On Nodal domains and spectral minimal partitions.

B. Helffer (Univ Paris-Sud and CNRS)

(After V. Bonnaillie-Noël, B. Helffer, T. Hoffmann-Ostenhof, S. Terracini, G. Vial)

November 2008 in Copenhagen (Gerd Grubb's fest)

Given an open set Ω and a partition of Ω by k open sets ω_j, we can consider the quantity $\max_j \lambda(\omega_j)$ where $\lambda(\omega_j)$ is the ground state energy of the Dirichlet realization of the Laplacian in ω_j.If we denote by $\mathfrak{L}_k(\Omega)$ the infimum over all the k-partitions of $\max_j \lambda(\omega_j)$, a minimal k-partition is then a partition which realizes the infimum. Although the analysis is rather standard when $k = 2$ (we find the nodal domains of a second eigenfunction), the analysis of higher k's becomes non trivial and quite interesting. In this talk, we would like to discuss the properties of minimal spectral partitions, illustrate the difficulties by considering simple cases like the disc, the square or the sphere ($k = 3$) and will also exhibit the possible role of the hexagone in the asymptotic behavior as $k \to +\infty$ of $\mathfrak{L}_k(\Omega)$.

This work has started in collaboration with T. Hoffmann-Ostenhof and has been continued (published or in progress) with the coauthors mentioned above.

We consider mainly two-dimensional Laplacians operators in bounded domains. We would like to analyze the relations between the nodal domains of the eigenfunctions of the Dirichlet Laplacians and the partitions by k open sets D_i which are minimal in the sense that the maximum over the D_i's of the ground state energy of the Dirichlet realization of the Laplacian in D_i is minimal.

Let Ω be a regular bounded domain ($C^{(1,+)}$ i.e. $C^{(1,\alpha)}$ for some $\alpha > 0$)

Let us consider the Laplacian $H(\Omega)$ on a bounded regular domain $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary condition. We denote by $\lambda_j(\Omega)$ the increasing sequence of its eigenvalues and by u_j some associated orthonormal basis of eigenfunctions. We know that the groundstate u_1 can be chosen to be strictly positive in Ω, but the other eigenfunctions u_k must have zerosets. We define for any function $u \in C_0^0(\overline{\Omega})$

$$N(u) = \overline{\{x \in \Omega \mid u(x) = 0\}} \tag{1}$$

and call the components of $\Omega \setminus N(u)$ the nodal domains of u. The number of nodal domains of such a function will be called $\mu(u)$.

We now introduce the notions of partition and minimal partition.

Definition 1

Let $1 \leq k \in \mathbb{N}$. We will call **partition** (or k-partition for indicating the cardinal of the partition) of Ω a family $\mathcal{D} = \{D_i\}_{i=1}^k$ of mutually disjoint sets such that

$$\cup_{i=1}^k D_i \subset \Omega\,. \tag{2}$$

We call it **open** if the D_i are open sets of Ω, **connected** if the D_i are connected.
We denote by $\mathfrak{O}_k$ the set of open connected partitions.

Sometimes (at least for the proof) we have to relax this definition by considering measurable sets for the partitions.

We now introduce the notion of spectral minimal partition sequence.

Definition 2

For any integer $k \geq 1$, and for $\mathcal{D}$ in $\mathfrak{O}_k$, we introduce

$$\Lambda(\mathcal{D}) = \max_i \lambda(D_i). \tag{3}$$

Then we define

$$\mathfrak{L}_k = \inf_{\mathcal{D}\in\mathfrak{O}_k} \Lambda(\mathcal{D}). \tag{4}$$

and call $\mathcal{D} \in \mathfrak{O}_k$ minimal if $\mathfrak{L}_k = \Lambda(\mathcal{D})$.

Remark A

If $k = 2$, it is rather well known (see [HeHO1] or [CTV3]) that $\mathfrak{L}_2$ is the second eigenvalue and the associated minimal 2-partition is a nodal partition, i.e. a partition whose elements are the nodal domains of some eigenfunction corresponding to λ_2.

We discuss roughly the notion of regular and strong partition.

Definition 3

A partition $\mathcal{D} = \{D_i\}_{i=1}^{k}$ of Ω in $\mathfrak{O}_k$ is called strong if

$$\mathrm{Int}\,(\overline{\cup_i D_i}) \setminus \partial\Omega = \Omega\,. \tag{5}$$

Attached to a strong partition, we associate a closed set in $\overline{\Omega}$:

Definition 4 : "Nodal set"

$$N(\mathcal{D}) = \overline{\cup_i\,(\partial D_i \cap \Omega)}\,. \tag{6}$$

$N(\mathcal{D})$ plays the role of the nodal set (in the case of a nodal partition).

This leads us to introduce the set $\mathcal{R}(\Omega)$ of regular partitions (or nodal like) through the properties of the associated closed set.

Definition 5

(i) There are finitely many distinct $x_i \in \Omega \cap N$ and associated positive integers ν_i with $\nu_i \geq 2$ s. t. near each of the x_i, N is the union of $\nu_i(x_i)$ smooth curves with one end at x_i and s. t. in the complement of these points in Ω, N is locally diffeomorphic to a regular curve.
(ii) $\partial\Omega \cap N$ consists of a (possibly empty) finite set of points z_i, s.t. at each z_i, ρ_i, with $\rho_i \geq 1$ lines hit the boundary. Moreover, $\forall z_i \in \partial\Omega$, then N is near z_i the union of ρ_i distinct smooth half-curves which hit z_i.
(iii) N has the equal angle meeting property

By equal angle meeting property, we mean that the half curves cross with equal angle at each critical point of N and also at the boundary together with the tangent to the boundary.

Figure 1

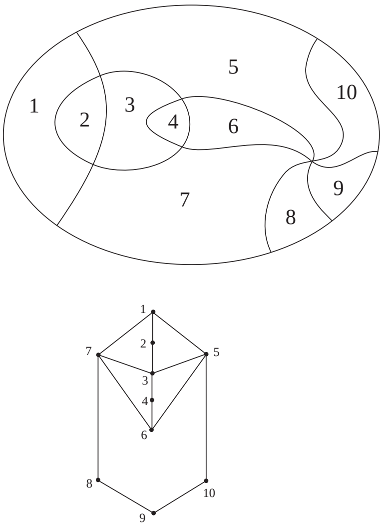
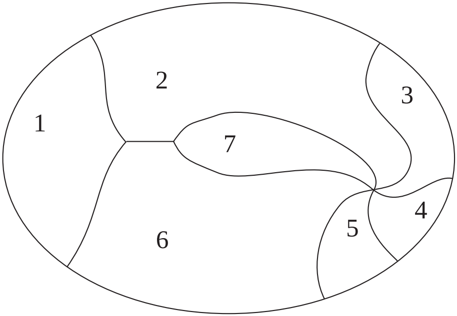


Figure: An example of regular strong bipartite partition with associated graph.

Figure 2



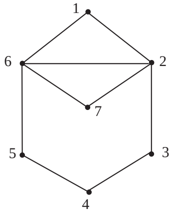


Figure: An example of regular strong nonbipartite partition with

It has been proved by Conti-Terracini-Verzini [CTV1, CTV2, CTV3] that

Theorem 1

For any k, there exists a minimal regular k-partition.

It has been proved by Conti-Terracini-Verzini [CTV1, CTV2, CTV3] that

Theorem 1

For any k, there exists a minimal regular k-partition.

Other proofs of somewhat weaker version of this statement have been given by Bucur, Henrot, F. H. Lin- Caffarelli.

This result is completed by (see Helffer–Hoffmann-Ostenhof–Terracini [HeHOTe]) :

Theorem 2

Any minimal k-partition has a regular representative.

A natural question is whether a minimal partition is the nodal partition induced by an eigenfunction.

For the next theorem we need some additional definitions.

We say that D_i, D_j are neighbors or $D_i \sim D_j$, if $D_{i,j} := \mathrm{Int}\,(\overline{D_i \cup D_j}) \setminus \partial\Omega$ is connected.

For the next theorem we need some additional definitions.

We say that D_i, D_j are neighbors or $D_i \sim D_j$, if $D_{i,j} := \mathrm{Int}\,(\overline{D_i \cup D_j}) \setminus \partial\Omega$ is connected.

We associate to each $\mathcal{D}$ a graph $G(\mathcal{D})$ by associating to each D_i a vertex and to each pair $D_i \sim D_j$ an edge.

For the next theorem we need some additional definitions.

We say that D_i, D_j are neighbors or $D_i \sim D_j$, if $D_{i,j} := \mathrm{Int}\,(\overline{D_i \cup D_j}) \setminus \partial\Omega$ is connected.

We associate to each $\mathcal{D}$ a graph $G(\mathcal{D})$ by associating to each D_i a vertex and to each pair $D_i \sim D_j$ an edge.

We will say that the graph is bipartite if it can be colored by two colors (two neighbours having two different colors).

For the next theorem we need some additional definitions.

We say that D_i, D_j are neighbors or $D_i \sim D_j$, if $D_{i,j} := \mathrm{Int}\,(\overline{D_i \cup D_j}) \setminus \partial\Omega$ is connected.

We associate to each $\mathcal{D}$ a graph $G(\mathcal{D})$ by associating to each D_i a vertex and to each pair $D_i \sim D_j$ an edge.

We will say that the graph is bipartite if it can be colored by two colors (two neighbours having two different colors).

We recall that the graph associated to a collection of nodal domains of an eigenfunction is always bipartite.

We have now the following converse theorem :

Theorem 3

If the graph of the minimal partition is bipartite this is a nodal partition.

A natural question is now to determine how general is the previous situation.

Surprisingly this only occurs in the so called Courant-Sharp situation.
The Courant nodal theorem says :

Theorem 4

Let $k \geq 1$, λ_k be the k-th eigenvalue and $E(\lambda_k)$ the eigenspace of $H(\Omega)$ associated to λ_k. Then, $\forall u \in E(\lambda_k) \setminus \{0\}$, $\mu(u) \leq k$.

Then we say that

Definition 6

u is Courant-sharp if

$$u \in E(\lambda_k) \setminus \{0\} \quad \text{and} \quad \mu(u) = k \,.$$

For any integer $k \geq 1$, we denote by L_k the smallest eigenvalue whose eigenspace contains an eigenfunction with k nodal domains. We set $L_k = \infty$, if there are no eigenfunctions with k nodal domains.
In general, one can show, that

$$\lambda_k \leq \mathfrak{L}_k \leq L_k \, . \tag{7}$$

The last goal consists in giving the full picture of the equality cases :

Theorem 5

Suppose $\Omega \subset \mathbb{R}^2$ is regular.
If $\mathfrak{L}_k = L_k$ or $\mathfrak{L}_k = \lambda_k$ then

$$\lambda_k = \mathfrak{L}_k = L_k \, .$$

In addition, one can find in $E(\lambda_k)$ a Courant-sharp eigenfunction.

Remarks B

1. For the one dimensional case the standard Sturm-Liouville theory gives
$$L_k = \mathfrak{L}_k = \lambda_k \ , \ \forall k \geq 1 \ . \tag{8}$$
2. $$\mathfrak{L}_1 = L_1 = \lambda_1, \tag{9}$$
   (by the property of the ground state) and (we recall) that
   $$\mathfrak{L}_2 = L_2 = \lambda_2, \tag{10}$$
   by the orthogonality of u_2 to the ground state combined with Courant's nodal Theorem.
3. The sequence $(\mathfrak{L}_k)_{k\in\mathbb{N}}$ is strictly increasing.

Using Theorem 5, it is now easier to analyze the situation for the disk or for rectangles (at least in the irrational case), since we have just to check for which eigenvalues one can find associated Courant-sharp eigenfunctions.
For a rectangle of sizes a and b, the spectrum is given by $\pi^2(m^2/a^2 + n^2/b^2)$ ($(m,n) \in (\mathbb{N}^*)^2$).
The first remark is that all the eigenvalues are simple if $\frac{a^2}{b^2}$ is irrational. We assume

$$(a/b)^2 \text{ is irrational.}$$

So we can associate to each eigenvalue $\lambda_{m,n}$, an (essentially) unique eigenfunction $u_{m,n}$ such that $\mu(u_{m,n}) = nm$. Given $k \in \mathbb{N}^*$, the lowest eigenvalue corresponding to k nodal domains is given by

$$L_k = \pi^2 \inf_{mn=k} (m^2/a^2 + n^2/b^2)\,.$$

In the irrational case, $\lambda_{m,n}$ cannot lead to a Courant-sharp situation if $\inf(m,n) \geq 3$ or if $\inf(m,n) \geq 2$ and m or n larger than 4. So there are only very few cases to analyze by hand, for which the answer can depend on $\frac{a}{b}$.

In the case of the square, it is not to difficult to see that $\mathfrak{L}_3$ is strictly less than L_3. We observe indeed that λ_4 is Courant-sharp, so $\mathfrak{L}_4 = \lambda_4$, and there is no eigenfunction corresponding to $\lambda_2 = \lambda_3$ with three nodal domains (by Courant's Theorem).

Restricting to the half-rectangle and assuming that there is a minimal partition which is symmetric with one of the symmetry axes of the square perpendicular to two opposite sides, one is reduced to analyze a family of Dirichlet-Neumann problems.

Numerical computations performed by V. Bonnaillie-Noël and G. Vial lead to a natural candidate for a symmetric minimal partition.
See http://www.bretagne.ens-cachan.fr/math/Simulations/MinimalPartitions/

Figure 3

Figure: Trace on the half-square of the candidate for the 3-partition of the square. The complete structure is obtained from the half square by symmetry with respect to the horizontal axis.

Here we describe some unpublished results [HeHO4] on the possible “topological” types of 3-partitions.

Proposition A

Let Ω be simply-connected and consider a minimal 3-partition $\mathcal{D} = (D_1, D_2, D_3)$ associated to $\mathfrak{L}_3$ and suppose that it is not bipartite.
Let

- $X(\mathcal{D})$ the singular points of $N(\mathcal{D}) \cap \Omega$
- $Y(\mathcal{D}) = N(\mathcal{D}) \cap \partial\Omega$.

Then there are three cases.

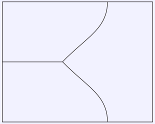
(a)

$X(\mathcal{D})$ consists of one point x with a meeting of three half-lines ($\nu(x) = 3$) and $Y(\mathcal{D})$ consists of

- either three y_1, y_2, y_3 points with $\rho(y_1) = \rho(y_2) = \rho(y_3) = 1$,
- or two points y_1, y_2 with $\rho(y_1) = 2,\ \rho(y_2) = 1$,
- or one point y with $\rho(y) = 3$.

Here, for $y \in \partial\Omega$, $\rho(y)$ is the number of half-lines ending at y.

Type (a) (first subcase)



(b)

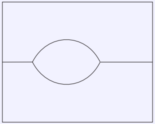
$X(\mathcal{D})$ consists of two distinct points x_1, x_2 so that $\nu(x_1) = \nu(x_2) = 3$ and $Y(\mathcal{D})$ consists

- either of two points y_1, y_2 such that

$$\rho(y_1) + \rho(y_2) = 2$$

- or of one point y with $\rho(y) = 2$.

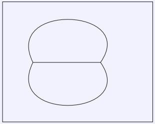
Type (b), second subcase



(c)

$X(\mathcal{D})$ consists again of two distinct points x_1, x_2 with
$\nu(x_1) = \nu(x_2) = 3$,
but $Y(\mathcal{D}) = \emptyset$.

Type (c)

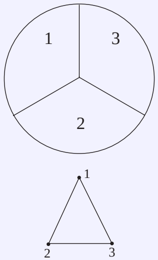


The proof of Proposition A relies essentially on Euler formula together with the property that the associated graph should be a triangle.

The proof of Proposition A relies essentially on Euler formula together with the property that the associated graph should be a triangle.
This leads (with some success) to analyze the minimal partition with some topological type. If in addition, we introduce some symmetries, this leads to guess some candidates for minimal partitions.

In the case of the disk, we have no proof that the minimal partition is the "Mercedes star". But if we assume that the minimal 3-partition is of type (a), then a double covering argument shows that it is indeed the Mercedes star.

The logo Mercedes and associated graph



In the case of the square, we have no proof that the candidate described by Figure 3 is a minimal 3-partition.

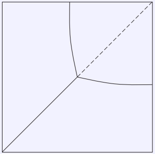
In the case of the square, we have no proof that the candidate described by Figure 3 is a minimal 3-partition.

But if we assume that the minimal partition is of type (a) and has the symmetry, then numerical computations lead to the Figure 3. Numerics suggest more : the center of the square is the critical point of the partition.

Once this property is accepted, a double covering argument shows that this is the projection of a nodal partition on the covering

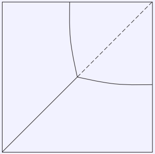
One can also try to look for a minimal partition having the symmetry with respect to the diagonal.

Figure 5 : Another candidate

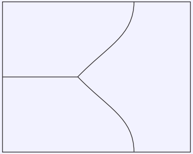


One can also try to look for a minimal partition having the symmetry with respect to the diagonal.

Figure 5 : Another candidate

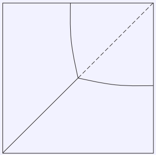


to compare with the former

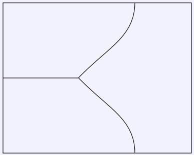


One can also try to look for a minimal partition having the symmetry with respect to the diagonal.

Figure 5 : Another candidate



to compare with the former



THIS LEADS TO THE SAME VALUE OF $\Lambda(\mathcal{D})$.

So this strongly suggests that there is a continuous family of minimal 3-partitions of the square.
This can be explained by a double covering argument, which is analogous to the argument of isospectrality of Jakobson-Levitin-Nadirashvili-Polterovich [JLNP] and Levitin-Parnovski-Polterovich [LPP].

This is an alternative approach to the double covering approach.

One considers the Aharonov-Bohm Laplacian in the square minus its center $\dot{\Omega} = \Omega \setminus \{0\}$, with the singularity of the potential at the center and normalized flux $\frac{1}{2}$.
The magnetic potential takes the form

$$\mathbf{A}(x,y) = (A_1, A_2) = \alpha \left(-\frac{y}{r^2}, \frac{x}{r^2} \right) \, . \tag{11}$$

This is an alternative approach to the double covering approach.

One considers the Aharonov-Bohm Laplacian in the square minus its center $\dot{\Omega} = \Omega \setminus \{0\}$, with the singularity of the potential at the center and normalized flux $\frac{1}{2}$.
The magnetic potential takes the form

$$\mathbf{A}(x,y) = (A_1, A_2) = \alpha \left(-\frac{y}{r^2}, \frac{x}{r^2}\right) \ . \tag{11}$$

We know that the magnetic field vanishes and in any cutted domain (such that it becomes simply connected) one has

$$\mathbf{A}(x,y) = \alpha\, d\theta\,, \tag{12}$$

where

$$z = x + iy = r \exp i\theta \ . \tag{13}$$

This is an alternative approach to the double covering approach.

One considers the Aharonov-Bohm Laplacian in the square minus its center $\dot{\Omega} = \Omega \setminus \{0\}$, with the singularity of the potential at the center and normalized flux $\frac{1}{2}$.
The magnetic potential takes the form

$$\mathbf{A}(x,y) = (A_1, A_2) = \alpha \left(-\frac{y}{r^2}, \frac{x}{r^2} \right) \,. \tag{11}$$

We know that the magnetic field vanishes and in any cutted domain (such that it becomes simply connected) one has

$$\mathbf{A}(x,y) = \alpha \, d\theta \,, \tag{12}$$

where

$$z = x + iy = r \exp i\theta \,. \tag{13}$$

Then the flux condition reads

$$\alpha = \frac{1}{2} \,. \tag{14}$$

So the Aharonov-Bohm operator in any open set $\Omega \subset \mathbb{R}^2 \setminus \{0\}$ is the Friedrichs extension starting from $C_0^\infty(\Omega)$ and the associated differential operator is

$$-\Delta_{\mathbf{A}} := (D_x - A_1)^2 + (D_y - A_2)^2 \,. \tag{15}$$

So the Aharonov-Bohm operator in any open set $\Omega \subset \mathbb{R}^2 \setminus \{0\}$ is the Friedrichs extension starting from $C_0^\infty(\Omega)$ and the associated differential operator is

$$-\Delta_{\mathbf{A}} := (D_x - A_1)^2 + (D_y - A_2)^2 \,. \tag{15}$$

In the case of the square, the operator commutes with the $\frac{\pi}{2}$ rotation.

So the Aharonov-Bohm operator in any open set $\Omega \subset \mathbb{R}^2 \setminus \{0\}$ is the Friedrichs extension starting from $C_0^\infty(\Omega)$ and the associated differential operator is

$$-\Delta_{\mathbf{A}} := (D_x - A_1)^2 + (D_y - A_2)^2 \,. \tag{15}$$

In the case of the square, the operator commutes with the $\frac{\pi}{2}$ rotation.
In the case of rectangles, it commutes with the symmetries with respect to the main axes but these symmetries should be quantized by antilinear operators,

$$\Sigma_1 u(x,y) = i\,\overline{u(-x,y)}\,.$$

and

$$\Sigma_2 u(x,y) = \overline{u(x,-y)}\,.$$

This operator is preserving “real” functions in the following sense. We say (cf Helffer–M. and T. Hoffmann-Ostenhof–Owen) that a function u is K-real, if it satisfies

$$Ku = u\,, \tag{16}$$

where K is an anti-linear operator in the form

$$K = \exp i\theta\, \Gamma\,, \tag{17}$$

where

$$\Gamma u = \bar{u}\,. \tag{18}$$

This operator is preserving "real" functions in the following sense. We say (cf Helffer–M. and T. Hoffmann-Ostenhof–Owen) that a function u is K-real, if it satisfies

$$Ku = u\,, \tag{16}$$

where K is an anti-linear operator in the form

$$K = \exp i\theta\,\Gamma\ , \tag{17}$$

where

$$\Gamma u = \bar{u}\ . \tag{18}$$

The fact that $(-\Delta_{\mathbf{A}})$ preserves K-real eigenfunctions is an immediate consequence of

$$K \circ (-\Delta_{\mathbf{A}}) = (-\Delta_{\mathbf{A}}) \circ K\ . \tag{19}$$

It is easy to find a basis of K-real eigenfunctions. These eigenfunctions (which can be identified to real antisymmetric eigenfunctions of the Laplacian on a suitable double covering of the square) have a nice nodal structure,

- which is locally the same inside the punctured square as the nodal set of real eigenfunctions of the Laplacian,
- with the specific property that the number of lines arriving at the origine should be odd.

More generally a path of index one around the origine should always meet an odd number of nodal lines.

Lemma B

The multiplicity of any eigenvalue is at least 2.

Lemma B

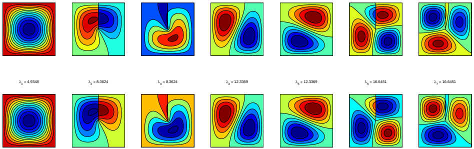
The multiplicity of any eigenvalue is at least 2.

Proposition B

The following problems have the same eigenvalues :

- The Dirichlet problem for the Aharonov-Bohm operator on the punctured square.
- The Dirichlet-Neumann problem on the upper-half square.
- The Dirichlet-Neumann problem on the left-half square.
- The Dirichlet-Neumann problem on the upper diagonal-half square.

The spectrum on the double covering of the square



The cutoff is done on the half-line $\{x = 0\,,\ y > 0\}$. The first line describes the nodal sets of the first seven eigenfunctions on the first sheet.
This picture has been obtained by V. Bonnaillie-Noël.

Remarks

- The guess for the square is that any nodal partition of a third K-real eigenfunction gives a minimal 3-partition.
- In the case of the general rectangle, Proposition B holds true except the last item but this is no more related to the 3-partition problem.

Remarks

- The guess for the square is that any nodal partition of a third K-real eigenfunction gives a minimal 3-partition.
- In the case of the general rectangle, Proposition B holds true except the last item but this is no more related to the 3-partition problem.

All the results or observations around the square and the rectangle arise from discussions, preliminary manuscripts written by or in collaboration with V. Bonnaillie-Noël, T. Hoffmann-Ostenhof, S. Terracini, G. Verzini or G. Vial.

The problem for k large.

We mention two conjectures. The first one is that

Conjecture 1

The limit of $\mathfrak{L}_k(\Omega)/k$ as $k \to +\infty$ exists.

The second one is that this limit is more explicitly given by

Conjecture 2

$$|\Omega| \lim_{k \to +\infty} \frac{\mathfrak{L}_k(\Omega)}{k} = \lambda_1(Hexa_1)\ .$$

This last conjecture says in particular that the limit is independent of Ω if Ω is a regular domain.

Of course the optimality of the regular hexagonal tiling appears in various contexts in Physics. It is easy to show the upper bound in Conjecture 2 and Faber-Krahn gives a weaker lower bound involving the first eigenvalue on the disk. But we have at the moment no idea of any approach for proving this in our context. We have explored in [BHV] numerically why this conjecture looks numerically reasonable.

The problem on the sphere

Let us mention two interesting conjectures on S^2.

We parametrize S^2 by the spherical coordinates $(\theta, \phi) \in [0, \pi] \times [-\pi, \pi]$ with $\theta = 0$ corresponding to the north pole, $\theta = \frac{\pi}{2}$ corresponding to the equator and $\theta = \pi$ corresponding to the south pole.

There is a particular partition of S^2 corresponding to cutting S^2 by the half-hyperplanes $\phi = 0, \frac{2\pi}{3}, -\frac{2\pi}{3}$. We call this partition the Y-partition.

The conjecture due to C. Bishop is :

Conjecture

The Y-partition gives a minimal 3-partition for S^2 when minimizing $\sum_{j=1}^{3} \lambda(D_j)$ over all the 3-partitions of S^2 .

The conjecture due to C. Bishop is :

Conjecture

The Y-partition gives a minimal 3-partition for S^2 when minimizing $\sum_{j=1}^{3} \lambda(D_j)$ over all the 3-partitions of S^2 .

Actually we can have the same conjecture for $\max_j \lambda(D_j)$.

Conjecture

The Y-partition gives a minimal 3-partition for S^2 when minimizing $\max_j \lambda(D_j)$ over all the 3-partitions of S^2 .

This conjecture is actually a consequence of the first conjecture but could be easier to prove.
The techniques developed in the previous parts give some insight on the second conjecture which has some similarity with the Mercedes star conjecture.

We have seen that for the disk the minimal 4-partition for $\max_j \lambda(D_j)$ consists simply in the complement in the disk of the two perpendicular axes.
One could think that a minimal 4-partition of S^2 could be what is obtained by cutting S^2

- either by the two planes $\phi = 0$ and $\theta = \frac{\pi}{2}$
- or by the two planes $\phi = 0$ and $\phi = \frac{\pi}{2}$.

This is actually excluded by the following theorem.

Theorem 6

The minimal 4-partition on S^2 cannot be a nodal partition.

M. Abramowitz and I. A. Stegun.
Handbook of mathematical functions,
Volume 55 of Applied Math Series. National Bureau of Standards, 1964.

G. Alessandrini.
Critical points of solutions of elliptic equations in two variables.
Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 14(2):229–256 (1988).

G. Alessandrini.
Nodal lines of eigenfunctions of the fixed membrane problem in general convex domains.
Comment. Math. Helv., 69(1):142–154, 1994.

A. Ancona, B. Helffer, and T. Hoffmann-Ostenhof.
Nodal domain theorems à la Courant.
Documenta Mathematica, Vol. 9, p. 283-299 (2004)

V. Bonnaillie-Noël, B. Helffer and G. Vial.

Preprint 2007.

M. Conti, S. Terracini, and G. Verzini.
An optimal partition problem related to nonlinear eigenvalues.
Journal of Functional Analysis 198, p. 160-196 (2003).

M. Conti, S. Terracini, and G. Verzini.
A variational problem for the spatial segregation of reaction-diffusion systems.
Indiana Univ. Math. J. 54, p. 779-815 (2005).

M. Conti, S. Terracini, and G. Verzini.
On a class of optimal partition problems related to the Fucik spectrum and to the monotonicity formula.
Calc. Var. 22, p. 45-72 (2005).

B. Helffer and T. Hoffmann-Ostenhof.
Converse spectral problems for nodal domains. Extended version.
Preprint Sept. 2005, mp_arc 05343.

B. Helffer and T. Hoffmann-Ostenhof.

Converse spectral problems for nodal domains.
Preprint April 2006. *Moscow Mathematical Journal*.

B. Helffer and T. Hoffmann-Ostenhof.
On nodal patterns and spectral optimal partitions.
Unpublished Notes, December 2005.

B. Helffer and T. Hoffmann-Ostenhof.
On minimal partitions for the disk.
Unpublished Notes.

B. Helffer, T. Hoffmann-Ostenhof and S. Terracini.
To appear in Annales Institut Henri Poincaré
(Analyse-nonlinéaire).

D. Jakobson, M. Levitin, N. Nadirashvili, I. Polterovic.
Spectral problems with mixed Dirichlet-Neumann boundary
conditions: isospectrality and beyond.
J. Comput. Appl. Math. 194, 141-155, 2006.

M. Levitin, L. Parnovski, I. Polterovich.
Isospectral domains with mixed boundary conditions

arXiv.math.SP/0510505b2 15 Mar2006.

A. Pleijel.
Remarks on Courant's nodal theorem.
Comm. Pure. Appl. Math., 9: 543–550, 1956.

B. Helffer: Département de Mathématiques, Bat. 425, Université Paris-Sud, 91 405 Orsay Cedex, France.

email: Bernard.Helffer@math.u-psud.fr