Semi-classical analysis and Harper's equation
Four lectures in Nanjing University

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Abstract:
If the first mathematical results were obtained more than 30 years ago with the interpretation of the celebrated Hofstadter butterfly proposed in 1976, more recent experiments in Bose-Einstein theory suggest new questions.
I will present a survey on rather old results of Helffer-Sjöstrand (at the end of the eighties) based on an illuminating strategy proposed by the physicist M. Wilkinson in 1985. This leads to the proof of the Cantor structure of the spectrum for the Harper model for a some specific family of irrational fluxes (hence a very particular case of the ten Martinis conjecture of M. Kac popularized by B. Simon and proven recently (the proof was achieved in 2009) by A. Avila, S. Jitomirskaya and coauthors) but also a detailed presentation of the structure of the spectrum.

In these four lectures, we will present how semi-classical analysis appears in the analysis of this problem. It appears actually in two connected ways:

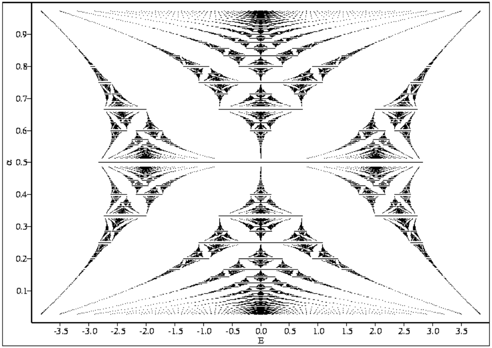
- First, when analyzing the bottom of the spectrum of a Schrödinger operator with constant magnetic field and electric periodic potential, the Harper's model is, in various asymptotic regimes, the right approximation of an effective Hamiltonian.
- Secondly, the analysis of the spectrum of the Harper's model can be done for some fluxes by semi-classical analysis.

We hope to give a flavor of the tools used in this context together with precise references.
If time permits, we will discuss more recent results and still remaining open problems.
End of Abstract

Introduction

The spectral properties of a charged particle in a two-dimensional system submitted to a periodic electric potential and a uniform magnetic field crucially depend on the arithmetic properties of the number α representing the magnetic flux quanta through the elementary cell of periods, see e.g. [Bel] for a description of various models.
Since the works by Azbel [Az] and Hofstadter [Hof] it is generally believed that for irrational α the spectrum is a Cantor set, that is a nowhere dense (the interior of the closure is empty) and perfect set (closed $+$ no isolated point), and the graphical presentation of the dependence of the spectrum on α shows a fractal behavior known as the Hofstadter butterfly.

The **Hofstadter's butterfly** is obtained in the following way. We put on the vertical axis the parameter proportional to the flux $\alpha = \frac{h}{2\pi} \in [0,1]$ and on the horizontal line $y = \alpha$ the union over θ of the spectra of the family $H_\alpha(\theta)$. The picture results of computations for rational α's.



Let us consider more generally the family of operators on $\ell^2(\mathbb{Z})$

$$(H_{\lambda,\alpha} u)_n = \frac{1}{2}(u_{n+1} + u_{n-1}) + \lambda \cos 2\pi(\theta + n\alpha) u_n \,.$$

Different names for this operator are given including Harper or Almost-Mathieu.

If $\alpha = \frac{p}{q}$ is rational the spectrum consists of the union of q intervals possibly touching at the end point. If α is irrational the spectrum is independent of θ and:

Ten Martini Theorem

The spectrum of the almost Mathieu operator $H_{\lambda,\alpha}$ is a Cantor set for all irrational α and for all $\lambda \neq 0$.

The Ten Martini conjectures was popularized by B. Simon in reference to some offer of M. Kac.

Computations for $\lambda \neq 1$ are proposed in a "numerical" paper of Guillement-Helffer-Treton [GHT].

After intensive efforts (we can mention Azbel (1964), Bellissard-Simon (1982), Van Mouche (1989), Helffer-Sjöstrand (1989), Puig (2004), Avila-Krikorian (2008)) this Cantor set structure was rigorously proved in 2009 by Avila-Jitomirskaya for all irrational values of α (see [AvJi] and references therein) for the models

$$u \mapsto (H_\alpha(\lambda,\theta)u)_n = \frac{1}{2}\left(u_{n+1} + u_{n-1}\right) + \lambda \cos(2\pi(\alpha n + \theta))u_n \,.$$

with $\lambda > 0$.
Unfortunately Mark Kac died before to know that he has to buy these ten Martini.

Only few results are available for other models. Traditionally, a couple of semiclassical methods plays an important role in the analysis of the two-dimensional magnetic Schrödinger operators with periodic potentials, see e.g. [BDP] for a review. In particular, the bottom part of the spectrum for strong magnetic fields can be described up to some extent using the tunnelling asymptotics. We will discuss this point in the first part of our lectures.
But physicists have no problems to use these results without to come back to the initial problem.

Coming back to mathematics, a more detailed analysis (Helffer and Sjöstrand – HSHarper1,HSHarper2,HSHarper3–in the years 1988-1990) shows that the study of some parts of the spectrum for the Schrödinger operator with a magnetic field and a periodic electric potentials reduces to the spectral problem for an operator pencil of one-dimensional quasiperiodic pseudodifferential operators.

Under some symmetry conditions for the electric potentials, the operator pencil reduces to the study of small perturbation of the continuous analog of the almost-Mathieu (=Harper) operator, which allowed one to carry out a rather detailed iterative analysis for special values of α.
In particular, in several asymptotic regimes a Cantor structure of the spectrum was proved.
This involved a pseudo-differential calculus, whose relevance in this context was predicted by the physicist Wilkinson (from United Kingdom) in the middle of the eighties.
End of Introduction

Preliminary properties and first meeting with the pseudo-differential calculus

We are interested in $\cup_\theta \sigma(H(\theta))$.
We observe that (with $h = 2\pi\alpha$)

$$H(\theta) = H(\theta+1) \text{ and } H(\theta+h) \text{ is unitary equivalent to } H(\theta)\,.$$

This implies that if $\alpha \notin \mathbb{Q}$, then the spectrum is independent of θ and secondly that

$$\cup_\theta \sigma(H(\theta)) = \sigma(\widetilde{H})\,,$$

where $\widetilde{H} : L^2(\mathbb{Z} \times [0,h)) \mapsto L^2(\mathbb{Z} \times [0,h))$ is defined by

$$\left(\widetilde{H}u\right)(\cdot,\theta) = H(\theta)u(\cdot,\theta)\,.$$

If we identify $L^2(\mathbb{Z} \times [0,h))$ with $L^2(\mathbb{R})$ by

$$u(k,\theta) = \widetilde{u}(\theta + hk)$$

the operator $\widetilde{H}$ becomes

$$\widetilde{H} = \frac{1}{2}(\tau_h + \tau_{-h}) + \lambda \cos x$$

where τ_h is the translation operator:

$$\tau_h v(x) = v(x-h)\,.$$

If we observe that $\tau_h = \exp ihD_x$, we can rewrite $\widetilde{H}$ as a h-pseudodifferential operator

$$\cos hD_x + \lambda \cos x$$

whose h-symbol is $\cos \xi + \lambda \cos x$.

In this last formalism, the Aubry duality is obtained by using a h-Fourier transform.

$$\mathcal{F}_h u(\xi) = (2\pi h)^{-\frac{1}{2}} \int e^{-ix\xi/h} u(x)\, dx\,.$$

By conjugation, the operator becomes

$$\lambda \cos(hD_\xi) + \cos x = \lambda(\cos(hD_\xi) + \frac{1}{\lambda}\cos x)\,.$$

We recall that the h-quantization of a symbol $p(x, \xi, h)$ with values in $M_n(\mathbb{C})$ is the pseudo-differential operator defined over $L^2(\mathbb{R}; \mathbb{C}^n)$ by

$$\left((\mathrm{Op}_h^W p)u\right)(x) = \frac{1}{2\pi h} \iint_{\mathbb{R}^2} e^{i\frac{(x-y)\xi}{h}} p\left(\frac{x+y}{2}, \xi, h\right) u(y)\, dy\, d\xi\,. \tag{1}$$

- One remark on renormalization. If $\tau = \tau_{2\pi}$ and $\hat{\tau}$ is the multiplication operator by $e^{2\pi i x/h}$, then $\widetilde{H}$ commutes with τ and $\hat{\tau}$.
  An important point is that τ and $\hat{\tau}$ do not necessarily commute.

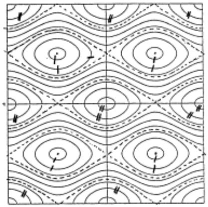
$$\tau\,\hat{\tau} = \exp(-i(2\pi)^2/h)\,\hat{\tau}\,\tau = \exp -i\tilde{h}\,\hat{\tau}\,\tau\,,$$

  with

$$(2\pi)/h = k + \tilde{h}/(2\pi)\,.$$

- The analysis of the energy levels (see next slide) of the symbol will play an important role.

Energy levels

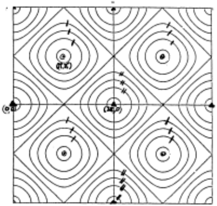


$-$: $-(1-\lambda)<E<(1-\lambda)$

$\nmid$: $-(1+\lambda)<E<-(1-\lambda)$

$\#$: $1-\lambda \leq E \leq 1+\lambda$

Pour $\lambda = 1$ on trouve une structure plus simple :



$-$: $E=0$ $\quad$ $\odot$: $E=-2$ $\quad$ $\nmid$: $-2<E<0$ $\quad$ Δ : $E=2$ $\quad$ $\#$: $0<E<2$

The lectures will present four connected points (in an order not yet decided)

- From Schrödinger equation to Harper's model.
- The rational case for the Harper's model. We will discuss various aspects of the Hofstadter model for fluxes close to a rational. The material comes from Wilkinson, Sokoloff, Bellissard and HSHarper2.
- Some hints for the renormalization procedure[1] in the irrational case leading to the proof of the Cantor structure. The material comes from HSHarper1 and HSHarper3.
- Discuss some conjectures around the wings. The material comes from a paper of Helffer-Kerdelhué-Sjöstrand [HKS] but used the semi-classical analysis near a rational developed in HSHarper2.

[1] Note that there is another renormalization procedure proposed by V. Buslaev and A. Fedotov which will not be discussed in these lectures

Semi-classical analysis of the Schrödinger operator.

Our semi-classical treatment of the Harper model was strongly inspired by the techniques introduced in the semi-classical analysis for Schrödinger: harmonic approximation, WKB construction, Agmon estimates, formula for the splitting.
The theory in this case is easier to explain. So we will start with a presentation of these techniques. This has also the advantage to show how the Harper operator is a good approximation of the problem of the Schrödinger operator with magnetic field and electric potential at the bottom.

The magnetic Schrödinger Operator

Our main object of interest is the Schrödinger operator with magnetic field and electric potential on a riemannian manifold, but in this talk we will mainly consider, except for specific toy models, a magnetic field

$$\beta = \mathrm{curl} \mathbf{A}$$

on a regular domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$) associated with a magnetic potential $\mathbf{A}$ (vector field on Ω), which (for normalization) satisfies :

$$\mathrm{div} \mathbf{A} = 0 \,.$$

We start from the closed quadratic form $Q_{h,A,V}$

$$W_0^{1,2}(\Omega) \ni u \mapsto Q_{h,A,V}(u) := \int_\Omega |(-ih\nabla + \mathbf{A})u(x)|^2 \, dx + \int V|u(x)|^2 \, dx \tag{2}$$

Let $P^D(h, \mathbf{A}, V, \Omega)$ be the self-adjoint operator associated to $Q_{h,A,V}$ and let $\lambda_1^D(h, \mathbf{A}, V, \Omega)$ be the corresponding groundstate energy.

Motivated by various questions we consider the connected problems in the asymptotic $h \to +0$.

- Pb 1 Determine the structure of the bottom of the spectrum : gaps, typically between the first and second eigenvalue.
- Pb2 Find an effective Hamiltonian which through standard semi-classical analysis can explain the complete spectral picture including tunneling.

The case when the magnetic field is constant

The first results are known from Landau at the beginning of the quantum Mechanics.

In the case in $\mathbb{R}^d$ ($d = 2, 3$), the models are more explicitly

$$h^2 D_x^2 + (hD_y - x)^2 \,,$$

($\beta(x, y) = 1$) and

$$h^2 D_x^2 + (hD_y - x)^2 + h^2 D_z^2 \,,$$

($\beta(x, y, z) = (0, 0, 1)$) and we have:

$$\inf \sigma(\mathcal{H}(\mathbf{A}, h, \mathbb{R}^d)) = h|\beta| \,.$$

The effect of an electric potential

2D with some electric one well potential (Helffer-Sjöstrand (1987) =[HSWell1]).

We add an electric potential.

$$h^2 D_x^2 + (hD_y - bx)^2 + V(x,y)\,.$$

V creating a well at a minimum of V : $(0,0)$. (V tending to $+\infty$ at ∞).
When $b = 0$, the analysis of the spectrum at the bottom is performed at the beginning of the eighties independently by B. Simon on one side and B. Helffer and J. Sjöstrand on the other side. In a paper in the Annales Ecole Normale di Pisa [HSWellmag], the authors show how to treat the case when b is small.

Harmonic approximation in the non-degenerate case

$$h^2 D_x^2 + (hD_y - bx)^2 + \frac{1}{2} < (x,y)|\mathrm{Hess}\, V(0,0)|(x,y)) \,.$$

$$\lambda_1(h) \sim \alpha h \,.$$

The electric potential plays the dominant role and determines the localization of the ground state. As mentioned to us by E. Lieb, this computation is already done by Fock at the beginning of the quantum mechanics.

Decay of the eigenfunctions and applications

As already seen when comparing the spectrum of the harmonic oscillator and of the Schrödinger operator, it could be quite important to know **a priori** how the eigenfunction attached to an eigenvalue $\lambda(h)$ decays in the classically forbidden region (that is the set of the x's such that $V(x) > \lambda(h)$). The Agmon estimates give a very efficient way to control such a decay. We refer to Helffer-LNM or to the original papers in the beginning of the eighties of Helffer-Sjöstrand or Simon for details and complements.

Let us start with very weak notion of localization. For a family $h \mapsto \psi_h$ of L^2-normalized functions defined in Ω, we will say that the family ψ_h lives (resp. fully lives) in a closed set U of $\overline{\Omega}$ if for any neighborhood $\mathcal{V}(U)$ of U,

$$\lim_{h\to 0} \int_{\mathcal{V}(U)\cap\Omega} |\psi_h|^2 dx > 0 \,,$$

respectively

$$\lim_{h\to 0} \int_{\mathcal{V}(U)\cap\Omega} |\psi_h|^2 dx = 1 \,.$$

For example one expects that the groundstate of the Schrödinger operator $-h^2\Delta + V(x)$ fully lives in $V^{-1}(\inf V)$. Similarly, one expects that, if $\overline{\lim}_{h\to 0}\lambda(h) \leq E < \inf \sigma_{ess}(P_{h,V}) - \epsilon_0$ (for $\epsilon_0 > 0$ small enough) and ψ_h is an eigenvector associated to $\lambda(h)$, then ψ_h will fully live in $V^{-1}(]-\infty, E])$.
Of course the above is very heuristic but there are more accurate mathematical notions like the frequency set (see the book of D. Robert) permitting to give a mathematical formulation to the above vague statements.

Once we have determined a closed set U, where ψ_h fully lives (and hopefully the smallest), it is interesting to discuss the behavior of ψ_h outside U, and to measure how small ψ_h decays in this region.

To illustrate the discussion, one can start with the very explicit example of the harmonic oscillator. The ground state $x \mapsto ch^{-\frac{1}{4}} \exp -\frac{x^2}{h}$ of $-h^2 \frac{d^2}{dx^2} + x^2$ lives at 0 and is exponentially decaying in any interval $[a, b]$ such that $0 \notin [a, b]$. This is this type of result that we want to recover but WITHOUT having an explicit expression for ψ_h.

Energy inequalities

The main but basic tool is a very simple identity attached to the Schrödinger operator $P_{h,A,V}$.

Proposition: Energy identity

Let Ω be a bounded open domain in $\mathbb{R}^m$ with C^2 boundary. Let $V \in C^0(\bar{\Omega};\mathbb{R})$, $A \in C^0(\bar{\Omega};\mathbb{R}^m)$ and ϕ a real valued lipschitzian function on $\bar{\Omega}$. Then, for any $u \in C^2(\bar{\Omega};\mathbb{C})$ with $u_{/\partial\Omega} = 0$, we have

$$\begin{aligned}\int_\Omega |\nabla_{h,A}(\exp\tfrac{\phi}{h}\, u)|^2\, dx + \int_\Omega (V - |\nabla\phi|^2) \exp\tfrac{2\phi}{h}\, |u|^2\, dx = \\ \Re\left(\int_\Omega \exp\tfrac{2\phi}{h}\, (P_{h,A,V}\, u)(x) \cdot \overline{u(x)}\, dx\right) .\end{aligned} \tag{3}$$

The Agmon distance

The Agmon metric attached to an energy E and a potential V is defined as $(V-E)_+ dx^2$ where dx^2 is the standard metric on $\mathbb{R}^n$. This metric is degenerate and is identically 0 at points living in the "classical" region: $\{x \mid V(x) \leq E\}$. Associated to the Agmon metric, we define a natural distance

$$(x,y) \mapsto d_{(V-E)_+}(x,y)$$

by taking the infimum :

$$d_{(V-E)_+}(x,y) = \inf_{\gamma \in \mathcal{C}^{1,pw}([0,1];x,y)} \int_0^1 [(V(\gamma(t)) - E)_+]^{\frac{1}{2}} |\gamma'(t)| dt \;, \tag{4}$$

where $\mathcal{C}^{1,pw}([0,1];x,y)$ is the set of the piecewise (pw) C^1 paths in $\mathbb{R}^n$ connecting x and y. When there is no ambiguity, we shall write more simply $d_{(V-E)_+} = d$.

Similarly to the Euclidean case, we obtain the following properties

- Triangular inequality

$$|d(x', y) - d(x, y)| \leq d(x', x) \ , \ \forall x, x', y \in \mathbb{R}^m \ . \tag{5}$$

- $$|\nabla_x d(x, y)|^2 \leq (V - E)_+(x) \ , \tag{6}$$

  almost everywhere.

We observe that the second inequality is satisfied for any derived distance like

$$d(x, U) = \inf_{y \in U} d(x, y) \ .$$

If $U = \{x \mid V(x) \leq E\}$, $d(x, U)$ measures the distance to the classical region.
All these notions being expressed in terms of metrics, they can be easily extended on manifolds.

Decay of eigenfunctions for the Schrödinger operator.

When u_h is a normalized eigenfunction of the Dirichlet realization in Ω satisfying $P_{h,A,V}u_h = \lambda_h u_h$ then the energy identity gives roughly that $\exp\frac{\phi}{h}\, u_h$ is well controlled (in L^2) in a region

$$\Omega_1(\epsilon_1, h) = \{x \mid V(x) - |\nabla\phi(x)|^2 - \lambda_h > \epsilon_1 > 0\}\ ,$$

by $\exp\left(\sup_{\Omega\setminus\Omega_1}\frac{\phi(x)}{h}\right)$. The choice of a suitable ϕ (possibly depending on h) is related to the Agmon metric $(V-E)_+\, dx^2$, when $\lambda_h \to E$ as $h \to 0$. The typical choice is $\phi(x) = (1-\epsilon)d(x)$ where $d(x)$ is the Agmon distance to the "classical" region $\{x \mid V(x) \leq E\}$. In this case we get that the eigenfunction is localized inside a small neighborhood of the classical region and we can measure the decay of the eigenfunction outside the classical region by

$$\exp(1-\epsilon)\frac{d(x)}{h}\ u_h = \mathcal{O}(\exp\frac{\epsilon}{h})\ , \tag{7}$$

for any $\epsilon > 0$.

More precisely we get for example the following theorem

Theorem: localization of eigenfunctions

Let us assume that V is C^∞, semibounded and satisfies

$$\lim\inf_{|x|\to\infty} V > \inf V = 0 \tag{8}$$

and

$$V(x) > 0 \; |x| \neq 0 \,. \tag{9}$$

Let u_h be a (family of L^2-) normalized eigenfunctions such that

$$P_{h,A,V} u_h = \lambda_h u_h \,, \tag{10}$$

with $\lambda_h \to 0$ as $h \to 0$. Then for all ϵ and all compact $K \subset \mathbb{R}^m$, there exists a constant $C_{\epsilon,K}$ such that for h small enough

$$||\nabla_{h,A}(\exp\frac{d}{h}\cdot u_h)||_{L^2(K)} + ||\exp\frac{d}{h}\cdot u_h||_{L^2(K)} \leq C_{\epsilon,K}\exp\frac{\epsilon}{h}\,. \tag{11}$$

Remarks

When V has a unique non degenerate minimum the estimate can be improved when $\lambda_h \in [0, C_0 h]$, by taking

$$\phi = d - Ch \inf(\log(\frac{d}{h}), \log C)\,.$$

We observe indeed that V, d and $|\nabla d|^2$ are equivalent in the neighborhood of the well.
It is also possible to control the decay of the eigenfunction at ∞.
This was actually the initial goal of S. Agmon.

First application

We can compare different Dirichlet problems corresponding to different open sets Ω_1 and Ω_2 containing a unique well U attached to an energy E. If for example $\Omega_1 \subset \Omega_2$, one can prove the existence of a bijection b between the spectrum of $P_{(h,\Omega_1)}$ in an interval $I(h)$ tending (as $h \to 0$) to E and the corresponding spectrum of $P_{(h,\Omega_2)}$ such that $|b(\lambda) - \lambda| = \mathcal{O}(\exp -S/h)$ (under a weak assumption on the spectrum at $\partial I(h)$).
Here S is chosen such that

$$0 < S < d_{(V-E)_+}(\partial\Omega_1, U) \, .$$

This can actually be improved (using more sophisticated perturbation theory) as $\mathcal{O}(\exp -2S/h)$.

Second application: the symmetric double well problem

Once the harmonic approximation is done, it is possible to construct an orthonormal basis of the spectral space attached to a given interval $I(h) := [\inf V, \inf V + Ch]$ (C avoiding the eigenvalues of the approximating harmonic oscillators at each minimum), each of the elements of the basis being exponentially localized in one of the wells.
The computation of the matrix of the operator in this basis using WKB approximation leads to the so-called "interaction matrix" (See Dimassi-Sjöstrand or Helffer LNM).

We consider the case with two wells, say U_1 and U_2. We assume that there is a symmetry[2] g in $\mathbb{R}^m$, such that $g^2 = Id$, $gU_1 = U_2$, and such that the corresponding action on $L^2(\mathbb{R}^m)$ defined by $gu(x) = u(g^{-1}x)$ commutes with the Laplacian. In addition $gV = V$.

We now define reference one well problems by introducing :

$$M_1 = \mathbb{R}^m \setminus B(U_2, \eta)\ ,\ M_2 = \mathbb{R}^m \setminus B(U_1, \eta)\ .$$

With this choice, we have $gM_1 = M_2$. The parameter $\eta > 0$ is free but can always be chosen arbitrarily small. We denote by ϕ_j the corresponding ground state of the Dirichlet realization of $-h^2\Delta + V$ in M_j and corresponding to the ground state energy $\lambda_{M_1} = \lambda_{M_2}$. According to our result on the decay, these eigenfunctions decay like $\tilde{O}(\exp -\frac{d(x,U_j)}{h})$, where $\tilde{O}(f)$ roughly[3] means $\exp \frac{\epsilon}{h} \cdot \mathcal{O}_\epsilon(f)$ for all $\epsilon > 0$ as $h \to 0$. We can of course keep the relation

$$g\phi_1 = \phi_2\ .$$

[2]Typically, we take $g = -I$ and $m = 2$

[3]More precisely, for any $\epsilon > 0$, one can choose above $\eta > 0$ such that...

Let us now introduce θ_j, which is equal to 1 on $B(U_j, \frac{3}{2}\eta)$ and with support in $B(U_j, 2\eta)$. We introduce

$$\chi_1 = 1 - \theta_2 \; , \; \chi_2 = 1 - \theta_1 \; ,$$

and we can also keep the symmetry condition :

$$g\chi_1 = \chi_2 \; .$$

Our approximate eigenspace will be generated by

$$\psi_j = \chi_j \phi_j \; , \; (j = 1, 2) \; ,$$

which satisfies

$$S_h \psi_j = \lambda_M \psi_j + r_j \; ,$$

with

$$r_j = h^2 (\Delta \chi_j) \phi_j + 2h^2 (\nabla \chi_j) \cdot (\nabla \phi_j) \; .$$

We note that the "smallness" of r_j can be immediately controlled using the decay estimates in $B(U_j, 2\eta) \setminus B(U_j, \frac{3}{2}\eta)$.

In order to construct an orthonormal basis of the eigenspace F corresponding to the two lowest eigenvalues near λ_M, we first project our basis ψ_j which was not far to be orthogonal and introduce:

$$v_j = \Pi_F \psi_j \ .$$

The resolvent formula shows that $v_j - \psi_j$ can be made very small (at least $\exp -\frac{S}{h}$ with $S < d(U_1, U_2)$ by chosing $\eta > 0$ small enough). More precisely, we have the following comparison.

Lemma

$$(v_j - \psi_j)(x) = \tilde{O}(\exp -\frac{\delta_j(x)}{h}) \ , \qquad (12)$$

in $\mathbb{R}^m \setminus B(U_{\hat{j}}, 4\eta)$, where $\widehat{1} = 2$, $\widehat{2} = 1$ and

$$\delta_j(x) = d(x, U_{\hat{j}}) + d(U_1, U_2) \ .$$

Proof

Our starting point is :

$$P_{h,M_j}\psi_j = \lambda_{M_j}\psi_j + r_j \ .$$

where

$$\operatorname{supp} r_j \subset B(U_{\hat{j}}, 2\eta) \ ,$$

and

$$r_j = \tilde{O}(\exp -\frac{d(x,U_j)}{h}) \ .$$

We have $v_j - \Pi_F \psi_j \in F^\perp$ and the spectral theorem gives already the estimate

$$||v_j - \pi_F \psi_j|| = \tilde{O}(\exp - \frac{d(U_1, U_2)}{h}) \,. \tag{13}$$

For a suitable contour Γ_h in $\mathbb{C}$ containing the interval $I(h)$ and remaining at a suitable distance of the spectrum

$$d(\Gamma_h, \sigma(P_h)) \geq \frac{1}{C_\epsilon} \exp - \frac{\epsilon}{h}, \forall \epsilon > 0 \,, \tag{14}$$

we can write :

$$v_j - \psi_j = \frac{1}{2\pi} \int_{\Gamma_h} (\lambda_M - z)^{-1} (P_h - z)^{-1} r_j dz \,.$$

We observe by a property of the resolvent deduced from Agmon estimates that:

$$\begin{aligned}(P_h - z)^{-1} r_j &= \tilde{O}(\sup_{y \in \mathrm{supp} r_j} \exp -\tfrac{[d(x,y)+d(y,U_j)]}{h}) \\ &= \tilde{O}(\exp -\tfrac{\delta_j(x)}{h})\,.\end{aligned}$$

The separation assumption (14) permits to get the same property for $v_j - \psi_j$:

$$v_j - \psi_j = \tilde{O}(\exp -\frac{\delta_j(x)}{h})\,.$$

This is indeed an improvment of the control in L^2.
We notice that :

$$\delta_j(x) \geq d(x, U_j) \ ,$$

What we see here is that the improved estimate does not lead to improvements near $U_{\hat{j}}$, where we have modified ϕ_j into ψ_j by introducing a cut-off function but that the improvement is quite significative when keeping a large distance (in comparison with η) with $U_{\hat{j}}$.

We then orthonormalize by the Gram-Schmidt procedure.

$$e_j = \sum_k (V^{-\frac{1}{2}})_{jk} v_k \ ,$$

with

$$V_{ij} = \langle v_i \mid v_j \rangle \ .$$

We note that

$$V_{ij} - \delta_{ij} = \mathcal{O}(\exp - \frac{S}{h}) \ .$$

At each step, we control the difference $e_j - \psi_j$, which satisfies also (12).
The matrix we would like to analyze is then simply the two by two matrix

$$M_{ij} = \langle (P_h - \lambda_M) e_i \mid e_j \rangle \ .$$

The eigenvalues of this matrix measure the dispersion of the two eigenvalues around λ_M.

We observe that symmetry considerations lead to :

$$M_{12} = M_{21} \text{ and } M_{11} = M_{22} \ .$$

So the eigenvalues are easy to compute and corresponding eigenvectors are $\frac{1}{\sqrt{2}}(1,1)$ and $\frac{1}{\sqrt{2}}(-1,+1)$. As soon as we have the main behavior of M_{12}, we can deduce that the eigenvalues are simple and that the splitting between the two eigenvalues is given by $2|M_{12}|$.
It remains to explain how one can compute M_{12}. The analysis of the decay permits to show that

$$M_{12} = \frac{1}{2}\left(\langle r_2\ ,\ \psi_1\rangle + \langle r_1\ ,\ \psi_2\rangle\right) + \mathcal{R}_{12}\ , \tag{15}$$

with

$$\mathcal{R}_{12} = \mathcal{O}(\exp -\frac{2S}{h})\ , \tag{16}$$

for a suitable choice of $\eta > 0$ small enough.

An integration by parts leads (observing that $\nabla\chi_1 \cdot \nabla\chi_2 \equiv 0$ for our choice of η) to the formula

$$M_{12} = h^2 \int \chi_1(\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2)\nabla\chi_2 + \mathcal{R}_{12}\,. \qquad (17)$$

A priori informations on the decay permit to restrict the integration in the right hand side of (17) to the set $\{d(x,U_1) + d(x,U_2) \leq d(U_1,U_2) + a\}$ for some $a > 0$.
A computation based on the Stokes Lemma gives then the existence of $\epsilon_0 > 0$ such that:

$$M_{12} = h^2 \int_\Gamma [\phi_2\partial_n\phi_1 - \phi_1\partial_n\phi_2]d\nu_\Gamma + \mathcal{O}(\exp -\frac{S_{12}+\epsilon_0}{h})\,. \qquad (18)$$

Here $S_{12} = d(U_1, U_2)$ and Γ is an open piece of hypersurface defined in the neighborhood of the minimal geodesic $\mathrm{geod}(U_1, U_2)$ between the two points U_1 and U_2, that we assume for simplification to be unique and ∂_n denotes the normal derivative to Γ, positively oriented from U_1 to U_2 .
The last step is to observe that in a neighborhhood of the intersection γ_{12} of Γ with $\mathrm{geod}(U_1, U_2)$, one can replace the function ϕ_j (or ψ_j) modulo $\mathcal{O}(h^\infty) \exp -\frac{d(x,U_j)}{h}$ by its WKB approximation $h^{-\frac{m}{4}} a_j(x,h) \exp -\frac{d(x,U_j)}{h}$.

This leads finally to

$$
\begin{aligned}
M_{12} = h^{1-\frac{m}{2}} \exp -\tfrac{d(U_1,U_2)}{h} \times \\
\times \int_\Gamma \exp -\tfrac{(d(x,U_1)+d(x,U_2)-d(U_1,U_2))}{h} \times \\
\times (a_1(x,0)a_2(x,0)(\partial_n d(x,U_1) - (\partial_n d(x,U_2)) + \mathcal{O}(h))\, d\nu_\Gamma\,,
\end{aligned}
\tag{19}
$$

where $d\nu_\Gamma$ is the induced measure on Γ.
With natural generic additional assumptions saying that the map

$$
\Gamma \ni x \mapsto (d(x,U_1) + d(x,U_2) - d(U_1,U_2))
$$

vanishes exactly at order 2 at γ_{12}, this finally leads to the formula giving the splitting after use of the Laplace integral method.

Schrödinger operators with magnetic potentials and periodic potential

For operators $H = \sum_{j=1}^{2}(hD_{x_j} - A_j)^2 + V$ with periodic potentials V,

$$V(x_1 + 2\pi, x_2) \equiv V(x_1, x_2 + 2\pi) \equiv V(x_1, x_2)\ ,$$

and constant (or periodic) magnetic fields

$$\mathrm{Curl}\, \vec{A} = B\ ,$$

it can be shown in several asymptotic regimes that the study of some parts of the spectrum reduces to a non-linear spectral problem of the above type.

This is for example the case (see LNP Sonderborg) when

- B is large. This is the so called strong magnetic field regime. V appears then as a perturbation. One can ask about how V perturbs the Landau spectrum.
- B is small. We can then consider the problem as a perturbation of the case $B = 0$. One can discuss in this context the so called Peierls substitution. see [HSHarper1], [HSHarper3] and [HSHarper4] and HS-Sonderborg and earlier contribution by physicists (see in [Bel] and references therein).

Semi-classical analysis together with weak magnetic field regime

We study in a semi-classical regime the Schrödinger operator $P_{h,A,V}$, defined as the self-adjoint extension in $L^2(\mathbb{R}^2)$ of the operator given in $C_0^\infty(\mathbb{R}^2)$ by

$$P^0_{h,A,V} = (hD_{x_1} - A_1(x))^2 + (hD_{x_2} - A_2(x))^2 + V(x)\,, \qquad (20)$$

where $D_{x_j} = \frac{1}{i}\partial_{x_j}$. Our goal is to study the spectrum of $P_{h,A,V}$ as a function of A and the semi-classical parameter $h > 0$, when V has its minima in the lattice and both V and $B = \nabla \wedge A$ are invariant by the symmetries of the lattice.

Let us explain the setting of our problem. A 2-dimensional Bravais lattice is the set of points spanned over $\mathbb{Z}$ by the vectors of a basis $\{\nu_1, \nu_2\}$ of $\mathbb{R}^2$. A fundamental domain of the Bravais lattice can be chosen in the form

$$\mathcal{V} = \left\{ t_1\nu_1 + t_2\nu_2 \,;\, (t_1, t_2) \in [0,1]^2 \right\}. \tag{21}$$

We consider the square lattice ($\nu_1 = (1,0)$ and $\nu_2 = (0,1)$) but other lattices are interesting (triangular lattice, Kagome lattice ...). For the square lattice, the map $\kappa : \mathbb{R}^2 \to \mathbb{R}^2$ here below is given by

$$\kappa(x_1, x_2) = (-x_2, x_1) \tag{22}$$

and note that $\kappa(\Gamma) = \Gamma$.
For $j = 1, 2$ consider the translations $t_j(x) = x - \nu_j$ and define in the affine group of the plane

$$\mathcal{G} = \text{the subgroup generated by } \kappa,\ t_1 \text{ and } t_2\,. \tag{23}$$

Setting $(gu)(x) = u(g^{-1}(x))$ for $g \in \mathcal{G}$, we define a group action of $\mathcal{G}$ on $C^\infty(\mathbb{R}^2)$ which can be extended as an unitary action on $L^2(\mathbb{R}^2)$.

Hypothesis V1

The electric potential V is a real nonnegative C^∞ function such that

$$
\begin{aligned}
& g\,V = V \quad \text{for all } g \in \mathcal{G} \\
& V \geq 0 \quad \text{and} \quad V(x) = 0 \text{ if and only if } x \in \Gamma\,, \\
& \mathrm{Hess}\,V(x) > 0 \quad \forall x \in \Gamma\,.
\end{aligned}
\tag{24}
$$

Hence, we have a unique minimum in each fundamental cell. The case with more than one minimum is also interesting.

Alternative formalism

We associated with the magnetic vector potential $A = (A_1, A_2)$ the 1-form

$$\omega_A = A_1 dx_1 + A_2 dx_2 \,. \tag{25}$$

The magnetic field B is then associated with the 2-form obtained by taking the exterior derivative of ω_A:

$$\sigma_B := d\omega_A = B(x) dx_1 \wedge dx_2 \,. \tag{26}$$

In the case of $\mathbb{R}^2$, we identify this 2-form with B. The flux of B through a fundamental domain $\mathcal{V}$ of Γ is then given by

$$\eta = \int_{\mathcal{V}} d\omega_A \,. \tag{27}$$

Hypothesis B1

The magnetic potential A is a C^∞ vector field such that the corresponding magnetic 2-form σ_B satisfies

$$g\sigma_B = \sigma_B \quad \text{for all } g \in \mathcal{G}\,. \tag{28}$$

This is automatically satisfied when B is constant. Note indeed that the symplectic 2-form $dx_1 \wedge dx_2$ is preserved.

In the case when $A = 0$ (see for example Chapter XIII.16 in Reed-Simon Vol. 4), the spectrum of $P_{h,A,V}$ is continuous and composed of bands. This is done by the so-called Floquet (Bloch) theory. Each band is the image of a Floquet eigenvalue $\lambda_j(\theta_1, \theta_2)$. A semi-classical analysis is possible and one can in particular have the asymptotics for the first eigenvalue and on the width of the first band (Outassourt, Simon).

The general case, even when the magnetic field is constant, is very delicate. The spectrum of $P_{h,A,V}$ depends crucially on the normalized flux of the magnetic field through a fundamental domain of the lattice, given by

$$\gamma = \frac{\eta}{h}\,. \tag{29}$$

Under the previous assumptions, we can define the magnetic translations T_1 and T_2 associated with ν_1 and ν_2. They have the form

$$(T_j u)(x) = \exp i \frac{\phi_j(x)}{h} u(x - \nu_j)$$

and commute with $P_{h,A,V}$.
ϕ_j is determined (modulo a constant) by

$$A(x - \nu_j) - A(x) = -\nabla \phi_j$$

We observe indeed that the left hand side has curl equal to zero by the assumption of invariance of B.
What is quite important is that T_1 and T_2 do not necessarily commute. We have actually

$$T_1 T_2 = \exp i\gamma \, T_2 \, T_1 \, .$$

Hence the Floquet theory (as explained for example in Reed-Simon IV) cannot be done. Note however that if $\frac{\gamma}{2\pi} = \frac{p}{q}$, some Floquet theory can be applied by using T_1 and T_2^q which are commuting.

Depending on the arithmetic properties of $\gamma/2\pi$, the spectrum can indeed become very singular (Cantor structure). To approach this problem, we are often lead to the study of limiting models in different asymptotic regimes, such as discrete operators defined over $\ell^2(\mathbb{Z}^2, \mathbb{C})$, or equivalently, as we will see later, pseudo-differential operators defined on $L^2(\mathbb{R}, \mathbb{C})$.

The discrete magnetic translations τ_1 and τ_2 are defined on $\ell^2(\mathbb{Z}^2)$ by

$$(\tau_1 v)_{n,m} = v_{n-1,m}\,, \qquad (\tau_2 v)_{n,m} = e^{-i\gamma n} v_{n,m-1}\,. \tag{30}$$

Following the ideas in HSHarper1, §9; we analyze the restriction of $P_{h,A,V}$ to a spectral space associated with the bottom of its spectrum, and we show the existence of a matrix of this space that keeps the symmetries of V and B with respect to $\mathcal{G}$.

In order to state our first theorem, let us explain more in detail this procedure. First of all, the harmonic approximation shows the existence of an exponentially small (with respect to h) band in which one part of the spectrum (including the bottom) is confined. We name this part the *low lying spectrum*. The rest of the spectrum is separated by a gap of size h/C.

Consider $\delta \in (0, 1/8)$ and a non negative radial smooth function χ, such that $\chi = 1$ in $B(0, \delta/2)$ and $\operatorname{supp} \chi \subset B(0, \delta)$. For any $m \in \Gamma$ define

$$V_m(\cdot) = \sum_{n \in \Gamma \setminus \{m\}} \chi(\cdot - n) \tag{31}$$

and

$$P_m = P + V_m \,. \tag{32}$$

All the P_m are unitary equivalent and

$$v = \liminf_{|x|\to\infty} V_m(x) \tag{33}$$

does not depend on m. The spectrum of P_m is discrete in the interval $[0, v)$. The first eigenvalue of P_m is simple and we denote it by $\lambda(h)$. There exists then $\epsilon_0 > 0$ such that $\sigma(P_m) \cap I(h) = \{\lambda(h)\}$, where $I(h) = [0, h(\lambda_{har,1} + \epsilon_0)]$ and $\lambda_{har,1}$ is the first eigenvalue of the operator associated with P_m by the harmonic approximation when $h = 1$. We define

$$\Sigma = \text{ the spectral space associated with } I(h)\,. \tag{34}$$

We denote by d_V the Agmon distance associated with the metric $V\,dx^2$. We then have:

Theorem: Effective hamiltonian

Under above assumptions, there exists $h_0 > 0$ such that for $h \in (0, h_0)$ there exists a basis of Σ in which $P_{h,A,V}\big|\Sigma$ has the matrix

$$\lambda(h) I + W_\gamma\,; \tag{35}$$

where for all $n, m \in \ \Gamma$ and $\beta \in \mathbb{Z}^2$, W_γ satisfies

$$\begin{array}{rcl} (W_\gamma)_{n,m} & = & \overline{(W_\gamma)_{m,n}} \\ (W_\gamma)_{n,m} & = & e^{-i\frac{\gamma}{2}(m-n)\wedge\beta}(W_\gamma)_{(n+\beta),(m+\beta)} \\ (W_\gamma)_{n,m} & = & (W_\gamma)_{\kappa(m,n)}\,. \end{array} \tag{36}$$

Moreover, there is $C > 0$ such that for every $\epsilon > 0$ there exists $h_\epsilon > 0$, such that for $h \in (0, h_\epsilon)$

$$|(W_\gamma)_{n,m}| \leq C \exp\left(-\frac{(1-\epsilon)d_V(m,n)}{h}\right)\,. \tag{37}$$

The coefficients of W_γ are related to the interaction between different sites of the lattice. Our next result concerns the study of this matrix, when we only keep the main terms for the Agmon distance. In order to estimate these terms, we need additional hypothesis. Here we assume (see HSHarper1 for more details):

Hypotheses

A. The nearest neighbors for the Agmon distance are the same of those for the Euclidean distance.

B. Between two nearest neighbors $m, n \in \Gamma$ there exists an unique minimal geodesic $\omega_{m,n}$ for the Agmon metric.

C. $\omega_{m,n}$ in non degenerate in the sense that there is a point $x_0 \in \omega_{m,n} \setminus \{m, n\}$ such that $x \mapsto d_V(x, m) + d_V(x, n) - d_V(m, n)$ restricted to a transverse line to $\omega_{m,n}$ at x_0 has a non degenerate local minimum at x_0.

Under these hypotheses, we will estimate the main terms in the case of a weak and constant magnetic field $B = hB_0$, given by the gauge

$$A(x_1, x_2) = \frac{hB_0}{2}(-x_2, x_1)\,, \quad B_0 > 0\,. \tag{38}$$

Theorem

There exists $C > 0$, $b_0 > 0$ and $h_0 > 0$ such that for $h \in (0, h_0)$,

$$W_\gamma = \rho\left(\hat{W}_\gamma + R_\gamma\right) \tag{39}$$

theorem continued

with

$$\rho = h^{\frac{1}{2}}\, b_0\, e^{-\frac{d_V\left(m_{(0,0)},m_{(0,1)}\right)}{h}}\, (1 + \mathcal{O}(h)) \;, \tag{40}$$

$$\hat{W}_\gamma = \tau_1 + \tau_1^* + \tau_2 + \tau_2^* \;, \tag{41}$$

and R_γ a relatively small term (see what was done for the double well problem).

The rational case

In order to compute the spectrum of $\hat{W}_\gamma$, we can start with the case when $\gamma/(2\pi)$ is a rational number. This is obtained by using the Floquet theory.
For $p, q \in \mathbb{N}^*$ we define the matrices $J_{p,q}, K_q \in \mathcal{M}_q(\mathbb{C})$ by

$$\begin{aligned} J_{p,q} &= \mathrm{diag}(\exp\left(2i\pi(j-1)p/q\right)) \\ (K_q)ij &= \begin{cases} 1 & \text{if } j = i+1 \pmod{q} \\ 0 & \text{if not} \end{cases} \end{aligned} \quad . \tag{42}$$

Note that

$$J_{p,q} = J_{1,q}^p \, .$$

Theorem

Let $\gamma = 2\pi p/q$ with $p, q \in \mathbb{N}^*$ relatively primes and denote by σ_γ the spectrum of $\hat{W}_\gamma$. We have

$$\sigma_\gamma = \bigcup_{\theta_1,\theta_2 \in [0,1]} \sigma(M_{p,q,\theta_1,\theta_2}) \,, \tag{43}$$

where $M_{p,q,\theta_1,\theta_2}$ is given by

$$M_{p,q,\theta_1,\theta_2} = e^{i\theta_2} J_{p,q} + e^{-i\theta_2} J_{p,q} + e^{i\theta_1} K_q + e^{-i\theta_1} K_q^* \,. \tag{44}$$

The bands are recovered by looking at the eigenvalues of $M_{p,q,\theta_1,\theta_2}$. These bands do not overlap and do not touch except possibly at the center (Van Mouche). See below for complement.

Pseudo-differential operators and Harper's equation

In –HSHarper1,HSHarper2,HSHarper3– (1988-1990) a machinery was developed for an iterative semiclassical analysis of a special class of pseudodifferential operators. One was concerned with the non-linear spectral problem (or, in other words, with the spectral problem for an operator pencil). Namely, for a family of self-adjoint operators $A(\mu)$ depending $\mu \in \mathbb{R}$ the μ-spectrum $\mu\text{-spec}\, A(\mu)$ denotes the set of all μ such that $0 \in \operatorname{Spec} A(\mu)$. The simplest case being the family $A - \mu$.

Quantization

Let $L : \mathbb{R}^2 \to \mathbb{R}$ be a periodic smooth function, $L(x, \xi + 2\pi; \mu, h) = L(x + 2\pi, \xi; \mu, h) = L(x, \xi; \mu, h)$. Here μ and h are real parameters. By the Weyl quantization procedure one can assign to L an operator $\hat{L}_h(\mu)$ in $L^2(\mathbb{R})$ by

$$\hat{L}_h(\mu) f(x) = \frac{1}{2\pi h} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\xi(x-y)/h} L\big(\frac{x+y}{2}, \xi; \mu, h\big) f(y) d\xi \, dy. \tag{45}$$

This operator belongs to $\mathcal{L}(L^2(\mathbb{R}))$, is selfadjoint if the symbol is real. There is for this class of operators a symbolic calculus, which is quite simple if one admits errors in $\mathcal{O}(h)$ (more sophisticated if one works modulo $\mathcal{O}(h^\infty)$). See the books of Hörmander for the basic notions or Dimassi-Sjöstrand and Zworski for the semi-classical aspects.

The operator $\hat{L}_h$ obtained is referred to as the Weyl h-quantization of L, and quantum Hamiltonians resulting from periodic symbols are often called Harper-like operators.

In particular, the symbol $L(x,\xi) := \cos x + \cos \xi$ produces the Harper operator on the real line,

$$\hat{L}_h f(x) = \frac{f(x+h) + f(x-h)}{2} + \cos x \, f(x). \qquad (46)$$

Symbols associated with some discrete operators

We consider a bounded linear operator C_h acting on $\ell^2(\mathbb{Z}^2)$ given by an infinite matrix $\big(C(p,q)\big)$, $p,q \in \mathbb{Z}^2$, satisfying

$$C(p+k, q+k) = e^{-ihk_2(p_1-q_1)} C(p,q), \quad p,q,k \in \mathbb{Z}^2, \tag{47}$$

with some $h > 0$.

Proposition A

Let C_h be a bounded self-adjoint operator in $\ell^2(\mathbb{Z}^2)$ with the property (47) and satisfying $|C(p,q)| \leq ae^{-b|p-q|}$ for some $a, b > 0$ and all $p, q \in \mathbb{Z}^2$. Then the spectrum of C_h coincides with the spectrum of the Weyl h-quantization of the symbol T given by

$$T(x,\xi) = \sum_{m,n\in\mathbb{Z}} c(m,n)e^{-imnh/2}e^{i(mx+n\xi)}, \tag{48}$$

where $c(m,n) = C\big((0,0),(m,n)\big)$, $m, n \in \mathbb{Z}$.

A third point of view

We start with

$$C_h f(m,n) = \frac{1}{2}\left(e^{ihn} f(m+1,n) + e^{-ihn} f(m-1,n)\right) \\ +\frac{1}{2}\left(f(m,n-1) + f(m,n+1)\right).$$

In this case, we can come back to the family of operators on $\ell^2(\mathbb{Z})$ by introducing

$$u(m,\theta) = \sum_n e^{in\theta} f(m,n).$$

In this way we come back to H_θ.

The spectrum of C_h is the union over $\theta \in (0, 2\pi)$ of the spectra of H_θ.

Consider now the general case. By assumption, $C(p,q) = \exp\big(ihp_2(q_1 - p_1)\big)\, c(q-p)$ for any $p, q \in \mathbb{Z}^2$, hence

$$C_h f(p) = \sum_{q\in\mathbb{Z}^2} e^{ihp_2(q_1-p_1)} c(q-p) f(q)$$
$$= \sum_{q\in\mathbb{Z}^2} e^{ihp_2 q_1} c(q) f(p+q).$$

Therefore, C_h commutes with the shift $f(p_1,p_2) \mapsto f(p_1+1,p_2)$, and the Floquet-Bloch theory is applicable.

Let us introduce the functions

$$\mathbb{R} \ni \varphi \mapsto b_n(\varphi) = \sum_{k\in\mathbb{Z}} c(k,n)e^{ik\varphi}, \quad n \in \mathbb{Z}, \quad \varphi \in \mathbb{R}.$$

All these functions are 2π-periodic and analytic in a complex neighborhood of $\mathbb{R}$. Consider a family of operators acting in $\ell^2(\mathbb{Z})$,

$$C_h(\theta)g(m) = \sum_{n\in\mathbb{Z}} b_n(mh+\theta)g(m+n), \quad m \in \mathbb{Z}, \quad \theta \in \mathbb{R}\,,$$

which satisfies

$$C_h(\theta) = C_h(\theta + 2\pi)\,.$$

Therefore, by the Floquet-Bloch theory, one has

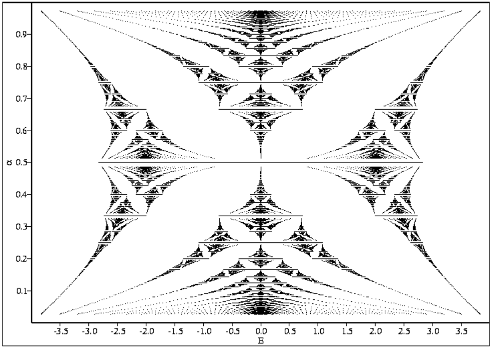
$$\mathrm{Spec}\, C_h = \bigcup_{\theta\in[0,2\pi)} \mathrm{Spec}\, C_h(\theta)\,.$$

Furthermore, for any θ the operators $C_h(\theta)$ and $C_h(\theta+h)$ are unitarily equivalent, $C_h(\theta+h) = SC_h(\theta)S^{-1}$, where S is the shift in $\ell^2(\mathbb{Z})$, $Sf(n) = f(n+1)$, which implies $\mathrm{Spec}\, C_h = \bigcup_{\theta\in[0,h)} \mathrm{Spec}\, C_h(\theta)$.
This coincides with the spectrum of the following operator T_h acting in $L^2\big(\mathbb{Z}\times[0,h)\big)$

$$T_h u(m,\theta) = C_h(\theta)u_\theta(m), \quad u_\theta(m) = u(m,\theta), \quad m\in\mathbb{Z}.$$

Coming back to Harper

In the case of the symbol $(x, \xi) \mapsto \cos x + \cos \xi$ we get the **Hofstadter's butterfly**
On the vertical axis the parameter proportional to the flux $\alpha = \frac{h}{2\pi} \in [0, 1]$. On the horizontal line $y = \alpha$ the union over θ of the spectra of the family $C_h(\theta)$. The picture results of computations for rational α's.



The hamiltonian point of view permits to explain the behavior of the spectrum as $\alpha \mapsto 0$

The first statement

The first statement was established in HSHarper1.

Theorem

For $\epsilon_0 > 0$, there exists $C_0 > 0$ such that if $h/(2\pi) \in (0,1) \setminus \mathbb{Q}$ and

$$h/(2\pi) = 1/(q_1 + 1/(q_2 + 1/q_3 + \cdots)))$$

with $q_j \in \mathbb{Z}$ and $|q_j| \geq C_0$, we have:

- The smallest closed interval J containing the spectrum $\sigma(H)$ has the form $[-2 + \mathcal{O}(1/|q_1|), 2 + \mathcal{O}(1/|q_1|)]$,
- $$\sigma(H) \subset \cup_{N_- \leq j \leq N_+} J_j$$

  where the J_j are closed intervals of positive length with $\partial J_j \subset \sigma(H)$,
- J_{j+1} is on the right of J_j at a distance of order $1/|q_1|$,

Theorem continued

- J_0 has length $2\epsilon_0 + \mathcal{O}(1/|q_1|)$ and contains 0 at a distance $\mathcal{O}(1/|q_1|)$ of its center
- The other bands have width $e^{-C(j)|q_1|}$ with $C(j)$ of order 1
- For $j \neq 0$, if κ_j denotes the affine function sending J_j onto $[-2, +2]$, then

$$\kappa_j(J_j \cap \sigma(P)) \subset \cup_k J_{j,k} \,,$$

where the $J_{j,k}$ have the same properties as the J_j with q_1 replaced by q_2 and so on.

Remark[4]

This theorem is used by Bourgain in order to give cases for which the integrated density of states is not Hölder.

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[4] Thanks to Q. Zhou for this remark.

Sketch of the proof of the renormalization theorem–step 1

The analysis proposed by Wilkinson is based on a WKB analysis.
As already mentioned the analysis of the spectrum of the Harper model is equivalent (isospectrality) with the spectral analysis of the pseudo-differential operator:

$$\cos hD_x + \cos x$$

on $L^2(\mathbb{R})$.
The symbol is $\cos x + \cos \xi$. For a given energy E, the wells are the connected components of $p^{-1}(E)$.
For each of these components connected, we can construct some quasimode states and the "approximate" spectrum is correct modulo $\mathcal{O}(h^\infty)$.

A few words on WKB solutions

This has a long story for the $1D$-Schrödinger operator

$$-h^2 \frac{d^2}{dx^2} + V(x)\,.$$

We assume that $V(x) \geq 0 = V(0)$. If for some $E_0 > 0$, $V^{-1}(-\infty, E_0)$ is connected, bounded and if ∇V is not critical except at the minimum of V where V is assumed to be non degenerate. Then the whole spectrum in $(0, E_0)$ can be obtained modulo $\mathcal{O}(h^\infty)$ by the socalled generalized Bohr-Sommerfeld condition which reads

$$f(\lambda_n(h), h) = (n + \frac{1}{2})h\,.$$

The first step for getting this rule is to try to construct solution of the type $a(x,h)\exp \pm i\frac{\phi}{h}$ with energy E this is posssible except at $V^{-1}(\{E\})$. We have first to solve in $V^{-1}(-\infty, E)$ the so called equation

$$\phi'(x)^2 = E - V(x)\,.$$

This is when trying to match together these locally defined solution that we get that this is only pssible for some h-dependent values of E.

In the case of the Harper model, if $E \in (-2,2)$, $E \neq 0$, we can perform a similar analysis whose first step is to solve

$$\cos\phi'(x) + \cos x = E\,.$$

One observes that there are many local solutions (if $\phi(x)$ is a solution $\phi(x) + 2\pi kx$ for $k \in \mathbb{Z}$ is another solution, $\phi(x + 2\pi m)$ for $m \in \mathbb{Z}$ is another solution (in atranslated interval).

Near each of these values $\lambda_n(h)$, we can construct a basis (close to orthonormal) of the spectral space of the Harper equation associated with the interval $(\lambda_n(h) - Ch^2, \lambda_n(h) + Ch^2)$. This is not too difficult for $\lambda_n(h)$ avoiding the critical value $E = 0$ of the symbol. The eigenvalues are indeed approximately given by a Bohr-Sommerfeld formula (as done for $-h^2 \frac{d^2}{dx^2} + V(x)$) and the eigenvalues are well separated $\lambda_{n+1} - \lambda_n(h) \geq \frac{1}{C} h$.
This basis is obtained by functions which are well localized in each of the wells, and are deduced from each other by translations (commuting with the Harper equation) exchanging the different wells.
This is quite similar with what we have done in the analysis of the Schrödinger operator with electric potential but this time we are dealing with "microlocal" wells i.e. defined in $T^*\mathbb{R}$.

But these wells interact by the socalled tunneling effects and the Harper model restricted to this spectral space is described in this basis by an infinite matrix which is not diagonal. For h small, Wilkinson gives how heuristically, by analyzing the interactions, one gets an operator on $\ell^2(\mathbb{Z}^2)$, which can be identified as a h^{new}-pseudodifferential operator (hence with a new semi-classical parameter) which is quite close to a new Harper model.
For having the complete structure of the spectrum, we have just to iterate the procedure.
One can hope that the procedure will work if the succession of h's which are obtained is sufficiently small. The sequence of h's is given by the expansion as a continuous fraction of $\alpha = \frac{h}{2\pi}$.
This actually does not work so easily ! The construction of the basis is difficult in the neighborhood of the critical point and this is why in [HSHarper1] we get only a partial result, where we avoid at each step a small zone. At this stage, we do not get the Cantor structure of the spectrum.
The complete solution was only obtained in [HS3] about one year later.

In [HSHarper3], in order to treat the Harper operator and perturbations of it occuring in a renormalization procedure, the following notion was introduced.

Definition

A symbol $L(x, \xi; \mu, h)$ will be called of strong type I if the following conditions are satisfied for all $h \in (0, h_0)$ with some $h_0 > 0$:

(a) L depends analytically on $\mu \in [-4, 4]$.

(b) There exists $\varepsilon > 0$ such that

(b1) $L(x, \xi; \mu, h)$ is holomorphic in

$$D_\varepsilon = \left\{(\mu, x, \xi) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C} : |\mu| \leq 4, |\Im x| < \frac{1}{\varepsilon}, |\Im \xi| < \frac{1}{\varepsilon}, \right\},$$

(b2) for $(\mu, x, \xi) \in D_\varepsilon$, there holds

$$\left| L(x, \xi; \mu, h) - (\cos x + \cos \xi - \mu) \right| \leq \varepsilon.$$

Continuation of the definition

(c) The following symmetry conditions hold:

$$L(x,\xi;\mu,h) = L(\xi,x;\mu,h) = L(x,-\xi;\mu,h)$$
$$L(x,\xi;\mu,h) = L(x+2\pi,\xi;\mu,h) = L(x,\xi+2\pi;\mu,h).$$

By $\varepsilon(L)$ we will denote the minimal value of ε for which the above conditions hold.

The final result reads

Theorem HS

Let $L(\mu, h)$ be a strong type I symbol. There exist ϵ_0 , C s. t. if $\varepsilon(L) \leq \epsilon_0$ and if

$$\frac{h}{2\pi} = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \dots}}}$$

with $n_j \geq C$, then the μ-spectrum of the associated operators $\hat{L}_h(\mu)$ is a zero measure Cantor set.

In particular, this applies to the spectrum of the Harper's model. But the theorem says also that this is stable by perturbations respecting all the symmetries.

Critical points

The analysis in the interval J_0 is more delicate. For $E = 0$, the wells are no more compact and the previous construction does not work at all. The renormalization is much more involved. We need a microlocal analysis of the model $h^2 D_x^2 - x^2$ and the renormalized operator is no more an Harper's model but a 2×2 system of h^{new}-pseudodifferential operator whose principal symbol is

$$Q(x,\xi) = \begin{pmatrix} b + \bar{a}e^{-i\xi} & \bar{b} + ae^{ix} \\ b + \bar{a}e^{-ix} & b + \bar{a}e^{i\xi} \end{pmatrix}$$

Fortunately, one can show that there are at the end four models permitting to complete the analysis after the first normalization.

Analysis near a rational – continued

This was the object of [HS2] which is inspired by previous works of Wilkinson, Sokoloff, Bellissard ... The main point is that the analysis of the spectrum for $\alpha = \frac{p}{q} + h$ can be obtained by analyzing a $q \times q$-system of h-pseudodifferntial operators with principal symbol $M_{p,q}(x,\xi)$. Except at the energy 0 where two bands may touch, the basic point is that we have the so-called Chamber's formula:

$$\mathrm{Det}(M_{p,q}(\theta_1,\theta_2) - z) = f_{p,q}(z) + 2(-1)^{q+1}(\cos q\theta_1 + \cos q\theta_2)\,, \tag{49}$$

where $z \mapsto f_{p,q}(z)$ is a polynomial of degree q with nice properties permitting for example to show that the bands do not overlap and can only touch at their end. This is a result of Van Mouche that they do not touch (except when q is even) at the center.

A typical example is for $q = 2$, where we get the matrix

$$M_{1,2}(\theta_1, \theta_2) = \begin{pmatrix} \cos\theta_1 & \cos\theta_2 \\ \cos\theta_2 & -\cos\theta_1 \end{pmatrix} \tag{50}$$

The eigenvalues are

$$\lambda_\pm(\theta_1, \theta_2) = \pm\sqrt{\cos^2\theta_1 + \cos^2\theta_2}$$

A semi-classical analysis of $M_{1,2}(hD_x, x)$ is possible including at the touching point. The harmonic approximation is replaced by a Dirac approximation

$$\begin{pmatrix} hD_x & x \\ x & -hD_x \end{pmatrix}$$

On the non-overlapping of the bands

In the case of the Harper model, the non overlapping of the bands has been proved in Bellissard-Simon who refers for one part to a general argument to Reed-Simon. The fact that except at the center for q even, the bands do not touch has been proven by P. Van Mouche. We show below that the non overlapping of the bands is a general property each time that we have a Chamberss formula but the "non touching" property is specific of the Harper model.

lemma

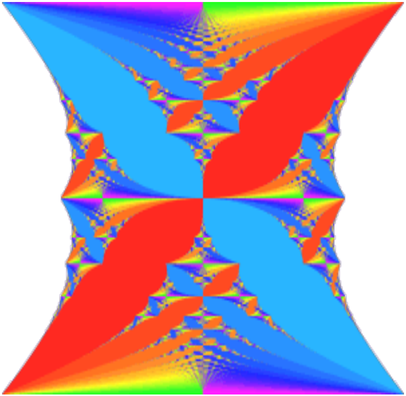
Let $f(\lambda)$ be a real polynomial of degree q, such that, for any $\mu \in I =]a, b[$, $f(\lambda) = \mu$ has q real solutions. Then $f'(\lambda) \neq 0$, for any λ such that $f(\lambda) = \mu \in I$.

About the paper "The Hofstadter butterfly revisited" by Helffer-Kerdelhué-Sjöstrand"

This is not a semi-classical analysis but an incomplete (based on conjectures) description of the spectrum and of the density of states in the gaps. Because the reference is in french, we give below a translation of some of the statements.

The gaps in the spectrum.

This is the "colored" butterfly realized in 2003 by Y. Avron and his team.



A perturbative theorem near $\lambda = 0$.

To follow [HKS] we consider instead

$$\ell^2(\mathbb{Z}) \ni u \mapsto \left(\hat{H}_\alpha(\lambda,\theta)u\right)(n) := \lambda(\Delta^{dis} u)(n) + \cos 2\pi(\alpha n + \theta)u(n)\,, \tag{51}$$

where Δ^{dis} is the discrete Laplacian.
Hence, one has to play with the Aubry duality to come back to the model $H_\alpha(\lambda,\theta)$.
For $\lambda = 0$ the spectrum of $\hat{H}_\alpha(\lambda,\theta)$ is of course the closure of the set of eigenvalues $(\cos 2\pi(\alpha n + \theta))$ with $n \in \mathbb{Z}$. For $\lambda \neq 0$, we would like to show the existence of gaps in the spectrum.
Given some positive integer ℓ, we assume that the following condition

$$\alpha \in NR(\ell) := (0,1) \setminus \cup_{j=1}^{2\ell} \mathbb{N}\frac{1}{j}\,. \tag{52}$$

The first observation is that under this condition, if $n \in \mathbb{Z}$, $\theta \in \mathbb{R}$ satisfy $\cos 2\pi(\alpha n + \theta) = \cos 2\pi(\alpha(n+\ell) + \theta)$, then there exists $k \in \mathbb{Z}$ such that

$$2\pi\left(\alpha(n+\ell) + \theta\right) = 2\pi k - 2\pi(\alpha n + \theta)\,. \tag{53}$$

This implies

$$\cos 2\pi(\alpha n + \theta) = (-1)^k \cos \pi\alpha\ell\,. \tag{54}$$

We now introduce

$$E_{\alpha,\ell} = \cos(\pi\alpha\ell) \tag{55}$$

and observe that under condition (52) we have

$$E_{\alpha,\ell} \notin \{0, 1, -1\}\,, \tag{56}$$

$$\text{for } \ell' \in \{1, 2, \ldots, \ell - 1\}\,,\ |E_{\alpha,\ell'}| \neq |E_{\alpha,\ell}|. \tag{57}$$

and

$$\cos 2\pi(\alpha n + \theta) \neq \cos 2\pi(\alpha(n - \ell) + \theta)\,. \tag{58}$$

All the results below will hold for fixed ℓ and α in a compact subset $\mathcal{A}$ of $NR(\ell)$ uniformly for λ small enough and $\theta \in \mathbb{R}$. The aim is to show the existence of a gap in the spectrum of $\hat{H}^\lambda$ tending to $E_{\alpha,\ell}$ as $\lambda \to 0$.
For $\epsilon > 0$, we introduce

$$A_{\epsilon,\theta,\alpha} = \{n \in \mathbb{Z}\,;\, \alpha n + \theta \pm \frac{\alpha\ell}{2} \in \mathbb{Z} + [-\epsilon, +\epsilon] \text{ for one sign } \} \quad (59)$$

and we observe that $A_{\epsilon,\theta,\alpha}$ is a union of pairs $(n_j, n_j + \ell)$ with $n_j \in \mathbb{Z}$.

Lemma

There exists $\epsilon(\ell, \mathcal{A}) > 0$ and $C = C(\ell, \mathcal{A}) > 0$ such that, for $\epsilon \in (0, \epsilon(\ell, \mathcal{A}))$, either $A_{\epsilon,\theta,\alpha} = \emptyset$ or $A_{\epsilon,\theta,\alpha} = \cup_{j\in\mathbb{Z}}(n_j, n_j + \ell)$ where $n_{j+1} \geq n_j + 2\ell + 1\,.$
Moreover,

$$|\cos(2\pi(\alpha(n+\theta)) - E_{\alpha,\ell}| \begin{cases} \geq \frac{\epsilon}{C}\,, & \text{if } n \in \mathbb{Z} \setminus A_{\epsilon,\theta,\alpha} \\ \leq \epsilon C\,, & \text{if } n \in A_{\epsilon,\theta,\alpha} \end{cases} \quad (60)$$

We now introduce what is called a Grushin problem (which is a variant of a Schur complements method). For $\epsilon > 0$ small enough, we introduce $\mathcal{P}^\lambda(z)$ in $\mathcal{L}(\ell^2(\mathbb{Z}) \times \ell^2(A_\epsilon))$ by

$$\mathcal{P}^\lambda(z) = \begin{pmatrix} \hat{H}^\lambda - z & i \\ \pi & 0 \end{pmatrix}, \tag{61}$$

where $\pi : \ell^2(\mathbb{Z}) \mapsto \ell^2(A_\epsilon)$ is the restriction operator and $i = \pi^* : \ell^2(A_\epsilon) \mapsto \ell^2(\mathbb{Z})$ is the natural injection given by

$$(iu)(n) = \begin{cases} u(n) & \text{if } n \in A_\epsilon \\ 0 & \text{if } n \notin A_\epsilon \end{cases}$$

For $\lambda = 0$, one can see $\mathcal{P}^0(z)$ as a direct sum parametrized by $\mathbb{Z}$ of scalar operators (when $n \in \mathbb{Z} \setminus A_\epsilon$) or 2×2 matrices when $n, n + \ell \in A_\epsilon$. In the first case the scalar is $\cos 2\pi(\alpha n + \theta) - z$ and when $n, n + \ell \in A_\epsilon$ the matrix is

$$\begin{pmatrix} \cos 2\pi(\alpha n + \theta) - z & 1 \\ 1 & \cos 2\pi(\alpha(n+\ell) + \theta) - z \\ & \end{pmatrix} .$$

If $|z - E_{\alpha,\ell}| \leq \frac{1}{2C}\epsilon$ (with the C as in the previous lemma), $\mathcal{P}^0$ is invertible and its inverse reads

$$\begin{aligned} \mathcal{P}^0(z)^{-1} &:= \begin{pmatrix} E_0(z) & E_0^+ \\ E_0^- & E_0^{-+} \end{pmatrix} \\ &= \begin{pmatrix} (1-\pi)\left((H^0 - z)_{\mathbb{Z}\setminus A_\epsilon}\right)^{-1}(1-\pi) & i \\ \pi & (z - H^0)_{/A_\epsilon} \end{pmatrix} . \end{aligned} \tag{62}$$

We immediately get for a new constant $\hat{C} = \hat{C}(\ell, \mathcal{A}) \geq C$ that

$$||\mathcal{P}^0(z)^{-1}|| \leq \hat{C}\epsilon^{-1} . \tag{63}$$

With a new constant $\check{C} \geq \hat{C}$, we immediately deduce that $\left(\mathcal{P}^\lambda - z\right)$ is invertible for $|\lambda| \leq \frac{\epsilon}{\check{C}}$ and $|z - E_{\alpha,\ell}| \leq \frac{1}{\check{C}}\epsilon$, with in addition the control

$$||\mathcal{P}^\lambda(z)^{-1}|| \leq \check{C}\epsilon^{-1}\,. \tag{64}$$

This inverse is indeed given by the Neumann series

$$\mathcal{E}_\lambda(z) = \mathcal{P}^\lambda(z)^{-1} = \sum_{j\geq 0}(-\lambda)^j \mathcal{P}^0(z)^{-1}\left(\begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}\mathcal{P}^0(z)^{-1}\right)^j\,. \tag{65}$$

Writing $\mathcal{E}_\lambda(z)$ in the form

$$\mathcal{E}_\lambda(z) = \begin{pmatrix} E_\lambda(z) & E_\lambda^+(z) \\ E_\lambda^-(z) & E_\lambda^{-+}(z) \end{pmatrix}$$

it is important to note (this is the interest of the Grushin method) that for z and λ satisfying the above assumptions

$$z \in \sigma(\hat{H}^\lambda) \text{ iff } 0 \in \sigma(E_\lambda^{-+}(z))\,. \tag{66}$$

We have for $E_\lambda^{-+}(z)$ the following expression

$$\begin{aligned} & E_\lambda^{-+}(z) \\ & = (z-\hat{H}^0)_{/A_\epsilon} \\ & + \sum_{j\geq 1}(-\lambda)^j \pi \left(\Delta(1-\pi)\left((z-\hat{H}^0)_{/\mathbb{Z}\setminus A_\epsilon}\right)^{-1}(1-\pi)\right)^{j-1}\Delta i\,. \end{aligned} \tag{67}$$

If $(n,m)\in(A_\epsilon)^2$ the element of the matrix of $E_\lambda^{-+}(z)$ is denoted by $E_\lambda^{-+}(z)(n,m)$. We observe from the above expression that

$$|\partial_z^k E_\lambda^{-+}(z)(n,m)| \leq C_{\epsilon,k}\lambda^{|n-m|}\,. \tag{68}$$

Let $n, n+\ell \in A_\epsilon$ and consider the 2×2 block matrix
$\begin{pmatrix} E_\lambda^{-+}(z)(n,n) & E_\lambda^{-+}(z)(n,n+\ell) \\ E_\lambda^{-+}(z)(n+\ell,n) & E_\lambda^{-+}(z)(n+\ell,n+\ell) \end{pmatrix}$.
It has the form

$$\begin{pmatrix} \hat{\lambda}_{n,\lambda,z} & \mu_{n,\lambda,z} \\ \mu_{n,\lambda,z} & \hat{\lambda}_{n+\ell,\lambda,z} \end{pmatrix} + \mathcal{O}_\epsilon(\lambda^{\ell+1})\,, \tag{69}$$

where the remainder corresponds to the contribution of

$$+\sum_{j\geq \ell+1}(-\lambda)^j \pi \left(\Delta(1-\pi)\left((z-\hat{H}^0)_{/\mathbb{Z}\setminus A_\epsilon}\right)^{-1}(1-\pi)\right)^{j-1}\Delta i$$

in the above formula.
We first look at the two terms on the diagonal that we now write $\hat{\lambda}_{n,\lambda,z}(\theta)$ and $\hat{\lambda}_{n+\ell,\lambda,z}(\theta)$ to recall now the dependence on θ which will now play a role.
These elements are still well defined when we replace θ by a variable $\tilde{\theta}$ varying in the largest integral $\mathcal{J}_n$ containing θ with the property that $n, n+\ell \in A_{\epsilon_0,\tilde{\theta}}$, where ϵ_0 is fixed, small but satisfying $\epsilon_0 >> \epsilon$. More explicitly, if $k \in \mathbb{Z}$ is such that

$$n\alpha + \theta + \frac{\alpha\ell}{2} \in k \in [-\epsilon, +\epsilon]\,,$$

then $\hat{\lambda}_{n,\lambda,z}(\tilde{\theta})$ and $\hat{\lambda}_{n+\ell,\lambda,z}(\tilde{\theta})$ are well defined for $|\tilde{\theta} - \theta_0| \leq \epsilon_0$, where θ_0 is defined by

$$n\alpha + \theta_0 + \frac{\alpha\ell}{2} = k\,.$$

Let $\delta(\tilde{\theta}) = \tilde{\theta} - \theta_0$. Then $|\delta(\theta)| \le \epsilon$. Note that θ_0 depends on n but all the estimates below will be uniform with respect to n. For $\tilde{\theta} = \theta_0$, n and $n+\ell$ are in a symmetric situation for the map $\tilde{n} \mapsto \cos 2\pi(\alpha\tilde{n} + \tilde{\theta})$:

$$\cos 2\pi(\alpha(n+\nu) + \theta_0) = \cos 2\pi\alpha(\nu - \frac{\ell}{2})\,,\ \forall \nu \in \mathbb{R}\,, \tag{70}$$

which implies

$$\cos 2\pi(\alpha n + \theta_0) = \cos 2\pi(\alpha(n+\ell) + \theta_0)\,. \tag{71}$$

By symmetry arguments we get

$$\hat{\lambda}_{n,\lambda,z}(\theta_0) = \hat{\lambda}_{n+\ell,\lambda,z}(\theta_0) := z - E_{\alpha,\ell,\lambda,z}\,, \tag{72}$$

where

$$\begin{cases} E_{\alpha,\ell,\lambda,z} & = E_{\alpha,\ell} + \mathcal{O}(\lambda^2)\,, \\ \partial_z^p E_{\alpha,\ell,\lambda,z} & = \mathcal{O}_p(\lambda^2)\,, \end{cases} \tag{73}$$

and $E_{\alpha,\ell,\lambda,z}$ is independent of n.

Let us also observe that the dependence on n in the expression of $\hat{\lambda}_{n,\lambda,z}(\tilde{\theta})$ and $\hat{\lambda}_{n+\ell,\lambda,z}(\tilde{\theta})$ appears only through $\tilde{\theta}-\theta_0$.
From (69), we also get the information

$$\partial_{\tilde{\theta}}\hat{\lambda}_{n,\theta,z}(\theta_0) = q + \mathcal{O}(\lambda^2)\,,\ \partial_{\tilde{\theta}}\hat{\lambda}_{n+\ell,\theta,z}(\theta_0) = -q + \mathcal{O}(\lambda^2)\,, \tag{74}$$

where

$$q = 2\pi\sin(\pi\alpha\ell) \neq 0\,. \tag{75}$$

With $\delta = \theta - \theta_0$, we obtain by a Taylor expansion

$$\begin{cases} \hat{\lambda}_{n,\lambda,z}(\theta) & = z - E_{\alpha,\ell,\lambda,z} + q\delta + \mathcal{O}(\lambda^2|\delta| + |\delta|^2)\,, \\ \hat{\lambda}_{n+\ell,\lambda,z}(\theta) & = z - E_{\alpha,\ell,\lambda,z} - q\delta + \mathcal{O}(\lambda^2|\delta| + |\delta|^2)\,. \end{cases} \tag{76}$$

We now introduce

$$w(z,\lambda) = z - E_{\alpha,\ell,\lambda,z}\,, \tag{77}$$

and we observe that

$$|w(z,\lambda)| \sim |z - z(\lambda)|\,, \tag{78}$$

where $z(\lambda) = E_{\alpha,\ell} + \mathcal{O}(\lambda^2)$ is independent of n.
For $\mu_{n,\lambda}$, we have the explicit formula

$$\mu_{n,\lambda} = (-\lambda/2)^{\ell} / \left(\prod_{j=1}^{\ell-1} (\,(\cos(2\pi(n+j)\alpha+\theta) - z))\right)\,, \tag{79}$$

but we will only use

$$|\mu_{n,\ell}| \sim \lambda^{\ell}\,. \tag{80}$$

The eigenvalues of the first term of (69) are given by

$$\hat{\lambda}_{\pm} = \left(\hat{\lambda}_n + \hat{\lambda}_{n+\ell}\right)/2 \pm \sqrt{\mu^2 + \left((\hat{\lambda}_n - \hat{\lambda}_{n+\ell})/2\right)^2}\,. \tag{81}$$

Hence we get

$$(\hat{\lambda}_+ + \hat{\lambda}_-)/2 = w(z,\lambda) + \mathcal{O}(\lambda^2|\delta| + |\delta|^2)\,, \tag{82}$$

and

$$\begin{aligned} (\hat{\lambda}_+ - \hat{\lambda}_-)/2 \quad &= \sqrt{\mu^2 + \left((\hat{\lambda}_n - \hat{\lambda}_{n+\ell})/2\right)^2} \\ &\sim \lambda^\ell + |\delta| + \mathcal{O}(\lambda^2|\delta| + |\delta|^2) \\ &\sim \lambda^\ell + |\delta| \text{ if } \epsilon \text{ and } \lambda \text{ are small enough} \quad . \end{aligned} \tag{83}$$

If we impose the condition

$$|z - z(\lambda)| \leq \lambda^\ell / D \tag{84}$$

with $D > 0$ large enough, we get

$$|w(z,\lambda)| \leq \lambda^\ell / \tilde{D}\,,$$

with $\tilde{D}$ as large as we want (through the choice of D).
Comparing (82) and (83), we see that, we can choose $\tilde{D}$ such that

$$\hat{\lambda}_+ - \hat{\lambda}_- >> (\hat{\lambda}_+ + \hat{\lambda}_-)/2\,,$$

hence we get

$$\inf(|\hat{\lambda}_+|, |\hat{\lambda}_-|) \geq \frac{1}{C}(\lambda^\ell + |\delta|)\,,\ \forall z \in (z(\lambda) - \frac{1}{D}\lambda^\ell, z(\lambda) + \frac{1}{D}\lambda^\ell)\,.$$

The first block in (69) therefore admits an inverse of norm $\mathcal{O}(1))\frac{1}{\lambda^\ell+|\delta|}$. The interval $(z(\lambda)-\frac{1}{D}\lambda^\ell, z(\lambda)+\frac{1}{D}\lambda^\ell)$ is independent of n and the result is obtained for fixed ϵ small enough. The perturbation term in (69) is controlled in $\mathcal{L}(\ell^2(A_\epsilon))$ as $\mathcal{O}(\lambda^{\ell+1})$. Hence we obtain the invertibility of $E_\lambda^{-+}(z)$ for $z\in(z(\lambda)-\frac{1}{D}\lambda^\ell, z(\lambda)+\frac{1}{D}\lambda^\ell)$ and λ small enough depending only on ℓ and $\mathcal{A}$.

We have finally obtained

Theorem (Helffer-Kerdelhué-Sjöstrand)

There exists $\lambda_0 = \lambda_0(\ell, \mathcal{A})$ and $C = C(\ell, \alpha)$, s. t. for all $\lambda \in (0, \lambda_0)$, there exists

$$z_\ell(\lambda, \alpha) = E_{\alpha,\ell} + O(\lambda^2)$$

such that for all $\theta \in \mathbb{R}$, we have

$$(z_\ell(\lambda, \alpha) - \frac{\lambda^\ell}{C}, z_\ell(\lambda) + \frac{\lambda^\ell}{C}) \cap \sigma(\hat{H}_\alpha(\lambda, \theta)) = \emptyset\,, \tag{85}$$

and the same result holds near $-z_\ell(\lambda, \alpha)$.

This gives a lower bound for the gap which appears to be optimal in the rational case due to some explicit computations given by P. Van Mouche in his analysis of the spectrum of Harper in the rational case (CMP 1989).

It is also proven under the same conditions as in the theorem:

Proposition

The value of the integrated density of states in the gap around $z_\ell(\lambda, \alpha)$ is given by

$$\rho^\lambda((-\infty, z_\ell(\lambda, \alpha)]) = 1 - \mathrm{dist}(\ell\alpha, 2\mathbb{Z}) = |2\{\frac{\ell\alpha}{2}\} - 1|\,. \qquad (86)$$

Some conjectural analysis of the wings

We denote by Σ_α^λ the spectrum of $\frac{1}{1+\lambda}\hat{H}^\lambda$. We then consider in $Q = [0,1] \times [-1,+1]$ the set

$$\Sigma^\lambda := \cup_\alpha (\alpha, \Sigma_\alpha^\lambda)\,.$$

It is well known that Σ^λ is closed and we call wing (in [HKS] this is called "fuseau") a connected component of the complementary $\complement\Sigma^\lambda$ of Σ^λ in Q. Hence by definition a wing is open.
The aim is to discuss the structure of the wings. We then observe the following properties:

Property P1

(P1) For any $\alpha_0 \in (0,1)$, the line $\alpha = \alpha_0$ cuts a wing $\mathcal{W}^\lambda$ on an (possibly empty) open interval $I(\alpha_0, \lambda)$.

Property P2

For any wing $\mathcal{W}$, there exists $\alpha_\pm(\mathcal{W})$ such that

$$\mathcal{W} \subset (\alpha_-, \alpha_+) \times [-1, +1]\,,\ 0 \leq \alpha_- < \alpha_+ \leq 1\,,\ \alpha_\pm \in \pi(\overline{\mathcal{W}})\,,$$

where π denotes the projection $\mathbb{R}^2 \ni (\alpha, E) \mapsto \alpha$.

This is obtained immediately by connectedness.

For an interval J in $\mathbb{R}^+$ we introduce

Conjecture C3 (J)

For any $\lambda \in J$ and for any wing $\mathcal{W}^\lambda$, the points $\alpha_\pm$ are rational.

This conjecture was open in the 90's. It would be interesting to know if it can be proved by more recent results (see below Avila-You-Zhou).

Property P4 : Continuity

If, for $J = (0, \lambda_1)$, Conjecture $C3(J)$ holds, then, for any $\lambda \in I$, the boundary of the wing $\mathcal{W}^\lambda$ is continuous.

By continuity, we mean that if we write $I(\alpha, \lambda) = (f_-(\alpha, \lambda), f_+(\alpha, \lambda))$ the maps $(\alpha_-, \alpha_+) \ni \alpha \mapsto f_\pm(\alpha, \lambda)$ belong to $C^0([\alpha_-, \alpha_+])$ with $f_-(\alpha_\pm, \lambda) = f_+(\alpha_\pm, \lambda)$. Note that Elliott gets $C^{\frac{1}{3}}_{loc}(\alpha_-, \alpha_+)$. What was not clearly established before [HKS] is the continuity at the ends of the interval.
The proof in [HKS] (stated for $J = (0, +\infty)$) is based on Assumption $C3(J)$ and a deformation argument in λ starting of what we know by perturbation theory for $\lambda = 0$ (as explained in Remark 3.5 in [HKS]) gives that $f_-(\alpha_\pm, \lambda) = f_+(\alpha_\pm, \lambda)$ is at the end of a band. The semiclassical proof "near a rational" which is also proposed gives a good hint for what is going on but is unfortunately incomplete when $0 < \lambda < 1$.

Property P5

If for $J = (0, \lambda_1)$, Conjecture $C3(J)$ holds, for all $\alpha \in (0, 1)$, $I(\alpha, \lambda)$ is either empty for all $\lambda \in (0, +\infty)$ or never closes for all $\lambda \in (0, +\infty)$.

Property P6

If, for $J = (0, \lambda_1)$, Conjecture $C3(J)$ holds, $\alpha_\pm(\mathcal{W}_\lambda)$ is independent of λ for $\lambda \in J$.

The discussion can be summarized by the following statement:

Theorem

If for $J = (0, \lambda_1)$, Conjecture $C3(J)$ holds, then for any "continuous" family $(\mathcal{W}_\lambda)_{\lambda \in J}$, there exist two rationals $\alpha_\pm$ (with $0 \leq \alpha_- < \alpha_+ \leq 1$) such that

$$\forall \lambda \in J\,, \pi(\overline{\mathcal{W}_\lambda}) = [\alpha_-, \alpha_+]\,,$$

and two functions $f_\pm$ defined on $(\alpha_-, \alpha_+) \times (0, \lambda_1)$ such that

- $f_\pm(\alpha, \lambda) \in C^0([\alpha_-, \alpha_+] \times [0, \lambda_1))\,,$
- $|f_\pm(\alpha, \lambda) - f_\pm(\alpha, \lambda')| \leq 2|\lambda - \lambda'|\,,$
- $f_-(\alpha, \lambda) < E < f_+(\alpha, \lambda), \alpha \in [\alpha_-, \alpha_+)$ iff $(\alpha, E) \in \mathcal{W}_\lambda\,,$
- $f_-(\alpha_\pm, \lambda) = f_+(\alpha_\pm, \lambda)$ belongs to the end of a band of $\Sigma^\lambda_{\alpha_\pm}$.

End of theorem

Moreover, if the integrated density of states in $\mathcal{W}^\lambda$ is given by $k(\alpha, \mu) = m\alpha + n$, then

1. $\alpha_\pm \in \cup_{j \le 2|m|} \mathbb{N}\{\frac{1}{j}\} \cup (0,1)$
2. $(\alpha_-, \alpha_+) \cap \left(\cup_{j \le 2|m|} \mathbb{N}\{\frac{1}{j}\} \cup (0,1) \right) = \emptyset$

Remark

Conjecture $C3(\lambda_1)$ implies for $\lambda \in (0, \lambda_1)$ the "dry" or "strong" form of the ten Martinis conjecture which was formulated by B. Simon in the form

Conjecture

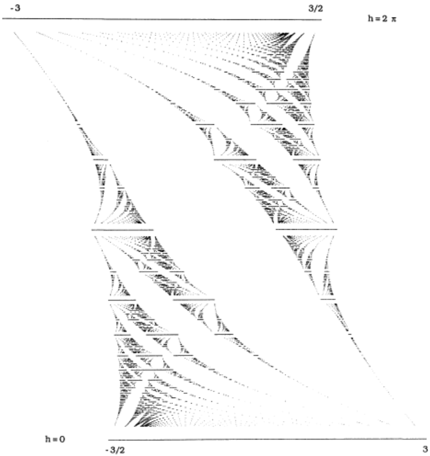
For all $\lambda \neq 0$, all $\alpha \notin \mathbb{Q}$ and any integers m, n with $0 < m\alpha + n < 1$, there exists a gap for which the IDS in the gap satisfies

$$k(\alpha, \cdot) = m\alpha + n\,.$$

Note that this conjecture is now proved for $\lambda \in (0, 1)$ by Avila-You-Zhou. Hence Conjecture $C3(+\infty)$ appears to be a stronger form of the "dry ten Martinis conjecture" (and probably equivalent).

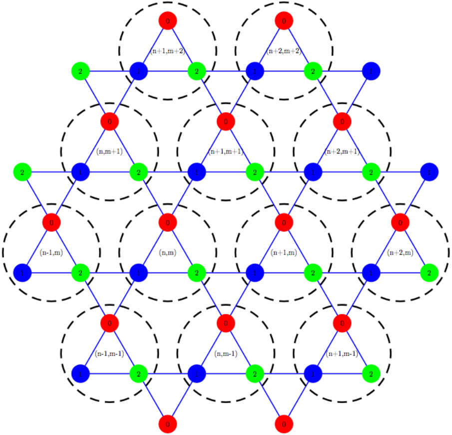
Other lattices

One can consider other lattices: the hexagonal Hofstadter butterfly (after Kerdelhué, Kreft-Seiler, Claro,....)

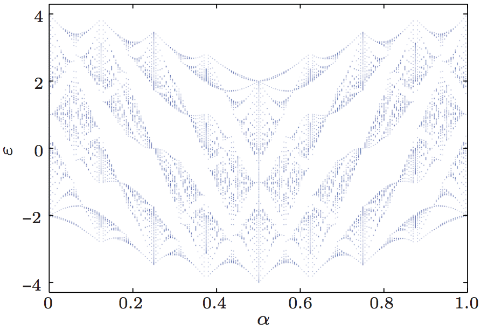


Other examples

J. Royo-Letelier and P. Kerdelhué (see [Hou]) have analyzed rigorously the case of a Kagome lattice.



The Kagome butterfly



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