Nodal sets for eigenfunctions on the square and on the sphere: A. Stern's analysis revisited.
(after P. Bérard and B. Helffer)

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at the occasion of the 75-th birthday of Gerd Grubb.

Introduction

The celebrated nodal domain theorem by Courant [26] says that the number of nodal domains of an eigenfunction associated with a k-th eigenvalue of the Dirichlet Laplacian (eigenvalues listed in increasing order) should be less than or equal to k. Pleijel [17] proved that equality holds only for finitely many values of k. In this case we speak of the Courant sharp situation (see [14, 15] for the connection of this property with the question of minimal spectral partitions).

If we look at the square, it is immediate that the first, second and fourth eigenvalues are Courant sharp. We will first analyze the statement by Pleijel saying that there are no other cases.

In the case of the sphere, it is possible to prove (Leydold-Karpushin) that the only cases where it occurs are the first and second eigenvalues.

We mainly discuss some results of Antonie Stern [20] who was a PhD student of R. Courant and defended her PhD in 1924, see [37]. These results concern the square and the sphere.
Although containing interesting arguments toward a proof, it is probably not true that her proofs are complete.
In 1977, Hans Lewy (another former student of R. Courant at about the same time as A. Stern) apparently forgetting (to mention) the work of A. Stern proves rigorously the results concerning the sphere.

The following biographical information has been indicated to us by Annette Vogt[1].

> *Antonie (Toni) Stern (1892 Dortmund - after 1967 Israel) studied mathematics. In 1925 she received the doctoral degree (Dr. phil.) at the Göttingen University, her advisor was Richard Courant. Obviously she could not find an academic position as a female mathematician, but she was a member of the German Mathematical Society (DMV) from 1926 until 1939 when she managed to escape Nazi Germany, and went into exile to Palestine, where her sister Ilse (b. 1900) was living already since 1924.*

[1] A. Vogt. "Wissenschaftlerinnen in Kaiser-Wilhelm-Instituten. A-Z." Berlin 2008. Veröffentlichungen aus dem Archiv zur Geschichte der Max-Planck-Gesellschaft, Bd. 12, 2. erw. Aufl., p. 180.

After her thesis, she changed her scientific field, and she became a researcher in the Kaiser Wilhelm Institute for Applied Physiology (occupational physiology) in Dortmund, from 1929 until 1933. Because of the Nazi's, she had to leave the KWI in late 1933. She was born in a Jewish family, the antisemitic laws in Nazi Germany were introduced in April 1933. What she did between 1933 and 1938 is not known. In the end of 1938 (after the November 1938 pogrom in Germany, called "Reichskristallnacht"), she emigrated from Dortmund to Palestine....

Some motivation coming from the minimal partitions

Before to present these results, we recall briefly the link with the problem of minimal spectral k-partitions in the two-dimensional case.

Some motivation coming from the minimal partitions

Before to present these results, we recall briefly the link with the problem of minimal spectral k-partitions in the two-dimensional case.
We consider mainly the Dirichlet Laplacian in a bounded domain $\Omega \subset \mathbb{R}^2$. We assume that Ω is sufficiently regular say with C^∞ boundary.

In [15] with T. Hoffmann-Ostenhof and S. Terracini, we have started to analyze the relations between the nodal domains of the real-valued eigenfunctions of this Laplacian and the partitions of Ω by k open sets D_i which are minimal in the sense that the maximum over the D_i's of the ground state energy (= lowest eigenvalue) of the Dirichlet realization of the Laplacian $H(D_i)$ in D_i is minimal.

We denote by $\lambda_j(\Omega)$ the increasing sequence of its eigenvalues and by u_j some associated orthonormal basis of real-valued eigenfunctions. The groundstate u_1 can be chosen to be strictly positive in Ω, but the other eigenfunctions u_k must have zerosets. For any real-valued $u \in C_0^0(\overline{\Omega})$, we define the zero set as

$$N(u) = \overline{\{x \in \Omega \mid u(x) = 0\}} \tag{1}$$

and call the components of $\Omega \setminus N(u)$ the nodal domains of u. The number of nodal domains of u is called $\mu(u)$. These $\mu(u)$ nodal domains define a k-partition of Ω, with $k = \mu(u)$.

We recall that the Courant nodal theorem says that, for $k \geq 1$, and if λ_k denotes the k-th eigenvalue and $E(\lambda_k)$ the eigenspace of $H(\Omega)$ associated with λ_k, then, for all real-valued $u \in E(\lambda_k) \setminus \{0\}$, $\mu(u) \leq k$. A theorem due to Pleijel [17] in 1956 says that this cannot be true when the dimension (here we consider the $2D$-case) is larger than one.

Minimal spectral partitions

We now introduce for $k \in \mathbb{N}$ $(k \geq 1)$, the notion of k-partition.
We will call k-**partition** of Ω a family $\mathcal{D} = \{D_i\}_{i=1}^{k}$ of mutually disjoint open connected sets in Ω. We denote by $\mathfrak{O}_k(\Omega)$ this set.
A spectral minimal partition sequence is defined by

Definition

For any integer $k \geq 1$, and for $\mathcal{D}$ in $\mathfrak{O}_k(\Omega)$, we set

$$\Lambda(\mathcal{D}) = \max_i \lambda(D_i). \tag{2}$$

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D} \in \mathfrak{O}_k} \Lambda(\mathcal{D}). \tag{3}$$

and call $\mathcal{D} \in \mathfrak{O}_k$ a minimal k-partition if $\mathfrak{L}_k = \Lambda(\mathcal{D})$.

If $k = 2$, it is rather well known that $\mathfrak{L}_2 = \lambda_2$ and that the associated minimal 2-partition is a **nodal partition**, i.e. a partition whose elements are the nodal domains of some eigenfunction corresponding to λ_2.

One can show (Conti-Terracini-Verzini [9, 10, 11] and Helffer–Hoffmann-Ostenhof–Terracini [15]) that minimal spectral partitions always exist, are actually as regular[2] as the nodal sets of an eigenfunction.

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[2] up to set of capacity 0

We say that D_i, D_j are **neighbors** or $D_i \sim D_j$, if $D_{ij} := \mathrm{Int}\,(\overline{D_i \cup D_j}) \setminus \partial\Omega$ is connected. We associate with each $\mathcal{D}$ a **graph** $G(\mathcal{D})$ by associating with each D_i a vertex and to each pair $D_i \sim D_j$ an edge. We will say that the graph is **bipartite** if it can be colored by two colors (two neighbors having two different colors). We recall that the graph associated with a collection of nodal domains of an eigenfunction is always bipartite.

Main results in the $2D$ case

A natural question is whether a minimal partition of Ω is a nodal partition, i.e. the family of nodal domains of an eigenfunction of $H(\Omega)$.

We have first the following converse theorem ([15]):

Theorem BP

If the minimal partition is bipartite this is a nodal partition.

A natural question is now to determine how general this previous situation is.

Surprisingly this only occurs in the so called Courant-sharp situation. We say that:

Definition Courant-sharp

A pair (u, λ_k) is Courant-sharp if
$u \in E(\lambda_k) \setminus \{0\}$ and $\mu(u) = k$.

An eigenvalue is
called Courant-sharp if there exists an associated Courant-sharp pair.

For any integer $k \geq 1$, we denote by $L_k(\Omega)$ the smallest eigenvalue whose eigenspace contains an eigenfunction of $H(\Omega)$ with k nodal domains. We set $L_k(\Omega) = \infty$, if there are no eigenfunctions with k nodal domains. In general, one can show, that

$$\lambda_k(\Omega) \leq \mathfrak{L}_k(\Omega) \leq L_k(\Omega) \,. \tag{4}$$

The last result gives the full picture of the equality cases :

Theorem 3

Suppose $\Omega \subset \mathbb{R}^2$ is regular.
If $\mathfrak{L}_k = L_k$ or $\mathfrak{L}_k = \lambda_k$ then

$$\lambda_k = \mathfrak{L}_k = L_k \,.$$

In addition, one can find a Courant-sharp pair (u, λ_k).

This is therefore interesting to determine for a given open set all the Courant sharp cases. This is what we want to do in the case of the square.

Pleijel's theorem revisited

Pleijel's theorem as stated in the introduction is the consequence of a more precise theorem which gives a link between Pleijel's theorem and minimal partitions. The classical proof is indeed going through the proposition

Proposition 1

$$\limsup_{n\to+\infty} \frac{\mu(\phi_n)}{n} \leq \frac{4\pi}{A(\Omega)\liminf_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k}}\,, \tag{5}$$

where $\mu(\phi_n)$ is the cardinal of the nodal components of $\Omega \setminus N(\phi_n)$ and then to establishing a lower bound for $A(\Omega)\liminf_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k}$.
We now focus our analysis on the square.

On Pleijel's analysis for the square

Consider the rectangle $\mathcal{R}(a,b) =]0, a\pi[\times]0, b\pi[$. The eigenvalues are given by

$$\hat{\lambda}_{m,n} = (\frac{m^2}{a^2} + \frac{n^2}{b^2})\,,\ \ m, n \geq 1,$$

with a corresponding basis of eigenfunctions given by

$$\phi_{m,n}(x,y) = \sin\frac{mx}{a}\,\sin\frac{ny}{b}\,.$$

It is easy to determine the Courant sharp eigenvalues when b^2/a^2 is irrational (see for example [15]). The rational case is more difficult. In [17], Pleijel claims that in the case of the square, the Dirichlet eigenvalue λ_k is Courant sharp if and only if $k = 1, 2, 4$. His proof involves the reduction to the analysis of the cases $k = 5, 7, 9$, and does not seem well justified for this last point ; he indeed refers to the book by Courant-Hilbert [27] where only pictures are presented, actually extracted from an older book by Pockels [35].

Let us consider the general question of analyzing the zero set of the Dirichlet eigenfunctions for the square $\mathcal{S}$. We have:

$$\phi_{m,n}(x,y) = \phi_m(x)\phi_n(y)\,, \ \text{ with } \phi_m(t) = \sin(m\pi t)\,.$$

Due to multiplicities, we have (at least) to consider the family of eigenfunctions,

$$(x,y) \mapsto \Phi_{m,n}(x,y,\theta) := \cos\theta\,\phi_{m,n}(x,y) + \sin\theta\,\phi_{n,m}(x,y)\,,$$

with $m,n \geq 1$, and $\theta \in [0,\pi[$.

In Pleijel's analysis [17] of the Courant sharp property for $\mathcal{S}$, it is shown that it is enough to consider the eigenvalues λ_5, λ_7 and λ_9 with correspond respectively to the pairs $(m,n) = (1,3)$, $(m,n) = (2,3)$ and $(m,n) = (1,4)$.

80 Ueber die Gleichung: $\Delta u + k^2 u = 0.$

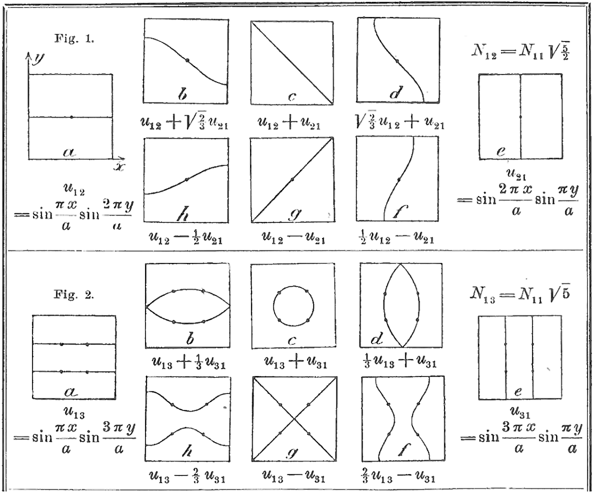

Figure: Nodal sets, Dirichlet eigenvalues λ_2 and λ_5 (Pockels, [35]).

Pleijel's reduction argument

Let us briefly recall Pleijel's argument. Let $N(\lambda) := \#\{n \mid \lambda_n < \lambda\}$ be the counting function. Using a covering of $\mathbb{R}^2$ by the squares $]k, k+1[\times]\ell, \ell+1[$, he first establishes the estimate

$$N(\lambda) > \frac{\pi}{4}\lambda - 2\sqrt{\lambda} - 1\,. \tag{6}$$

For any n such that $\lambda_{n-1} < \lambda_n$, we have $N(\lambda_n) = n - 1$, and

$$n > \frac{\pi}{4}\lambda_n - 2\sqrt{\lambda_n}\,. \tag{7}$$

On the other hand, if λ_n is Courant sharp, the Faber-Krahn inequality gives the necessary condition

$$\frac{\lambda_n}{n} \geq \frac{\mathbf{j}^2}{\pi}$$

or

$$\frac{n}{\lambda_n} \leq \pi \mathbf{j}^{-2} \sim 0.545\,. \tag{8}$$

Recall that $\pi \mathbf{j}^2$ is the ground state energy of the disk of area 1.

Combining (7) and (8), leads to the inequality

$$\lambda_n < 68\,. \tag{9}$$

After re-ordering the values $m^2 + n^2$, we get the spectral sequence for $\lambda_n \leq 68$,

It remains to analyze, among the eigenvalues which are less than 68, the ones which could be Courant sharp, and hence satisfy (8). Computing the quotients $\frac{n}{\lambda_n}$ in the list, this leaves us with the eigenvalues λ_5, λ_7 and λ_9.
For these last three cases, Pleijel refers to pictures in Courant-Hilbert [27], actually reproduced from Pockel [35], see above. Although the choice of pictures suggests that some theoretical analysis is involved, one cannot see any systematic analysis, the difficulty being that we have to analyze the nodal sets of eigenfunctions living in two-dimentional eigenspaces. Hence one has to give a detailed proof that eigenvalues λ_5, λ_7 and λ_9 are not Courant sharp.

Of course we know that $\Phi_{m,n}$ has mn nodal components (this corresponds to the "product" situation with $\theta = 0$ or $\theta = \frac{\pi}{2}$). However, we have already mentioned that the number of nodal domains for a linear combination of two given independent eigenfunctions can be smaller or larger than the number of nodal domains of the given eigenfunctions.

The three cases left by Pleijel

Behind all the computations we have the property that, for $x \in]0, \pi[$,

$$\sin mx = \sqrt{1-u^2}\, U_{m-1}(u)\,, \tag{10}$$

where U_{m-1} is the Chebyshev polynomial of second type and $u = \cos x$.

First case : eigenvalue λ_5 $((m, n) = (1, 3))$.

We look at the zeroes of $\Phi_{1,3}(x, y, \theta)$. Let,

$$\cos x = u\,,\ \cos y = v\,. \tag{11}$$

This is a C^∞ change of variables from the square $]0, \pi[\times]0, \pi[$ onto $]-1, +1[\times]-1, +1[$. In these coordinates, the zero set of $\Phi_{1,3}(x, y, \theta)$ inside the square is given by

$$\cos\theta\,(4v^2 - 1) + \sin\theta\,(4u^2 - 1) = 0\,. \tag{12}$$

Except the two easy cases when $\cos\theta = 0$ or $\sin\theta = 0$, which can be analyzed directly (product situation), we immediately get that the only possible critical point is $(u, v) = (0, 0)$, i.e. $(x, y) = (\frac{\pi}{2}, \frac{\pi}{2})$, and that this can only occur for $\cos\theta + \sin\theta = 0$, i.e. for $\theta = \frac{\pi}{4}$.

This analysis shows rigorously that the number of nodal domains is 2, 3 or 4 as claimed in [17], and numerically observed in the Figures above. As a matter of fact, we have a complete description of the situation.

Second case: eigenvalue λ_7 $((m,n) = (2,3))$.

We look at the zeros of $\Phi_{2,3}(x,y,\theta)$. We first observe that

$$\Phi_3(x,y,\theta) = \sin x \sin y \times$$
$$\times \left(2\cos\theta \cos x(\cos 2y + 2\cos^2 y) + 2\sin\theta \cos y(\cos 2x + 2\cos^2 x)\right) .$$

In the coordinates (11), this reads:

$$\Phi_3(x,y,\theta) = 2\sqrt{1-u^2}\sqrt{1-v^2}\left(u\cos\theta(4v^2-1) + v\sin\theta(4u^2-1)\right) . \tag{13}$$

We have to look at the solutions of:

$$\Psi_{2,3}(u,v,\theta) := u(4v^2-1)\cos\theta + v(4u^2-1)\sin\theta = 0 \,, \tag{14}$$

inside $[-1,+1] \times [-1,+1]$.

Analysis at the boundary.

Due to symmetries, it suffices to consider the values $\theta \in [0, \frac{\pi}{2}]$ and the boundaries $u = -1$ and $v = -1$.

At the boundary $u = -1$, we get:

$$-\cos\theta\,(4v^2 - 1) + 3v\sin\theta = 0\,, \tag{15}$$

with the condition that $v \in [-1, +1]$. An analysis shows that the zero set of $\Psi_{2,3}$ always hits the boundary at six points.

Critical points.

We now look at the critical points of $\Psi_{2,3}$. We get two equations:

$$(4v^2 - 1)\cos\theta + 8uv\sin\theta = 0\,, \tag{16}$$

and

$$8uv\ \cos\theta + (4u^2 - 1)\ \sin\theta = 0\,. \tag{17}$$

The critical points on the zero set of $\Psi_{2,3}$ are the common solutions of (14), (16), and (17).

If $\cos\theta \sin\theta \neq 0$, we obtain that $u = v = 0$, and these equations have no common solution. It follows that we have no interior critical point on their nodal set.
Hence the lines cannot intersect each other.

The number of nodal domains is four (delimited by three non intersecting lines) or six in the product case. Hence the maximal number of nodal domains is six.
Hence we are not in a Courant sharp situation.

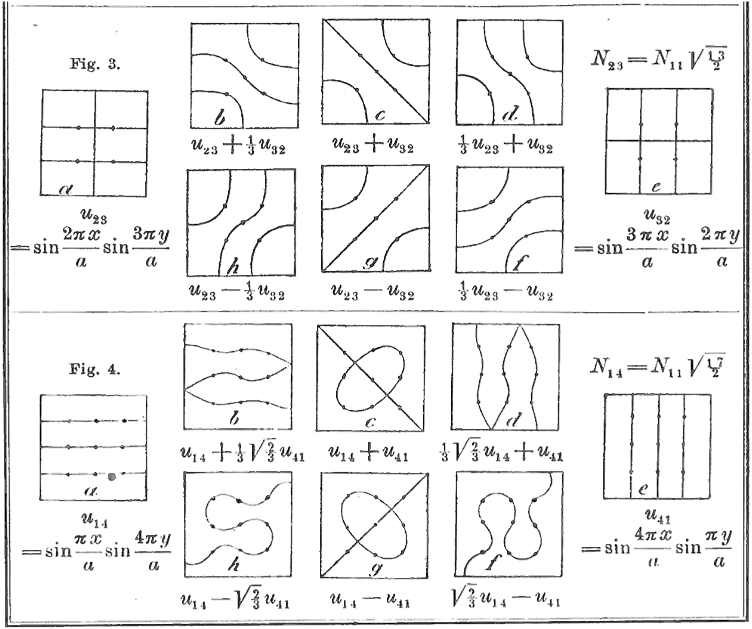

Figure: Nodal sets, Dirichlet eigenvalues λ_7 and λ_9 (Pockels, [35]).

Third case : eigenvalue λ_9 $((m,n) = (1,4))$.

We look at the zeros of $\Phi_{1,4}(\cdot,\cdot,\theta)$. Here we can write

$$\Phi_{1,4}(x,y,\theta) = 4\sin x \sin y\, \Psi_{1,4}(u,v,\theta)$$

with

$$\Psi_{1,4}(u,v,\theta) := \cos\theta\, v(2v^2-1) + \sin\theta\, u(2u^2-1)\,.$$

Hence, we have to analyze the equation

$$\cos\theta\, v(2v^2-1) + \sin\theta\, u(2u^2-1) = 0\,. \tag{18}$$

At the boundary.

Due to the symmetries, the zero set of $\Psi_{1,4}$ hits parallel boundaries at symmetrical points. For $u = \pm 1$ these points are given by:

$$v(2v^2 - 1) \pm \tan\theta = 0\,.$$

If we start from $\theta = 0$, we first have three zeroes: $0, \pm\frac{1}{\sqrt{2}}$.
Looking at the derivative, we have a double point when $v = \pm\frac{1}{\sqrt{6}}$, which corresponds to $\tan\theta = \frac{\sqrt{2}}{3\sqrt{3}}$.
For larger values of θ, we have only one point till $\tan\theta = 1$.
Hence, there are 3, 2, 1 or 0 solutions satisfying $v \in [-1, +1]$.
The analogous equation for $v = \pm 1$ appears with $\cot\theta$ instead of $\tan\theta$, so that the boundary analysis depends on the comparison of $|\tan\theta|$ with $\frac{\sqrt{2}}{3\sqrt{3}}$, 1 and $\frac{3\sqrt{3}}{\sqrt{2}}$. When the points disappear on $u = \pm 1$, they appear on $v = \pm 1$.
Finally, the maximal number of points along the boundary is six counting with multiplicities.

At the critical points. The critical points of $\Psi_{1,4}$ satisfy:

$$\cos\theta\,(6v^2-1)=0\,, \tag{19}$$

and

$$\sin\theta\,(6u^2-1)=0\,. \tag{20}$$

If we exclude the "product" case, the only critical points are determined by $u^2=\frac{1}{6}$ and $v^2=\frac{1}{6}$. Plugging these values in (18), we obtain that interior critical points on the zero set of $\Psi_{1,4}$ can only appear when:

$$\cos\theta\ \pm\sin\theta\ =0\,. \tag{21}$$

Hence, we only have to look at $\theta=\frac{\pi}{4}$ and $\theta=\frac{3\pi}{4}$. Because of symmetries, it suffices to consider $\theta=\frac{\pi}{4}$:

$$\Psi_{1,4}(u,v,\frac{\pi}{4})=\frac{1}{\sqrt{2}}(u+v)(2(u-\frac{v}{2})^2+\frac{3}{2}v^2-1)\,.$$

The zero set is the union of an ellipse contained in the square and a straight lines, with two intersection points. It follows that the function $\Phi_{1,4}(x,y,\frac{\pi}{4})$ has four nodal domains. Figure 3 shows the deformation of the nodal set of $\Phi_{1,4}(x,y,\theta)$ for $\theta\lesssim\frac{\pi}{4}$ close to $\frac{\pi}{4}$.

Hence we have proved that the maximum number of nodal domains is 4.

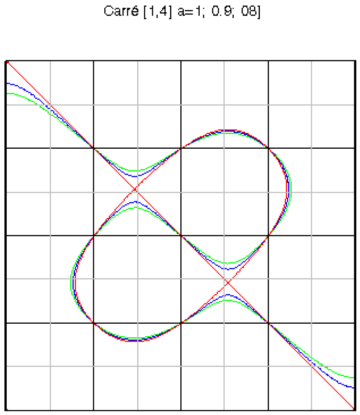

Figure: Eigenvalue λ_9, deformation of the nodal set near $\theta = \frac{\pi}{4}$.

Let us summarize what we have so far obtained for the eigenfunctions associated with λ_9.

- We have determined the aspect of the nodal set of $\Phi_{1,4}$ when $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$, and these are the only cases for which the interior part of the nodal set hits the boundary at the vertices.
- When $\theta \neq \frac{\pi}{4}$ or $\frac{3\pi}{4}$, the nodal set of $\Phi_{1,4}$ has no interior critical point and hence no self-intersection, and that it hits the boundary at 2 or 6 points counting multiplicities.
- All nodal sets must contain the lattice points $(i\frac{\pi}{4}, j\frac{\pi}{4})$ for $1 \leq i, j \leq 3$. This implies, for energy considerations, that the nodal sets cannot contain any closed component avoiding these lattice points.

It remains to prove that the maximal number of nodal domains for $\Phi_{1,4}$ is 4, as suggested by the patterns in Figure 2, and hence that λ_9 is not Courant sharp.

Remark

Figures 2 and 3 indicate that for some values of θ the function $\Phi_{1,4}(x, y, \theta)$, has exactly two nodal domains. This phenomenon was studied by Antonie Stern [20] who claims that for all $k \geq 2$, there are eigenfunctions associated with the Dirichlet eigenvalue $1 + 4k^2$ of the square $[0, \pi]^2$, with exactly two nodal domains. This is what we want to analyze now more carefully.

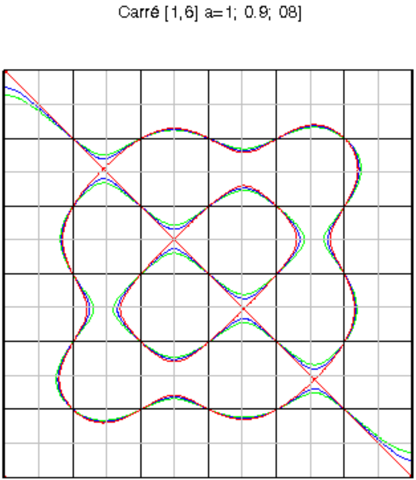

Figure: Eigenvalue λ_{23} ($(m,n) = (1,6)$), deformation of the nodal set near $\theta = \frac{\pi}{4}$.

The observations of A. Stern

The general topic of A. Stern's thesis is the asymptotic behaviour of eigenvalues and eigenfunctions. In Part I, she studies the nodal sets of eigenfunctions of the Laplacian in the square (with Dirichlet boundary conditions) or on the sphere. As before, the eigenvalues are listed in increasing order, with multiplicities.

As we have seen in the previous sections, Pleijel's theorem [17] states that for a plane domain, there are only finitely many Courant sharp Dirichlet eigenvalues. For the square with Dirichlet boundary conditions, A. Stern claims that there are infinitely many eigenvalues with an associated eigenfunction having exactly two nodal domains,

[E1]…Im eindimensionalen Fall wird nach den Sätzen von Sturm[3] *das Intervall durch die Knoten der nten Eigenfunktion in n Teilgebiete zerlegt. Dies Gesetz verliert seine Gültigkeit bei mehrdimensionalen Eigenwertproblemen, …es läßt sich beispielweise leicht zeigen, daß auf der Kugel bei jedem Eigenwert die Gebietszahlen 2 oder 3 auftreten, und daß bei Ordnung nach wachsenden Eigenwerten auch beim Quadrat die Gebietszahl 2 immer wieder vorkommt.*
[Q1]…Wir wollen nun zeigen, daß beim Quadrat die Gebietszahl zwei immer wieder auftritt.

[3] Journal de Mathématiques, T.1, 1836, p. 106-186, 269-277, 375-444

Theorem Sq1

Let D be the unit square in $\mathbb{R}^2$, and Δ the non-positive Laplacian with Dirichlet boundary conditions. Then, for any integer m, there exists an eigenfunction u of $-\Delta$, associated with the eigenvalue $(4m^2 + 1)\pi^2$, whose nodal set inside the square consists of a single simple closed curve. As a consequence, u has exactly two nodal domains.

Theorem [Sq1] is stated without proof in Courant-Hilbert [27, p. 455], with a reference to Stern's thesis [20], and illustrated by two figures taken from [20].

We first deal with the case of the square, The following theorem summarizes the main assertions of A. Stern in the case of the square, see quotation supra and,

[Q2]...Wir betrachten die Eigenwerte

$$\lambda_n = \lambda_{2r,1} = 4r^2 + 1\,,\ r = 1, 2, \ldots$$

und die Knotenlinie der zugehörige Eigenfunktion

$$u_{2r,1} + u_{1,2r} = 0\,,$$

für die sich, wie leicht mittels graphischer Bilder nachgewiesen werden kann, die Figur 7 ergibt.

[Q3]. . . Laßen wir nur μ von $\mu = 1$ aus abnehmen, so lösen sich die Doppelpunkte der Knotenlinie alle gleichzeitig und im gleichem Sinne auf, und es ergibt sich die Figur 8. Da die Knotenlinie aus einem Doppelpunktlosen Zuge besteht, teilt sich das Quadrat in zwei Gebiete und zwar geschieht dies für alle Werte $r = 1, 2, \ldots$, also Eigenwerte $\lambda_n = \lambda_{2r,1} = 4r^2 + 1$.

Theorem

For any $r \in \mathbb{N}$, consider the family $\Phi_{1,2r}(x,y,\theta)$ of eigenfunctions of the Laplacian in the square $[0,\pi]^2$, associated with the Dirichlet eigenvalue $1+4r^2$,

$$\Phi_{1,2r}(x,y,\theta) := \cos\theta \sin x \sin(2ry) + \sin\theta \sin(2rx) \sin y.$$

Then,

1. for $\theta = \frac{\pi}{4}$, the nodal pattern of Φ is as shown in the figures (below left).
2. for $\theta < \frac{\pi}{4}$, and θ sufficiently close to $\frac{\pi}{4}$ the double points all disappear at the same time and in a similar manner as in the figures (below right). The nodal set consists of a line with no double point. It divides the square in two domains.

Comments. Although this is not stated explicitly in the thesis of A. Stern, one can infer that

- The eigenfunction $\Phi(x, y, \frac{\pi}{4})$ has $2r$ nodal domains and $2r - 2$ double points,
- For θ close to and different from $\frac{\pi}{4}$, the nodal sets consists of the boundary of the square and a *connected* simple curve from one point of the boundary to a symmetric point. This curve divides the domain into two connected components.

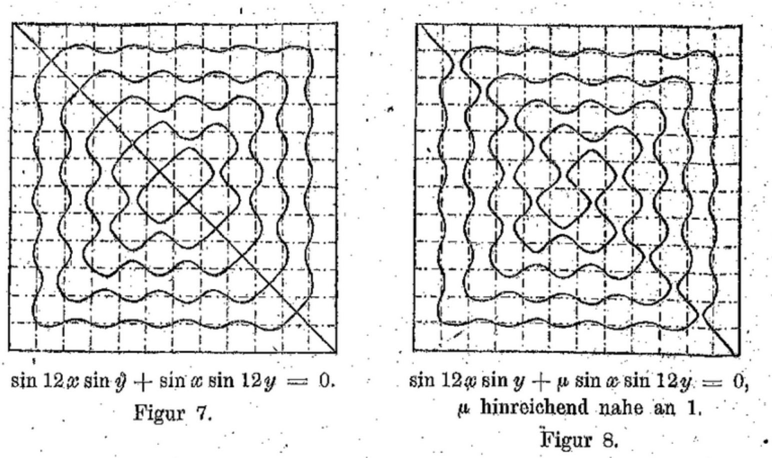

Figure: Case $r = 6$, nodal pattern for $\theta = \frac{\pi}{4}$ and θ close to $\frac{\pi}{4}$, facsimile from [20]

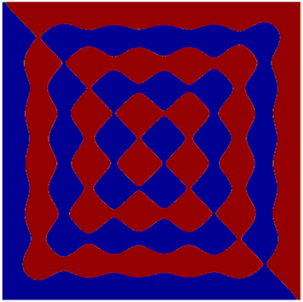

Figure: Nodal domains, courtesy Virginie Bonnaillie-Noël [25]

Property P1

Let ϕ and ψ be two linearly independent eigenfunctions associated with the same eigenvalue for the square $\mathcal{S}$. Let μ be a real parameter, and consider the family of eigenfunctions $\phi_\mu = \psi + \mu\phi$. Let $N(\phi)$ denote the nodal set of the eigenfunction ϕ.

1. Consider the domains in $\mathcal{S} \setminus N(\phi) \cup N(\psi)$ in which $\mu\,\phi\,\psi > 0$ and hatch them ('schraffieren'). Then the nodal set $N(\phi_\mu)$ avoids the hatched domains,
2. The points in $N(\phi) \cap N(\psi)$ belong to the nodal set $N(\phi_\mu)$ for all μ,

These properties are clear. A. Stern also mentions the following.

Property P2

The nodal set $N(\phi_\mu)$ depends continuously on μ.

which is rather clear near regular point, but not so clear near multiple points.
Finally, A. Stern mentions her use of a graphical method which may have been classical at her time, and could explain the amazing quality of her pictures. On this occasion, she also gives the idea of looking at the intersections of the nodal set $N(\phi_\mu)$ with horizontal or vertical lines.

All in all, the arguments given by A. Stern seem very sketchy and we have found necessary to write the details in the same spirit as for Pleijel's statement. The complete proof is based on:

1. Complete determination of the multiple points of $N(\Phi^{\frac{\pi}{4}})$;
2. Absence of multiple points in $N(\Phi^{\theta})$, when θ is different from $\frac{\pi}{4}$, and close to $\frac{\pi}{4}$;
3. Connectedness of the nodal set $N(\Phi^{\theta})$, or why there are no other components, e.g. closed inner components, in the nodal set.

Sketch of the proof of Stern's Theorem

Consider the eigenvalue $\hat{\lambda}_{1,R} := 1 + R^2$ for the square $\mathcal{S}$ with Dirichlet boundary conditions, and the eigenfunction

$$\Phi^\theta(x,y) =: \Phi(x,y,\theta) := \cos\theta \, \sin x \, \sin(Ry) + \sin\theta \, \sin(Rx) \, \sin y \,,$$

for $\theta \in [0,\pi[$.

Introduce the Q-squares,

$$Q_{i,j} :=]\frac{i\pi}{R}, \frac{(i+1)\pi}{R}[\times]\frac{j\pi}{R}, \frac{(j+1)\pi}{R}[\,, \text{ for } 0 \le i,j \le R-1\,,$$

and the lattice,

$$\mathcal{L} := \left\{ (\frac{i\pi}{R}, \frac{j\pi}{R}) \mid 1 \le i,j \le R-1 \right\}\,.$$

The basic idea is to start from the analysis of a given nodal set, e.g. from the nodal set for $\theta = \frac{\pi}{4}$, and then to use a perturbation argument.

Here are the key points.

1. One defines checkerboards by Q-squares (depending on the sign of $\cos\theta$), whose black squares do not contain any nodal point of Φ^θ.
2. The lattice $\mathcal{L}$ is contained in $N(\Phi^\theta)$ for all θ.
3. Determine the possible *critical zeroes* of the eigenfunction Φ^θ, both in the interior of the square or on the boundary and note that the points in $\mathcal{L}$ are not critical zeroes.
4. Determine whether critical zeroes are degenerate or not and their order when they are degenerate.

5. Determine how critical zeroes appear or disappear when θ varies, and how the nodal set $N(\Phi^\theta)$ evolves. For this purpose, make a local analysis in the square $Q_{i,j}$, depending on whether it is contained in $\mathcal{S}$ or touches the boundary,
6. Determine the nodal sets of the eigenfunctions associated with the eigenvalue $\hat{\lambda}_{1,R}$ for $\theta = \frac{\pi}{4}$ and $\frac{3\pi}{4}$, and prove a separation lemma in the $Q_{i,j}$ to determine whether the medians of this Q-square meet the nodal set of Φ^θ when $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$.
7. Prove that the nodal set $N(\Phi^\theta)$ does not contain any closed component.

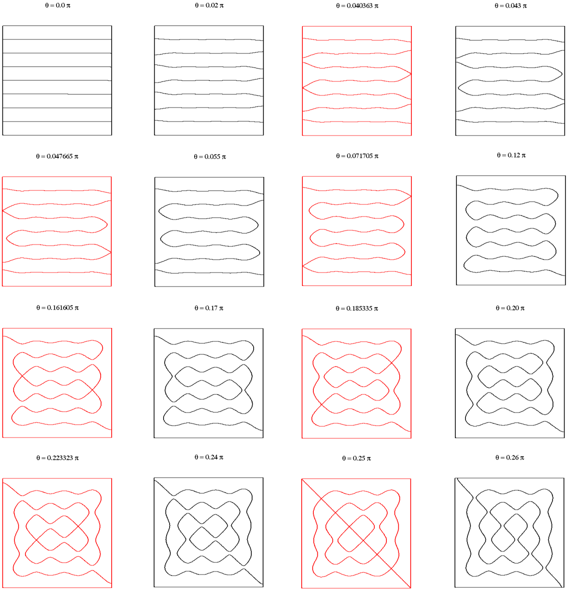

Figure: Typical nodal patterns for the eigenvalue $(1, 8)$

The case of the sphere: A. Stern (1925), H. Lewy (1977)

Let us now go back to another chapter of Antonie Stern's 1925 thesis [20], written under the supervision of Richard Courant.

Pleijel's theorem has been generalized to surfaces by J. Peetre [33], see also [24]. For example, only finitely many eigenvalues of the sphere are Courant sharp. A. Stern claims that there is always a spherical harmonic with exactly three nodal domains (resp. with exactly two nodal domains), when the degree is odd (resp. even), see [20], Einleitung, citation [E1] supra and

> *[K1] . . . Zunächst wollen wir zeigen, daß es zu jedem Eigenwert Eigenfunktionen gibt, deren Nullinien die Kugelfläche nur in zwei oder drei Gebiete teilen.*
> *[K2] . . . ebenso wollen wir jetzt zeigen, daß die Gebietszahl drei bei allen Eigenwerten*
>
> $$\lambda_n = 2r(2r+1) \qquad r = 1, 2, \cdots$$
>
> *immer wieder vorkommt.*

Theorem SP1

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$, and Δ the non-positive spherical Laplacian. For any odd integer ℓ, there exists a spherical harmonic, of degree ℓ, whose nodal set consists of a single simple closed curve. As a consequence, u has exactly two nodal domains.

Theorem SP2

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$, and Δ the non-positive spherical Laplacian. For any even integer $\ell \geq 2$, there exists a spherical harmonic, of degree ℓ, whose nodal set consists of two disjoint simple closed curves. As a consequence, u has exactly three nodal domains.

Theorems [SP1] and [SP2] do not seem to be mentioned in Courant-Hilbert [27]. On the other hand, Stern's results on spherical harmonics appear in the 1977 paper [31] by Hans Lewy (Theorems 1 and 2), without any reference to A. Stern.

Stern's proofs are far from being complete, but she provides nice geometric arguments and figures.

> *[I1] Legen wir die beiden Knotenliniensysteme übereinander und schraffieren wir die Gebiete, in denen beide Funktionen gleiches Verzeichen haben, so kann die Knotenlinie der Kugelfunktion*
>
> $$P_{2r+1}^{2r+1}(\cos\vartheta)\cos(2r+1)\varphi + \mu P_{2r+1}(\cos\vartheta)\,, \quad \mu > 0$$
>
> *nur in der nichtschraffierten Gebieten verlaufen*
>
> *[I3] und zwar für hinreichend kleine* μ *in beliebiger Nachbarschaft der Knotenlinien von* $P_{2r+1}^{2r+1}(\cos\vartheta)\cos(2r+1)\varphi$*, d. h. der* $2r+1$ *Meridiane, da sich bei stetiger Änderung von* μ *das Knotenliniensystem stetig ändert ....*
>
> *[I2] Da die Knotenlinie ferner durch die* $2(2r+1)^2$ *Schnittpunkte der Nullinien der beiden obenstehenden Kugelfunktionen gehen muß ...*

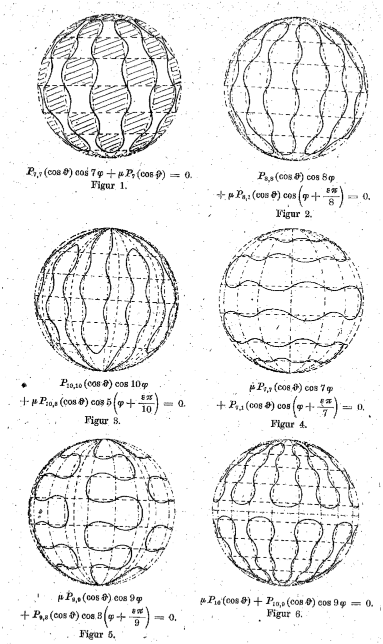

Figure: From Stern's PHD.

We can now state the following quantitative version of A. Stern's first theorem.
Following Stern [20], we consider the one-parameter family of spherical harmonics,

$$H^{\mu,\ell} = W_\ell + \mu \, Z_\ell \, , \tag{22}$$

which may be written in spherical coordinates as

$$h^{\mu,\ell}(\vartheta, \varphi) = \sin^\ell(\vartheta) \, \sin(\ell\varphi) + \mu \, P_\ell(\cos\vartheta) \, . \tag{23}$$

Together with P. Bérard, we have established:

Proposition

Assume that $0 < \mu < \mu_c(\ell)$.

1. When ℓ is odd, the nodal set $N(H^{\mu,\ell})$ is a unique regular simple closed curve and hence, the eigenfunction $H^{\mu,\ell}$ has exactly two nodal domains.
2. When ℓ is even, the nodal set $N(H^{\mu,\ell})$ is the union of ℓ regular disjoint simple closed curves and hence, the eigenfunction $H^{\mu,\ell}$ has exactly $(\ell + 1)$ nodal domains.

Note that $\mu_c(\ell)$ is rather explicit and associated with the zeroes of Legendre polynomials. The proof of H. Lewy was only perturbative.

Critical values

For $\mu > 0$, the *critical* values of μ for which the zero set of $H^{\mu,\ell}$ has critical points are:

$$\mu_i(\ell) = \frac{\sin^\ell(\vartheta_i(\ell-1))}{|P_\ell(\cos\vartheta_i(\ell-1))|}, \tag{24}$$

for $1 \leq i \leq \ell - 1$.

They are well-defined because the denominators do not vanish, since the zeros of the Legendre polynomials P_ℓ and $P_{\ell-1}$ are intertwined.

For the value μ_i, $H^{\mu_i,\ell}$ has finitely many critical zeros. Note that the values $\mu_i(\ell)$ are positive. Observing the parity of the P_ℓ, it suffices to consider the values $\mu_i(\ell)$ for $1 \leq i \leq [\frac{\ell}{2}]$, where $[\frac{\ell}{2}]$ denotes the integer part of $\frac{\ell}{2}$.

$\mu_c(\ell) > 0$ is then defined by:

$$\mu_c(\ell) = \inf_{1\leq i\leq[\frac{\ell}{2}]} \mu_i(\ell), \tag{25}$$

where the positive values $\mu_i(\ell)$ are given by (24).

Bifurcations

Variation of μ when $\ell = 3$.

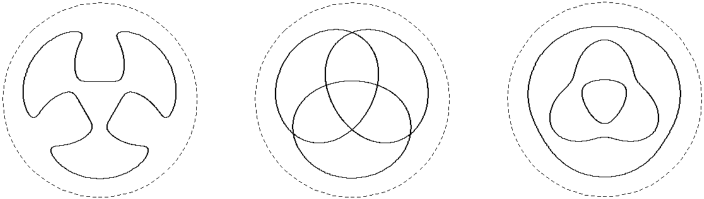

Figure: From Bérard-Helffer.

V. I. Arnold et al.
Some unsolved problems in the theory of differential equations and mathematical physics.
Russian Math. Surveys 44(4), 157-171, (1989)

P. Bérard and B. Helffer.
Square case. To be completed.

P. Bérard and B. Helffer.
Sphere case. To be completed.

G. Blum, S. Gnutzmann, and U. Smilansky.
Nodal domain statistics: A criterion for quantum chaos,
Phys. Rev. Lett. 88 (2002), 114101-114104.

B. Bourdin, D. Bucur, and E. Oudet.
Optimal partitions for eigenvalues.
SIAM J.Sci. Comp. 31(6) 4100-4114, (2009).

J. Bourgain.
On Pleijel's nodal domain theorem.
arXiv:1308.4422v1 [math.SP] 20 Aug 2013.

V. Bonnaillie-Noël, B. Helffer and G. Vial.
Numerical simulations for nodal domains and spectral minimal partitions.
ESAIM Control Optim. Calc.Var. DOI:10.1051/cocv:2008074 (2008).

L.A. Caffarelli and F.H. Lin.
An optimal partition problem for eigenvalues.
Journal of scientific Computing **31**(1/2), DOI: 10.1007/s10915-006-9114.8 (2007).

M. Conti, S. Terracini, and G. Verzini.
An optimal partition problem related to nonlinear eigenvalues.
Journal of Functional Analysis **198**, 160–196 (2003).

M. Conti, S. Terracini, and G. Verzini.
A variational problem for the spatial segregation of reaction-diffusion systems.
Indiana Univ. Math. J. **54**, 779–815 (2005).

M. Conti, S. Terracini, and G. Verzini.

On a class of optimal partition problems related to the Fučik spectrum and to the monotonicity formula.
Calc. Var. **22**, 45–72 (2005).

T. C. Hales.
The honeycomb conjecture.
Disc. Comp. Geom. 25 (2001), p. 1-22.

B. Helffer.
On nodal domains and spectral minimal partitions: a survey.
Milan Journal of Mathematics **78**(2), 575–590 (2010).

B. Helffer, T. Hoffmann-Ostenhof.
Remarks on two notions of spectral minimal partitions.
Adv. Math. Sci. Appl.20 (1), p. 249-263, (2010).

B. Helffer, T. Hoffmann-Ostenhof, S. Terracini.
Nodal domains and spectral minimal partitions.
Ann. Inst. H. Poincaré Anal. Non Linéaire **26**, 101–138 (2009).

J. Leydold.

On the number of nodal domains of spherical harmonics.
Topology, 35, 301-321, 1996

A. Pleijel.
Remarks on Courant's nodal theorem.
Comm. Pure. Appl. Math. **9**, 543–550 (1956) .

I. Polterovich.
Pleijel's nodal domain theorem for free membranes.
Proceeding of the AMS, Volume 137, Number 3, March 2009, Pages 10211024.

U. Smilansky and R. Sankaranarayanan.
Nodal domain distribution of rectangular drums,
arXiv:nlin/0503002, 1-3.

S. Steinerberger.
Geometric uncertainty principle with an application to Pleijel's estimate.
arXiv:1306.3103v3 [math.MG] 6 Aug 2013

T. Hoffmann-Ostenhof.

Geometric Aspects of Spectral Theory, Open Problem Session (xiv) page 2068-2069, Oberwolfach Reports 33, 2012

P. Bérard.
Inégalités isopérimétriques et applications. Domaines nodaux des fonctions propres.
SEDP (Polytechnique) (1981-1982) Exposé 11. Numdam

P. Bérard and B. Helffer.
Eigenfunctions on the sphere: Courant's property, and A. Stern's and H. Lewy's analyses
In preparation.

P. Bérard and D. Meyer.
Inégalités isopérimétriques et applications.
Ann. Sci. École Norm. Sup. 15 (1982), 513-541.

V. Bonnaillie-Noël.
Pictures of nodal domains for the square with Dirichlet boundary conditions. Available here

R. Courant.

Ein allgemeiner Satz zur Theorie der Eigenfunktionen selbstadjungierter Differentialausdrücke,
Nachr. Ges. Göttingen (1923), 81-84.

R. Courant and D. Hilbert.
Methods of Mathematical Physics, Vol. 1.
New York (1953).

G. Gauthier-Shalom and K. Przybytkowski.
Description of a nodal set on ^{2}.
2006 McGill University Research Report (unpublished).

B. Helffer, T. Hoffmann-Ostenhof.
A review on large k minimal spectral k-partitions and Pleijel's Theorem.
submitted.

B. Helffer, T. Hoffmann-Ostenhof, and S. Terracini.
Nodal domains and spectral minimal partitions.
Ann. Inst. H. Poincaré Anal. Non Linéaire **26**, 101–138 (2009).

H. Lewy.

On the minimum number of domains in which the nodal lines of spherical harmonics divide the sphere.
Comm. Partial Differential Equations 2 (1977), no. 12, 1233–1244.

W. Magnus, F. Oberhettinger and R.P. Soni.
"Formulas and Theorems for the Special Functions of Mathematical Physics." Third Edition.
Berlin: Springer-Verlag, 1966.

J. Peetre.
Remarks on Courant's nodal theorem.
Comm. Pure. Appl. Math. 9 (1956), 543–550

Å. Pleijel.
Remarks on Courant's nodal theorem.
Comm. Pure. Appl. Math. **9**, 543–550 (1956) .

F. Pockels.
Über die partielle Differentialgleichung $\Delta u = k^2 u = 0$ and deren Auftreten in mathematischen Physik.

Historical Math. Monographs. Cornell University. (Originally Teubner- Leipzig 1891.) Cornell University

A. Stern.
Bemerkungen über asymptotisches Verhalten von Eigenwerten und Eigenfunctionen.
Diss. Göttingen 1925. Extracts available here

A. Vogt.
" Wissenschaftlerinnen in Kaiser-Wilhelm-Instituten. A-Z."
Veröffentlichungen aus dem Archiv zur Geschichte der Max-Planck-Gesellschaft, Bd. 12. Berlin 2008, 2. erw. Aufl.