Spectral flow for pair compatible equipartitions.

Bernard Helffer (Université de Nantes and CNRS)

(in collaboration with Mikael Persson Sundqvist (Lund university))

November 2020

Abstract

We show that a recent spectral flow approach proposed by Berkolaiko–Cox–Marzuola for analyzing the nodal deficiency of the nodal partition associated to an eigenfunction can be extended to more general partitions. To be more precise, we work with spectral equipartitions that satisfy a pair compatible condition. Nodal partitions and spectral minimal partitions are examples of such partitions.

Along the way, we discuss, using former collaborations with M. and T. Hoffmann-Ostenhof, M. Owen, V. Bonnaillie, S. Terracini, different approaches to the Dirichlet-to-Neumann operators: via Aharonov–Bohm operators, via a double covering argument, and via a slitting of the domain. For lack of time, we will focus on the first approach.

This work is in collaboration with M. Persson Sundqvist (University of Lund).

Main goals

We consider the Dirichlet Laplacian $-\Delta_\Omega = -\Delta$ in a bounded domain $\Omega \subset \mathbb{R}^2$, where $\partial\Omega$ is piecewise C^1.

Our goal is to analyze the relations between spectral properties of this Laplacian and partitions $\mathcal{D}$ of Ω by k open sets $\{D_i\}_{i=1}^k$, which are spectral equipartitions in the sense that:
In each D_i's the ground state energy $\lambda_1(D_i)$ of the Dirichlet realization of the Laplacian in D_i is the same;
In addition they satisfy a pair compatibility condition (PCC):
For any pair of neighbors D_i, D_j , there is a linear combination of the ground states in D_i and D_j which is an eigenfunction of the Dirichlet problem in $\mathrm{Int}(\overline{D_i \cup D_j})$.

Nodal partitions and minimal partitions are typical examples of these PCC-equipartitions.
A difficult question is to recognize which PCC-equipartitions are minimal. This problem has been solved in the bipartite case (which corresponds to the Courant sharp situation) but the problem remains open in the general case.

Our main goal is to extend the construction and analysis of the spectral flow and Dirichlet-to-Neumann operators, which was done for nodal partitions in Berkolaiko-Cox-Marzuola [BCM], to spectral equipartitions that satisfy PCC.

The construction of [BCM]

Let $\Omega \subset \mathbb{R}^2$ and λ_* be some eigenvalue of the Dirichlet Laplacian $-\Delta_\Omega$, with corresponding eigenfunction ϕ_*.
We define

$$\Gamma = \{x \in \Omega \ : \ \phi_*(x) = 0\}\,,$$

and

$$\Omega_\pm = \{x \in \Omega \ : \ \pm\,\phi_*(x) > 0\}\,.$$

Let k_* be the the minimal label of λ_* and $\nu(\phi_*)$ the number of connected components of the set $\Omega \setminus \Gamma$.

The Dirichlet-to-Neumann operator

Assume that $E \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary (nodal domains and later our more general partitions) have this property), and that λ is not in the spectrum of $-\Delta_E$. Given g on ∂E, let u be the unique solution to

$$\begin{cases} -\Delta u = \lambda u & \text{in } E, \\ u = g & \text{on } \partial E. \end{cases}$$

Then the Dirichlet-to-Neumann operator $\Lambda_E(\lambda)$ is defined as an unbounded operator on $L^2(\partial E)$

$$\Lambda_E(\lambda) g := \frac{\partial u}{\partial \nu},$$

where ν is a unit normal vector pointing out of E.

Theorem BCM

If $\epsilon > 0$ is sufficiently small, then

$$\begin{aligned} k_* - \nu(\phi_*) \quad &= 1 - \dim\ker(-\Delta_\Omega - \lambda_*) \\ &+ \mathrm{Mor}\Big(R_{\Gamma_\Omega,\Gamma}\left(\Lambda_{\Omega_+}(\lambda_* + \epsilon) + \Lambda_{\Omega_-}(\lambda_* + \epsilon)\right) \mathbf{i}_{\Gamma,\Gamma_\Omega}\Big), \end{aligned} \tag{1}$$

where

- Mor counts the number of negative eigenvalues of an operator (the so-called Morse index of the operator),
- $\Gamma_\Omega = \Gamma \cup \partial\Omega$,
- i_{Γ,Γ_Ω} is the extension by 0 operator from Γ to Γ_Ω,
- $R_{\Gamma_\Omega,\Gamma}$ is the restriction to Γ operator.

The operator

$$\Lambda(\Gamma, \lambda_* + \epsilon) := R_{\Gamma \cup \partial\Omega, \Gamma} \left(\Lambda_{\Omega_+}(\lambda_* + \epsilon) + \Lambda_{\Omega_-}(\lambda_* + \epsilon)\right) \mathbf{i}_{\Gamma, \Gamma \cup \partial\Omega}$$

is considered as an unbounded operator on $L^2(\Gamma)$.
Actually it is defined through a quadratic form with form domain $H^{\frac{1}{2}}(\Gamma)$ (see later).

Remark
The nodal deficiency $k_* - \nu(\phi_*)$ is non-negative due to Courant's nodal theorem.

Spectral flow for a family with delta potentials on Γ

To characterize the negative eigenvalues of $\Lambda(\Gamma, \lambda_* + \epsilon)$ it is fruitful to study the family of operators $-\Delta_{\Omega,\sigma}$, $0 \leq \sigma < +\infty$, induced by the bilinear form

$$\mathfrak{B}_\sigma(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \sigma \int_\Gamma u \, v \, ds, \quad u, v \in H_0^1(\Omega).$$

Indeed, if we denote by $\{\lambda_k(\sigma)\}_{k=1}^{+\infty}$ the set of eigenvalues of $-\Delta_{\Omega,\sigma}$, in increasing order, then [BCM] shows that if $\epsilon > 0$ is sufficiently small, then $-\sigma$ is an eigenvalue of $\Lambda(\Gamma, \lambda_* + \epsilon)$ if, and only if, $\lambda_* + \epsilon = \lambda_k(\sigma)$ for some $k \in \mathbb{N}$.

They also show that each analytic branch of the eigenvalues is increasing with σ. Moreover, as $\sigma \to +\infty$, the eigenvalues $\lambda_k(\sigma)$ converges to the eigenvalues of $-\Delta_{\Omega,+\infty}$ which is the Laplacian in Ω with Dirichlet boundary conditions imposed on $\partial\Omega \cup \Gamma$.

Due to the construction, the eigenvalue λ_* is in fact the lowest eigenvalue of $-\Delta_{\Omega,+\infty}$, with multiplicity $\nu(\phi_*)$.

Thus,

$$\lim_{\sigma\to+\infty} \lambda_k(\sigma) \begin{cases} = \lambda_*, & \text{if } 1 \le k \le \nu(\phi_*), \\ > \lambda_*, & \text{if } k > \nu(\phi_*). \end{cases}$$

By the definition of k_*, the operator $-\Delta_{\Omega,0} = -\Delta_\Omega$ has exactly $\le k_* - 1 + \dim\ker(-\Delta_\Omega - \lambda_*)$ eigenvalues $\le \lambda_*$, and so exactly $k_* - 1 + \dim\ker(-\Delta_\Omega - \lambda_*) - \nu(\phi_*)$ of them will pass $\lambda_* + \epsilon$, where $\epsilon > 0$ is sufficiently small.

Examples: Equipartitions of the unit circle

We assume that N is odd (N even corresponds to a nodal situation) and consider an N-equipartition $\mathcal{D}$

$$k(\mathcal{D}) = N$$

of the unit circle.
Then we consider the angular part of the Laplacian, $-\frac{d^2}{d\theta^2}$, with Dirichlet conditions at each sub-dividing point.

Each interval have length $\Theta = 2\pi/N$, and the smallest eigenvalue—the energy of the partition—is given by $\mathcal{E}(\mathcal{D}) = (N/2)^2$.
This partition is NOT a nodal partition associated with an eigenfunction of $-\frac{d^2}{d\theta^2}$.

One should instead consider the magnetic operator on the circle is given by

$$T = -\Big(\frac{d}{d\theta} - \frac{i\,\pi}{2}\Big)^2 ,$$

and its spectrum consist of eigenvalues $\left\{\left(\frac{2n-1}{2}\right)^2\right\}_{n=1}^{+\infty}$, each with multiplicity two,

$$\dim \ker \Big[T - \Big(\frac{2n-1}{2}\Big)^2\Big] = 2.$$

In particular, the minimal label $\ell(\mathcal{D})$ of the eigenvalue $\mathcal{E}(\mathcal{D}) = (N/2)^2$ is

$$\ell(\mathcal{D}) = N.$$

Alternately, one can consider $-\frac{d^2}{d\theta^2}$ on $]0, 2\pi[$ but with antiperiodic condition.
The last possibility is to consider $-\frac{d^2}{dx^2}$ on $\mathbb{R}/(4\pi\mathbb{Z})$ restricted to the functions satisfying $u(x + 2\pi) = -u(x)$.

As in the nodal case, we look for a formula with the form

$$\ell(\mathcal{D}) - k(\mathcal{D}) = 1 - \dim\ker\big(T - \mathcal{E}(\mathcal{D})\big) + \tau(\epsilon, \mathcal{D}), \qquad (2)$$

where $\tau(\epsilon, \mathcal{D})$ denotes the number of negative eigenvalues of some Dirichlet-to-Neumann operator to be discussed below.

In fact, since $\ell(\mathcal{D}) = N$, $k(\mathcal{D}) = N$, $\dim\ker\big(T - \mathcal{E}(\mathcal{D})\big) = 2$, we need to check that

$$\tau(\epsilon, \mathcal{D}) = 1.$$

First we compute the Dirichlet-to-Neumann operator and the associated 2×2 matrix M_λ which associates with the solution u of

$$-\frac{d^2}{d\theta^2} u = \lambda u\,,\ u(0) = u_0\,,\ u(\Theta) = u_1\,,$$

the pair

$$(v_0, v_1) = (-u'(0), u'(\Theta))\,.$$

This leads to

$$\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = M_\lambda \begin{bmatrix} u_0 \\ u_1 \end{bmatrix},$$

where M_λ is the matrix

$$M_\lambda = \begin{bmatrix} \sqrt{\lambda}\cot(\sqrt{\lambda}\Theta) & -\dfrac{\sqrt{\lambda}}{\sin(\sqrt{\lambda}\Theta)} \\ -\dfrac{\sqrt{\lambda}}{\sin(\sqrt{\lambda}\Theta)} & \sqrt{\lambda}\cot(\sqrt{\lambda}\Theta) \end{bmatrix} = \begin{bmatrix} \alpha(\lambda) & \beta(\lambda) \\ \beta(\lambda) & \alpha(\lambda) \end{bmatrix},$$

We continue in the same way along the circle. With

$$(u_k, u_{k+1}) = (u(k\Theta), u((k+1)\Theta))$$

and

$$(v_k, v_{k+1}) = (-u'(k\Theta), u'((k+1)\Theta)),$$

we find

$$\begin{bmatrix} v_k \\ v_{k+1} \end{bmatrix} = M_\lambda \begin{bmatrix} u_k \\ u_{k+1} \end{bmatrix}, \qquad 0 \leq k \leq N-1.$$

But when we come to (u_N, v_N) we have walked around the circle, and are back at the point we started. We are in a magnetic situation. We shoud replace (u_N, v_N) by $(-u_0, -v_0)$.
Thus, our Dirichlet-t- Neumann operator which associates with $(u_0, u_1, \ldots, u_{N-1})$ the N-tuple $(v_0, v_1, \ldots, v_{N-1})$, is given by the matrix

$$\mathcal{M}_\lambda := \frac{1}{2} \begin{bmatrix} 2\alpha(\lambda) & \beta(\lambda) & 0 & 0 & \cdots & -\beta(\lambda) \\ \beta(\lambda) & 2\alpha(\lambda) & \beta(\lambda) & 0 & \cdots & 0 \\ 0 & \beta(\lambda) & 2\alpha(\lambda) & \beta(\lambda) & \cdots & 0 \\ 0 & 0 & \beta(\lambda) & 2\alpha(\lambda) & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \beta(\lambda) \\ -\beta(\lambda) & 0 & 0 & \cdots & \beta(\lambda) & 2\alpha(\lambda) \end{bmatrix}.$$

The eigenvalues of $\mathcal{M}_\lambda$ are given by

$$\mu_k = \alpha - \beta \cos(2k\pi/N) \, , \; k = 0, \ldots, N-1 \, . \qquad (3)$$

The lowest one is $\mu_0 = \alpha(\lambda) - \beta(\lambda)$, and this eigenvalue is negative if $\sqrt{\lambda} = N/2 + \epsilon$, with $\epsilon > 0$ sufficiently small. The other eigenvalues are positive.
We conclude that if $\sqrt{\lambda} = N/2 + \epsilon$, with $\epsilon > 0$ sufficiently small, then the matrix $\mathcal{M}_\lambda$ has exactly 1 negative eigenvalue. This means that Formula (2) is indeed true.

Equipartitions: Notation and definitions

We consider a bounded connected open set Ω in $\mathbb{R}^2$. A *k-partition* of Ω is a family $\mathcal{D} = \{D_i\}_{i=1}^k$ of mutually disjoint, connected, open sets in Ω such that $\overline{\Omega} = \overline{\cup_{i=1}^k D_i}$. We denote by $\mathfrak{O}_k(\Omega)$ the set of k-partitions of Ω. If $\mathcal{D} = \{D_i\}_{i=1}^k \in \mathfrak{O}_k(\Omega)$ and the eigenvalues $\lambda_1(D_i)$ of the Dirichlet Laplacian in D_i are equal for $1 \leq i \leq k$, we say that the partition $\mathcal{D}$ is a *spectral equipartition*. We give two examples of how such partitions occur.

Nodal partitions

We denote by $\{\lambda_j(\Omega)\}_{j=1}^{+\infty}$ the increasing sequence of eigenvalues of the Dirichlet Laplacian in Ω and by $\{u_j\}_{j=1}^{+\infty}$ some associated orthonormal basis of real-valued eigenfunctions.

For a function $u \in C^0(\overline{\Omega})$, we define the *zero set* of u as

$$N(u) = \overline{\{x \in \Omega \mid u(x) = 0\}},$$

and call the components of $\Omega \setminus N(u)$ the *nodal domains* of u. We denote the number of nodal domains of u by $\mu(u)$. These $\mu(u)$ nodal domains define a k-partition of Ω, with $k = \mu(u)$.

Since an eigenfunction u_j, restricted to each nodal domain D_i satisfy the eigenvalue equation $-\Delta u_j = \lambda_j u_j$, with the Dirichlet boundary condition on ∂D_i, each nodal partition is indeed a spectral equipartition.
By the Courant nodal theorem, $\mu(u_j) \leq j$. We say that the pair (λ_j, u_j) is *Courant sharp* if $\mu(u_j) = j$, i.e. has nodal deficiency 0.

Minimal partitions

For any integer $k \geq 1$, and for $\mathcal{D}$ in $\mathfrak{O}_k(\Omega)$, we introduce the *energy of the partition*,

$$\mathcal{E}(\mathcal{D}) = \max_i \lambda_1(D_i).$$

Then we define

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D} \in \mathfrak{O}_k} \mathcal{E}(\mathcal{D})\,.$$

We call $\mathcal{D} \in \mathfrak{O}_k$ a *minimal spectral* k*-partition* if $\mathfrak{L}_k(\Omega) = \mathcal{E}(\mathcal{D})$.

If $k = 2$, $\mathfrak{L}_2(\Omega) = \lambda_2(\Omega)$ and the associated minimal 2-partition consists of the nodal domains of some second eigenfunction u_2 In general, every minimal spectral partition is an equipartition (see [HHOT2009]).

Regularity assumptions on partitions

Attached to $\mathcal{D}$, we associate the *boundary set* of the partition:

$$\mathcal{N}(\mathcal{D}) = \overline{\cup_i \left(\partial D_i \cap \Omega\right)}\,,$$

which plays the role of the nodal set (in the case of a nodal partition).

A partition $\mathcal{D}$ is said *regular* if $\mathbf{K} := \mathcal{N}(\mathcal{D})$ is regular in the following sense

(i) Except for finitely many critical points $\{x_\ell\} \subset K \cap \Omega$, K is locally diffeomorphic to a regular curve. In the neighd. of each x_ℓ K consists of a union of $\nu_\ell \geq 3$ smooth half-curves with one end at x_ℓ.

(ii) $K \cap \partial\Omega$ consists of a finite set of boundary points $\{z_m\}$. Moreover, in a neighd of each z_m, K is a union of ρ_m distinct smooth half-curves with one end at z_m.

(iii) K has the *equal angle meeting property*.

Nodal sets are regular (by Bers [Be1955]) and it is proven in [HHOT2009] that minimal partitions are also regular (modulo a set of capacity 0).

Example

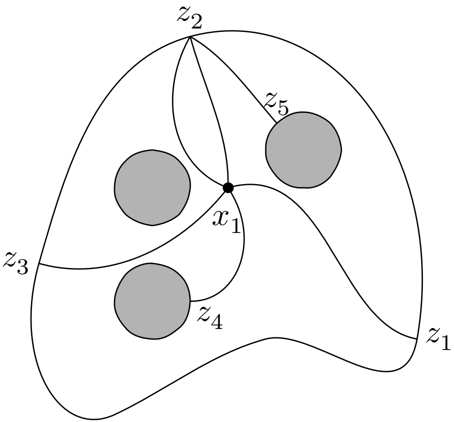

Figure: Partition of a set Ω with three holes. Non admisssible case.

Odd and even points

Given a partition $\mathcal{D}$ of Ω, we denote by $X^{\mathrm{odd}}(\mathcal{D})$ the set of odd critical points, i.e. points x_ℓ for which ν_ℓ is odd. When $\partial\Omega$ has one exterior boundary and m interior boundaries (corresponding to m holes), we should also impose (see [HHOO1999]) that an odd number of lines arrives at some component of the interior boundary. We should distinguish between the odd interior boundaries and the even interior boundaries.

To simplify in this talk we assume from now on that Ω is simply connected.
Hence we have no interior boundaries (no holes).

Pair compatibility condition

Given a partition $\mathcal{D} = \{D_i\}$ of Ω, we say that D_i and D_j are *neighbors*, which we write $D_i \sim D_j$, if the set $D_{ij} := \mathrm{Int}(\overline{D_i \cup D_j}) \setminus \partial\Omega$ is connected.
We associate with $\mathcal{D}$ a graph $G(\mathcal{D})$ by associating with each D_i a vertex and to each pair $D_i \sim D_j$ an edge.
We recall that a graph is said to be *bipartite* if its vertices can be colored by two colors so that all pairs of neighbors have different colors.
We say that $\mathcal{D}$ is *admissible* (or bipartite) if the associated graph $G(\mathcal{D})$ is bipartite.

Nodal partitions are always admissible, since the eigenfunction changes sign when going from one nodal domain to a neighbor nodal domain.

Compatibility condition between neighbors

Let $\mathcal{D} = \{D_i\}_{i=1}^k$ be a regular equipartition of energy $\lambda := \mathcal{E}(\mathcal{D})$. Given $D_i \sim D_j$, $\mathcal{E}(\mathcal{D})$ is the groundstate energy $-\Delta_{D_i}$ and $-\Delta_{D_j}$. There is, however, in general no way to construct a function u_{ij} in the domain of $-\Delta_{D_{ij}}$ s. t. $u_{ij} = c_i u_i$ in D_i and $u_{ij} = c_j u_j$ in D_j.

Definition of PCC

A regular equipartition $\mathcal{D} = \{D_i\}_{i=1}^k$ satisfies the *pair compatibility condition*, (for short PCC), if, for any pair (i, j) such that $D_i \sim D_j$, there is an eigenfunction $u_{ij} \neq 0$ of $-\Delta_{D_{ij}}$ s. t. $-\Delta_{D_{ij}} u_{ij} = \lambda u_{ij}$, and where the nodal set of u_{ij} is given by $\partial D_i \cap \partial D_j$.

Nodal partitions and spectral minimal partitions satisfy the PCC. Hence it is quite natural to consider the equipartitions satisfying this property.

Admissible k-partitions and Courant sharp eigenvalues

It has been proved by Conti–Terracini–Verzini [13, 14, 15] and Helffer–T. Hoffmann-Ostenhof–Terracini [HHOT2009], that, for any $k \in \mathbb{N}$, there exists a minimal regular k-partition.

It is also proven (see [HHO2007], [HHOT2009]) that if the graph of a minimal partition is bipartite, then this partition is nodal.

A natural question was to determine how general the previous situation is. Surprisingly this only occurs in the Courant sharp situation, i.e. when the nodal deficiency is 0.

For any $k \geq 1$, we denote by $L_k(\Omega)$ the smallest eigenvalue of L_Ω, whose eigenspace contains an eigenfunction with k nodal domains. In general, one can show that

$$\lambda_k(\Omega) \leq \mathfrak{L}_k(\Omega) \leq L_k(\Omega) \ .$$

The following result gives the full picture of the equality cases:

Theorem HHOT

Suppose that $\Omega \subset \mathbb{R}^2$ is smooth and that $k \in \mathbb{N}$. If $\mathfrak{L}_k(\Omega) = L_k(\Omega)$ or $\mathfrak{L}_k(\Omega) = \lambda_k(\Omega)$ then

$$\lambda_k(\Omega) = \mathfrak{L}_k(\Omega) = L_k(\Omega),$$

and one can find a Courant sharp eigenpair (λ_k, u_k).

The Aharonov–Bohm operator

Let $\Omega \subset \mathbb{R}^2$ be a bounded connected domain. We recall some definitions and results about the Aharonov–Bohm (AB) Hamiltonian with poles at a finite number of points. These results were initially motivated by the work of Berger–Rubinstein [BeRu] and further developed in [1, HHOO1999, 6, BH2011].
Following Helffer–Hoffmann-Ostenhof M.&T., Owen, we begin with the case of one pole.

Simply connected Ω, one AB pole

We assume one AB pole $X = (x_0, y_0) \in \Omega$ and introduce the magnetic vector potential

$$\mathbf{A}^X(x,y) = (A_1^X(x,y), A_2^X(x,y)) = \frac{\Phi}{2\pi}\left(-\frac{y-y_0}{r^2}, \frac{x-x_0}{r^2}\right),$$

with $\Phi \in \mathbb{R}$ and $r^2 = |x-x_0|^2 + |y-y_0|^2$.
The associated magnetic field vanishes identically in $\dot{\Omega}_X = \Omega \setminus \{X\}$. We introduce

$$\nabla_{\mathbf{A}^X} = \nabla - i\,\mathbf{A}^X,$$

and consider the self-adjoint AB Hamiltonian

$$T_{\mathbf{A}^X} = -(\nabla_{\mathbf{A}^X})^2.$$

This operator is defined as the Friedrichs extension associated with the quadratic form

$$C_0^{+\infty}(\dot{\Omega}_X) \ni u \mapsto \int_\Omega \left|\nabla_{\mathbf{A}^X} u\right|^2 dx.$$

We introduce next the multi-valued complex argument function

$$\phi_X(x,y) = \arg\big(x - x_0 + i\,(y - y_0)\big).$$

This function satisfies

$$\mathbf{A}_X = \frac{\Phi}{2\pi}\nabla\phi_X.$$

This implies that with the flux condition

$$\frac{\Phi}{2\pi} = \frac{1}{2}$$

one has

$$-\mathbf{A}^X = \mathbf{A}^X - \nabla\phi_X,$$

and that multiplication with the function $e^{i\,\phi_X}$, uni-valued in $\dot{\Omega}_X$, is a gauge transformation intertwining $T_{\mathbf{A}^X}$ and $T_{-\mathbf{A}^X}$.

The anti-linear operator $K_X : L^2(\Omega) \to L^2(\Omega)$, defined by

$$u \mapsto K_X u = \exp(i\,\phi_X)\bar{u}$$

satisfies

$$K_X T_{\mathbf{A}^X} = T_{\mathbf{A}^X} K_X .$$

We say that u is K_X-real, if it satisfies

$$K_X u = u$$

and we note that:
The operator $T_{\mathbf{A}^X}$ is preserving the K_X- real functions.

Simply connected Ω, several AB poles

We can extend our construction in the case of a configuration with ℓ distinct points $\mathbf{X} = \{X_j\}_{j=1}^{\ell}$ in Ω (putting a flux $\Phi = \pi$ at each of these points). We can just take as magnetic potential

$$\mathbf{A}^{\mathbf{X}} = \sum_{j=1}^{\ell} \mathbf{A}^{X_j}.$$

The corresponding AB Hamiltonian $T_{\mathbf{A}^{\mathbf{X}}}$ is defined as the Friedrichs extension, via the quadratic form in $C_0^{+\infty}(\dot{\Omega}_{\mathbf{X}})$, where $\dot{\Omega}_{\mathbf{X}} = \Omega \setminus \mathbf{X}$.

We also construct the anti-linear operator $K_{\mathbf{X}}$. As in the case of one AB pole, we can consider the (real) subspace of the $K_{\mathbf{X}}$-real functions in $L^2_{K_{\mathbf{X}}}(\dot{\Omega}_{\mathbf{X}})$, and our operator as an unbounded selfadjoint operator in $L^2_{K_{\mathbf{X}}}(\dot{\Omega}_{\mathbf{X}})$.

It was shown in [HHOO1999] (Helffer–Hoffmann-Ostenhof (M.&T.), Owen) that the nodal set of a $K_{\mathbf{X}}$-real eigenfunction has the same structure as the nodal set of a real-valued eigenfunction of the Dirichlet Laplacian
except that an odd number of half-lines meet at each pole.

Equipartitions and nodal partitions of AB Hamiltonians

Let $\mathcal{D}$ be a regular k-equipartition with energy $\mathcal{E}(\mathcal{D}) = \mathfrak{l}_k(\Omega)$ satisfying the Pair Compatibility Condition.

We denote by $\mathbf{X} = X^{\mathrm{odd}}(\mathcal{D}) = \{X_j\}_{j=1}^{\ell}$ the critical points of the boundary set $\mathcal{N}(\mathcal{D})$ of the partition for which an odd number of half-curves meet.
For this family of points $\mathbf{X}$, it is shown in [HHOT2009] that $\mathfrak{l}_k(\Omega)$ is an eigenvalue of the AB Hamiltonian associated with $\dot{\Omega}_{\mathbf{X}}$, and we can explicitly construct the corresponding eigenfunction with k nodal domains D_i.

Proof of [HHOT] statement.

There exists a family $\{u_i\}_{i=1}^k$ of functions such that u_i is a ground state of $-\Delta_{D_i}$ and $u_i - u_j$ is a second eigenfunction of $-\Delta_{D_{ij}}$ when $D_i \sim D_j$ (here we have extended u_i and u_j by 0 outside of D_i and D_j, respectively, and we recall that $D_{ij} = \mathrm{Int}(\overline{D_i \cup D_j})$).

The claim is that one can find a sequence $\epsilon_i(x)$ of $\mathbb{S}^1$-valued functions, where ϵ_i is a suitable square root of $\exp(i\phi_{\mathbf{X}})$ in D_i, such that $\sum_i \epsilon_i(x) u_i(x)$ is an eigenfunction of the AB Hamiltonian $T_{\mathbf{A}^{\mathbf{X}}}$ associated with the eigenvalue $\mathfrak{L}_k(\Omega) = \mathcal{E}(\mathcal{D})$.

The Berkolaiko–Cox–Marzuola construction in the Aharonov–Bohm approach

Thus, let $\mathcal{D}$ be a k-partition in Ω. We denote by $\Gamma = \mathcal{N}(\mathcal{D})$ the boundary set of the partition of energy $\mathfrak{l}_k(\Omega)$. We denote by m_k the multiplicity of $\mathfrak{l}_k(\Omega)$ as eigenvalue of the magnetic AB Hamiltonian $T_{\mathbf{A}^{\mathbf{X}}}$.

We consider the family $\{\mathfrak{B}_\sigma\}_{\sigma\in\mathbb{R}}$ of sesquilinear forms defined on the magnetic Sobolev space $H^1_{0,\mathbf{A}}(\Omega) \times H^1_{0,\mathbf{A}}(\Omega)$ (see Léna [24] and also [18]) by

$$(u, v) \mapsto \mathfrak{B}_\sigma(u, v) = \int_\Omega \nabla_{\mathbf{A}} u \cdot \overline{\nabla_{\mathbf{A}} v} + \sigma \int_\Gamma u\, \overline{v}\, dS_\Gamma ,$$

where $\mathbf{A} = \mathbf{A}^{\mathbf{X}}$ and dS_Γ is the induced measure on (each arc of) Γ.

We set $H^{1/2}_{\mathbf{A}}(\Gamma) := \oplus_i H^{1/2}_{\mathbf{A}}(\gamma_i)$, and writing

$$\int_\Gamma u\,\overline{v}\,dS_\Gamma = \sum_i \int_{\gamma_i} u\,\overline{v}\,dS_{\gamma_i},$$

we note that the sesquilinear form $\mathfrak{B}_\sigma$ is continuous on $H^1_{0,\mathbf{A}}(\Omega) \times H^1_{0,\mathbf{A}}(\Omega)$.
Associated with this sesquilinear form we have the corresponding magnetic-Robin AB Hamiltonian L_σ.

We also define $L_{+\infty}$ as the corresponding AB magnetic Schrödinger operator, with Dirichlet boundary conditions at $\partial\Omega \cup \Gamma$.

We now collect some properties of the operators L_σ.

Proposition

For each $-\infty < \sigma \leq +\infty$, L_σ has compact resolvent.
Moreover if $\sigma < +\infty$, then

$$D(L_\sigma) = \{u \in H^1_{0,\mathbf{A}}(\Omega) \,|\, (\nabla_{\mathbf{A}})^2 u \in L^2(\Omega) + \text{transmission conditions}\} .$$

The transmission condition is the following:
If $D_i \sim D_j$ and γ is a regular arc in $\partial D_i \cap \partial D_j$, then,

$$\nu_i \cdot \nabla_{\mathbf{A}} u_i + \nu_j \cdot \nabla_{\mathbf{A}} u_j = -\sigma u \text{ on } \gamma\,, \tag{4}$$

where ν_i is the exterior normal to D_i (at a point of γ) and u_i denotes the restriction of u to D_i.

Given $-\infty < \sigma \leq +\infty$, we denote by by $\{\lambda_n(\sigma)\}_{n\in\mathbb{N}}$ the increasing sequence of eigenvalues of L_σ, counted with multiplicity. These eigenvalues are piecewise analytic by Kato theory. Following [BCM], a perturbative argument shows that $\sigma \mapsto \lambda_n(\sigma)$ is either increasing or constant and the latter case only occurs when $\lambda_n(0)$ is an eigenvalue of $L_{+\infty}$.

Proposition

As $\sigma \to +\infty$,

$$\lambda_n(\sigma) \to \lambda_n(+\infty). \qquad (5)$$

Toward the construction of the magnetic Neumann-Poincaré operator

We can now construct the magnetic Neumann–Poincaré operator extending the construction of [BCM].
For each D_j, we consider ∂D_j. We introduce the magnetic Dirichlet–Neumann operator on ∂D_j which associates, for $\epsilon > 0$, to a function $h \in H^{1/2}_{\mathbf{A}}(\partial D_j)$, a solution u to

$$\begin{cases} T_{\mathbf{A}} u = (\mathfrak{l}_k + \epsilon) u & \text{in } D_j, \\ u = h & \text{on } \partial D_j \,. \end{cases} \tag{6}$$

We define a pairing of elements in $H^{-1/2}_{\mathbf{A}}(\partial D_j)$ and $H^{1/2}_{\mathbf{A}}(\partial D_j)$, inspired by how it is done in the non-magnetic case by the Green–Riemann formula.

If $v_0 \in H^{1/2}_{\mathbf{A}}(\partial D_j)$ there exists $w_0 \in H^1_{\mathbf{A}}(D_j)$ such that

$$\begin{cases} -(\nabla_{\mathbf{A}})^2 w_0 = 0 & \text{in } D_j \\ w_0 = v_0 & \text{on } \partial D_j. \end{cases}$$

The mapping $v_0 \mapsto w_0$ is continuous from $H^{1/2}_{\mathbf{A}}(\partial D_j)$ into $H^1_{\mathbf{A}}(D_j)$. Then, we set

$$\left\langle \nu_j \cdot \nabla_{\mathbf{A}} u, v_0 \right\rangle_{H^{-1/2}_{\mathbf{A}}(\partial D_j), H^{1/2}_{\mathbf{A}}(\partial D_j)} := -\langle \nabla_{\mathbf{A}} u, \nabla_{\mathbf{A}} w_0 \rangle + \langle (\nabla_{\mathbf{A}})^2 u, w_0 \rangle, \tag{7}$$

where ν_j is the exterior normal derivative to ∂D_j.

The reduced magnetic Dirichlet to Neuman operator

We then define, for each D_j, the reduced magnetic Dirichlet–Neumann operator on $H^{1/2}_{\mathbf{A}}(\partial D_j \cap \Omega)$ by restricting the magnetic Dirichlet–Neumann operator initially defined on $H^{1/2}_{\mathbf{A}}(\partial D_j)$ and using the identification

$$H^{1/2}_{\mathbf{A}}(\partial D_j \cap \Omega) \sim \hat{H}^{1/2}_{\mathbf{A}} := \{h \in H^{1/2}_{\mathbf{A}}(\partial D_j), h = 0 \text{ on } \partial\Omega \cap \partial D_j\}.$$

This gives

$$\Lambda_{j,\mathbf{A}}(\epsilon, \iota_k)h = \nu_j \cdot \nabla_A u|_{\partial D_j \cap \Omega}.$$

where u is the solution of (6) in D_j.

The Neumann-Poincaré operator

At this point the Neumann–Poincaré operator $\Lambda^{NP}_{\mathbf{A}}(\epsilon, \mathcal{D})$ is defined as an operator from $H^{1/2}_{\mathbf{A}}(\Gamma)$ into $H^{-1/2}_{\mathbf{A}}(\Gamma)$:

$$\Lambda^{NP}_{\mathbf{A}}(\epsilon, \mathcal{D}) = \sum_{j=1}^{k} \iota_j \Lambda_{j,\mathbf{A}}(\epsilon, \mathfrak{l}_k) r_j \, , \tag{8}$$

where r_j is the restriction of $H^{1/2}_{\mathbf{A}}(\Gamma)$ to $H^{1/2}_{\mathbf{A}}(\Omega \cap \partial D_j)$ and ι_j is the extension (by 0) of the operator from $H^{-1/2}_{\mathbf{A}}(\Omega \cap \partial D_j)$ to $H^{-1/2}_{\mathbf{A}}(\Gamma)$.

Proposition

The operator $\Lambda^{NP}_{\mathbf{A}}(\epsilon, \mathcal{D})$ is self-adjoint.

Following [BCM],
$\tau_{\mathbf{A}}(\epsilon, \mathcal{D})$ denotes the number of negative eigenvalues of $\Lambda_{\mathbf{A}}^{\mathrm{NP}}(\epsilon, \mathcal{D})$.

We introduce the partition deficiency of $\mathcal{D}$ as

$$\mathrm{Def}(\mathcal{D}) := \ell(\mathcal{D}) - k(\mathcal{D}),$$

where
$\ell(\mathcal{D})$ denotes the minimal labelling of the eigenvalue $\mathfrak{l}_k$ of the AB Hamiltonian $T_{\mathbf{A}}$,
and
$k(\mathcal{D})$ is the number of components of the partition $\mathcal{D}$.

We are ready to state our main result.

Main Theorem

Let $\mathcal{D}$ be a regular k-equipartition of Ω satisfying PCC with energy $\mathfrak{l}_k = \mathfrak{l}_k(\Omega)$ and $\mathbf{A} = \mathbf{A}^{\mathbf{X}}$ be the associated A-B potential. Then, for sufficiently small $\epsilon > 0$,

$$\mathrm{Def}(\mathcal{D}) = 1 - \dim \ker(T_{\mathbf{A}} - \mathfrak{l}_k) + \tau_{\mathbf{A}}(\epsilon, \mathcal{D}).$$

One ingredient in the proof is:

Lemma

Assume that $\sigma > 0$.
Then $-\sigma$ is an eigenvalue of $\Lambda_{\mathbf{A}}^{NP}(\epsilon, \mathcal{D})$ if, and only if, $\mathfrak{l}_k + \epsilon$ is an eigenvalue of L_σ. If this is the case, then the multiplicities agree.

Remark

It would be interesting to understand, like in the case of a nodal partition, the link between the zero deficiency property

$$1 - \dim \ker(T_{\mathbf{A}} - \mathfrak{l}_k) + \tau_{\mathbf{A}}(\epsilon, \mathcal{D}) = 0\,,$$

and the minimal partition property.

If we have a minimal k-partition then we are in the Courant sharp situation for the corresponding AB Hamiltonian $T_{\mathbf{A}}$, hence it has the zero deficiency property.

The converse is true as recalled above for a bipartite partition but wrong in general.
A counterexample is given for the square and $k = 5$ in Bonnaillie-Helffer.

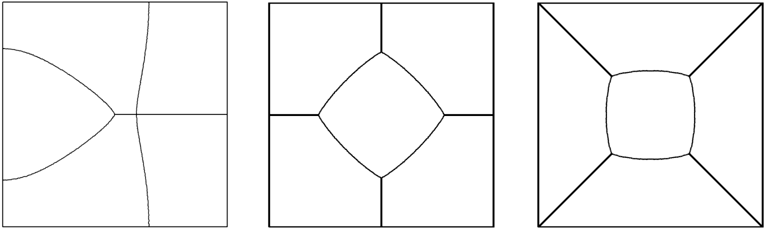

Figure: Three PCC 5-equipartitions with 0 deficiency index. The middle one has minimal energy among the three.

Conclusion

Other approaches

In the $(1D)$, we have described three approachs.

In the general case, we have explained in this talk only the magnetic approach.

The two other approaches are also interesting and correspond to

- Working on a double covering of $\Omega \setminus \{\mathbf{X}\}$.
- Working in $\Omega \setminus \hat{\Gamma}$ where $\hat{\Gamma}$ is a suitable subset of Γ.

Perspective

It remains unclear if this [BCM] formula will help for a better understanding of the nodal partitions and by extension of the (PCC)-equipartitions.

THANK YOU.

Bibliography

B. Alziary, J. Fleckinger-Pellé, P. Takáč.
Eigenfunctions and Hardy inequalities for a magnetic Schrödinger operator in $\mathbb{R}^2$.
Math. Methods Appl. Sci. **26**(13), 1093–1136 (2003).

W. Arendt and R. Mazzeo,
Spectral properties of the Dirichlet-to-Neumann operator on Lipschitz domains, Ulmer Seminare, Heft 12, 28–38 (2007).

W. Arendt and R. Mazzeo.
Friedlander's eigenvalue inequalities and the Dirichlet-to-Neumann semigroup.
Commun. Pure Appl. Anal. **11**(6), 2201–2212 (2012).

R. Band, G. Berkolaiko, H. Raz, and U. Smilansky.
The number of nodal domains on quantum graphs as a stability index of graph partitions.
Comm. Math. Phys. **311**(3), 815–838 (2012).

T. Beck, I. Bors, G. Conte, G. Cox, and J.L. Marzuola.

Limiting eigenfunctions of Sturm–Liouville operators subject to a spectral flow.
arXiv:2006.13839.

J. Berger, J. Rubinstein.
On the zero set of the wave function in superconductivity.
Comm. Math. Phys. **202**(3), 621–628 (1999).

G. Berkolaiko, G. Cox, and J.L. Marzuola.
Nodal deficiency, spectral flow, and the Dirichlet-to-Neumann map.
Lett. Math. Phys. **109**(7), 1611–1623 (2019).

L. Bers.
Local behaviour of solution of general linear elliptic equations.
Comm. Pure Appl. Math., **8**, 473–476 (1955).

V. Bonnaillie-Noël, B. Helffer.
Numerical analysis of nodal sets for eigenvalues of Aharonov–Bohm Hamiltonians on the square and application to minimal partitions.
Exp. Math., **20**(3) 304–322 (2011).

V. Bonnaillie-Noël, B. Helffer and T. Hoffmann-Ostenhof.
Spectral minimal partitions, Aharonov–Bohm Hamiltonians and application.
Journal of Physics A : Math. Theor. **42**(18), 185–203 (2009).

V. Bonnaillie-Noël, B. Helffer and G. Vial.
Numerical simulations for nodal domains and spectral minimal partitions.
ESAIM: Control, Optimisation and Calculus of Variations, **16**(1), 221–246 (2010).

V. Bonnaillie-Noël, B. Noris, M. Nys, and S. Terracini.
On the eigenvalues of Aharonov–Bohm operators with varying poles.
Analysis & PDE **7**(6), 1365–1395 (2014).

B. Bourdin, D. Bucur, and E. Oudet.
Optimal partitions for eigenvalues.
Siam Journal on Scientific Computing **31**(6), 4100–4114 (2009).

J. F. Brasche and M. Melgaard.
The Friedrichs extension of the Aharonov–Bohm Hamiltonian on a disc.
Integral Equations Operator Theory **52**, 419–436 (2005).

D. Bucur, G. Buttazzo, and A. Henrot.
Existence results for some optimal partition problems.
Adv. Math. Sci. Appl. **8**, 571–579 (1998).

L.A. Caffarelli and F.H. Lin.
An optimal partition problem for eigenvalues.
Journal of scientific Computing **31**(1/2), (2007)

M. Conti, S. Terracini, and G. Verzini.
An optimal partition problem related to nonlinear eigenvalues.
Journal of Functional Analysis **198**, 160–196 (2003).

M. Conti, S. Terracini, and G. Verzini.
A variational problem for the spatial segregation of reaction-diffusion systems.
Indiana Univ. Math. J. **54**, 779–815 (2005).

M. Conti, S. Terracini, and G. Verzini.
On a class of optimal partition problems related to the Fučik spectrum and to the monotonicity formula.
Calc. Var. **22**, 45–72 (2005).

L. Friedlander
The Dirichlet-to-Neumann operator for quantum graphs.
arXiv:1712.08223

S. Gallot, D. Hulin, and J. Lafontaine.
Riemannian Geometry.
Universitext (1991).

M. Goffeng and E. Schrohe.
Spectral flow of exterior Landau–Robin hamiltonians.
Journal of Spectral Theory. **7**(3), 847–879 (2017).

D.S. Grebenkov and B. Helffer.
On spectral properties of the Bloch-Torrey operator in two dimensions.
SIAM J. Math. Anal **50**(1), 622–676 (2018).

B. Helffer.
Lower bound for the number of critical points of minimal spectral k-partitions for k large.
Ann. Math. Qué., **41**(1), 111–118 (2017).

B. Helffer, T. Hoffmann-Ostenhof.
Converse spectral problems for nodal domains.
Mosc. Math. J. **7**(1), 67–84 (2007).

B. Helffer, T. Hoffmann-Ostenhof.
On a magnetic characterization of minimal spectral partitions.
J. Eur. Math. Soc. (JEMS), **1**, 461–470 (2010).

B. Helffer, T. Hoffmann-Ostenhof.
On spectral minimal partitions : the case of the disk.
CRM proceedings **52**, 119–136 (2010).

B. Helffer, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, M. P. Owen.
Nodal sets for ground states of Schrödinger operators with zero magnetic field in non-simply connected domains.

Comm. Math. Phys. **202**(3), 629–649 (1999).

B. Helffer, T. Hoffmann-Ostenhof, S. Terracini.
Nodal domains and spectral minimal partitions.
Ann. Inst. H. Poincaré Anal. Non Linéaire **26**, 101–138 (2009).

T. Hoffmann-Ostenhof, P. Michor, and N. Nadirashvili.
Bounds on the multiplicity of eigenvalues for fixed membranes.
Geom. Funct. Anal. **9**(6), 1169–1188 (1999).

M. Karpukhin, G. Kokarev, and I. Polterovich.
Multiplicity bounds for Steklov eigenvalues on Riemannian surfaces.
Ann. Inst. Fourier (Grenoble) **64**, 2481–2502 (2014).

C. Léna.
Eigenvalues variations for Aharonov–Bohm operators.
J. Math. Phys., **56**(1):011502,18, 2015.

B. Noris and S. Terracini.
Nodal sets of magnetic Schrödinger operators of Aharonov–Bohm type and energy minimizing partitions.

Indiana Univ. Math. J. **58**(2), 617–676 (2009).

A. Pleijel.
Remarks on Courant's nodal theorem.
Comm. Pure. Appl. Math. **9**, 543–550 (1956) .

B. Helffer: Laboratoire Jean Leray, Université de Nantes, France.

email: Bernard.Helffer@univ-nantes.fr