Bu|| &\leq a||Au|| + b||u|| \\ &\leq a||(A + i\lambda)u|| + \tfrac{b}{\lambda}||(A + i\lambda)u|| \\ &\leq (a + \tfrac{b}{\lambda})||(A + i\lambda)u|| . \end{aligned} \tag{10.3.6}$$

It is then enough to choose $\lambda > 0$ large enough such that

$$(a + \frac{b}{\lambda}) < 1 .$$

Writing

$$A + B - i\lambda = [I + B(A - i\lambda)^{-1}](A - i\lambda) , \tag{10.3.7}$$

it is easy to deduce the surjectivity using the lemma and the surjectivity of $(A - i\lambda)$.

Application.

As an application, let us treat the case of the Schrödinger operator with Coulomb potential.

Proposition 10.3.3.
The operator $-\Delta - \frac{1}{|x|}$ with domain $C_0^\infty(\mathbb{R}^3)$ is essentially selfadjoint.

We recall that the operator is well defined because $\frac{1}{r}$ belongs to $L^2_{loc}(\mathbb{R}^3)$.
We first observe a Sobolev type inequality.

Lemma 10.3.4. *:*
There exists a constant C such that for all $u \in H^2(\mathbb{R}^3)$, all $a > 0$ and all $x \in \mathbb{R}^3$, we have

$$|u(x)| \leq C(a||\Delta u||_0 + a^{-3}||u||_0) \; .$$

(Prove first the inequality[33] for all u with $x = 0$ and $a = 1$, then use translation and dilation.)
In the second step, we show that the potential $V = -\frac{1}{r}$ is a perturbation of the Laplacian.
There exists indeed a constant C such that for all $u \in H^2(\mathbb{R}^3)$ and all $b > 0$, we have

$$||Vu||_0 \leq C(b||\Delta u||_0 + b^{-3}||u||_0) \; .$$

For this proof we observe that, for any $R > 0$,

$$\int V(x)^2|u(x)|^2dx = \int_{|x|\leq R} V(x)^2|u(x)|^2dx + \int_{|x|\geq R} V(x)^2|u(x)|^2dx \; ,$$

and treat the first term of the right hand side using the Sobolev's type inequality by

$$\int_{|x|\leq R} V(x)^2|u(x)|^2dx \leq (\sup_{|x|\leq R} |u(x)|^2 \, dx) \int_{|x|\leq R} V(x)^2 \, dx \, ,$$

and the second term by the trivial estimate

$$\int_{|x|\geq R} V(x)^2|u(x)|^2dx \leq (\sup_{|x|\geq R} |V(x)|)^2 \int_{|x|\geq R} |u(x)|^2 \, dx \; .$$

Using the Sobolev inequality (actually it is enough to take $a = 1$), we finally obtain :

$$||Vu||^2 \leq CR(||\Delta u||^2 + ||u||^2) + \frac{1}{R^2}||u||^2 \, . \tag{10.3.8}$$

We obtained the expected estimate by considering R small enough.

Remark 10.3.5.
We note that the same proof shows that $-\Delta+V$ is essentially selfadjoint starting from $C_0^\infty(\mathbb{R}^m)$ if $V \in L^2+L^\infty$, that is if $V = V_1+V_2$ with $V_1 \in L^2$ and $V_2 \in L^\infty$.

10.4 Other criteria of selfadjointness for Schrödinger operators

We present in this subsection two criteria which are specific of the Schrödinger case. The first one seems due to Rellich (See [Sima]) and we present it in the easy case when the potential is regular. The second one permits to treat singular potentials and is due to Kato (cf [HiSi] or [Ro]).

The first theorem is adapted to operators which are already know to be positive on $C_0^\infty(\mathbb{R}^m)$.

[33] In this case, this is just the Sobolev's injection theorem of $H^2(\mathbb{R}^3)$ into $C_b^0(\mathbb{R}^3)$, where $C_b^0(\mathbb{R}^3)$ is the space of the continuous bounded functions.

Theorem 10.4.1.
A Schrödinger operator $T = -\Delta + V$ on $\mathbb{R}^n$ associated with a C^0 potential V, which is semibounded on $C_0^\infty(\mathbb{R}^n)$, is essentially selfadjoint. In other words, the Friedrichs extension is the unique selfadjoint extension starting from $C_0^\infty(\mathbb{R}^m)$.

This theorem is complementary to the second theorem (Theorem 10.4.4) which will be stated at the end of this subsection because we do not have to assume the positivity of the potential but only the semi-boundedness of the operator T.

Proof.
Let T be our operator. Possibly by adding a constant, we can assume that

$$< Tu, u >_{\mathcal{H}} \geq ||u||^2 \ , \ \forall u \in C_0^\infty(\mathbb{R}^m) \ . \tag{10.4.1}$$

Of course this inequality can be rewritten in the form :

$$||\nabla u||^2 + \int_{\mathbb{R}^m} V(x)|u(x)|^2 dx \geq ||u||^2 \ , \ \forall u \in C_0^\infty(\mathbb{R}^m) \ .$$

In this form, the inequality can be extended to the elements of $H^1_{comp}(\mathbb{R}^m)$, corresponding to the distributions of $H^1(\mathbb{R}^m)$ with compact support :

$$||\nabla u||^2 + \int_{\mathbb{R}^m} V(x)|u(x)|^2 dx \geq ||u||^2 \ , \ \forall u \in H^1_{comp}(\mathbb{R}^m) \ . \tag{10.4.2}$$

According to the general criterion of essential selfadjointness (cf Theorem 10.2.6), it is enough to verify that $R(T)$ is dense. Let us show this property.
Let $f \in L^2(\mathbb{R}^m)$, such that

$$< f, Tu >_{\mathcal{H}} = 0 \ , \ \forall u \in C_0^\infty(\mathbb{R}^m) \ . \tag{10.4.3}$$

We have to show that $f = 0$.
Because T is real, one can assume that f is real.
We first observe that (10.4.3) implies that : $(-\Delta + V)f = 0$ in $\mathcal{D}'(\mathbb{R}^m)$. A standard regularity theorem for the Laplacian[34] implies that $f \in H^2_{loc}(\mathbb{R}^m)$.
We now introduce a family of cutoff functions ζ_k by

$$\zeta_k := \zeta(x/k) \ , \ \forall k \in \mathbb{N} \ , \tag{10.4.4}$$

where ζ is a C^∞ function satisfying $0 \leq \zeta \leq 1$, $\zeta = 1$ on $B(0,1)$ and $\operatorname{supp} \zeta \subset B(0,2)$.
For any $u \in C^\infty$ and any $f \in H^2_{loc}$, we have the identity

$$\begin{aligned} &\int \nabla(\zeta_k f) \cdot \nabla(\zeta_k u) dx + \int \zeta_k(x)^2 V(x) u(x)\, f(x)\, dx \\ &\quad = \int |(\nabla \zeta_k)(x)|^2 u(x) f(x)\, dx + \sum_{i=1}^m \int \left(f(\partial_i u) - u(\partial_i f)\right)(x) \zeta_k(x) (\partial_i \zeta_k)(x)\, dx \\ &\quad + \langle f(x) \ , \ T\zeta_k^2 u \rangle \ . \end{aligned} \tag{10.4.5}$$

[34] This is the property that $f \in L^2_{loc}(\mathbb{R}^m)$, $\Delta f \in H^{-1}_{loc}(\mathbb{R}^m)$ implies that $f \in H^1_{loc}(\mathbb{R}^m)$ together with property that $f \in L^2_{loc}(\mathbb{R}^m)$, $\Delta f \in L^2_{loc}(\mathbb{R}^m)$ implies that $f \in H^2_{loc}(\mathbb{R}^m)$.

When f satisfies (10.4.3), we get :

$$\begin{aligned}&\int_{\mathbb{R}^m} \nabla(\zeta_k f)\cdot\nabla(\zeta_k u)dx + \int \zeta_k(x)^2 V(x)u(x)\,f(x)\,dx\\ &\quad = \int |(\nabla\zeta_k)(x)|^2 u(x)f(x)\,dx + \sum_{i=1}^m \int \left(f(\partial_i u) - u(\partial_i f)\right)(x)\zeta_k(x)(\partial_i\zeta_k)(x)\,dx\ ,\end{aligned} \tag{10.4.6}$$

for all $u \in C^\infty(\mathbb{R}^m)$.
This formula can be extended to functions $u \in H^1_{loc}$. In particular, we can take $u = f$.
We obtain

$$< \nabla(\zeta_k f)\ ,\ \nabla(\zeta_k f) > + \int \zeta_k^2 V(x)|f(x)|^2\,dx = \int |\nabla\zeta_k|^2|f(x)|^2 dx\ . \tag{10.4.7}$$

Using (10.4.1), (10.4.7) and taking the limit $k \to +\infty$, we get

$$\begin{aligned}||f||^2 &= \lim_{k\to+\infty} ||\zeta_k f||^2\\ &\leq \limsup_{k\to+\infty}\left(< \nabla(\zeta_k f)\ ,\ \nabla(\zeta_k f) > + \int \zeta_k^2 V(x)|f(x)|^2\,dx\right)\\ &= \limsup_{k\to+\infty}\int f(x)^2|(\nabla\zeta_k)(x)|^2\,dx = 0\ .\end{aligned} \tag{10.4.8}$$

This proves the theorem.

Remark 10.4.2.
When V is C^∞, we get, in the previous proof, that $f \in C^\infty$ and we immediately can prove (10.4.7) without going through the previous discussion.

Example 10.4.3.

- *If $V \geq 0$ and C^∞, then the Schrödinger operator $-\Delta + V$ with domain $C_0^\infty(\mathbb{R}^m)$ is essentially selfadjoint. The operator $-\Delta+V$ is indeed positive.*
- *If ϕ is C^∞ on $\mathbb{R}^m$, then the operators $-\Delta + |\nabla\phi|^2 \pm \Delta\phi$ are essentially selfadjoint. They are indeed positive on $C_0^\infty(\mathbb{R}^m)$. They can actually be written in the form $\sum_j Z_j^* Z_j$ with $Z_j = \partial_{x_j} \mp \partial_{x_j}\phi$. These operators appear naturally in statistical mechanics.*

Let us now mention, without proof, a quite general theorem due to Kato (See for example [Ro]).

Theorem 10.4.4.
Let V in $L^2_{loc}(\mathbb{R}^m)$ such that $V \geq 0$ almost everywhere on $\mathbb{R}^m$. Then $-\Delta + V$ with domain $C_0^\infty(\mathbb{R}^m)$ is essentially selfadjoint.

This theorem is based on the so called Kato's inequality.

Remark 10.4.5.
This last theorem may be extended to the case of the Schrödinger operator with magnetic regular potential A (See Subsection 7.5).

11 Non-selfadjoint case : Maximal accretivity and application to the Fokker-Planck operator

11.1 Accretive operators

We collect here some material on accretive operators. The references could be the books by Dautray-Lions (Vol. 5, Chapter XVII), Reed-Simon or the book of B. Davies. Let $\mathcal{H}$ be a complex (or real) Hilbert space.

Definition 11.1.1.
Let A be an unbounded operator in $\mathcal{H}$ with domain $D(A)$. We say that A is accretive if

$$Re\,\langle Ax \mid x\rangle_{\mathcal{H}} \geq 0\ ,\ \forall x \in D(A)\ . \tag{11.1.1}$$

Definition 11.1.2.
An accretive operator A is maximally accretive if it does not exist an accretive extension $\tilde{A}$ with strict inclusion of $D(A)$ in $D(\tilde{A})$.

Proposition 11.1.3.
Let A be an accretive operator with domain $D(A)$ dense in $\mathcal{H}$. Then A is closable and its closed extension $\overline{A}$ is accretive.

For the analysis of the Fokker-Planck operator, the following criterion, which extends the standard criterion of essential self-adjointness, will be the most suitable

Theorem 11.1.4.
For an accretive operator A, the following conditions are equivalent

1. *$\overline{A}$ is maximally accretive.*
2. *There exists $\lambda_0 > 0$ such that $A^* + \lambda_0 I$ is injective.*
3. *There exists $\lambda_1 > 0$ such that the range of $A + \lambda_1 I$ is dense in $\mathcal{H}$.*

Note that in this case $-\overline{A}$ is the infinitesimal generator of a contraction semi-group.

11.2 Application to the Fokker-Planck operator

We would like to show

Proposition 11.2.1.
Let V be a C^∞ potential on $\mathbb{R}^n$, then the closure $\overline{K}$ of the Fokker-Planck operator defined on $C_0^\infty(\mathbb{R}^{2n})$ by

$$K := -\Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2} + X_0\ , \tag{11.2.1}$$

where

$$X_0 := -\nabla V(x)\cdot\partial_v + v\cdot\partial_x \tag{11.2.2}$$

is maximally accretive.
Moreover K^* is also maximally accretive.

The idea is to adapt the proof that a semi-bounded Schrödinger operator with regular potential is essentially self-adjoint on $L^2(\mathbb{R}^n)$.

Proof:
We apply the abstract criterion taking $\mathcal{H} = L^2(\mathbb{R}^n\times\mathbb{R}^n)$ and $A = K$. The operators being real, we can consider everywhere real functions. The accretivity on $C_0^\infty(\mathbb{R}^{2n})$ is clear. We can then consider the closure $\overline{K}$.

Changing K in $T := K + (\frac{n}{2}+1)I$, we would like to show that its range is dense.
Let $f\in L^2(\mathbb{R}^m)$, with $m = 2n$, such that

$$< f \mid Tu >_{\mathcal{H}} = 0\ ,\ \forall u\in C_0^\infty(\mathbb{R}^m)\ . \tag{11.2.3}$$

We have to show that $f = 0$.
Because K is real, one can assume that f is real.
We first observe that (11.2.3) implies that :

$$(-\Delta_v + v^2/4 + 1 - X_0)f = 0\ ,\quad \text{in } \mathcal{D}'(\mathbb{R}^m)\ .$$

The standard hypoellipticity theorem for the Hörmander operators[35] of type 2 implies that $f\in C^\infty(\mathbb{R}^m)$.
We now introduce a family of cut-off functions $\zeta_k := \zeta_{k_1,k_2}$ by

$$\zeta_{k_1,k_2}(x,v) := \zeta(x/k_1)\zeta(v/k_2)\ ,\ \forall k\in\mathbb{N}^2\ , \tag{11.2.4}$$

where ζ is a C^∞ function satisfying $0\le\zeta\le 1$, $\zeta = 1$ on $B(0,1)$ and $\operatorname{supp}\zeta\subset B(0,2)$.
For any $u\in C_0^\infty$, we have the identity

$$\begin{array}{l}\int \nabla_v(\zeta_k f)\cdot\nabla_v(\zeta_k u)\ dxdv + \int \zeta_k(x,v)^2(v^2/4+1)u(x,v)\, f(x,v)\ dx\, dv\\ \quad + \int f(x,v)(X_0(\zeta_k^2 u))(x,v)\ dx\, dv\\ \qquad = \int |(\nabla_v\zeta_k)(x,v)|^2 u(x,v) f(x,v)\ dx\, dv\\ \qquad\quad + \sum_{i=1}^m \int \left(f(\partial_{v_i}u) - u(\partial_{v_i}f)\right)(x,v)\zeta_k(x,v)(\partial_{v_i}\zeta_k)(x,v)\ dx\, dv\\ \qquad\quad + \langle f(x,v)\mid T\zeta_k^2 u\rangle\ .\end{array} \tag{11.2.5}$$

When f satisfies (11.2.3), we get :

$$\begin{array}{l}\int_{\mathbb{R}^m} \nabla_v(\zeta_k f)\cdot\nabla_v(\zeta_k u)\ dxdv + \int \zeta_k^2(v^2/4+1)u(x,v)\, f(x,v)\ dx\, dv\\ \quad + \int f(x,v)(X_0(\zeta_k^2 u))(x,v) dx\, dv\\ \qquad = \int |(\nabla_y\zeta_k)(x)|^2 u(x) f(x,v)\ dx\, dv\\ \qquad\quad + \sum_{i=1}^m \int \left(f(\partial_{v_i}u) - u(\partial_{v_i}f)\right)(x,v)\zeta_k(x,v)(\partial_{v_i}\zeta_k)(x,v)\ dx\, dv\ ,\end{array} \tag{11.2.6}$$

[35] These operators are in the form $P = -\sum_{j=1}^k X_j^2 + X_0 + a(x)$, where the X_j are real vectorfield. If the X_j together with the brackets $[X_\ell, X_m]$ span at each point x the whole tangent space, then one can show that the corresponding operator is hypoelliptic. P is said hypoelliptic if for any $u\in\mathcal{D}'$ and any open set ω, $Pu\in C^\infty(\omega)$ implies that $u\in C^\infty(\omega)$.

for all $u \in C^\infty(\mathbb{R}^m)$. In particular, we can take $u = f$.
We obtain

$$\begin{aligned}
&< \nabla_v(\zeta_k f) \mid \nabla_v(\zeta_k f) > + \int \zeta_k^2 (v^2/4+1)|f(x,v)|^2\, dx\, dv \\
&+ \int f(x,v)(X_0(\zeta_k^2 f))(x,v)\, dx\, dv \\
&= \int |\nabla_v \zeta_k|^2 |f(x,v)|^2\, dx\, dv\,.
\end{aligned} \tag{11.2.7}$$

With an additional integration by part, we get

$$\begin{aligned}
&< \nabla_v(\zeta_k f) \mid \nabla_v(\zeta_k f) > + \int \zeta_k^2 (v^2/4+1)|f(x,v)|^2\, dx\, dv \\
&+ \int \zeta_k f(x,v)^2 (X_0 \zeta_k)(x,v)\, dx\, dv \\
&= \int |\nabla_v \zeta_k|^2 |f(x,v)|^2\, dx\, dv\,.
\end{aligned} \tag{11.2.8}$$

This leads to the existence of a constant C such that, for all k,

$$\begin{aligned}
&||\zeta_k f||^2 + \tfrac{1}{4}||\zeta_k v f||^2 \\
&\leq C\tfrac{1}{k_2^2}||f||^2 + C\tfrac{1}{k_1}||v\zeta_k f||\,||f|| + C\tfrac{1}{k_2}||\nabla V(x)\zeta_k f||\,||f||\,.
\end{aligned} \tag{11.2.9}$$

(The constant C will possibly be changed from line to line). This leads to

$$||\zeta_k\, f||^2 + \frac{1}{8}||\zeta_k\, v\, f||^2 \leq C(\frac{1}{k_2^2} + \frac{1}{k_1^2})||f||^2 + C(k_1)\frac{1}{k_2}||\zeta_k f||\,||f||\,, \tag{11.2.10}$$

where

$$C(k_1) = \sup_{|x|\leq 2k_1} |\nabla_x V(x)|$$

This implies

$$||\zeta_k\, f||^2 \leq C(\frac{\tilde{C}(k_1)}{k_2^2} + \frac{1}{k_1^2})||f||^2\,. \tag{11.2.11}$$

This finally leads to $f = 0$. For example, one can take first the limit $k_2 \to +\infty$, which leads to

$$||\zeta(\frac{x}{k_1})f||^2 \leq \frac{C}{k_1^2}||f||^2\,,$$

and then the limit $k_1 \to +\infty$.

11.3 Decay of the semi-group and ϵ-pseudospectra

We recall that for any $\epsilon > 0$, we define the ϵ-pseudospectra by

$$\Sigma_\epsilon(\mathcal{A}^D) = \{\lambda \in \mathbb{C} \,|\, ||(\mathcal{A}^D - \lambda)^{-1}|| > \frac{1}{\epsilon}\}\,, \tag{11.3.1}$$

with the convention that $||(\mathcal{A}^D - \lambda)^{-1}|| = +\infty$ if $\lambda \in \sigma(\mathcal{A}^D)$.
We have

$$\cap_{\epsilon>0}\Sigma_\epsilon(\mathcal{A}^D) = \sigma(\mathcal{A}^D)\,. \tag{11.3.2}$$

We define, for any accretive closed operator, for $\epsilon > 0$,

$$\widehat{\alpha}_\epsilon(\mathcal{A}) = \inf_{z \in \Sigma_\epsilon(\mathcal{A})} \text{Re}\, z\,. \tag{11.3.3}$$

We also define

$$\widehat{\omega}_0(\mathcal{A}) = \lim_{t \to +\infty} \frac{1}{t} \log || \exp -t\mathcal{A}|| \tag{11.3.4}$$

$$\widehat{\alpha}_\epsilon(\mathcal{A}) \leq \inf_{z \in \sigma(\mathcal{A})} \text{Re}\, z\,. \tag{11.3.5}$$

Theorem 11.3.1 (Gearhart-Prüss)**.**
Let $\mathcal{A}$ be a densely defined closed operator in an Hilbert space X such that $-\mathcal{A}$ generates a contraction semi-group and let $\widehat{\alpha}_\epsilon(\mathcal{A})$ and $\widehat{\omega}_0(\mathcal{A})$ denote the ϵ-pseudospectral abcissa and the growth bound of $\mathcal{A}$ respectively. Then

$$\lim_{\epsilon \to 0} \widehat{\alpha}_\epsilon(\mathcal{A}) = -\widehat{\omega}_0(\mathcal{A})\,. \tag{11.3.6}$$

We refer to Engel-Nagel for a proof.

This theorem is interesting because it reduces the question of the decay, which is basic in the question of the stability to an analysis of the ϵ-spectra of the operator.

11.4 Application : The complex Airy operator in $\mathbb{R}^+$

Here we mainly describe some results presented in Almog (article in Siam). We can then associate the Dirichlet realization $\mathcal{A}^D$ of the complex Airy operator $D_x^2 + ix$ on the half-line, whose domain is

$$D(\mathcal{A}^D) = \{u \in H_0^1(\mathbb{R}^+), x^{\frac{1}{2}} u \in L^2\,,\, (D_x^2 + i\,x)u \in L^2(\mathbb{R}^+)\}\,, \tag{11.4.1}$$

and which is defined (in the sense of distributions) by

$$\mathcal{A}^D u = (D_x^2 + i\,x)u\,. \tag{11.4.2}$$

Moreover, by construction, we have

$$\text{Re}\, \langle \mathcal{A}^D u \,|\, u\rangle \geq 0\,,\, \forall u \in D(\mathcal{A}^D)\,. \tag{11.4.3}$$

Again we have an operator, which is the generator of a semi-group of contraction, whose adjoint is described by replacing in the previous description $(D_x^2 + i\,x)$ by $(D_x^2 - i\,x)$, the operator is injective and as its spectrum contained in $\text{Re}\,\lambda > 0$. Moreover, the operator has compact inverse, hence the spectrum (if any) is discrete.

Using what is known on the usual Airy operator, Sibuya's theory and a complex rotation (see alternately what we said for the non self-adjoint harmonic oscillator), we obtain that the spectrum of $\mathcal{A}^D$ $\sigma(\mathcal{A}^D)$ is given by that

$$\sigma(\mathcal{A}^D) = \cup_{j=1}^{+\infty} \{\lambda_j\} \tag{11.4.4}$$

with

$$\lambda_j = \exp i\frac{\pi}{3}\, \mu_j \,, \tag{11.4.5}$$

the μ_j's being real zeroes of the Airy function satisfying

$$0 < \mu_1 < \cdots < \mu_j < \mu_{j+1} < \cdots \,. \tag{11.4.6}$$

It is also shown in Almog that the vector space generated by the corresponding eigenfunctions is dense in $L^2(\mathbb{R}^+)$.

We arrive now to the analysis of the properties of the semi-group and the estimate of the resolvent.
As before, we have, for $\mathrm{Re}\,\lambda < 0$,

$$||(\mathcal{A}^D - \lambda)^{-1}|| \leq \frac{1}{|\,\mathrm{Re}\,\lambda|}\,, \tag{11.4.7}$$

If $\mathrm{Im}\,\lambda < 0$ one gets also a similar inequality, so the main remaining question is the analysis of the resolvent in the set $\mathrm{Re}\,\lambda \geq 0\,,\ \mathrm{Im}\,\lambda \geq 0$, which corresponds to the numerical range of the symbol.

We recall that for any $\epsilon > 0$, we define the ϵ-pseudospectra by

$$\Sigma_\epsilon(\mathcal{A}^D) = \{\lambda \in \mathbb{C}\,|\,||(\mathcal{A}^D - \lambda)^{-1}|| > \frac{1}{\epsilon}\}\,, \tag{11.4.8}$$

with the convention that $||(\mathcal{A}^D - \lambda)^{-1}|| = +\infty$ if $\lambda \in \sigma(\mathcal{A}^D)$.
We have

$$\cap_{\epsilon>0}\Sigma_\epsilon(\mathcal{A}^D) = \sigma(\mathcal{A}^D)\,. \tag{11.4.9}$$

We define, for any accretive closed operator, for $\epsilon > 0$,

$$\widehat{\alpha}_\epsilon(\mathcal{A}) = \inf_{z\in\Sigma_\epsilon(\mathcal{A})} \mathrm{Re}\, z\,. \tag{11.4.10}$$

We also define

$$\widehat{\omega}_0(\mathcal{A}) = \lim_{t\to+\infty} \frac{1}{t}\log||\exp -t\mathcal{A}|| \tag{11.4.11}$$

$$\widehat{\alpha}_\epsilon(\mathcal{A}) \leq \inf_{z\in\sigma(\mathcal{A})} \mathrm{Re}\, z\,. \tag{11.4.12}$$

We apply Gearhart-Prüss theorem to our operator $\mathcal{A}_D$ and our main theorem is

Theorem 11.4.1.

$$\widehat{\omega}_0(\mathcal{A}_D) = -\ \mathit{Re}\,\lambda_1\,. \tag{11.4.13}$$

Using the first eigenfunction it is easy to see that

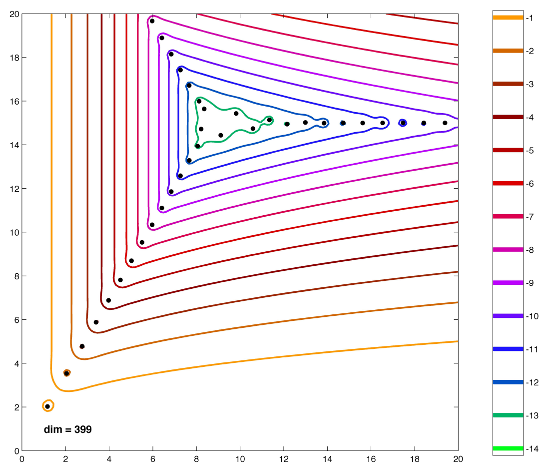
$$||\exp -t\mathcal{A}^D|| \geq \exp -\ \mathrm{Re}\,\lambda_1\, t\,. \tag{11.4.14}$$

Hence we have immediately

$$0 \geq \widehat{\omega}_0(\mathcal{A}^D) \geq -\ \mathrm{Re}\,\lambda_1\,. \tag{11.4.15}$$

To prove that $-\ \mathrm{Re}\,\lambda_1 \geq \widehat{\omega}_0(\mathcal{A}^D)$, it is enough to show the following lemma.

Figure 2: Complex Airy operator on the halfline : pseudospectra



Lemma 11.4.2.
For any $\alpha <$ Re λ_1, there exists a constant C such that, for all λ s.t. Re $\lambda \leq \alpha$

$$||(\mathcal{A}^D - \lambda)^{-1}|| \leq C\,. \tag{11.4.16}$$

Proof : We know that λ is not in the spectrum. Hence the problem is just a control of the resolvent as $|\text{ Im }\lambda| \to +\infty$. The case, when Im $\lambda < 0$ has already be considered. Hence it remains to control the norm of the resolvent as Im $\lambda \to +\infty$ and Re $\lambda \in [-\alpha, +\alpha]$.

This is indeed a semi-classical result ! The main idea is that when Im $\lambda \to +\infty$, we have to inverse the operator

$$D_x^2 + i(x - \text{ Im }\lambda) - \text{ Re }\lambda\,.$$

If we consider the Dirichlet realization in the interval $]0, \frac{\text{Im}\,\lambda}{2}[$ of $D_x^2 + i(x - \text{ Im }\lambda) - \text{ Re }\lambda$, it is easy to see that the operator is invertible by considering the imaginary part of this operator and that this inverse $R_1(\lambda)$ satisfies

$$||R_1(\lambda)|| \leq \frac{2}{\text{Im }\lambda}.$$

Far from the boundary, we can use the resolvent of the problem on the line for which we have a uniform control of the norm for Re $\lambda \in [-\alpha, +\alpha]$.

12 Discrete spectrum, essential spectrum

12.1 Discrete spectrum

We have already recalled in Proposition 8.5.4 a characterization of the spectrum. Let us now complete this characterization by introducing different spectra.

Definition 12.1.1.
If T is a selfadjoint operator, we shall call discrete spectrum of T the set

$$\sigma_{disc}(T) = \{\lambda \in \sigma(T) \ s.\ t.\ \exists \epsilon > 0, \dim range\,(E(]\lambda - \epsilon, \lambda + \epsilon[)) < +\infty\}\ .$$

With this new definition, we can say that, for a selfadjoint operator with compact resolvent, the spectrum is reduced to the discrete spectrum.
For a compact selfadjoint operator, the spectrum is discrete outside 0. We see in this case that the discrete spectrum is not closed.
Equivalently, let us observe now give another characterization :

Proposition 12.1.2.
Let T be a selfadjoint operator. A real λ is in the discrete spectrum if and only if :
λ is an isolated point in $\sigma(T)$ and if λ is an eigenvalue of finite multiplicity[36].

Proof.
If $\lambda \in \sigma_{disc}(T)$, we immediately see that there exists ϵ_0 such that, $\forall \epsilon$ such that $0 < \epsilon < \epsilon_0$, $E_{]\lambda-\epsilon,\lambda+\epsilon[}$ becomes a projector independent of ϵ with finite range. This is actually the projector $\Pi_\lambda = 1_{\{\lambda\}}(T)$ and we observe moreover $E_{]\lambda,\lambda+\epsilon_0[} = 0$ and $E_{]\lambda-\epsilon_0,\lambda[} = 0$. This shows that λ is an isolated point in $\sigma(T)$. Using the spectral representation of T, one immediately get that, if $x = \Pi_\lambda x$ ($x \neq 0$), then x is an eigenfunction of T. Moreover, one easily obtains that $(T - \lambda)$ is invertible on $R(I - \Pi_\lambda)$. One can indeed find a continuous bounded f such that $f(T)(T - \lambda)(I - \Pi_\lambda) = (I - \Pi_\lambda)$.

Conversely, let λ be isolated. The previous proof as already shown that in this case the range of Π_λ is an eigenspace. The assumption of finite multiplicity permits then to conclude.

Remark 12.1.3.
The discrete spectrum is not a closed set ! If we consider in $\mathbb{R}^3$, the Schrödinger operator with coulomb potential, the discrete spectrum is a sequence of eigenvalues tending to 0 but 0 does not belong to the discrete spectrum.

12.2 Essential spectrum

Definition 12.2.1.
The essential spectrum is the complementary in the spectrum of the discrete spectrum.

[36] that is with corresponding eigenspace of finite dimension.

Intuitively, a point of the essential spectrum corresponds

- either to a point in the continuous spectrum,
- or to a limit point of a sequence of eigenvalues with finite multiplicity,
- or to an eigenvalue of infinite multiplicity.

The discrete spectrum being composed of isolated points, we get

Proposition 12.2.2.
The essential spectrum of a selfadjoint operator T is closed in $\mathbb{R}$.

12.3 Basic examples:

1. The essential spectrum of a compact selfadjoint operator is reduced to 0.
2. The essential spectrum of an operator with compact resolvent is empty.
3. The Laplacian on $\mathbb{R}^n$ $-\Delta$ is a selfadjoint operator on $L^2(\mathbb{R}^n)$ whose domain is the Sobolev space $H^2(\mathbb{R}^n)$. The spectrum is continuous and equal to $\overline{\mathbb{R}^+}$. The essential spectrum is also $\overline{\mathbb{R}^+}$ and the operator has no discrete spectrum.
4. The Schrödinger operator with constant magnetic field ($B \neq 0$) in $\mathbb{R}^2$:
$$S_B := (D_{x_1} - \frac{Bx_2}{2})^2 + (D_{x_2} + \frac{Bx_1}{2})^2 \,, \tag{12.3.1}$$
with
$$D_{x_j} := \frac{1}{i}\partial_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j} \,.$$
The spectrum is formed with eigenvalues $(2k+1)|B|$ but the spectrum is not discrete because each eigenvalue is with infinite multiplicity. This will be treated in the next subsection.

12.4 On the Schrödinger Operator with constant magnetic field

The Schrödinger operator with magnetic field has been briefly introduced in Subsection 7.5. We have in particularly given examples where this operator was with compact resolvent. In this section, we analyze more in detail the properties of the Schrödinger operator with constant magnetic field in dimension 2 and 3. This appears to play an important role in Superconductivity theory.

12.4.1 Dimension 2

We would like to analyze the spectrum of :
$$S_B := (D_{x_1} - \frac{B}{2}x_2)^2 + (D_{x_2} + \frac{B}{2}x_1)^2 \,. \tag{12.4.1}$$

12.4.2 The case of $\mathbb{R}^2$

We first look at the selfadjoint realization in $\mathbb{R}^2$. Let us show briefly, how one can analyze its spectrum. We leave as an exercise to show that the spectrum (or the discrete spectrum) of two selfadjoints operators S and T are the same if there exists a unitary operator U such that $U(S \pm i)^{-1}U^{-1} = (T \pm i)^{-1}$. We note that this implies that U sends the domain of S onto the domain of T.
In order to determine the spectrum of the operator S_B, we perform a succession of unitary conjugations. The first one is called a gauge transformation. We introduce U_1 on $L^2(\mathbb{R}^2)$ defined, for $f \in L^2$ by

$$U_1 f = \exp iB \frac{x_1 x_2}{2} f \ . \tag{12.4.2}$$

It satisfies

$$S_B U_1 f = U_1 S_B^1 f \ , \ \forall f \in \mathcal{S}(\mathbb{R}^2) \ , \tag{12.4.3}$$

with

$$S_B^1 := (D_{x_1})^2 + (D_{x_2} + Bx_1)^2 \ . \tag{12.4.4}$$

Remark 12.4.1. .
U_1 is a very special case of what is called a gauge transform. More generally, we can consider $U = \exp i\phi$ where ϕ is C^∞. If $\Delta_A := \sum_j (D_{x_j} - A_j)^2$ is a general Schrödinger operator associated with the magnetic potential A, then $U^{-1}\Delta_A U = \Delta_{\tilde{A}}$ where $\tilde{A} = A + \operatorname{grad}\phi$. Here we observe that $B := \operatorname{rot} A = \operatorname{rot} \tilde{A}$. The associated magnetic field is unchanged in a gauge transformation. We are discussing in our example the very special case (but important!) when the magnetic potential is constant.

We have now to analyze the spectrum of S_B^1.
Observing that the operator is with constant coefficients with respect to the x_2-variable, we perform a partial Fourier transform with respect to the x_2 variable

$$U_2 = \mathcal{F}_{x_2 \mapsto \xi_2} \ , \tag{12.4.5}$$

and get by conjugation, on $L^2(\mathbb{R}^2_{x_1,\xi_2})$,

$$S_B^2 := (D_{x_1})^2 + (\xi_2 + Bx_1)^2 \ . \tag{12.4.6}$$

We now introduce a third unitary transform U_3

$$(U_3 f)(y_1, \xi_2) = f(x_1, \xi_2) \ , \quad \text{with } y_1 = x_1 + \frac{\xi_2}{B} \ , \tag{12.4.7}$$

and we obtain the operator

$$S_B^3 := D_y^2 + B^2 y^2 \ , \tag{12.4.8}$$

operating on $L^2(\mathbb{R}^2_{y,\xi_2})$.
The operator depends only on the y variable. It is easy to find for this operator

an orthonormal basis of eigenvectors. We observe indeed that if $f \in L^2(\mathbb{R}_{\xi_2})$, and if ϕ_n is the $(n+1)$-th eigenfunction of the harmonic oscillator, then

$$(x, \xi_2) \mapsto |B|^{\frac{1}{4}} f(\xi_2) \cdot \phi_n(|B|^{\frac{1}{2}} y)$$

is an eigenvector corresponding to the eigenvalue $(2n+1)|B|$. So each eigenspace has an infinite dimension. An orthonormal basis of this eigenspace can be given by the vectors $e_j(\xi_2)|B|^{\frac{1}{4}} \ \phi_n(|B|^{\frac{1}{2}} y)$ where e_j $(j \in \mathbb{N})$ is a basis of $L^2(\mathbb{R})$.
We have consequently an empty discrete spectrum. The eigenvalues are usually called Landau levels.

12.4.3 Magnetic Schrödinger operators in dimension 3

We only consider the Schrödinger operator with constant magnetic field in $\mathbb{R}^3$. After some rotation in $\mathbb{R}^3$, we arrive to the model :

$$P(h, \vec{b}) = D_{x_1}^2 + (D_{x_2} - bx_1)^2 + D_{x_3}^2 \ , \tag{12.4.9}$$

with

$$b = ||B|| \ .$$

This time, we can take the partial Fourier transform, with respect to x_2 and x_3 in order to get the operator

$$D_{x_1}^2 + (\xi_2 - bx_1)^2 + \xi_3^2 \ .$$

When $b \neq 0$, we can translate in the x_1 variable and get the operator on $L^2(\mathbb{R}^3)$

$$D_{y_1}^2 + (|b|y_1)^2 + \xi_3^2 \ .$$

It is then easy to see that the spectrum is $[|b|, +\infty[$. No point being isolated, we cannot expect any discrete spectrum.

12.5 Weyl's criterion:

We have already mentioned that the essential spectrum is a closed set. In order to determine the essential spectrum it is useful to have theorems proving the invariance by perturbation. The following characterization is in this spirit quite useful.

Theorem 12.5.1.
Let T be a selfadjoint operator. Then λ belongs to the essential spectrum if and only if there exists a sequence u_n in $D(T)$ with $||u_n|| = 1$ such that u_n tends weakly[37] *to 0 and $||(T-\lambda)u_n|| \to 0$ as $n \to \infty$.*

[37] We recall that we say that a sequence u_n in a separable Hilbert space $\mathcal{H}$ is weakly convergent if, for any g in $\mathcal{H}$, $< u_n | g >_{\mathcal{H}}$ is convergent. In this case, there exists a unique f such that $< u_n | g >_{\mathcal{H}} \to < f | g >_{\mathcal{H}}$ and $||u_n||$ is a bounded sequence.

Let us give a proof of the theorem (we also refer to [Ro], [HiSi] or [RS-IV]). The sequence appearing in the theorem is called a Weyl sequence. A point λ such that there exists an associated Weyl sequence is said to belong to the Weyl spectrum $W(T)$. Let us show the inclusion

$$W(T) \subset \sigma_{ess}(T)\ . \tag{12.5.1}$$

We have already seen that

$$W(T) \subset \sigma(T)\ . \tag{12.5.2}$$

Let us suppose by contradiction that $\lambda \in \sigma_{disc}(T)$. Let $\Pi_\lambda := E_{\{\lambda\}}$ be the associated spectral projector. We first observe that, Π_λ being finite range, hence compact, we have :

$$\Pi_\lambda u_n \to 0 \in \mathcal{H}\ . \tag{12.5.3}$$

Let us define

$$w_n = (I - \Pi_\lambda) u_n\ .$$

We get $||w_n|| \to +1$ and $(T-\lambda)w_n = (I-\Pi_\lambda)(T-\lambda)u_n \to 0$.
But $(T-\lambda)$ is invertible on $R(I-\Pi_\lambda)$, so we get $w_n \to 0$ and the contradiction. This shows the announced inclusion (12.5.1).
For the converse, we first observe that, if $\lambda \in \sigma_{ess}(T)$ then, for any $\epsilon > 0$, $\dim R(E_{]\lambda-\epsilon,\lambda+\epsilon[}) = +\infty$. Considering a decreasing sequence ϵ_n such that $\epsilon_n > 0$ and $\lim_{n\to+\infty} \epsilon_n = 0$, it is easy to obtain an orthonormal system u_n such that $u_n \in R(E_{]\lambda-\epsilon_n,\lambda+\epsilon_n[})$. With this last property, we immediately get[38] that $||(T-\lambda)u_n|| \to 0$.

Corollary 12.5.2.
The operator $-h^2\Delta + V$ with V a continuous function tending to 0 as $|x| \to \infty$ ($x \in \mathbb{R}^D$) has $\overline{\mathbb{R}^+}$ as essential spectrum.

For proving the inclusion of $\overline{\mathbb{R}^+}$ in the essential spectrum, we can indeed consider the sequence

$$u_n(x) = \exp(ix\cdot\xi) n^{-(D/2)} \cdot \chi((x-R_n)/n)$$

with $\chi \geq 0$ and supported in the ball $B(0,1)$ and equal to one on say $B(0,\frac{1}{2})$. The sequence R_n is chosen such that $|R_n|$ (for example $|R_n| = n^2$) tends to ∞ and such that the support of the u_n are disjoints.
This is a particular case of a Weyl sequence (called in [HiSi] a Zhislin sequence). The converse can be obtained by abstract analysis and the fact that we know that the essential spectrum of $-\Delta$ is $[0,+\infty[$. This idea will be formalized through the following notion of relative compactness.

Definition 12.5.3.
If T is a closed operator with a dense domain D_T, we shall say that the operator V is relatively compact with respect to T or T-compact if $D_T \subset D_V$ and if the image by V of a closed ball in D_T (for the graph-norm $u \mapsto \sqrt{||u||^2 + ||Tu||^2}$) is relatively compact in $\mathcal{H}$.

[38] See also what we have done for the proof of Proposition 8.5.4.

In other words, we shall say that V is T-compact, if, from each sequence u_n in D_T which is bounded in $\mathcal{H}$ and such that Tu_n is bounded in $\mathcal{H}$, one can extract a subsequence u_{n_i} such that Vu_{n_i} is convergent in $\mathcal{H}$. Here we recall (exercise) that when T is closed, $D(T)$ equipped with the graph norm is an Hilbert space.

Example 12.5.4.
If V is the multiplication operator by a continuous function V tending to 0 then V is $(-\Delta)$-compact. This is a consequence of Proposition 5.4.1 and of the uniform continuity of V on each compact.

Weyl's Theorem says

Theorem 12.5.5. .
Let T be a selfadjoint operator, and V be symmetric and T-compact, then $T + V$ is selfadjoint and the essential spectrum of $T + V$ is equal to the essential spectrum of T.

The first part can be deduced from what was discussed in the proof of Kato-Rellich's theorem (See Theorem 10.3.1). We observe indeed the following variant of Lions's Lemma :

Lemma 12.5.6.
If V is T-compact and closable, then, for any $a > 0$, there exists $b > 0$ such that

$$||Vu|| \leq a||Tu|| + b||u|| \ , \ \forall u \in D(T) \ . \tag{12.5.4}$$

Proof of the lemma.
The proof is by contradiction. If (12.5.4) is not true, then there exists $a > 0$ such that, $\forall n \in \mathbb{N}^*$, there exists $u_n \in D(T)$ such that

$$a||Tu_n|| + n||u_n|| < ||Vu_n|| \ . \tag{12.5.5}$$

Observing that $||Vu_n|| \neq 0$ and that the inequation is homogeneous, we can in addition assume that u_n satisfies the condition :

$$||Vu_n|| = 1 \ . \tag{12.5.6}$$

From these two properties we get that the sequence Tu_n is bounded and that $u_n \to 0$.
On the other hand, by T-compactness, we can extract a subsequence u_{n_k} such that Vu_{n_k} is convergent to v with $||v|| = 1$. But $(0, v)$ is in the closure of the graph of V. But V being closable, this implies $v = 0$ and a contradiction.

For the second part we can use the Theorem 12.5.1. If we take a Weyl's sequence u_n such that $u_n \to 0$ (weakly) and $(T - \lambda)u_n \to 0$ strongly, let us consider $(T + V - \lambda)u_n$. We have simply to show that one can extract a subsequence u_{n_k} such that $(T + V - \lambda)u_{n_k} \to 0$.

But Tu_n is a bounded sequence. By the T-compactness, we can extract a subsequence such that Vu_{n_k} converges strongly to some v in $\mathcal{H}$. It remains to show that $v = 0$. But here we can observe that for any $f \in D(T)$, we have

$$< v, f >= \lim_{k \to +\infty} < Vu_{n_k}, f >= \lim_{k \to +\infty} < u_{n_k}, Vf >= 0 \ .$$

Here we have used the symmetry of V and the weak convergence of u_n to 0. Using the density of $D(T)$ in $\mathcal{H}$, we obtain $v = 0$.
This shows that a Weyl sequence for T is a Weyl sequence for $T + V$. For the converse, one can intertwine the roles of T and $T + V$, once we have shown that V is $(T + V)$-compact. For this, we can use Lemma 12.5.6, and observe that the following inequality is true :

$$||Tu|| \leq \frac{1}{1-a} \left(||(T+V)u|| + b||u|| \right) \ . \tag{12.5.7}$$

This shows that if u_n is a sequence such that $(||u_n|| + ||(T + V)u_n||)$ is bounded, then this sequence has also the property that $(||u_n|| + ||Tu_n||)$ is bounded.

13 The max-min principle

13.1 Introduction

The max-min principle is an alternative way for describing the lowest part of the spectrum when it is discrete. It gives also an efficient way to localize these eigenvalues or to follow their dependence on various parameters.

13.2 On positivity

We first recall the following definition

Definition 13.2.1.
Let A be a symmetric operator. We say that A is positive (and we write $A \geq 0$), if

$$< Au, u > \geq 0 \; , \; \forall u \in D(A) \; . \tag{13.2.1}$$

The following proposition relates the positivity with the spectrum

Proposition 13.2.2.
Let A be a selfadjoint operator. Then $A \geq 0$ if and only if $\sigma(A) \subset [0, +\infty[$.

Proof.
It is clear that if the spectrum is in $\mathbb{R}^+$, then the operator is positive. This can be seen for example through the spectral representation :

$$< Au, u > = \int_{\lambda \in \sigma(A)} \lambda \; d||E_\lambda u||^2 \; .$$

Now, if $A \geq 0$, then, for any $a > 0$, $A + a$ is invertible. We have indeed

$$a||u||^2 \leq < (A + a)u, u > \leq ||(A + a)u|| \; ||u|| \; ,$$

which leads to

$$a||u|| \leq ||(A + a)u|| \; , \; \forall u \in D(A) \; . \tag{13.2.2}$$

This inequality gives the closed range and the injectivity. A being selfadjoint, we get also from the injectivity, the density of the image of $(A + a)$. This shows that $-a$ is not in the spectrum of A.

Example 13.2.3.
Let us consider the Schrödinger operator $P := -\Delta + V$, with $V \in C^\infty$ and semi-bounded, then

$$\sigma(P) \subset [\inf V, +\infty[\; . \tag{13.2.3}$$

13.3 Variational characterization of the discrete spectrum

Theorem 13.3.1.
Let A be a selfadjoint semibounded operator. Let $\Sigma := \inf \sigma_{ess}(A)$ and let us consider $\sigma(A) \cap]-\infty, \Sigma[$, described as a sequence (finite or infinite) of eigenvalues that we write in the form

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n \cdots .$$

Then we have

$$\lambda_1 = \inf_{\phi \in D(A), \phi \neq 0} ||\phi||^{-2} < A\phi, \phi > , \tag{13.3.1}$$

$$\lambda_2 = \inf_{\phi \in D(A) \cap K_1^\perp, \phi \neq 0} ||\phi||^{-2} < A\phi, \phi > , \tag{13.3.2}$$

and, for $n \geq 2$,

$$\lambda_n = \inf_{\phi \in D(A) \cap K_{n-1}^\perp, \phi \neq 0} ||\phi||^{-2} < A\phi, \phi > , \tag{13.3.3}$$

where

$$K_j = \oplus_{i \leq j} \; Ker\ (A - \lambda_i) .$$

Proof.

Step 1. Let us start, with the lowest eigenvalue. Let us define ${}^w idehat\mu_1(A)$ by

$$\widehat{\mu}_1(A) := \inf_{\phi \in D(A), \phi \neq 0} ||\phi||^{-2} < A\phi, \phi > \ . \tag{13.3.4}$$

If ϕ_1 is an eigenfunction associated to λ_1, we get immediately the inequality

$$\widehat{\mu}_1(A) \leq \lambda_1(A) . \tag{13.3.5}$$

Let us prove the converse inequality. Using the spectral theorem, one get immediately that $A \geq \inf \sigma(A)$.
So we get

$$\inf \sigma(A) \leq \widehat{\mu}_1(A) . \tag{13.3.6}$$

Now, if the spectrum below Σ is not empty, we get

$$\lambda_1(A) \leq \widehat{\mu}_1(A) .$$

We have consequently the equality. We have actually a little more.

We have indeed proved that, if $\widehat{\mu}_1(A) < \Sigma$, then, the spectrum below Σ is not empty, and the lowest eigenvalue is $\widehat{\mu}_1(A)$.

Step 2. The proof is by recursion, applying Step 1 to $A_{/D(A)\cap K_{n-1}^{\perp}}$.

This ends the proof of Theorem 13.3.1.

Example 13.3.2. *(Payne-Polya-Weinberger Inequality.)*
Let $P = -\Delta + V$ with $V \in C^\infty$ positive and $V \to +\infty$ as $|x| \to +\infty$.
Let us assume that V is even

$$V(x) = V(-x) \ . \tag{13.3.7}$$

Then λ_2 satisfies

$$\lambda_2 \leq \inf_{\phi \in Q(P), \phi \text{ odd}} < P\phi, \phi > \tag{13.3.8}$$

Let u_1 be the first normalized eigenfunction. We admit that the lowest eigenvalue of the Schrödinger operator is simple (variant of the Krein-Rutman's Theorem) and that the first eigenfunction can be chosen strictly positive, with exponential decay at ∞ together with ∇u_1 (this is a consequence of Agmon's inequality [Ag]). Then it is not difficult to verify that u_1 is even. Let us consider $v_j := x_j u_1$. v_j is in the form domain of P. We observe that

$$P(x_j u_1) = \lambda_1 x_j u_1 - 2\partial_j u_1 \ .$$

Taking the scalar product with $x_j u_1$, we then obtain

$$\begin{array}{ll} (\lambda_2 - \lambda_1)||x_j u_1||^2 & \leq -2 < \partial_j u_1, x_j u_1 > \\ & \leq ||u_1||^2 \\ & \leq 1 \ . \end{array} \tag{13.3.9}$$

We now use the uncertainty principle (1.2.11) and get :

$$(\lambda_2 - \lambda_1) \leq 4||\partial_j u_1||^2 \ . \tag{13.3.10}$$

On the other hand,

$$||\nabla u_1||^2 + \int_{\mathbb{R}^m} V(x)|u_1(x)|^2 dx = \lambda_1 \ , \tag{13.3.11}$$

and this gives

$$||\nabla u_1||^2 \leq \lambda_1 \ . \tag{13.3.12}$$

Putting the inequalities (13.3.9) and (13.3.12), we get, summing over j,

$$\lambda_2 - \lambda_1 \leq \frac{4}{m}\lambda_1 \ . \tag{13.3.13}$$

This inequality is not optimal, in the sense that for $m = 1$ and $V(x) = x^2$, we have $\lambda_2 - \lambda_1 = 2\lambda_1$.

Example 13.3.3.
Let us consider $S_h := -h^2\Delta + V$ on $\mathbb{R}^m$ where V is a C^∞ potential tending to 0 at ∞ and such that $\inf_{x\in\mathbb{R}^m} V(x) < 0$.
Then if $h > 0$ is small enough, there exists at least one eigenvalue for S_h. We note that the essential spectrum is $[0, +\infty[$. The proof of the existence of this eigenvalue is elementary. If x_{min} is one point such that $V(x_{min}) = \inf_x V(x)$, it is enough to show that, with $\phi_h(x) = \exp -\frac{\lambda}{h}|x - x_{min}|^2$, the quotient $\frac{<S_h\phi_h,\phi_h>}{||\phi_h||^2}$ tends as $h \to 0$ to $V(x_{min}) < 0$.

13.4 Max-min principle

We now give a more flexible criterion for the determination of the bottom of the spectrum and for the bottom of the essential spectrum. This flexibility comes from the fact that we do not need an explicit knowledge of the various eigenspaces.

Theorem 13.4.1.
Let $\mathcal{H}$ an Hilbert space of infinite dimension[39] *and A be a selfadjoint semi-bounded operator of domain $D(A) \subset \mathcal{H}$. Let us introduce*

$$\mu_n(A) = \sup_{\psi_1,\psi_2,\dots,\psi_{n-1}} \inf_{\left\{\begin{array}{l} \phi \in [span(\psi_1,\dots,\psi_{n-1})]^\perp; \\ \phi \in D(A) \text{ and } ||\phi|| = 1 \end{array}\right\}} \langle A\phi \mid \phi\rangle_{\mathcal{H}} . \qquad (13.4.1)$$

Then either
(a) $\mu_n(A)$ is the n-th eigenvalue when ordering the eigenvalues in increasing order (and counting the multiplicity) and A has a discrete spectrum in $]-\infty, \mu_n(A)]$
or
(b) $\mu_n(A)$ corresponds to the bottom of the essential spectrum. In this case, we have $\mu_j(A) = \mu_n(A)$ for all $j \geq n$.

Remark 13.4.2.
In the case when the operator has compact resolvent, case (b) does not occur and the supremum in (13.4.1) is a maximum. Similarly the infimum is a minimum. This explains the traditional terminology " Max-Min principle" for this theorem.

Proof.
If Ω is a borelian, let E_Ω be the projection-valued measure for A (see Remark 8.3.7).
We first prove that

$$\dim\ \left(RangeE_{]-\infty,a[}\right) < n \text{ if } a < \mu_n(A). \qquad (13.4.2)$$

$$\dim\ \left(RangeE_{]-\infty,a[}\right) \geq n \text{ if } a > \mu_n(A)\ . \qquad (13.4.3)$$

[39] In the case of a finite dimensional Hilbert space of dimension d, the minimax principle holds for $n \leq d$.

Notons que la conjonction de (13.4.2) et (13.4.3) montre que $\mu_n(A)$ est dans le spectre de A.

Step 1: Proof of (13.4.2).
Let a and n be given such that $a < \mu_n(A)$. Let us prove (13.4.2) by contradiction. If it was false, then we would have $\dim\ \big(Range(E_{]-\infty,a[})\big) \geq n$ and we could find an n-dimensional space $V \subset Range(E_{]-\infty,a[})$. Note now, that A being bounded from below, $Range(E_{]-\infty,a[})$ is included in $D(A)$.
So we can find an n-dimensional space $V \subset D(A)$, such that

$$\forall \phi \in V\ ,\ < A\phi\, ,\, \phi > \leq a||\phi||^2\ . \tag{13.4.4}$$

But then given any $\psi_1, \cdots, \psi_{n-1}$ in $\mathcal{H}$, we can find $\phi \in V \cap \{\psi_1, \cdots, \psi_{n-1}\}^\perp$ such that $||\phi|| = 1$ and $< A\phi\, ,\, \phi > \leq a$. Coming back to the definition, this shows that $\mu_n(A) \leq a$ and a contradiction.

Note that we have proved in this step the following proposition

Proposition 13.4.3.
Suppose that there exists a and an n-dimensional subspace $V \subset D(A)$ such that (13.4.4) is satisfied. Then we have the inequality :

$$\mu_n(A) \leq a\ . \tag{13.4.5}$$

Modulo the complete proof of the theorem, we obtain

Corollary 13.4.4.
Under the same assumption as in Proposition 13.4.3, if a is below the bottom of the essential spectrum of A, then A has at least n eigenvalues (counted with multiplicity).

Exercise 13.4.5.
In continuation of Example 13.3.3, show that for any $\epsilon > 0$ and any N, there exists $h_0 > 0$ such that for $h \in]0, h_0]$, S_h has at least N eigenvalues in $[\inf V, \inf V + \epsilon]$. One can treat first the case when V has a unique non degenerate minimum at 0.

Step 2 : Proof of (13.4.3).
Suppose that (13.4.3) is false. Then dim (Range $(E_{]-\infty,a[}) \leq n-1$, so we can find $(n-1)$ generators $\psi_1, \cdots, \psi_{n-1}$ of this space. Then any $\phi \in D(A) \cap span\{\psi_1, \cdots, \psi_{n-1}\}^\perp$ is in Range $(E_{[a,+\infty[})$, so

$$< A\phi\, ,\, \phi > \geq a||\phi||^2\ .$$

Therefore, coming back to the definition of $\mu_n(A)$, we get $\mu_n(A) \geq a$ in contradiction with our initial assumption.

Before to continue the proof, let us emphasize on one point.

Remark 13.4.6.
In the definition of $\mu_n(A)$, $\psi_1, \cdots, \psi_{n-1}$ *are only assumed to belong to the Hilbert space* $\mathcal{H}$.

Step 3 : $\mu_n(A) < +\infty$.
First the semi-boundedness from below of A gives a uniform lower bound.
Secondly, if $\mu_n(A) = +\infty$, this would mean by (13.4.2) that :
$\dim(Range(E_{]-\infty,a[})) < n$ for all a,
and consequently that $\mathcal{H}$ is finite dimensional. This is a contradiction, if $\mathcal{H}$ is infinite dimensional. But the finite case is trivial, we have indeed $\mu_n(A) \leq ||A||$, in this case.

As the statement of the theorem suggests, there are two cases to consider and this will be the object of the two next steps.
Step 4.
Let us first assume (with $\mu_n = \mu_n(A)$) that

$$\dim\ (Range(E_{]-\infty,\mu_n+\epsilon[})) = +\infty\ ,\ \forall \epsilon > 0\ . \tag{13.4.6}$$

We claim that, in this case, we are in the second alternative in the theorem. Using (13.4.2) and (13.4.6), we get indeed

$$\dim\ (Range(E_{]\mu_n-\epsilon,\mu_n+\epsilon[})) = +\infty\ ,\ \forall \epsilon > 0\ . \tag{13.4.7}$$

This shows that $\mu_n(A) \in \sigma_{ess}(A)$.
On the other hand, using again (13.4.2), we immediately get that $]-\infty, \mu_n(A)[$ does not contain any point in the essential spectrum. Thus $\mu_n(A) = \inf\{\lambda \mid \lambda \in \sigma_{ess}(A)\}$.

Let us show now that $\mu_{n+1} = \mu_n$ in this case. From the definition of the $\mu_k(A)$, it is clear that $\mu_{n+1} \geq \mu_n$, since one can take $\psi_n = \psi_{n-1}$.
But if $\mu_{n+1} > \mu_n$, (13.4.2) would also be satisfied for μ_{n+1}, and this is in contradiction with (13.4.6).
Step 5.

Let us now assume that

$$\dim\ (Range(E_{]-\infty,\mu_n+\epsilon_0[})) < \infty\ ,\ \ \text{for some } \epsilon_0 > 0\ . \tag{13.4.8}$$

Then it is clear, that the spectrum is discrete in $]-\infty, \mu_n + \epsilon_0[$. Then, for $\epsilon_1 > 0$ small enough,

$$Range(E_{]-\infty,\mu_n]}) = Range(E_{]-\infty,\mu_n+\epsilon_1[})\ ,$$

and by (13.4.3)

$$\dim\ (Range E_{]-\infty,\mu_n]}) \geq n\ . \tag{13.4.9}$$

So there are at least n eigenvalues $E_1 \leq E_2 \leq \cdots \leq E_n \leq \mu_n$ for A. If E_n were strictly less than μ_n, $\dim(Range E_{]-\infty,E_n]})$ would equal n in contradiction with

(13.4.2). Therefore $E_n = \mu_n$ and μ_n is an eigenvalue.

This ends the proof of Theorem 13.4.1.

A first natural extension of Theorem 13.4.1 is obtained by

Theorem 13.4.7.
Let A be a selfadjoint semibounded operator and $Q(A)$ its form domain [40] . *Then*

$$\mu_n(A) = \sup_{\psi_1,\psi_2,\dots,\psi_{n-1}} \inf_{\left\{\begin{array}{l} \phi \in [span(\psi_1,\dots,\psi_{n-1})]^\perp; \\ \phi \in Q(A) \text{ and } ||\phi|| = 1 \end{array}\right\}} \langle A\phi \mid \phi\rangle_{\mathcal{H}} \; . \quad (13.4.10)$$

Proof.
Let $\tilde{\mu}_n$ be the right hand side of (13.4.10). By imitating the proof of the previous theorem, we get that each $\tilde{\mu}_n$ obeys one of the two conditions. These conditions determine μ_n and consequently $\mu_n = \tilde{\mu}_n$.
One can also note (see Subsection 3.3) that, when constructing the Friedrichs extension, one has shown that the domain of the Friedrichs extension is dense in the form domain.

Applications

- It is very often useful to apply the max-min principle by taking the minimum over a dense set in $Q(A)$.
- The max-min principle permits to control the continuity of the eigenvalues with respect to parameters. For example the lowest eigenvalue $\lambda_1(\epsilon)$ of $-\frac{d^2}{dx^2} + x^2 + \epsilon x^4$ increases with respect to ϵ. Show that $\epsilon \mapsto \lambda_1(\epsilon)$ is right continuous on $[0, +\infty[$. (The reader can admit that the corresponding eigenfunction is in $\mathcal{S}(\mathbb{R})$ for $\epsilon \geq 0$).
- The max-min principle permits to give an upperbound on the bottom of the spectrum and the comparison between the spectrum of two operators. If $A \leq B$ in the sense that, $Q(B) \subset Q(A)$ and[41]

  $$< Au, u > \leq < Bu, u > \; , \; \forall u \in Q(B) \; ,$$

  then
  $$\lambda_n(A) \leq \lambda_n(B) \; .$$

  Similar conclusions occur if we have $D(B) \subset D(A)$.

[40] associated by completion with the form $u \mapsto \langle u|Au\rangle_{\mathcal{H}}$ initially defined on $D(A)$.
[41] It is enough to verify the inequality on a dense set in $Q(B)$.

Example 13.4.8. *(Comparison between Dirichlet and Neumann)*
Let Ω be a bounded regular connected open set in $\mathbb{R}^m$. Then the N-th eigenvalue of the Neumann realization of $-\Delta + V$ is less or equal to the N-th eigenvalue of the Dirichlet realization. It is indeed enough to observe the inclusion of the form domains.

Example 13.4.9. *(monotonicity with respect to the domain)*
Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^m$ two bounded regular open sets. Then the $n-th$ eigenvalue of the Dirichlet realization of the Schrödinger operator in Ω_2 is less or equal to the n-th eigenvalue of the Dirichlet realization of the Schrödinger operator in Ω_1. We observe that we can indeed identify $H_0^1(\Omega_1)$ with a subspace of $H_0^1(\Omega_2)$ by just an extension by 0 in $\Omega_2 \setminus \Omega_1$.
We then have

$$\begin{aligned}
\lambda_n(\Omega_2) \quad & = \sup_{\{\psi_1,\cdots,\psi_{n-1}\in L^2(\Omega_2)\}} \inf_{\left\{\begin{array}{l} \phi \in H_0^1(\Omega_2) \\ <\phi,\psi_j>_{L^2(\Omega_2)} \text{ and } {}_{||\phi||=1} \end{array}\right\}} ||\nabla \phi||^2_{L^2(\Omega_2)} \\
& \leq \sup_{\{\psi_1,\cdots,\psi_{n-1}\in L^2(\Omega_2)\}} \inf_{\left\{\begin{array}{l} \phi \in H_0^1(\Omega_1) \\ <\phi,\psi_j>_{L^2(\Omega_2)} \text{ and } {}_{||\phi||=1} \end{array}\right\}} ||\nabla \phi||^2_{L^2(\Omega_2)} \\
& = \sup_{\{\psi_1,\cdots,\psi_{n-1}\in L^2(\Omega_2)\}} \inf_{\left\{\begin{array}{l} \phi \in H_0^1(\Omega_1) \\ <\phi,\psi_j>_{L^2(\Omega_1)} \text{ and } {}_{||\phi||=1} \end{array}\right\}} ||\nabla \phi||^2_{L^2(\Omega_1)} \\
& = \sup_{\{\psi_1,\cdots,\psi_{n-1}\in L^2(\Omega_1)\}} \inf_{\left\{\begin{array}{l} \phi \in H_0^1(\Omega_1) \\ <\phi,\psi_j>_{L^2(\Omega_1)} \text{ and } {}_{||\phi||=1} \end{array}\right\}} ||\nabla \phi||^2_{L^2(\Omega_1)} \\
& = \lambda_n(\Omega_1)\,.
\end{aligned}$$

Note that this argument is not valid for the Neumann realization.

13.5 CLR inequality

In order to complete the picture, let us mention (confer [RS-IV], p. 101) that, if $m \geq 3$, then the following theorem due to Cwickel-Lieb-Rozenbljum is true :

Theorem 13.5.1.
There exists a constant L_m, such that, for any V such that $V_- \in L^{\frac{m}{2}}$, and if $m \geq 3$, the number of strictly negative eigenvalues of S_1 N_- is finite and bounded by

$$N_- \leq L_m \int_{V(x)\leq 0} (-V)^{\frac{m}{2}}\, dx\,. \tag{13.5.1}$$

This shows that when $m \geq 3$, we could have examples of negative potentials V (which are not identically zero) and such that the corresponding Schrödinger operator S_1 has no eigenvalues. A sufficient condition is indeed

$$L_m \int_{V\leq 0} (-V)^{\frac{m}{2}}\, dx < 1\,.$$

In the other direction, we have[42] the following results.

[42]These counterexamples come back (when $m = 1$ to Avron-Herbst-Simon [AHS] and when $m = 2$ to Blanchard-Stubbe [BS]).

Proposition 13.5.2.
Let V be in $C_0^\infty(\mathbb{R}^m)$ $(m=1,2)$. If

$$\int_{\mathbb{R}^m} V(x)\, dx < 0 \;, \tag{13.5.2}$$

then

Proof.
We just treat the case when $V \in C_0^\infty(\mathbb{R}^m)$.
We first observe that the the essential spectrum is $[0,+\infty[$. For the proof of the proposition, it is then enough to find $\psi \in D(S_1)$ such that

$$< S_1\psi, \psi >_{L^2(\mathbb{R}^m)} < 0 \;.$$

When $m=1$, taking $\psi_a = \exp -a|x|$, $a>0$, we find that

$$\int_{\mathbb{R}} |\psi_a'(x)|^2\, dx = a \;,$$

and

$$\lim_{a\to 0} \int_{\mathbb{R}} V(x)|\psi_a(x)|^2\, dx = \int_{\mathbb{R}} V(x)\, dx < 0 \;,$$

by the dominated convergence Theorem.
When $m=2$, we can take $\psi_a(x) = \exp -\frac{1}{2}|x|^a$, $a>0$, then

$$\int_{\mathbb{R}^2} ||\nabla\psi_a(x)||^2\, dx = \frac{\pi}{2} a \;,$$

and

$$\lim_{a\to 0} \int_{\mathbb{R}^2} V(x)|\psi_a(x)|^2\, dx = e^{-\frac{1}{2}} \int_{\mathbb{R}^2} V(x)\, dx < 0 \;.$$

13.6 Essential spectrum and Persson's Theorem

We refer to Agmon's book [Ag] for details.

Theorem 13.6.1.
Let V be a real-valued potential in the Kato-Rellich class[43]*, and let $H = -\Delta + V$ be the corresponding self-adjoint, semibounded Schrödinger operator with domain $H^2(\mathbb{R}^m)$. Then, the bottom of the essential spectrum is given by*

$$\inf \sigma_{ess}(H) = \Sigma(H) \;, \tag{13.6.1}$$

where

$$\Sigma(H) := \sup_{\mathcal{K}\subset\mathbb{R}^m} \left[\inf_{||\phi||=1} \{< \phi, H\phi > \; | \; \phi \in C_0^\infty(\mathbb{R}^m \setminus \mathcal{K})\} \right] \;, \tag{13.6.2}$$

where the supremum is over all compact subset $\mathcal{K} \subset \mathbb{R}^m$.

[43]Cf Theorem 10.3.1.

Essentially this is a corollary of Weyl's Theorem 12.5.5. We will indeed play with the fact that

Lemma 13.6.2.

$$\sigma_{ess}(H) = \sigma_{ess}(H + W) \ ,$$

for any regular potential W with compact support.

14 Exercises and Problems

We present in this section some exercises or problems proposed in the last years. They sometimes strongly intersect with the course.

14.1 Exercises

Exercise 14.1.1 (a natural problem in Bose-Einstein theory)**.**
Let $\omega > 0$.
Discuss in function of $\Omega \in \mathbb{R}$ *the semi-boundedness of the operator defined on* $\mathbb{S}(\mathbb{R}^2)$ *by*

$$H^\Omega := -\frac{1}{2}\Delta_{x,y} + \frac{1}{2}\omega^2 r^2 - \Omega L_z \,, \tag{14.1.1}$$

with

$$L_z = i(x\partial_y - y\partial_x) \,, \tag{14.1.2}$$

The answer can be found by showing that

$$\phi_{j,k}(x,y) = e^{\frac{\omega}{2}(x^2+y^2)} \, (\partial_x + i\partial_y)^j \, (\partial_x - i\partial_y)^k \left(e^{-\omega(x^2+y^2)}\right) \tag{14.1.3}$$

where j *and* k *are non-negative integers, is an eigenfunction of* H^Ω *and of* L_z.

Exercise 14.1.2. *(After Effros, Avron-Seiler-Simon).*
Let P *and* Q *two selfadjoint projectors in a Hilbert space* $\mathcal{H}$. *i) Let us assume that* $A = P - Q$ *is compact. Show that if* $\lambda \neq \pm 1$ *is in the spectrum, then* $-\lambda$ *is in the spectrum with the same multiplicity. For this, one can first show that with* $B = I - P - Q$,

$$A^2 + B^2 = I \;,\; AB + BA = 0 \,.$$

ii) Assume now that A *is in addition trace class, that is that the series* $|\mu_j|$, *where* μ_j *are the non zero eigenvalues of* A, *counted with multiplicity. Compute* $Tr\ A := \sum_j \mu_j$ *and show that it is an integer.*

Exercise 14.1.3. *(Temple's inequality).*
Let A *be a selfadjoint operator on an Hilbert space and* $\psi \in D(A)$ *such that* $||\psi|| = 1$.
Suppose that in some interval $]\alpha, \beta[$, $\sigma(A) \cap]\alpha, \beta[= \{\lambda\}$ *and that* $\eta = \langle \psi \mid A\psi \rangle$ *belongs to the interval* $]\alpha, \beta[$. *Then show that :*

$$\eta - \frac{\epsilon^2}{\beta - \eta} \leq \lambda \leq \eta + \frac{\epsilon^2}{\eta - \alpha} \,.$$

with :

$$\epsilon^2 = ||(A - \eta)\psi||^2 \,.$$

As a preliminary result, one can show that $(A-\alpha)(A-\lambda)$ *and* $(A-\beta)(A-\lambda)$ *are positive operators. Then apply the inequalities with* ψ.
Show that this inequality is an improvment if $\epsilon^2 \leq (\beta - \eta)(\eta - \alpha)$.
Compare with what is given by the spectral theorem or the minmax principle.

Exercise 14.1.4. .
Let $A(x_1, x_2) = (A_1(x_1, x_2)\,,\, A_2(x_1, x_2))$ *be a* C^∞ *vector field on* $\mathbb{R}^2$. *Let* V *be a* C^∞ *positive function on* $\mathbb{R}^2$.
Let $P := (D_{x_1} - A_1(x_1, x_2))^2 + (D_{x_2} - A_2(x_1, x_2))^2 + V(x)$ *the differential operator defined on* $C_0^\infty(\mathbb{R}^2)$.
a) Show that P *admits a selfadjoint extension in* $L^2(\mathbb{R}^2)$.
b) Show that P *is essentially selfadjoint.*

Exercise 14.1.5. .
We admit the results of Exercise 14.1.4. Show that the selfadjoint extension in $L^2(\mathbb{R}^2)$ *of*

$$T := -(\frac{d}{dx_1} - ix_2x_1^2)^2 - \frac{d^2}{dx_2^2} + x_2^2\,,$$

is with compact resolvent.

Exercise 14.1.6. .
Let V *be a* C^∞ *positive potential in* $\mathbb{R}^2$. *Let us consider, with* $B \in \mathbb{R} \setminus \{0\}$, *the operator*

$$P = D_{x_1}^2 + (D_{x_2} + B\,x_1)^2 + V(x)\,.$$

a) Recall briefly the spectrum of its selfadjoint extension in the case $V = 0$.
b) We assume that V *tends to* 0 *as* $|x| \to +\infty$. *Determine the essential spectrum of* $\bar{P}$.

Exercise 14.1.7. .
Let K *be a kernel in* $\mathcal{S}(\mathbb{R}^2)$ *which is strictly positive and symmetric.*
a) Show that the associated operator $\mathcal{K}$ *which is defined on* $\mathcal{S}(\mathbb{R})$ *by*

$$(\mathcal{K}u)(x) = \int_{\mathbb{R}} K(x, y)u(y)dy\,,$$

can be extended as a compact operator on $L^2(\mathbb{R})$. *b) Let* I *be an open interval in* $\mathbb{R}$ *and let us denote by* $\mathcal{K}_I$ *the operator on* $L^2(I)$ *defined by*

$$(\mathcal{K}_I u)(x) = \int_I K(x, y)u(y)dy\,.$$

Let λ_I^1 *be the largest eigenvalue of* $\mathcal{K}_I$. *Show that*

$$\lambda_I^1 \leq \lambda_{\mathbb{R}}^1\,.$$

Show that we have strict inequality when I *is not* $\mathbb{R}$.
c) Let u^1 *be a normalized eigenfunction of* $\mathcal{K}$ *associated with* $\lambda_{\mathbb{R}}^1$. *Using its restriction to* I, *show the inequality :*

$$\lambda_{\mathbb{R}}^1 \leq \lambda_I^1\,(1 - ||u^1||_{L^2(I^C)})^{-1}\,.$$

d) Let $I_n = [-n, n]$. *Show that* $\lambda_{I_n}^1$ *converges rapidly to* $\lambda_{\mathbb{R}}^1$ *as* $n \to +\infty$. *More precisely, show that, for all* $j \in \mathbb{N}$, *there exists a constant* C_j *such that :*

$$|\lambda_{\mathbb{R}}^1 - \lambda_{I_n}^1| \leq C_j\, n^{-j}\,,\ \forall n \in \mathbb{N}^*\,.$$

Exercise 14.1.8. .
Let us consider in $\Omega =]0,1[\times\mathbb{R}$, a positive C^∞ function V and let S_0 be the Schrödinger operator $S_0 = -\Delta + V$ defined on $C_0^\infty(\Omega)$.
(a) Show that S_0 admits a selfadjoint extension on $L^2(\Omega)$. Let S this extension.
(b) Determine if S is with compact resolvent in the following cases :

1. $V(x) = 0$,
2. $V(x) = x_1^2 + x_2^2$,
3. $V(x) = x_1^2$,
4. $V(x) = x_2^2$
5. $V(x) = (x_1 - x_2)^2$.

Determine the spectrum in the cases (1) and (4). One can first determine the spectrum of the Dirichlet realization (or of Neumann) of $-d^2/dx^2$ on $]0,1[$.

Exercise 14.1.9. .
We consider in $\mathbb{R}^2$ the operator defined on $\mathcal{S}(\mathbb{R}^2\,;\mathbb{C}^2)$ by

$$\mathcal{D}_0 = \alpha_1 D_{x_1} + \alpha_2 D_{x_2} + \alpha_3 \,.$$

Here the matrices α_j are hermitian 2×2 matrices such that :

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij} \,,$$

and we recall that $D_{x_j} = \frac{1}{i}\,\frac{\partial}{\partial_{x_j}}$ for $j = 1,2$.

a) Is $\mathcal{D}_0$ symmetric? semi-bounded ? It is suggested to use the Fourier transform.

b) Compute $\mathcal{D}_0^2$.

c) Show that $\mathcal{D}_0$ admits a selfadjoint extension $\mathcal{D}_1$ in $L^2(\mathbb{R}^2\,;\mathbb{C}^2)$, and determine its domain.

d) Determine the spectrum of $\mathcal{D}_1$.

e) We suppose that, for all $x \in \mathbb{R}^2$, $V(x)$ is a 2×2 hermitian matrix, with bounded C^∞ coefficients. Show that $\mathcal{D}_V = \mathcal{D}_0 + V$ admits a selfadjoint extension and determine its domain.

Exercise 14.1.10. .
Let H_a be the Dirichlet realization of $-d^2/dx^2 + x^2$ in $]-a,+a[$. Show that the lowest eigenvalue $\lambda_1(a)$ of H_a is strictly positive, monotonically decreasing as $a \to +\infty$ and tend exponentially fast to 1 as $a \to +\infty$. Give an estimate as fine as possible of $|\lambda_1(a) - 1|$.

Exercise 14.1.11. .
We consider on $C_0^\infty(\mathbb{R}^2)$, the operator

$$P_0 := (D_{x_1} - x_2)^2 + (D_{x_2} - x_1)^2 \,.$$

We recall that $D_{x_j} = \frac{1}{i}\partial_{x_j}$.

Show that its natural selfadjoint extension P is unitary equivalent to the operator $-\Delta$ (of domain H^2). Determine its spectrum and its essential spectrum.

Exercise 14.1.12. .
Show that one can associate to the differential operator on $C_0^\infty(\mathbb{R}\times]0,1[)$:

$$T_0 := (D_{x_1} - x_2 x_1^2)^2 + (D_{x_2})^2 \ ,$$

an unbounded selfadjoint operator T *on* $L^2(\mathbb{R}\times]0,1[)$ *whose spectrum is with compact resolvent.*

Exercise 14.1.13. .
Let ϕ *be a* C^2*- function on* $\mathbb{R}^m$ *such that* $|\nabla\phi(x)| \to +\infty$ *as* $|x| \to +\infty$ *and with uniformly bounded second derivatives. Let us consider the differential operator on* $C_0^\infty(\mathbb{R}^2)$ $-\Delta + 2\nabla\phi\cdot\nabla$*. We consider this operator as an unbounded operator on* $\mathcal{H} = L^2(\mathbb{R}^m, \exp -2\phi\ dx)$*. Show that it admits a selfadjoint extension and that its spectrum is discrete.*
We assume in addition that : $\int_{\mathbb{R}^m} \exp -2\phi dx < +\infty$*. Show that its lowest eigenvalue is simple and determine a corresponding eigenvector.*

Exercise 14.1.14. .
We consider in $\mathbb{R}^3$ *the differential operator* $S_0 := -\Delta - \frac{1}{r}$*, a priori defined on* $C_0^\infty(\mathbb{R}^3)$.
a) Show that the operator admits a selfadjoint extension S.
b) Show the continuous injection of $H^2(\mathbb{R}^3)$ *into the space of the Hölder functions* $C^s(\mathbb{R}^3)$*, with* $s \in]0, \frac{1}{2}[$*, and the compact injection for all compact* K *of* $C^s(K)$ *into* $C^0(K)$.
c) Determine the essential spectrum of S*. One possibility is to start with the analysis of* $S_\chi = -\Delta - \frac{\chi}{r}$ *where* χ *is* C^∞ *with compact support.*
d) Show using the minimax-principle that S *has at least one eigenvalue. One can try to minimize over a* $u \mapsto < S_0 u\ ,\ u\rangle/||u||^2$ *with* $u(x) = \exp -ar$.
e) Determine this lowest eigenvalue (using the property that the groundstate should be radial).

Exercise 14.1.15. .
a) Let g *be a continuous function on* $\mathbb{R}$ *such that* $g(0) = 0$*. Analyze the convergence of the sequence* $(g(t)u_n(t))_{n\geq 1}$ *in* $L^2(\mathbb{R})$ *where* $u_n(t) = \sqrt{n}\chi(nt)$ *and* χ *is a* C^∞ *function with compact support.*
b) let $f \in C^0([0,1];\mathbb{R})$*. Let* T_f *be the multiplication operator by* f *defined on* $L^2(]0,1[)$*:* $u \mapsto T_f u = fu$.
Determine the spectrum of T_f*. Discuss in function of* f *the possible existence of eigenvalues. Determine the essential spectrum of* T_f.

Exercise 14.1.16. .
Discuss in function of $\alpha \geq 0$ *the possibility of associating to the differential operator define on* $C_0^\infty(\mathbb{R}^3)$

$$-\Delta - r^{-\alpha}$$

a selfadjoint operator on $L^2(\mathbb{R}^3)$.

Exercise 14.1.17. .
Let Ω *be a non empty open subset in* $\mathbb{R}^d$ *and let us consider the multiplication*

operator on $L^2(\mathbb{R}^d)$ defined by the multiplication by χ_Ω where χ_Ω is equal to 1 in Ω and 0 outside. Determine the spectrum, the essential spectrum, the discrete spectrum.

Exercise 14.1.18. .
Show that the spectrum in $\mathbb{R}^2$ of $P = D_x^2 + x^2 + D_y^2$ is $[1, +\infty[$.

Exercise 14.1.19. .
Let $V \in \mathcal{S}(\mathbb{R}^d)$ be positive and let us consider the operator :

$$T := (-\Delta + 1)^{-\frac{1}{2}} V (-\Delta + 1)^{-\frac{1}{2}} .$$

a) Explain how to define $(-\Delta + 1)^{-\frac{1}{2}}$, as an operator on $L^2(\mathbb{R}^d)$.
b) Show that T is a bounded, selfadjoint, positive, compact operator on $L^2(\mathbb{R}^d)$.
c) Discuss the injectivity in function of V ?
d) Establish a link with the research of pairs (u, μ) in $H^2(\mathbb{R}^d) \times \mathbb{R}^+$ such that :

$$(-\Delta + 1 - \mu V)u = 0 .$$

Exercise 14.1.20. .
Let $\delta \in \mathbb{R}$.
(a) Show that the operator P_δ defined on $C_0^\infty(\mathbb{R}^2)$ by

$$P_\delta := D_x^2 + D_y^2 + x^2 y^4 + x^4 y^2 + \delta(x + y)$$

is semibounded.
One can first show the inequality :

$$\langle P_0 u \ , \ u \rangle \geq ||xu||^2 + ||yu||^2 .$$

(b) Show that there exists a natural selfadjoint extension of P_δ.
(c) What is the corresponding form domain ?
(d) Show that the selfadjoint extension is with compact resolvent.

Exercise 14.1.21. .
Let us consider in $\mathbb{R}^+$, the Neumann realization in $\mathbb{R}^+$ of $P_0(\xi) := D_t^2 + (t - \xi)^2$, where ξ is a parameter in $\mathbb{R}$. We would like to find an upper bound for $\Theta_0 = \inf_\xi \mu(\xi)$ where $\mu(\xi)$ is the smallest eigenvalue of $P_0(\xi)$. Following the physicist Kittel, one can proceed by minimizing $\langle P_0(\xi)\phi(\cdot;\rho) \mid \phi(\cdot;\rho)\rangle$ over the normalized functions $\phi(t;\rho) := c_\rho \exp -\rho t^2$ ($\rho > 0$). For which value of ξ is this quantity minimal ?? Deduce the inequality :

$$\Theta_0 < \sqrt{1 - \frac{2}{\pi}} .$$

Exercise 14.1.22. .
In the same spirit as in the previous exercise. Find an upper bound for the quartic operator $D_t^2 + \frac{1}{4}t^4$.
Using the comparison with an harmonic oscillator $D_t^2 + \alpha t^2 + \beta$, find an optimal

(with respect to the method) lower bound.
Using the comparison with $D_t^2 + V_\alpha(t)$ where $V_\alpha(t) = 0$ for $|t| \leq \alpha$ and $V_\alpha(t) = \frac{1}{4}\alpha^4$ for $|t| \geq \alpha$, and optimizing over α find an alternative lower bound (and compute it with the help of a computer).

Exercise 14.1.23. [44]
Let Ω be a bounded regular set in $\mathbb{R}^d$. Let us denote by λ_n ($n \geq 1$), (resp. μ_n) the sequence of eigenvalues of the Dirichlet problem for the Laplacian (resp. the Neumann problem). Show that

$$\mu_{n+1} \leq \lambda_n \, , \, \forall n \geq 1 \, .$$

One can use the minimax principle and analyze the quantity $\int_\Omega |\nabla u|^2 dx$ for

$$u = \sum_{j=1}^{n} \alpha_j \phi_j^D(x) + \beta \exp i\xi \cdot x \, ,$$

with $|\xi|^2 = \lambda_n$.
Is the inequality strict ?

We now consider the function

$$\xi \mapsto \chi(\xi) = \int_\Omega \exp i\xi \cdot x \, dx \, .$$

Show that if Ω is balanced (that is if $x \in \Omega$ iff $-x \in \Omega$), then χ is real.

We denote by $\kappa(\Omega)$ the function

$$\kappa(\Omega) = \inf\{\xi \in \mathbb{R}^d \, , \, \chi(\xi) = 0\} \, .$$

Show that if $\sqrt{\lambda_n(\Omega)} > 2\kappa(\Omega)$ then

$$\mu_{n+2}(\Omega) \leq \lambda_n(\Omega) \, .$$

This time one can use the minimax principle and analyze the quantity $\int_\Omega |\nabla u|^2 dx$ for

$$u = \sum_{j=1}^{n} \alpha_j \phi_j^D(x) + \beta_1 \exp i\xi_1 \cdot x + \beta_2 \exp i\xi_2 \cdot x \, ,$$

with $|\xi_1|^2 = |\xi_2|^2 = \lambda_n$ and $\chi(\xi_1 - \xi_2) = 0$.

[44] This is inspired by a paper of Benguria-Levitin-Parnovski [?] and by discussions with A. Laptev

14.2 Problems

Problem 14.2.1. .
Let us consider in the disk of $\mathbb{R}^2$ $\Omega := D(0,R)$ *the Dirichlet realization of the Schrödinger operator*

$$S(h) := -\Delta + \frac{1}{h^2} V(x) , \tag{14.2.1}$$

where V *is a* C^∞ *potential on* $\bar{\Omega}$ *satisfying :*

$$V(x) \geq 0 . \tag{14.2.2}$$

Here $h > 0$ *is a parameter.*

a) Show that this operator is with compact resolvent.
b) Let $\lambda_1(h)$ *be the lowest eigenvalue of* $S(h)$*. We would like to analyze the behavior of* $\lambda_1(h)$ *as* $h \to 0$*. Show that* $h \to \lambda_1(h)$ *is monotonically increasing.*
c) Let us assume that $V > 0$ *on* $\bar{\Omega}$*; show that there exists* $\epsilon > 0$ *such that*

$$h^2 \lambda_1(h) \geq \epsilon . \tag{14.2.3}$$

d) We assume now that $V = 0$ *in an open set* ω *in* Ω*. Show that there exists a constant* $C > 0$ *such that, for any* $h > 0$,

$$\lambda_1(h) \leq C . \tag{14.2.4}$$

One can use the study of the Dirichlet realization of $-\Delta$ *in* ω.
e) Let us assume that :

$$V > 0 \text{ almost everywhere in } \Omega . \tag{14.2.5}$$

Show that, under this assumption :

$$\lim_{h \to 0} \lambda_1(h) = +\infty . \tag{14.2.6}$$

One could proceed by contradiction supposing that there exists C *such that*

$$\lambda_1(h) \leq C , \ \forall h \text{ such that } 1 \geq h > 0 . \tag{14.2.7}$$

and establishing the following properties.

- *For* $h > 0$*, let us denote by* $x \mapsto u_1(h)(x)$ *an* L^2*-normalized eigenfunction associted with* $\lambda_1(h)$*. Show that the family* $u_1(h)$ $(0 < h \leq 1)$ *is bounded in* $H^1(\Omega)$.

- *Show the existence of a sequence* h_n $(n \in \mathbb{N})$ *tending to* 0 *as* $n \to +\infty$ *and* $u_\infty \in L^2(\Omega)$ *such that*
$$\lim_{n \to +\infty} u_1(h_n) = u_\infty$$
in $L^2(\Omega)$.

- *Deduce that :*

$$\int_\Omega V(x)\, u_\infty(x)^2\, dx = 0 \,.$$

- *Deduce that $u_\infty = 0$ and make explicite the contradiction.*

f) Let us assume that $V(0) = 0$; show that there exists a constant C, such that :

$$\lambda_1(h) \leq \frac{C}{h} \,.$$

g) Let us assume that $V(x) = \mathcal{O}(|x|^4)$ près de 0. Show that in this case :

$$\lambda_1(h) \leq \frac{C}{h^{\frac{2}{3}}} \,.$$

h) We assume that $V(x) \sim |x|^2$ near 0; discuss if one can hope a lower bound in the form

$$\lambda_1(h) \geq \frac{1}{C\,h} \,.$$

Justify the answer by illustrating the arguments by examples and counterexamples.

Problem 14.2.2. .
We consider on $\mathbb{R}$ and for $\epsilon \in I := [-\frac{1}{4}, +\infty[$ the operator $H_\epsilon = -d^2/dx^2 + x^2 + \epsilon|x|$.
a) Determine the form domain of H_ϵ and show that it is independent of ϵ.
b) What is the nature of the spectrum of the associated selfadjoint operator?
c) Let $\lambda_1(\epsilon)$ the smallest eigenvalue. Give rough estimates permitting to estimate from above or below $\lambda_1(\epsilon)$ independently of ϵ on every compact interval of I.
d) Show that, for any compact sub-interval J of I, there exists a constant C_J such that, for all $\epsilon \in J$, any L^2-normalized eigenfunction u_ϵ of H_ϵ associated with $\lambda_1(\epsilon)$ satisfies :

$$||u_\epsilon||_{B^1(\mathbb{R})} \leq C_J \,.$$

For this, on can play with : $\langle H_\epsilon u_\epsilon, u_\epsilon\rangle_{L^2(\mathbb{R})}$.
e) Show that the lowest eigenvalue is a monotonically increasing sequence of $\epsilon \in I$.
f) Show that the lowest eigenvalue is a locally Lipschitzian function of $\epsilon \in I$. One can use again the max-min principle.
g) Show that $\lambda(\epsilon) \to +\infty$, as $\epsilon \to +\infty$ and estimate the asymptotic behavior.
h) Discuss the same questions for the case $H_\epsilon = -d^2/dx^2 + x^2 + \epsilon x^4$ (with $\epsilon \geq 0$).

Problem 14.2.3. .
The aim of this problem is to analyze the spectrum $\Sigma^D(P)$ of the Dirichlet realization of the operator $P := (D_{x_1} - \frac{1}{2}x_2)^2 + (D_{x_2} + \frac{1}{2}x_1)^2$ in $\mathbb{R}^+ \times \mathbb{R}$.

1. *Show that one can a priori compare the infimum of the spectrum of P in $\mathbb{R}^2$ and the infimum of $\Sigma^D(P)$.*

2. *Compare $\Sigma^D(P)$ with the spectrum $\Sigma^D(Q)$ of the Dirichlet realization of $Q := D_{y_1}^2 + (y_1 - y_2)^2$ in $\mathbb{R}^+ \times \mathbb{R}$.*

3. *We first consider the following family of Dirichlet problems associated with the family of differential operators : $\alpha \mapsto H(\alpha)$ defined on $]0, +\infty[$ by :*

$$H(\alpha) = D_t^2 + (t - \alpha)^2 \ .$$

   Compare with the Dirichlet realization of the harmonic oscillator in $]-\alpha, +\infty[$.

4. *Show that the lowest eigenvalue $\lambda(\alpha)$ of $H(\alpha)$ is a monotonic function of $\alpha \in \mathbb{R}$.*

5. *Show that $\alpha \mapsto \lambda(\alpha)$ is a continuous function on $\mathbb{R}$.*

6. *Analyze the limit of $\lambda(\alpha)$ as $\alpha \to -\infty$.*

7. *Analyze the limit of $\lambda(\alpha)$ as $\alpha \to +\infty$.*

8. *Compute $\lambda(0)$. For this, one can compare the spectrum of $H(0)$ with the spectrum of the harmonic oscillator restricted to the odd functions.*

9. *Let $t \mapsto u(t; \alpha)$ the positive L^2-normalized eigenfunction associated with $\lambda(\alpha)$. Let us admit that this is the restriction to $\mathbb{R}^+$ of a function in $\mathcal{S}(\mathbb{R})$. Let, for $\alpha \in \mathbb{R}$, T_α be the distribution in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R})$ défined by*

$$\phi \mapsto T_\alpha(\phi) = \int_0^{+\infty} \phi(y_1, \alpha) u_\alpha(y_1) dy_1 \ .$$

   Compute QT_α.

10. *By constructing starting from T_α a suitable sequence of L^2-functions tending to T_α, show that $\lambda(\alpha) \in \Sigma^D(Q)$.*

11. *Determine $\Sigma^D(P)$.*

Problem 14.2.4. .
Let H_a be the Dirichlet realization of $-d^2/dx^2 + x^2$ in $]-a, +a[$.
(a) Briefly recall the results concerning the case $a = +\infty$.
(b) Show that the lowest eigenvalue $\lambda_1(a)$ of H_a is decreasing for $a \in]0, +\infty[$ and larger than 1.
(c) Show that $\lambda_1(a)$ tends exponentially fast to 1 as $a \to +\infty$. One can use a suitable construction of approximate eigenvectors.
(d) What is the behavior of $\lambda_1(a)$ as $a \to 0$. One can use the change of variable $x = ay$ and analyze the limit $\lim_{a \to 0} a^2 \lambda_1(a)$.
(e) Let $\mu_1(a)$ be the smallest eigenvalue of the Neumann realization in $]-a, +a[$. Show that $\mu_1(a) \leq \lambda_1(a)$.
(f) Show that, if u_a is a normalized eigenfunction associated with $\mu_1(a)$, then there exists a constant C such that, for all $a \geq 1$, we have :

$$||x u_a||_{L^2(]-a,+a[)} \leq C \ .$$

(g) Show that, for u in $C^2([-a,+a])$ and χ in $C_0^2(]-a,+a[)$, we have :

$$-\int_{-a}^{+a} \chi^2 u''(t)u(t)dt = \int_{-a}^{+a} |(\chi u)'(t)|^2 dt - \int_{-a}^{+a} \chi'(t)^2 u(t)^2 dt\,.$$

(h) Using this identity with $u = u_a$, a suitable χ which should be equal to 1 on $[-a+1, a-1]$, the estimate obtained in (f) and the minimax principle, show that there exists C such that, for $a \geq 1$, we have :

$$\lambda_1(a) \leq \mu_1(a) + Ca^{-2}\,.$$

Deduce the limit of $\mu_1(a)$ as $a \to +\infty$.

Problem 14.2.5.

We consider, for $c \in \mathbb{R}$, the differential operator which is defined on $C_0^\infty(\mathbb{R}^2)$ by

$$A_{0,c} = D_x^2 + (D_y - \frac{1}{2}x^2)^2 + icy\,,$$

and considered as an unbounded operator on $L^2(\mathbb{R}^2)$. We recall that $D_x = -i\partial_x$ et $D_y = -i\partial_y$. The problem consists in analyzing the spectral properties of $A_{0,c}$ as a function of $c \in \mathbb{R}$.

Part I

For $\eta \in \mathbb{R}$, we consider the unbounded operator on $L^2(\mathbb{R})$ $\mathfrak{h}_0(\eta)$ with domain $C_0^\infty(\mathbb{R})$ associated with the differential operator

$$\mathfrak{h}_0(\eta) := -\frac{d^2}{dx^2} + (\eta - \frac{x^2}{2})^2\,.$$

Ia) Show that, for any η, one can construct a selfadjoint extension of $\mathfrak{h}_0(\eta)$ and describe its domain.
Ib) Show that the operator has compact resolvent and deduce that the spectrum consists in a sequence Montrer que l'opérateur e of eigenvalues $\lambda_j(\eta)$ ($j \in \mathbb{N}^*$) tending to $+\infty$.
Ic) Show that $\lim_{\eta\to-\infty} \lambda_1(\eta) = +\infty$.
Id) Show that $\lambda_1(\eta) > 0$.
Ie) Show that $\eta \to \lambda_1(\eta)$ is continuous.
If) Show that $\eta \mapsto \lambda_j(\eta)$ is monotonically decreasing for $\eta < 0$.
Ig) We admit that $\lim_{\eta\to+\infty} \lambda_1(\eta) = +\infty$. Show that $\eta \mapsto \lambda_1(\eta)$ attains its infimum λ^* for (at least) one point.
Ih) We admit that $\eta \mapsto \lambda_1(\eta)$ is of multiplicity 1, of class C^1 and that one can, for any η, associate with $\lambda_1(\eta)$ an eigenfunction $u_1(\cdot,\eta)$ in $\mathcal{S}(\mathbb{R})$ with C^1 dependence of η such that $||u_1|| = 1$ and $u_1 > 0$.
Show that

$$\lambda_1'(\eta) = 2\int_{\mathbb{R}} (\eta - \frac{x^2}{2})\, u_1(x,\eta)^2\, dx\,.$$

Deduce that the critical points η_c of λ_1 satisfy $\eta_c > 0$.
Ii) Show that $u_1(\cdot, \eta)$ is even.
Ij) Show that, if η_c is a critical point of λ_1, then

$$I(\eta_c) := \int_0^{+\infty} x(\frac{x^2}{2} - \eta_c)\, u_1(x, \eta_c)^2\, dx \geq 0\,,$$

Ik) Computing differently $I(\eta)$, deduce that

$$\eta_c^2 \leq \lambda_1(\eta_c)\,.$$

Ik) Using Gaussian quasimodes for $\mathfrak{h}_1(0)$ determine an interval (as good as possible) in which λ_1 should have its minimum.

Part II

Here we suppose that $c = 0$ and we write A_0 for $A_{0,0}$.
IIa) Show that the operator A_0 is symmetric.
IIb) Show that one can construct its Friedrichs extension A_0^{Fried}, associated with a sesquilinear form to be defined precisely.
Describe the form domain and the domain of the operator.
IIc) Show that the operator is essentially selfadjoint.
IId) Show that

$$\sigma(A_0^{Fried}) \subset [\lambda^*, +\infty[\,.$$

It is suggested to use a partial Fourier transform with respect to y.
IIe) By constructing suitable families of approximate eigenfunctions, show that

$$\sigma(A_0^{Fried}) = [\lambda^*, +\infty[$$

IIf) Show that A_0^{Fried} is not with compact resolvent.

Part III

We now suppose that $c \neq 0$.
IIIa) Show that

$$Re\, \langle A_{0,c} u\,,\, u\rangle = \langle A_0 u\,,\, u\rangle\,,$$

and that

$$Im\, \langle A_{0,c} u\,,\, u\rangle = c\langle yu\,,\, u\rangle\,,$$

for all $u \in D(A_{0,c})$.
IIIb) Show that

$$Re\, \langle A_{0,c} u\,,\, u\rangle \geq \lambda^*\, ||u||^2\,,$$

for all $u \in D(A_{0,c})$.
IIIc) Show that $A_{0,c}$ is closable. We denote by $\overline{A_{0,c}}$ its closure. Recall how this operator is defined and describe its domain.
IIId) Show that, for $\lambda > -\lambda^*$, $\overline{A_{0,c}} + \lambda$ is injective and with closed range.
IIIe) Show that for $\lambda > -\lambda^*$, $\overline{A_{0,c}} + \lambda$ has dense range. **It is suggest to adapt**

the proof of IIc.

IIIf) Show that

$$\sigma(\overline{A_{o,c}}) \subset \{\lambda \in \mathbb{C}\,, \quad Re\,\lambda \geq \lambda^*\}\,.$$

The aim of questions IIIg to IIIk is to show that $\overline{A_{o,c}}$ has, for $c \neq 0$, a compact resolvent.

IIIg) Show that, for any compact set $K \subset \mathbb{R}^2$, there exists a constant C_K such that, for any $u \in C_0^\infty(\mathbb{R}^2)$ with compact support in K, we have

$$||u||^2_{H^1(\mathbb{R}^2)} \leq C_K \left(\; Re\,\langle A_{0,c} u\,,\, u\rangle + ||u||^2_{L^2} \right)\,.$$

IIIh) Show that, for all $u \in C_0^\infty(\mathbb{R}^2)$, with compact support in $\{y > 0\}$ or in $\{y < 0\}$, we have

$$\int |y||u(x,y)|^2 \, dxdy \leq \frac{1}{|c|} \mid \; Im\,\langle A_{0,c} u\,,\, u\rangle|\,.$$

By using a partition of unity, deduce that

$$|||y|^{\frac{1}{2}} u||^2 \leq C \left(||A_0 u||^2_{L^2} + ||u||^2_{L^2} \right)\,.$$

IIIi) Show that, for all $u \in C_0^\infty(\mathbb{R}^2)$, with compact support in $\{x > 0\}$ or in $\{x < 0\}$, we have

$$\int |x||u(x,y)|^2 \, dxdy \leq \quad Re\,\langle A_{0,c} u\,,\, u\rangle|\,.$$

By using a partition of unity, deduce that there exists a constant C such that, for all $u \in C_0^\infty(\mathbb{R}^2)$, we have

$$|||x|^{\frac{1}{2}} u||^2 \leq C \left(||A_0 u||^2_{L^2} + ||u||^2_{L^2} \right)\,.$$

IIIj) Deduce that $\overline{A_{0,c}}$ has a compact resolvent.

IIIk) Show that if (u,λ) is a spectral pair for $\overline{A_{0,c}}$, then, for all $a \in \mathbb{R}$, $(u_a, \lambda - ica)$, where u_a is defined by $u_a(x,y) = u(x, y+a)$, is a spectral pair.

IIIl) Deduce that the spectrum of $\overline{A_{0,c}}$ is empty.

IIIm) Show that $\overline{A_{0,c}} = A^*_{0,-c}$.

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