On Harper's equation for the Kagome lattice (after Kerdelhué–Royo-Letelier and Helffer-Kerdelhué-Royo-Letelier)

In honor of G. Nenciu for his 70-th birthday

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Abstract:
If the first mathematical results were obtained more than 30 years ago with the interpretation of the celebrated Hofstadter butterfly, more recent experiments in Bose-Einstein theory suggest new questions. I will start with a partial survey on old results (Helffer-Sjöstrand, Puig, Avila-Jitomirskaya-Krikorian,...) and then discuss more recent questions related to generalized butterflies (Dalibard and coauthors, Hou, Kerdelhué–Royo-Letelier). These new questions are strongly related to Harper on triangular or hexagonal lattices (in connection with the now very popular graphene). Our historics is focused on the mathematical results.

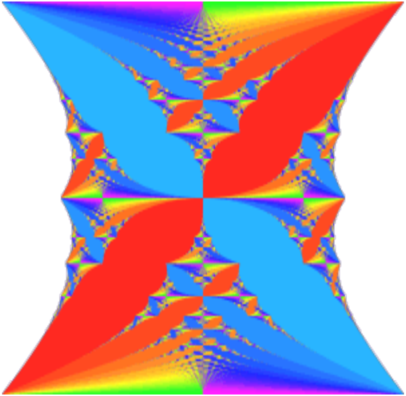
Introduction

The spectral properties of a charged particle in a two-dimensional system submitted to a periodic electric potential and a uniform magnetic field crucially depend on the arithmetic properties of the number γ representing the magnetic flux quanta through the elementary cell of periods, see e.g. [Bel] for a description of various models.

Since the works by Azbel [Az] and Hofstadter [Hof] it is generally believed that for irrational $\alpha = \gamma/2\pi$ the spectrum is a Cantor set, that is a nowhere dense (the interior of the closure is empty) and perfect set (closed $+$ no isolated point), and the graphical presentation of the dependence of the spectrum on γ shows a fractal behavior known as the Hofstadter butterfly.

The gaps in the spectrum.

This is the "colored" butterfly realized in 2003 by Y. Avron and his team.



After intensive efforts this was rigorously proved recently (Ten Martinis conjecture) for all irrational values of α for the discrete Hofstadter model, i.e. the discrete magnetic Laplacian admitting a reduction to the almost Mathieu equation, see [AvJi] and references therein.

Only few results are available for other models. Traditionally, various semiclassical methods play an important role in the analysis of the two-dimensional magnetic Schrödinger operators with periodic potentials, see e.g. [BDP] for a review. In particular, the bottom part of the spectrum for strong magnetic fields can be described up to some extent using the tunnelling asymptotics, Wannier functions where G. Nenciu was strongly involved,... and this leads to simpler models like Harper.
Usually physicists have no problems to use these results without to come back to the initial problem.

Coming back to mathematics, a more detailed analysis (Bellissard, Helffer and Sjöstrand [HS1, HS2, HS3]) shows that the study of some parts of the spectrum for the Schrödinger operator with a magnetic field and a periodic electric potentials reduces to the spectral problem for an operator pencil of one-dimensional quasiperiodic pseudodifferential operators.

Under some symmetry conditions for the electric potentials, the operator pencil reduces to the study of small perturbation of the continuous analog of the almost-Mathieu (=Harper) operator, which allowed one to carry out a rather detailed iterative analysis for special values of α.
In particular, in several asymptotic regimes a Cantor structure of the spectrum was proved.
This involved a pseudo-differential calculus, whose relevance in this context was predicted by the british physicist Wilkinson.

Pseudo-differential operators

In [HS1, HS2, HS3] (1988-1990) a machinery was developed for an iterative semiclassical analysis of a special class of pseudodifferential operators. One was concerned with the non-linear spectral problem (or, in other words, with the spectral problem for an operator pencil). Namely, for a family of self-adjoint operators $A(\mu)$ depending $\mu \in \mathbb{R}$ the μ-spectrum $\mu\text{-spec}\, A(\mu)$ denotes the set of all μ such that $0 \in \operatorname{Spec} A(\mu)$. The simplest case being the family $A - \mu$.

Quantization

Let $L : \mathbb{R}^2 \to \mathbb{R}$ be a periodic smooth function, $L(x, \xi + 2\pi; \mu, h) = L(x + 2\pi, \xi; \mu, h) = L(x, \xi; \mu, h)$. Here μ and h are real parameters. By the Weyl quantization procedure one can assign to L an operator $\hat{L}_h(\mu)$ in $L^2(\mathbb{R})$ by

$$\hat{L}_h(\mu) f(x) = \frac{1}{2\pi h} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\xi(x-y)/h} L\left(\frac{x+y}{2}, \xi; \mu, h\right) f(y) d\xi \, dy. \quad (1)$$

The operator $\hat{L}_h$ obtained is referred to as the Weyl h-quantization of L, and quantum Hamiltonians resulting from periodic symbols are often called Harper-like operators.

In particular, the symbol $L(x, \xi) := \cos x + \cos \xi$ produces the Harper operator on the real line,

$$\hat{L}_h f(x) = \frac{f(x+h) + f(x-h)}{2} + \cos x \, f(x). \tag{2}$$

In [HS3], in order to treat the Harper operator and perturbations of it occuring in a renormalization procedure, the following notion was introduced.

Definition

A symbol $L(x, \xi; \mu, h)$ will be called of strong type I if the following conditions are satisfied for all $h \in (0, h_0)$ with some $h_0 > 0$:

(a) L depends analytically on $\mu \in [-4, 4]$.

(b) There exists $\varepsilon > 0$ such that

  (b1) $L(x, \xi; \mu, h)$ is holomorphic in

$$D_\varepsilon = \left\{(\mu, x, \xi) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C} : |\mu| \leq 4, |\Im x| < \frac{1}{\varepsilon}, |\Im \xi| < \frac{1}{\varepsilon}, \right\},$$

  (b2) for $(\mu, x, \xi) \in D_\varepsilon$, there holds

$$\left| L(x, \xi; \mu, h) - (\cos x + \cos \xi - \mu) \right| \leq \varepsilon.$$

Continuation of the definition

(c) The following symmetry conditions hold:

$$L(x,\xi;\mu,h) = L(\xi,x;\mu,h) = L(x,-\xi;\mu,h)$$
$$L(x,\xi;\mu,h) = L(x+2\pi,\xi;\mu,h) = L(x,\xi+2\pi;\mu,h).$$

By $\varepsilon(L)$ we will denote the minimal value of ε for which the above conditions hold.

In [HS1, HS2, HS3] a detailed analysis was performed for pseudodifferential operators associated with strong type I symbols. One of the results was

Theorem 1

Let $L(\mu, h)$ be a strong type I symbol. There exist ϵ_0 , C s. t. if $\varepsilon(L) \leq \epsilon_0$ and if $(2\pi)^{-1}h$ is an irrational admitting a representation as a continuous fraction

$$\frac{h}{2\pi} = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \dots}}}$$

with $n_j \geq C$, then the μ-spectrum of the associated operators $\hat{L}_h(\mu)$ is a zero measure Cantor set.

In particular, this applies to the spectrum of the Harper's model. But the theorem says also that this is stable by perturbations respecting all the symmetries.

Schrödinger operators with magnetic potentials

For operators $H = \sum_{j=1}^{2}(\hbar D_{x_j} - A_j)^2 + V$ with periodic potentials V,

$$V(x_1 + 2\pi, x_2) \equiv V(x_1, x_2 + 2\pi) \equiv V(x_1, x_2)\ ,$$

and constant (or periodic) magnetic fields

$$\mathrm{Curl}\,\vec{A} = B\ ,$$

it was shown in several asymptotic regimes that the study of some parts of the spectrum reduces to a non-linear spectral problem of the type above.
We will see later that we can consider other symmetries.

This is for example the case for

- B^{-1}-pseudodifferential operators with symbols close to $V(x, \xi)$ (see for example [HS4] which treats the strong magnetic case)
- B-pseudodifferential operators with symbols to the first Floquet eigenvalue of the Schrödinger operator without magnetic field (Peierls substitution) (corresponding to the case of the weak magnetic field, see [HS1], [HS3] and [HS4] and earlier contributions by mathematicians and physicists (see the surveys by J. Bellissard in [Bel], G. Nenciu in [Ne2], J. Sjöstrand [Sj] and references therein).

Hence, strong type I operators appear for strong magnetic field when considering potentials V close to $\cos x_1 + \cos x_2$.
Moreover in the semi-classical limit $\hbar \to 0$ or in the tight binding situation, it can be shown (case of a square lattice) that—up to the multiplication by an exponentially small term corresponding to the tunneling— the lowest Floquet eigenvalue is close to $(\cos\theta_1 + \cos\theta_2)$.
Here it is important to assume the symmetry for V $V(-x_2, x_1) = V(x_1, x_2)$, an assumption of non degenerate minima for V (one for each cell) and a geometric assumption on the geodesics for neighboring wells (the geometry is the Agmon metric $(V - \min V)dx^2$).

Symbols associated with some discrete operators

It is well known that the spectrum of the operator (2) as a set coincides with the spectrum of the discrete magnetic Laplacian acting on $\ell^2(\mathbb{Z}^2)$, see e.g. [HS1],

$$C_h f(m,n) = e^{ihn} f(m+1,n) + e^{-ihn} f(m-1,n) + f(m,n-1) + f(m,n+1).$$

More generaly consider a bounded linear operator C_h acting on $\ell^2(\mathbb{Z}^2)$ given by an infinite matrix $\big(C(p,q)\big)$, $p,q \in \mathbb{Z}^2$, satisfying

$$C(p+k, q+k) = e^{-ihk_2(p_1-q_1)} C(p,q), \quad p,q,k \in \mathbb{Z}^2, \qquad (3)$$

with some $h > 0$.

Proposition A

Let C_h be a bounded self-adjoint operator in $\ell^2(\mathbb{Z}^2)$ with the property (3) and satisfying $|C(p,q)| \le ae^{-b|p-q|}$ for some $a, b > 0$ and all $p, q \in \mathbb{Z}^2$. Then the spectrum of C_h coincides with the spectrum of the Weyl h-quantization of the symbol T given by

$$T(x,\xi) = \sum_{m,n\in\mathbb{Z}} c(m,n)e^{-imnh/2}e^{i(mx+n\xi)}, \qquad (4)$$

where $c(m,n) = C\big((0,0),(m,n)\big)$, $m, n \in \mathbb{Z}$.

A third point of view

Let us return to the initial operator C_h. By assumption, $C(p,q) = \exp\left(ihp_2(q_1 - p_1)\right)c(q-p)$ for any $p, q \in \mathbb{Z}^2$, hence

$$
\begin{aligned}
C_h f(p) &= \sum_{q\in\mathbb{Z}^2} e^{ihp_2(q_1-p_1)} c(q-p) f(q) \\
&= \sum_{q\in\mathbb{Z}^2} e^{ihp_2 q_1} c(q) f(p+q).
\end{aligned}
$$

Therefore, C_h commutes with the shift $f(p_1, p_2) \mapsto f(p_1+1, p_2)$, and the Floquet-Bloch theory is applicable.

We get a family of operators acting in $\ell^2(\mathbb{Z})$,

$$C_h(\theta)g(m) = \sum_{n\in\mathbb{Z}} b_n(mh+\theta)g(m+n), \quad m \in \mathbb{Z}, \quad \theta \in \mathbb{R}\,,$$

which satisfies

$$C_h(\theta) = C_h(\theta + 2\pi)\,.$$

Therefore, by the Floquet-Bloch theory, one has

$$\mathrm{Spec}\, C_h = \bigcup_{\theta\in[0,2\pi)} \mathrm{Spec}\, C_h(\theta)\,.$$

Furthermore, for any θ the operators $C_h(\theta)$ and $C_h(\theta+h)$ are unitarily equivalent, $C_h(\theta+h) = SC_h(\theta)S^{-1}$, where S is the shift in $\ell^2(\mathbb{Z})$, $Sf(n) = f(n+1)$, which implies
$\mathrm{Spec}\, C_h = \bigcup_{\theta\in[0,h)} \mathrm{Spec}\, C_h(\theta)$.
This coincides with the spectrum of the following operator T_h acting in $L^2(\mathbb{Z}\times[0,h))$

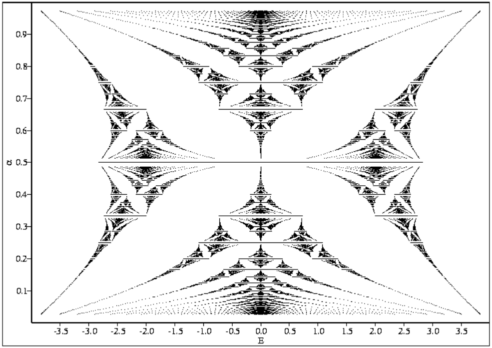
$$T_h u(m,\theta) = C_h(\theta)u_\theta(m), \quad u_\theta(m) = u(m,\theta), \quad m\in\mathbb{Z}.$$

Finally when $h/2\pi$ is irrational, the spectrum is independent of θ.

The Hofstadter butterfly

In the case of the symbol $(x, \xi) \mapsto \cos x + \cos \xi$ we get the **Hofstadter's butterfly**

On the vertical axis the parameter proportional to the flux $\alpha = \frac{h}{2\pi} \in [0, 1]$. On the horizontal line $y = \alpha$ the union over θ of the spectra of the family $C_h(\theta)$. The picture results of computations for rational α's.



The hamiltonian point of view permits to explain the behavior of the spectrum as $\alpha \mapsto 0$ or more generally as $\alpha \to \frac{p}{q}$.

Cantor structure

Let us consider more generally the family of operators on $\ell^2(\mathbb{Z})$

$$(H_{\lambda,\alpha}u)_n = u_{n+1} + u_{n-1} + 2\lambda \cos 2\pi(\theta + n\alpha)u_n \,.$$

Different names for this operator are given including Harper or Almost-Mathieu.

If $\alpha = \frac{p}{q}$ is rational the spectrum consists of the union of q intervals possibly touching at the end point. If α is irrational the spectrum is independent of θ and:

Ten Martini Theorem

The spectrum of the almost Mathieu operator $H_{\lambda,\alpha}$ is a Cantor set for all irrational α and for all $\lambda \neq 0$.

Previously, we were discussing the case $\lambda = 1$. The Ten Martini conjectures was popularized by B. Simon in reference to some offer of M. Kac.
Computations for $\lambda \neq 1$ are proposed in a "numerical" paper of Guillement-Helffer-Treton [GHT].

Historics
Azbel (1964), Bellissard-Simon (1982), Van Mouche (1989), Helffer-Sjöstrand (1989), Puig (2004), Avila-Krikorian (2008), Avila-Jitomirskaya (2009).
Unfortunately Mark Kac died before to know that he has to buy these ten Martini.

Other examples

We first mentione the triangular case

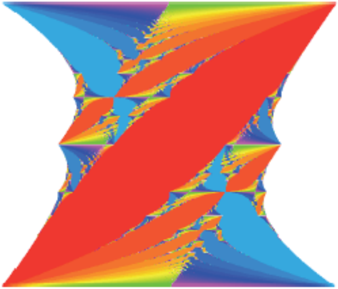


Figure 1: A phase diagram for the Hofstadter model on a triangular lattice where the flux through the down triangles $\Phi_d = \pi/2$. The vertical axis is the total flux Φ. The horizontal axis is the chemical potential. The colors represent the Chern numbers. The model is inversion symmetric, see sec. A.1.

Figure: Picture by J. E. Avron, O. Kenneth and G. Yeshoshua (2013).

One should add the graphene case (or hexagonal case).

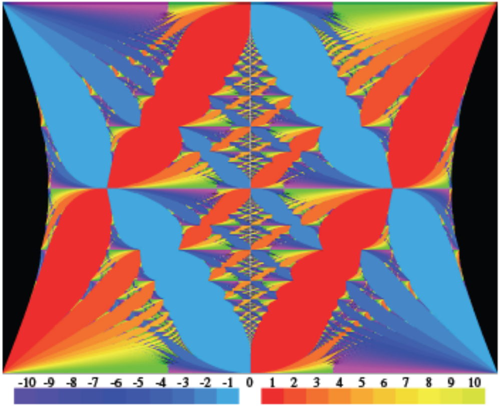


Figure 5: The colored Hofstadter butterfly for the honeycomb lattice, as obtained by the method of this paper. The vertical axis is the magnetic flux per unit cell Φ ranging from 0 to 1. The horizontal axis is the Fermi energy ranging from -3 to 3. The colors represent the Chern numbers. The resolution of this figure is 1920×1440 and the maximal value of q is $q_{max} = 720$.

Figure: The colored Hofstadter butterfly for the Honeycomb lattice by A. Agazzi, J.-P. Eckmann, and G.M. Graf (2014) .

In her thesis J. Royo-Letelier has started (see [Hou]) to analyze rigorously the case of a Kagome lattice. This was extended in a paper in collaboration with P. Kerdelhué. Questions around the Chambers's formula have been analyzed by Helffer-Kerdelhué-Royo-Letelier. This involves new semi-classical problems related to "flat" bands.

Kagome lattice

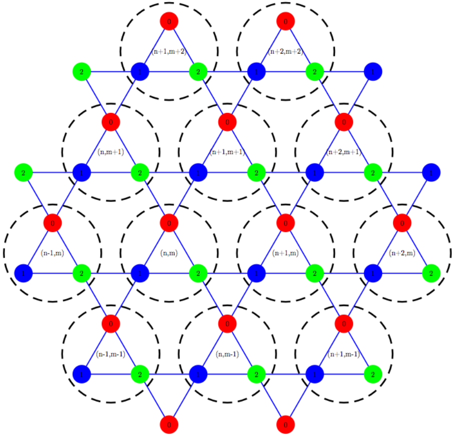
The kagome lattice is not a Bravais lattice, but is a discrete subset of $\mathbb{R}^2$ invariant under translations along a triangular lattice and containing three points per fundamental domain of this lattice. Each point of the lattice has four nearest neighbours for the Euclidean distance. The word *kagome* means a bamboo-basket (kago) woven pattern (me) and it seems that the lattice was named by the Japanese physicist K. Husimi .

Let $\Gamma_{\triangle}$ be the triangular lattice spanned by $\mathcal{B} = \{2\nu_1, 2\nu_2\}$, where

$$\nu_\ell = r^{\ell-1}(1,0)$$

and r is the rotation of angle $\frac{\pi}{3}$ and center the origin.
The kagome lattice can be seen as the union of three conveniently translated copies of $\Gamma_{\triangle}$:

$$\Gamma = \left\{ m_{\alpha,\ell} = 2\alpha_1\nu_1 + 2\alpha_2\nu_2 + \nu_\ell \, ; \, (\alpha_1, \alpha_2) \in \mathbb{Z}^2 \, , \, \ell = 1, 3, 5 \right\}.$$


(n+1,m+2)
(n+2,m+2)
(n,m+1)
(n+1,m+1)
(n+2,m+1)
(n-1,m)
(n,m)
(n+1,m)
(n+2,m)
(n-1,m-1)
(n,m-1)
(n+1,m-1)

Coming from a Schrödinger operator

As in the Harper model there is an electric potential whose minima are on a Kagome lattice. Moreover there are examples obtained with trigonometric polynomials. This means that they can be obtained by a combination of lasers.
Remembering the definitions of the vectors ν_j, we denote by $\nu^\perp$ the vector deduced from ν by a rotation of $\frac{\pi}{2}$ and for $j \in \{1,3,5\}$ we define

$$\mu_j = \sqrt{3}\,\nu_j^\perp\,.$$

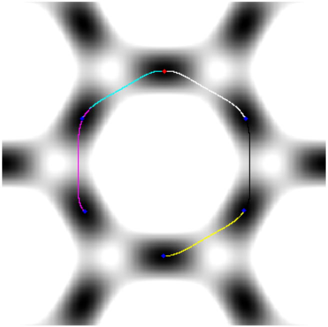
For $j = 1,3,5$ we set $\phi_j = 3\pi/2$ and define the potentials $V_j : \mathbb{R}^2 \to \mathbb{R}$ as

$$V_j(x) = \left[\cos\left(x \cdot \pi\mu_j + \phi_j\right) + 2\cos\left(\frac{x \cdot \pi\mu_j + \phi_j}{3}\right)\right]^2\,,$$

and V as

$$V = -V_1 - V_3 - V_5\,.$$

V has local minima at the points of the kagome lattice.



The minima appear on the center of the black zone around an hexagon. The maximum at the center of the hexagon. Each minimum has four nearest neighbors (for the Agmon distance). These minima are leaving on a kagome lattice (subset of an hexagonal lattice). The figure is invariant by the double triangular lattice.

Analysis of the rational case and Chambers formula

Once a semi-classical (or tight-binding) approximation is done, involving a tunneling analysis and a construction of Wannier functions we arrive (modulo a controlled smaller error) in the case of a square lattice to the so-called Harper model, which is defined on $\ell^2(\mathbb{Z}^2, \mathbb{C})$ by

$$(Hu)_{m,n} := \frac{1}{2}(u_{m+1,n} + u_{m-1,n}) + \frac{1}{2} e^{i\gamma m} u_{m,n+1} + \frac{1}{2} e^{-i\gamma m} u_{m,n-1}\,,$$

where γ denotes the flux of the constant magnetic field through the fundamental cell of the lattice.
When $\frac{\gamma}{2\pi}$ is a rational, a Floquet theory permits to reduce the analysis to the analysis of the eigenvalues of a family of $q \times q$ matrices depending on a parameter $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$.

More precisely, when

$$\gamma = 2\pi p/q\,, \tag{5}$$

where $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$ are relatively prime, the two following matrices play an important role:

$$J_{p,q} = \mathrm{diag}(e^{i(j-1)\gamma})\,, \tag{6}$$

and

$$(K_q)_{jk} = 1 \;\text{ if } k \equiv j+1[q]\,,\ 0 \text{ else.} \tag{7}$$

In the case of Harper, the family of matrices is

$$M_H(\theta_1, \theta_2) = e^{i\theta_1} J_{p,q} + e^{-i\theta_1} J_{p,q}^* + e^{i\theta_2} K_q + e^{-i\theta_2} K_q^* \,. \qquad (8)$$

The Hofstadter butterfly is then obtained as a picture in the rectangle $(-2, +2) \times [0, 1]$. A point $(\lambda, \gamma/2\pi)$ is in the picture if there exists θ such that
$\det(M_H(\theta_1, \theta_2) - \lambda) = 0$ for some $\frac{p}{q}$ with $p/q \in [0, 1]$ ($q \leq 50$).
The Chambers formula gives a very elegant formula for this determinant:

$$\det(M_H(\theta_1, \theta_2) - \lambda) = f_{p,q}^H(\lambda) + (-1)^q (\cos q\theta_1 + \cos q\theta_2) \,, \qquad (9)$$

where f^H is a polynomial of degree q.

Many other models have been considered. In the case of a triangular lattice, the second model is, according to [Ke] (see also [Avetal]),

$$\begin{aligned} M_T(\theta_1, \theta_2, \phi) &= e^{i\theta_1} J_{p,q} + e^{-i\theta_1} J^*_{p,q} + e^{i\theta_2} K_q + e^{-i\theta_2} K^*_q \\ &\quad + e^{i\phi} e^{i(\theta_1-\theta_2)} J_{p,q} K^*_q + e^{-i\phi} e^{i(\theta_2-\theta_1)} K_q J^*_{p,q} \end{aligned} \tag{10}$$

with $\phi = -\gamma/2$.

The Chambers formula in this case takes the form

$$\begin{aligned} &\det(M_T(\theta_1, \theta_2, \phi) - \lambda) \\ &= f^T_{p,q}(\lambda) + (-1)^{q+1} \left(\cos q\theta_1 + \cos q\theta_2 + \cos q(\theta_2 - \theta_1 - \phi)\right) . \end{aligned} \tag{11}$$

In the case of the hexagonal lattice, which appears also in the analysis of the graphene, we have to analyze

$$M_G(\theta_1,\theta_2) := \begin{pmatrix} 0 & I_q + e^{i\theta_1} J_{p,q} + e^{i\theta_2} K_q \\ I_q + e^{-i\theta_1} J^*_{p,q} + e^{-i\theta_2} K^*_q & 0 \end{pmatrix} \tag{12}$$

We denote by P_G the characteristic polynomial of M_G. The resulting spectrum is given in Figure 3.

Finally, inspired by the physicist Hou, [KR] have shown that for the kagome lattice, the following approximating model is relevant. we consider the matrix:

$$M_K(\theta_1,\theta_2,\omega) = \begin{pmatrix} 0 & A(\theta_1,\theta_2,\omega) & B(\theta_1,\theta_2,\omega) \\ A^*(\theta_1,\theta_2,\omega) & 0 & C(\theta_1,\theta_2,\omega) \\ B^*(\theta_1,\theta_2,\omega) & C^*(\theta_1,\theta_2,\omega) & 0 \end{pmatrix}, \tag{13}$$

with

$$\begin{aligned} A(\theta_1,\theta_2,\omega) &= e^{i(\omega+\frac{\gamma}{8})}(e^{-i\theta_1}J^*_{p,q} + e^{-i\frac{\gamma}{2}}e^{-i(\theta_1-\theta_2)}J^*_{p,q}K_q) \\ B(\theta_1,\theta_2,\omega) &= e^{-i(\omega+\frac{\gamma}{8})}(e^{-i\theta_1}J^*_{p,q} + e^{-i\theta_2}K^*_q) \\ C(\theta_1,\theta_2,\omega) &= e^{i(\omega+\frac{\gamma}{8})}(e^{-i\frac{\gamma}{2}}e^{i(\theta_1-\theta_2)}J_{p,q}K^*_q + e^{-i\theta_2}K^*_q)\,. \end{aligned} \tag{14}$$

Here ω is a parameter appearing in the model (most of the physicists consider without justification the case $\omega = 0$). We refer to [KR] for a discussion of this point.

The trigonometric polynomial

$$(x, \xi) \mapsto p^{\triangle}(x, \xi) = \cos x + \cos \xi + \cos(x - \xi) \qquad (15)$$

which was playing an important role in the analysis of the triangular Harper model (see Claro-Wannier [CW] and Kerdelhué [Ke]) will also appear in our analysis.
We denote by $P_K(\theta_1, \theta_2, \lambda)$ the characteristic polynomial $\det(\lambda I_{3q} - M(\theta_1, \theta_2))$.
We prove that for a model considered by Hou [Hou] , there exists a formula which is similar to the one obtained by Chambers [Ch] for the Harper model. (see also Helffer-Sjöstrand [HS1], [HS2], Bellissard-Simon [BelSim], C. Kreft [Kr], I. Avron (and coauthors) [Avetal]).

The first statement is probably well known in the physical literature.

Theorem [Graphene]

$$P_G(\theta_1, \theta_2, \lambda) = (-1)^q \det(M_T(\theta_1, \theta_2, 0) + 3 - \lambda^2)\,. \qquad (16)$$

The second statement was to our knowledge unobserved.

Theorem [Kagome]

There exists a polynomial Q_ω of degree $3q$, with real coefficients, depending on γ and possibly on ω, but not on (θ_1, θ_2), such that

$$
\begin{aligned}
&P_K(\theta_1, \theta_2, \omega, \lambda) \\
&= Q_\omega(\lambda) + 2p^\triangle(q(\theta_1 + p\pi), q(\theta_2 + p\pi)) \left(\lambda + 2\cos(3\omega - \tfrac{\gamma}{8})\right)^q .
\end{aligned} \tag{17}
$$

Corollary

A flat band exists if and only if

$$Q_\omega(-2\cos(3\omega - \frac{\gamma}{8})) = 0 .$$

Let us illustrate by some examples mainly extracted of [KR].
In the case when $q = 1$ and $p = 0$, one finds, for the Hou's model:

$$P(\theta_1, \theta_2, \lambda) = -\lambda^3 + 6\lambda + 4\cos(3\omega) + 2\,(\lambda + 2\cos(3\omega))\, p^{\triangle}(\theta_1, \theta_2)\,.$$

Hence, we have in this case:

$$Q_\omega(\lambda) = -\lambda^3 + 6\lambda + 4\cos(3\omega)\,.$$

The condition for a flat band reads:

$$Q_\omega(-2\cos(3\omega)\ ) = 0\,,$$

which takes the simple form: $(\cos 3\omega)^3 - \cos 3\omega = 0\,.$
Hence $\cos 3\omega = 0$ or $\cos 3\omega = \pm 1$. So the "flat bands" appear only for discrete value of ω, including the particular case $\omega = 0$, mostly considered in the physical literature. Note that in [KR], it is proved only that $\omega \to 0$ as a function of the initial semi-classical parameter.

We now consider other examples:

- For the triangular model, for $p/q = 1/6$, the spectrum is given by :

$$\lambda^6 - 18\lambda^4 - 12\sqrt{3}\lambda^3 + 45\lambda^2 + 36\sqrt{3}\lambda + 6 - 2p^{\triangle}(6\theta_1, 6\theta_2) = 0\,.$$

  $Q(\lambda)$ satisfies $Q(-\sqrt{3}) = Q'(-\sqrt{3}) = 0$. The second gap is closed.

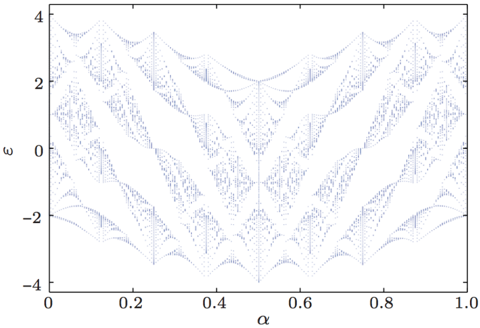
- For the Graphene model, for $p/q = 1/2$, the spectrum is given by

$$\lambda^4 - 6\lambda^2 + 3 - 2(\cos(2\theta_1) + \cos(2\theta_2) - \cos(2(\theta_1 - \theta_2)))$$

  The bands are $[-\sqrt{6}, -\sqrt{3}]$, $[-\sqrt{3}, 0]$, $[0, \sqrt{3}]$ and $[\sqrt{3}, \sqrt{6}]$.

- For the Hou-model, as shown in [KR] for $\omega = \pi/8$ and $p/q = 3/2$, the bands are $\{-2\}$ (with multiplicity 2), $[1 - \sqrt{6}, 1 - \sqrt{3}]$, $[1 - \sqrt{3}, 1]$, $[1, 1 + \sqrt{3}]$ and $[1 + \sqrt{3}, 1 + \sqrt{6}]$.

The Kagome butterfly



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