# Spectral Theory for the Bloch-Torrey operator

Bernard Helffer
(after Almog, Grebenkov, Helffer, Henry, Moutal,...)

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We consider the semi-classical Bloch-Torrey operator in $L^2(\Omega, \mathbb{R}^k)$ where $\Omega$ is a domain in $\mathbb{R}^d$ and $k \in \{1, \dots, 3\}$.
If the simplest model is some closed realization of

$$-h^2\Delta + ix_1$$

the original model of Bloch-Torrey corresponds to $k = d = 3$ and reads

$$-\Delta \otimes I_3 + \vec{b}(x)\times$$

where $\vec{b}(x)$ is the magnetic field.

After some general discussion of the qualitative spectral analysis of these models, our aim is to analyze, for various domains and in the limit $h \to 0$, the left margin of the spectrum and to establish resolvent estimates.

The presented results have been obtained in collaboration with Y. Almog, D.S. Grebenkov and N. Moutal in continuation of earlier works by Y. Almog and R. Henry devoted to the semi-classical analysis of $-h^2\Delta + iV(x)$ where $V$ is a $C^\infty$ potential.

# The Bloch-Torrey operator

We consider a simplified version of the Bloch-Torrey equation [26] (Torrey 1956), that is commonly used to model Diffusion-Weighted Magnetic Resonance Imaging (DW-MRI).
*It describes the diffusion-precession of spin-bearing particles in nuclear magnetic resonance experiments and helps in understanding the intricate relation between the geometric structure of a studied sample (domain) and the measured signal.*
Here I just repeat what my physicist collaborator D. Grebenkov says !
It assumes the form

$$\partial_t \vec{m} = -\gamma \, \vec{b} \times \vec{m} + D\Delta \vec{m} \,. \tag{1}$$

This time-dependent equation describes the evolution in time of a vector field $\vec{m}$ on $\mathbb{R}^3$, representing the divergence free magnetization vector under the action of an external magnetic field $\vec{b}$.
$\gamma$ and $D$ are non zero physical parameters.

To obtain any information on the semigroup associated with (1), we need to analyze the resolvent of a suitable realization of the differential operator $-D\Delta \otimes I_3 + \gamma \vec{b}(x) \times$ .
After dilation and a change of notation we write

$$B_\epsilon(x, d_x) := -\epsilon^2 \Delta + \vec{b}(x) \times , \qquad (2)$$

and as $\epsilon \to$ this can be seen as a semi-classical problem.
Considering what has been done for the simpler model $-\epsilon^2\Delta + iV(x)$, we should first analyzed toy models corresponding to linearized versions of the operator and typically

$$-\Delta + i\vec{v} \cdot \vec{x}$$

in $\mathbb{R}^d$ or in $\mathbb{R}^d_+$.
The spectral properties are immediately related to the spectral properties of the realizations of the so called complex Airy operator in $\mathbb{R}$ and $\mathbb{R}^+$ :

$$A(x, \frac{d}{dx}) := -\frac{d^2}{dx^2} + ix$$

The spectrum is empty in the first case and related to the zeros of the Airy function in the second case.

Coming back to the general Torrey situation. The problem is much more complex and will not be solved in the most general situations.
We begin by considering the general problem in $\mathbb{R}^3$ and associate with this differential operator a closed realization $\mathcal{B}_\epsilon$. It is natural to ask the following questions :

- Is the operator maximally accretive.
- Can we characterize the domain of the operators ?
- When do we have compact resolvent ?
- Can we localize the spectrum ?
- In view of tha analysis of the decay of the associated semi-group, can we control the resolvent ?
- What can we do in the semi-classical limit ?

If there is a hudge litterature for $-\epsilon^2 \Delta + iV(x)$, we are not aware of many contributions in this new context, except some generalities which do not take in account the specific structure of our matrix-valued potential $M$ (see nevertheless M. Kunze, L. Lorenzi, A. Maichine, and A. Rhandi [23, 24]).

Note that this problem is not self-adjoint because $M(x)$ is antisymmetric !
Let us consequently look first at simpler situations.

# A model (1D)

Let us first consider the case when $\vec{b}(x)$ depends only on one variable (say $x_1$).
In this case, we apply a partial Fourier transform in the $x_2$ and $x_3$ direction which leads to the following family of $(1D)$ operators depending on $((\xi_2, \xi_3) \in \mathbb{R}^2)$

$$B_\epsilon\left(x_1, \frac{d}{dx_1}, \xi_2, \xi_3\right) := -\epsilon^2 \frac{d^2}{dx_1^2} \otimes I + \begin{pmatrix} \epsilon^2(\xi_2^2 + \xi_3^2) & -b_3 & b_2 \\ b_3 & \epsilon^2(\xi_2^2 + \xi_3^2) & -b_1 \\ -b_2 & b_1 & \epsilon^2(\xi_2^2 + \xi_3^2) \end{pmatrix} . \quad (3)$$

What can we say about this $1D$ model ?

In the Airy approximation spirit, it is natural to consider the reduction to the case $\xi_2 = \xi_3 = 0$) and to linearize $\vec{b}$

$$\vec{b} = (\beta_0, 0, \beta_3 x_1)\,.$$

with $\beta_3 \neq 0$.
In the case $\beta_0 = 0$, there is a big simplification (see below), the problem is non trivial for $\beta_0 \neq 0$ and we will need a rather complicate analysis for answering to the general questions above.

# A relatively simple $2D$ model

More generally, for $\vec{b} = (0, 0, b_3(x_1, x_2))$ then the skew-symmetric matrix associated with $\vec{b}\times$ is given by

$$\mathbf{M} = \begin{pmatrix} 0 & -b_3 & 0 \\ b_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4}$$

In a new basis of $\mathbb{C}^3$ the operator becomes

$$\widetilde{\mathcal{B}}_\epsilon := \begin{pmatrix} -\epsilon^2\Delta + ib_3 & 0 & 0 \\ 0 & -\epsilon^2\Delta - ib_3 & 0 \\ 0 & 0 & -\epsilon^2\Delta \end{pmatrix} . \tag{5}$$

Obviously, in this basis $-\epsilon^2\Delta + \vec{m}$ can be considered as three separate scalar operators. The spectral properties of $-\epsilon^2\Delta + ib$ have been considered in Almog (2008), Henry (2015), Almog-Henry (2016), Almog-Grebenkov-Helffer (2016–2020).

Note that if we define $-\epsilon^2\Delta + \mathbf{M}$ on $L^2(\mathbb{R}^3, \mathbb{R}^3)$ for $\vec{b} = (0, 0, x_1)$ we obtain that the spectrum is $\mathbb{R}_+$ (which is precisely $\sigma(-\Delta)$ on $L^2(\mathbb{R}^3)$) given that $\sigma(-\epsilon^2\Delta + ix_1) = \emptyset$ on $L^2(\mathbb{R}^3)$.

# Properties of Schrödinger operators with matrix-valued potentials (after Helffer-Nourrigat, Almog-Helffer)

One can generalize (2) by considering the operator

$$\mathcal{P}(x, d_x) := -\Delta \otimes I_d + M(x)\,, \qquad (6)$$

where $M \in C^\infty(\mathbb{R}^k, M_d(\mathbb{R}))$.
With

$$M_s = \frac{1}{2}(M + M^t)\,,\; M_{as} = \frac{1}{2}(M - M^t)\,.$$

we assume

$$(Ass.1) \quad M_s \geq 0\,.$$

# Accretivity

We can extend maximal accretivity results that have been established for the selfadjoint operator $-\Delta + V$ (see [11]) and also for two interesting non-selfadjoint operators : the Fokker-Planck operator [17] and the complex Schrödinger operator $-\Delta + iV$ [16].

**Proposition A (maximal accretivity)**

Let $\mathcal{P}$ (above) be defined on $C_0^\infty(\mathbb{R}^k, \mathbb{R}^d)$ and satisfy (Ass.1). Then, its closure, under the graph norm, denoted by $\mathbf{P}$, is maximally accretive as an unbounded operator in $L^2(\mathbb{R}^k, \mathbb{R}^d)$. Moreover,

$$D(\mathbf{P}) \subset H^1(\mathbb{R}^k, \mathbb{R}^d)\,.$$

# Essential spectrum

If $\dim \ker M(x) > 0$ for all $x \in \mathbb{R}^k \setminus K$ as in the case $M_s \equiv 0$ we can show that the resolvent is not compact.
The proof is based on the construction of an infinite orthonormal sequence of approximate solutions.
This is indeed true and as an application we get for the Bloch-Torrey model

**Proposition ES**

Suppose that for $|x| \geq R$ it holds that $\vec{b} \neq 0$

$$|d_x^\alpha b_j(x)| \leq C\,|\vec{b}(x)|, \forall \alpha \text{ s.t } 1 \leq |\alpha| \leq 2\,, j = 1, 2, 3\,. \tag{7}$$

Then the resolvent of $\mathcal{B}_\epsilon$ is not compact.
Moreover $\mathbb{R}_+ \subseteq \sigma(\mathbf{P})$, where $\mathbb{R}_+ := [0, +\infty)$.

# Particular case

This applies for

$$\vec{b} = Ax + \vec{f}\,,$$

where $A \neq 0$ is a $d \times k$ matrix and $\vec{f} \in \mathbb{R}^d$.

Note that if $A = 0$, then

$$\sigma(\mathbf{P}) = \mathbb{R}_+ \cup \{\mathbb{R}_+ + i|\vec{f}|\} \cup \{\mathbb{R}_+ - i|\vec{f}|\}\,.$$

Of particular interest is the case where $k = 1$ and $A = \hat{\mathbf{i}}_3$ and hence $\mathbb{R}_+$ is in the essential spectrum of $\mathcal{B}_\epsilon$.

# Maximal estimates

A natural question is the effective description of $D(\mathbf{P})$, currently defined as the closure of $C_0^\infty(\mathbb{R}^k,\mathbb{R}^d)$ under the graph norm.
We can still determine the domain in various different cases. In particular, when looking at the Bloch-Torrey equation, we get as a corollary of a more general theorem

**Theorem ME1**

Let $d=k=3$ and $M=\vec{b}\times$. Then, if $\exists C>0$ and $\exists R>0$ s.t. $\forall |x|\geq R$, $\vec{b}(x)\neq 0$ and

$$|d_x^\alpha b_j(x)|\leq C\,|\vec{b}(x)|, \forall \alpha \text{ s.t } 1\leq |\alpha|\leq 2, j=1,2,3\,, \tag{8}$$

then

$$D(\mathbf{P})=\{\vec{u}\in H^2(\mathbb{R}^3,\mathbb{R}^3)\,,\,\vec{b}\times\vec{u}\in L^2(\mathbb{R}^3,\mathbb{R}^3)\}\,.$$

# Characterization of the domain when two components of $\vec{b}$ are bounded

In this case, we can use the results in [18] (Helffer-Nourrrigat 2018), obtained for the scalar operator $-\Delta + iV(x)$.

**Theorem ME2 (Almog-Helffer)**

Let $\mathcal{B}_\epsilon$ ($\epsilon > 0$) where $b_1, b_2 \in L^\infty(\mathbb{R}^k)$ and $b_3 \in C^{r+1}(\mathbb{R}^k)$ ($r \geq 1$) satisfies

$$\max_{|\beta|=r+1} |D_x^\beta b_3(x)| \leq C_0 \sqrt{\sum_{|\alpha|\leq r} |D_x^\alpha b_3(x)|^2 + 1}\,. \tag{9}$$

Then

$$D(\mathcal{B}_1) = \{\vec{u} \in H^2(\mathbb{R}^k, \mathbb{R}^3)\,,\ \vec{b} \times \vec{u} \in L^2(\mathbb{R}^k, \mathbb{R}^3)\}\,. \tag{10}$$

# Proof

After a change of basis $M$ assumes the form

$$\tilde{M} := \begin{pmatrix} -ib_3 & 0 & -(b_1 - ib_2)/\sqrt{2} \\ 0 & ib_3 & -(b_1 + ib_2)/\sqrt{2} \\ (b_1 + ib_2)/\sqrt{2} & (b_1 - ib_2)/\sqrt{2} & 0 \end{pmatrix} .$$

and

$$\tilde{\mathcal{B}}_1 := -\Delta \otimes I_3 + \tilde{M} .$$

For $\vec{\tilde{u}} \in L^2(\mathbb{R}^k, \mathbb{R}^d)$ satisfying $\tilde{\mathcal{B}}_1 \vec{\tilde{u}} = \vec{\tilde{f}} \in L^2(\mathbb{R}^k, \mathbb{R}^3)$,

$$\begin{array}{l} -\Delta \tilde{u}_1 - ib_3 \tilde{u}_1 = \tilde{g}_1 \\ -\Delta \tilde{u}_2 + ib_3 \tilde{u}_2 = \tilde{g}_2 \\ -\Delta \tilde{u}_3 = \tilde{g}_3 \,, \end{array} \tag{11}$$

where

$$\begin{array}{ll} \tilde{g}_1 & = \tilde{f}_1 + \frac{b_1 - ib_2}{\sqrt{2}} \tilde{u}_3 , \\ \tilde{g}_2 & = \tilde{f}_2 + \frac{b_1 + ib_2}{\sqrt{2}} \tilde{u}_3 \\ \tilde{g}_3 & = \tilde{f}_3 - \frac{b_1 + ib_2}{\sqrt{2}} \tilde{u}_1 - \frac{b_1 - ib_2}{\sqrt{2}} \tilde{u}_2 \,. \end{array} \tag{12}$$

Clearly, $\tilde{g}_i \in L^2(\mathbb{R}^k)$ ($i \in \{1, 2, 3\}$).

By standard elliptic estimates in the third line $\tilde{u}_3 \in H^2(\mathbb{R}^k)$.

For the two first lines the scalar theorem relative to $-\Delta \pm ib_3$ implies

$$\vec{\tilde{u}} \in H^2(\mathbb{R}^k, \mathbb{R}^3) \,,\; b_3 \tilde{u}_1 \in L^2(\mathbb{R}^k) \,,\; b_3 \tilde{u}_2 \in L^2(\mathbb{R}^k) \,.$$

# Resolvent estimates

We can now obtain "rough" estimates for the resolvent of $\mathcal{B}_1$, using known resolvent estimates obtained for $-\Delta \pm ib_3$.

Better estimates are actually obtained for the particular case where $\vec{b}$ is affine (see below).
The resolvent equation $(\tilde{\mathcal{B}}_1 - \lambda)\tilde{u} = \tilde{f}$, takes the form (dropping the accent in the sequel) :

$$f_1 = (-\Delta - ib_3 - \lambda)u_1 - \frac{b_1 - ib_2}{\sqrt{2}}u_3 \tag{13a}$$

$$f_2 = (-\Delta + ib_3 - \lambda)u_2 - \frac{b_1 + ib_2}{\sqrt{2}}u_3 \tag{13b}$$

$$f_3 = (-\Delta - \lambda)u_3 + \frac{b_1 + ib_2}{\sqrt{2}}u_1 + \frac{b_1 - ib_2}{\sqrt{2}}u_2\,. \tag{13c}$$

Assuming that $\lambda \notin \sigma(-\Delta \pm ib_3)$ we write

$$R_\pm(\lambda) = (-\Delta \mp ib_3 - \lambda)^{-1}$$

to obtain for $u_3$ :

$$((-\Delta - \lambda) - cR_-(\lambda)\bar{c} - \bar{c}R_+(\lambda)c)\, u_3 = f_3 - cR_+(\lambda)f_1 - \bar{c}R_-(\lambda)f_2\,, \quad (14)$$

where

$$c := (b_1 + ib_2)/\sqrt{2}\,.$$

Assuming further $\lambda \notin \mathbb{R}^+$ we can attempt to estimate the norm of the well-defined bounded operator

$$K_\lambda := \frac{1}{2}(-\Delta - \lambda)^{-1} \left( cR_-(\lambda)\bar{c} + \bar{c}R_+(\lambda)c \right) . \tag{15}$$

To proceed further we assume

**Assumption**

For an interval $I$, $\exists s < 1$, $D_1 > 0$ and $D_2 > 0$ such that, if $\operatorname{Re}\lambda \in I$ and $|\operatorname{Im}\lambda| \geq D_1$, then

$$\|R_\pm(\lambda)\| \leq D_2 \, |\operatorname{Im}\lambda|^s .$$

The above bound applies for $I = (-\infty, \tau)$ for every $\tau \in \mathbb{R}$, and $b_3(x) = x_1$ (in which case $s = 0$) or $b_3(x) = x_1^2$ (where $s = -\frac{1}{3}$) (see my Cambridge book (2013)).

Given the above assumption we can obtain the following resolvent estimate

### Proposition RE

Let $b_3 \in C^r(\mathbb{R}^k)$ satisfy previous assumption, for some $I$. Then, $\exists C_1 > 0$ and $\exists C_2 > 0$ s. t. $\forall \lambda \in \mathbb{C}$ satisfying $\mathrm{Re}\,\lambda \in I$ and $|\mathrm{Im}\,\lambda| \geq C_1$, it holds that $\lambda \notin \sigma(\mathcal{B}_1)$ and

$$\|(\mathcal{B}_1 - \lambda)^{-1}\| \leq C_2\, |\mathrm{Im}\,\lambda|^s\,. \tag{16}$$

# Point spectrum

**Proposition PS**

Let $\Omega = \mathbb{C} \setminus (\mathbb{R}_+ \cup \sigma(-\Delta \pm ib_3))$. Under the previous assumptions, it holds for $\lambda \in \Omega$ :

1. $\lambda \in \sigma(\mathcal{B}_1)$ if and only if $-1 \in \sigma(K_\lambda)$.
2. If $-\Delta \pm ib_3$ has a compact resolvent, then $\sigma(\mathcal{B}_1) \cap \Omega$ consists in isolated eigenvalues of $\mathcal{B}_1$ of finite multiplicity.

# Remarks

- One can establish Proposition PS as soon as $\bar{c}R_+(\lambda)c$ is compact.
- One can go further in the case $k = 1$ and $b_3(x) = x$.
  In this case, $\sigma(-\epsilon^2\Delta \pm ib_3) = \emptyset$ and $\mathbb{R}_+ \subset \sigma(B_1)$ and the proposition implies that :
  $\sigma(B_\epsilon) \cap (\mathbb{C} \setminus \mathbb{R}_+)$ is discrete.
- Assuming $k = 1$, $b_1$ constant, $b_2 = 0$, and $b_3(x) = x$, we will obtain (Almog-Helffer) much more precise results for $\mathcal{B}_\epsilon$ in the asymptotic regime $\epsilon \to 0$.
  This is actually hard work ! We will only give the main results.

# Main statements for a simplified model

In the case where $\mathcal{B}_\epsilon$ is defined in $\mathbb{R}$ we obtain

**Theorem : localization of the spectrum**

Let $\mathcal{B}_\epsilon$ be defined by

$$B_\epsilon\left(x, \frac{d}{dx}\right) := -\epsilon^2 \frac{d^2}{dx^2} + \vec{b}\times \,, \tag{17}$$

with $\vec{b} = (1, 0, x)$.
Let for $\epsilon > 0, n \in \mathbb{N}^*$, $\kappa_n^0(\epsilon) := i + \frac{2n-1}{2}(1+i)\epsilon$.
Then we have :

- $\Lambda \in \sigma(\mathcal{B}_\epsilon) \Leftrightarrow \overline{\Lambda} \in \sigma(\mathcal{B}_\epsilon)$.
- $\mathbb{R}_+ \subset \sigma(\mathcal{B}_\epsilon)$.
- Then $\forall N \in \mathbb{N}^*\ \exists \epsilon_0$ and $\hat{C}$, s.t. $\forall 0 < \epsilon \leq \epsilon_0$, $\exists \{\kappa_n(\epsilon)\}_{n=1}^N \subset \sigma(\mathcal{B}_\epsilon)$ s.t.
$$\left|\kappa_n(\epsilon) - \kappa_n^0(\epsilon)\right| \leq \hat{C}\epsilon^2 \,, \ \text{for } n = 1 \ldots, N \,. \tag{18}$$

## Theorem : estimates for the resolvent.

Under the same assumptions,

- Let $\varrho > 0$, $\hat{R} > 0$ and $N_\varrho = \left[\frac{2\varrho+1}{2}\right]$, and

$$\begin{aligned} &\mathbf{D}(\hat{R}, \varrho, \epsilon) \\ &= \{\Lambda \in \mathbb{C} \setminus \cup_{n=1}^{N_\varrho} \big(B(\kappa_n^0(\epsilon), \hat{R}\epsilon^2) \cup B(\overline{\kappa_n^0(\epsilon)}, \hat{R}\epsilon^2)\big)\} \\ &\quad \cap \{\mathrm{Re}\,\Lambda \leq \varrho\epsilon\} \cap \{\mathrm{Im}\,\Lambda \neq 0\}\,. \end{aligned} \tag{19}$$

Then, $\exists C > 0$ and $\exists \hat{R}_0 > 1$ s. t. $\forall \hat{R}_0 < \hat{R} < [\sqrt{2}\epsilon]^{-1}$ and $\Lambda \in \mathbf{D}(\hat{R}, \varrho, \epsilon)$ it holds that

$$\|(\mathcal{B}_\epsilon - \Lambda)^{-1}\| \leq C\left(1 + \frac{\epsilon^{2/3}}{|\mathrm{Im}\,\Lambda|^2} + \frac{1}{\hat{R}\epsilon^{\frac{5}{3}}}\right). \tag{20}$$

# Part II : The Bloch-Torrey equation in the case with boundary.

Consider a bounded smooth open set $\mathcal{O} \subset \mathbb{R}^k$ and the Dirichlet realization of $\mathcal{P}$ in $\mathcal{O}$ denoted by $\mathbf{P}^{\mathcal{O}}$. For $M \in C^2(\mathcal{O}, M_d(\mathbb{R}))$, $\mathbf{P}^{\mathcal{O}}$ is a bounded perturbation of $-\Delta$ and hence $D(\mathbf{P}^{\mathcal{O}}) = H^2(\mathcal{O}, \mathbb{R}^d) \cap H^1_0(\mathcal{O}, \mathbb{R}^d)$, and $\mathbf{P}^{\mathcal{O}}$ has a compact resolvent.

Nevertheless, in the presence of a small parameter $\epsilon$, i.e. when

$$\mathbf{P}^{\mathcal{O}} = -\epsilon^2 \Delta + M$$

the behaviour of spectrum and the resolvent in the limit $\epsilon \to 0$ could probably be understood from the analysis of linearized operators acting on $\mathbb{R}^k$.

This is precisely the case in the two dimensional setting when $\mathbf{P}^{\mathcal{O}}$ is equivalent to the Dirichlet realization of $-\epsilon^2 \Delta + iV$ (considered inn the works of Almog, Henry, Almog-Henry, Almog-Grebenkov-Helffer ... [19, 2, 8]).

# The scalar Bloch-Torrey equation

We would like to analyze the behavior as $h \to 0$ of the spectrum of the semi-classical realization $\mathcal{A}_h^D$ of $-h^2\Delta + ix$.
The main result is :

**Theorem**

We have

$$\lim_{h\to 0} \frac{1}{h^{2/3}} \inf\{\mathrm{Re}\, \sigma(\mathcal{A}_h^D)\} = \frac{|a_1|}{2}, \tag{21}$$

where $a_1 < 0$ is the rightmost zero of the Airy function **Ai**.
Moreover, for every $\varepsilon > 0$, there exist $h_\varepsilon > 0$ and $C_\varepsilon > 0$ such that

$$\forall h \in (0, h_\varepsilon), \quad \sup_{\substack{\gamma \le \frac{|a_1|}{2} \\ \nu \in \mathbb{R}}} \|(\mathcal{A}_h^D - (\gamma - \varepsilon)h^{2/3} - i\nu)^{-1}\| \le \frac{C_\varepsilon}{h^{2/3}}. \tag{22}$$

In particular the spectrum of $\mathcal{A}_h^D$ is not empty.

This holds for the interior problem and the exterior problem. One can also consider Neumann, Robin or transmission problems.

In all these studies it appears that there is a localization of the "first" eigenfunctions at the boundary and more specifically at points where $\nabla V$ is parallel to the normal.

This can be understood starting from the analysis of our toy models $-\Delta + i\vec{v}\cdot\vec{x}$ in $\mathbb{R}^2$ and $\mathbb{R}^2_+$.

Note, and this explains the above mentioned localization, in the second case that the spectrum is empty except in the case when $\vec{v} = (1,0)$ i.e. is parallel to the normal to the boundary $\{x_1 = 0\}$.

We consider the Bloch-Torrey operator in a periodically perforated domain ($d = 2$),

$$\Omega = \mathbb{R}^d \setminus \{\cup_{\gamma \in \mathbb{Z}^d} H_\gamma\} ,$$

where

$$H_\gamma = \{\mathbf{x} \in \mathbb{R}^2 \ : \ \mathbf{x} - \gamma \in H_0\}$$

and

$$H_0 \subset (-1/2, 1/2)^2$$

is a domain with a piecewise smooth boundary such that all the $H_\gamma$ are disjoint.

A typical example of $H_0$ is a ball $B(0, r)$ of radius $r < \frac{1}{2}$ centered at the origin.

One of the major difficulties in the definition and study of such non-self-adjoint operators is that the potential $ix_1$ is not periodic, unbounded and changing sign. We focus on the Dirichlet realization $\mathcal{A}^D$.

# Periodic problems in the non self adjoint case

1. What about the spectral application of the Floquet theory for non selfadjoint operators. If the Floquet decomposition exists, do we have always
$$\sigma(\mathcal{A}^D) = \overline{\cup \sigma(\mathcal{A}^D_q)} \quad ?$$
2. Do we have $\overline{\rho(\mathcal{A}^D)} = \mathbb{C}$ ? Is $\rho(\mathcal{A}^D)$ connected ?
3. Is the link between the existence of a Weyl's sequence and spectrum correct in the non self-adjoint case ?
4. Another difficulty is that the Torrey operator in a periodic structure is NOT periodic in the two directions.
5. Is the spectrum non empty ?

# Partial answers

1. For the first item, we can start with a Floquet theory in the $x_2$-direction.
   We can only prove

**Proposition F1**

$$\overline{\cup_{q_2 \in \mathbb{R}} \sigma(\mathcal{A}^D_{q_2})} \subset \sigma(\mathcal{A}^D)\,,$$

2. The questions are open but probably yes (in 2D). In the self-adjoint case the property is evidently true.

   An instructive model in $1D$ is to consider the $2$-steps model on the line $-\frac{d^2}{dx^2} + V(x)$, where $V(x)$ has period $2$ and is defined on $[0,2[$ by $V(x) = 0\,,\ \text{ for } x \in [0,1[\,,\ V(x) = L\,,\ \text{ for } x \in [1,2[\,.$
   For this model with $\mathrm{Re}\, L > 0$ large, one can show by explicit computations that the first bands are simply deformed when considering the parameter $L$ complex.

- [3] No in general but yes in our particular case.
- [5] Yes in the semi-classical result.

# Partial answer to item 4

We observe that

$$\tau_1 \circ \mathcal{A}^D_{q_2} = (\mathcal{A}^D_{q_2} - i) \circ \tau_1 \,. \tag{23}$$

Hence

$$K_{q_2} := \exp(-2\pi \mathcal{A}^D_{q_2}) \,, \tag{24}$$

commutes with the first translation $\tau_1$.
One can then do the Floquet theory for $K_{q_2}$ leading a priori to a family $K_{q_2,q_1}$.

## Proposition

If $\mu$ is an eigenvalue of $K_{q_2,0}$ with corresponding eigenfunction $u_0$, then $\frac{1}{2\pi}\log\mu + i\mathbb{Z}$ belongs to the spectrum of $\mathcal{A}^D_{q_2}$ and, for each $k$, we can construct starting from $u_0$ an eigenfunction $u_{\lambda_k}$ of $\mathcal{A}^D_{q_2}$ associated with $\lambda_k := \frac{1}{2\pi}\log\mu + ik$.

# The heuristics behind the proof.

Heuristically, we start from an eigenfunction $u_0$ and associate with it

$$u_{q_1} = \exp\Big(-q_1(\mathcal{B}^D_{q_2} - \lambda_0)\Big)\, u_0\,, \tag{25}$$

which satisfies the Floquet condition relative to $q_1$.
Defining $u$ by

$$u = \frac{1}{2\pi}\int_0^{2\pi} u_{q_1}\, dq_1\,. \tag{26}$$

we obtain formally :

$$\begin{aligned}
(\mathcal{B}^D_{q_2} - \lambda_0) u \ & = \big(\tfrac{1}{2\pi}\int_0^{2\pi} (\mathcal{B}^D_{q_2} - \lambda_0)\exp(-p(\mathcal{B}^D_{q_2} - \lambda_0))\, dq_1\big)\, u_0 \\
& = \big(\tfrac{1}{2\pi}\int_0^{2\pi} (-\tfrac{d}{dp}\exp(-q_1(\mathcal{B}^D_{q_2} - \lambda_0)))\, dq_1\big)\, u_0 \\
& = \tfrac{1}{2\pi}\big(I - \exp(-2\pi(\mathcal{B}^D_{q_2} - \lambda_0))\big)\, u_0 \\
& = 0\,.
\end{aligned}$$

We finally observe that $u$ is not $0$, we have indeed formally

$$\Sigma_n \tau_1^n u = u_0\,.$$

It remains to transform this formal proof into a correct one !

# The end

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- M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. Dover, 1972.
- Y. Almog, D. Grebenkov, and B. Helffer, *On a Schrödinger operator with a purely imaginary potential in the semiclassical limit*. Comm. Partial Differential Equations, 44 (2019), 1542–1604.
- Y. Almog, B. Helffer and X. B. Pan, *Superconductivity near the normal state in a half-plane under the action of a perpendicular electric current and an induced magnetic field*. Trans. AMS 365 (2013), 1183-1217.
- Y. Almog, *The stability of the normal state of superconductors in the presence of electric currents*. SIAM Journal on Mathematical Analysis, 40 (2008), pp. 824–850.
- Y. Almog and B. Helffer, *On the stability of laminar flows between plates*. ArXiv :1908.06328, (2019).
- Y. Almog, B. Helffer, and X.-B. Pan, *Superconductivity near the normal state under the action of electric currents and induced magnetic fields in* $\mathbb{R}^2$. Comm. Math. Phys., 300 (2010), 147–184.

——, *Superconductivity near the normal state in a half-plane under the action of a perpendicular electric current and an induced magnetic field*. Trans. Amer. Math. Soc., 365 (2013), 1183–1217.

Y. Almog and R. Henry.
*Spectral analysis of a complex Schrödinger operator in the semiclassical limit.*
SIAM J. Math. Anal. 44 (2016), pp. 2962–2993.

P. T. Callaghan, *Principles of nuclear magnetic resonance microscopy*, Oxford University Press on Demand, 1993.

E. B. Davies,
*Linear Operators and their Spectra*. vol. 106 of Cambridge Studies in Advanced Mathematics,
Cambridge University Press, Cambridge, 2007.

B. Helffer, *Semiclassical analysis, Witten Laplacians, and statistical mechanics.*
Series on Partial Differential Equations and Applications, Vol. 1 (2002).

D. S. Grebenkov.

*NMR Survey of Reflected Brownian Motion.*
Rev. Mod. Phys. 79 (2007), pp. 1077–1137.

D. S. Grebenkov,
*Diffusion MRI/NMR at high gradients : challenges and perspectives.*
(accepted to Micro. Meso. Mater., 2017).

D. S. Grebenkov, B. Helffer.
*On spectral properties of the Bloch-Torrey operator in two dimensions.*
Submitted. Preprint : http ://arxiv.org/abs/1608.01925.

D.S. Grebenkov, B. Helffer and N. Moutal.
On the spectral properties of the Bloch-Torrey equation in infinite periodically perforated domains.
ArXiv 2020.

B. Helffer, *Spectral Theory and its Applications*. vol. 139 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2013.

B. Helffer, F. Nier,
*Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians.*

Lecture Notes in Mathematics, 1862. Springer-Verlag, Berlin, 2005.

B. Helffer, J. Nourrigat,
*On the domain of a magnetic Schrödinger operator with complex electric potential.*
Analysis and operator theory, 149–165, Springer Optim. Appl., 146, Springer, Cham, 2019.

R. Henry, *On the semiclassical analysis of Schrödinger operators with purely imaginary electric potentials in a bounded domain*. ArXiv preprint arXiv :1405.6183.

T. Kato, *Perturbation Theory for Linear Operators*. Springer, third edition, 1980.

P. Kuchment.
*Floquet Theory for Partial Differential Equations.*
Operator Theory : Advances and Applications, vol. 60, Birkhäuser Verlag, Basel, 1993.

P. Kuchment.
An overview of periodic elliptic operators.
*Bull. Amer. Math. Soc.* **53** (2016), pp. 343-414.

M. Kunze, L. Lorenzi, A. Maichine, and A. Rhandi, *$L^p$-theory for Schrödinger systems*, Math. Nachr., 292 (2019), pp. 1763–1776.

M. Kunze, A. Maichine, and A. Rhandi, *Vector-valued Schrödinger operations in $L^p$-spaces*, Discrete Contin. Dyn. Syst. Ser. S, 13 (2020), pp. 1529–1541.

N. Moutal, A. Moutal, and D. S. Grebenkov.
Diffusion NMR in periodic media : efficient computation and spectral properties.
*J. Phys. A : Math. Theor.* **53** (2020), 325201.

H.C. Torrey,
*Bloch equations with diffusion terms.*
Phys. Rev. Vol. 104, no 3 (Nov. 1956), 563–565.