Preliminaries around Helffer-Nourrigat Conjecture

Bernard Helffer[2]

March 2024

[2]Laboratoire de Mathématiques Jean Leray, Nantes Université (France).

Introduction

The conjecture of Helffer-Nourrigat, which was formulated in its local version (1979), has been solved recently (2022) by Ianivos Androulidakis, Omar Mohsen and Robert Yuncken.
Two years later (2024) Omar Mohsen proves the microlocal version of this conjecture.
In this introductory talk[4], I would like to come back to the Rockland conjecture which I prove with J. Nourrigat in 1979. The solution of the Rockland conjecture was the starting point for the formulation of the local and microlocal version of the so-called HN-conjecture devoted to the maximal hypoellipticity of Polynomial of vector fields. Many particular cases were analyzed in the Birkhäuser book of 1985 and in later works of J. Nourrigat (1985-90).
In this talk I mainly follow the Chapter II of my book with J. Nourrigat but also add later contribution mainly coming from the polish school in the nineties.

[4] The talk was given before a talk by Omar Mohsen.

A few definitions

Lie Algebra.

A Lie algebra $\mathcal{G}$ on $\mathbb{R}$ is a vector space on $\mathbb{R}$ together with a bilinear map (Lie-Bracket)

$$\mathcal{G} \times \mathcal{G} \ni (x, y) \mapsto [x, y] \in \mathcal{G}\,,$$

such that

- $\forall x \in \mathcal{G}$, $[x, x] = 0\,,$
- Jacobi Identity holds:

$$\forall x, y, z \in \mathcal{G}, [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0\,.$$

Graded Lie Algebras

We only consider **graded Lie Algebras**, i.e. which can be written, for some $r \in \mathbb{N} \setminus \{0\}$, as a direct sum

$$\mathcal{G} = \oplus_{j=1}^{r} \mathcal{G}_j \,,$$

with

$$[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j} \text{ if } i + j \leq r$$

and

$$[\mathcal{G}_i, \mathcal{G}_j] = 0 \text{ if } i + j > r \,.$$

In addition, we assume that $\mathcal{G}$ is stratified, i.e. generated by $\mathcal{G}_1$.

We denote by $\mathcal{G}^{r,p}$ the maximal stratified algebra of rank r with p generators.

G and $\mathcal{G}$

One can put on $\mathcal{G}$ a group structure by using the Campbell-Hausdorff formula

$$a \circ b := a + b + \frac{1}{2}[a, b] + \frac{1}{12}([a, [a, b]] - [b, [a, b]]) + \cdots$$

In the nilpotent case, this Campbell-Hausdorff formula becomes finite. We can also write

$$G = \exp \mathcal{G}\,,$$

and we have with $g_1 = \exp a_1$ and $g_2 = \exp a_2$

$$g_1 \cdot g_2 = \exp(a_2 \circ b_2)\,.$$

The elements of $\mathcal{G}$ can be identified with the left invariant vector fields on the group G. (We can first identify $\mathcal{G}$ and the tangent space at the neutral element e $T_e G$.)

Enveloping Algebra

The enveloping algebra $\mathcal{U}(\mathcal{G})$ can be defined in the stratified case as the space of the non commutative polynomials in the form

$$P = \sum_{\alpha} a_\alpha Y^\alpha$$

where $Y^\alpha = Y_{\alpha_1} Y_{\alpha_2} \cdots Y_{\alpha_k}$, Y_i ($i = 1, \dots, p$) is a basis of $\mathcal{G}_1$, $\alpha_\ell \in \{1, \dots, p\}$ and $a_\alpha \in \mathbb{C}$.

We have a natural family of dilations defined by

$$\delta_t(\sum_{j=1}^{r} a_j) = \sum_{j=1}^{r} t^j a_j \,,\; a_j \in \mathcal{G}_j \,.$$

Using this dilation, we can introduce the subspace $\mathcal{U}_m(\mathcal{G})$ of the homogeneous elements

$$\delta_t P = t^m P \,.$$

For example, the operator $\sum_{i=1}^{p} Y_i^2$ belongs to $\mathcal{U}_2(\mathcal{G})$. Notice that it can also be considered as a left invariant operator on G, which is a particular case of the Hörmander operator.

Examples

We focus on two particular Lie Algebra.

Heisenberg group.
A basis of its Lie Algebra is given by Y_1, Y_2, Z, $[Y_1, Y_2] = Z$.
In exponential coordinates

$$Y_1 = \partial_{u_1} - \frac{1}{2} u_2 \partial_{u_3}\,,\; Y_2 = \partial_{u_2} + \frac{1}{2} u_1 \partial_{u_3}\,,\; Z = \partial_{u_3}\,.$$

Here $p = 2$, $r = 2$.

Engel group.
Y_1, Y_2, Z, W, $[Y_1, Y_2] = Z$, $[Y_1, [Y_1, Y_2] = W$, with
$[Y_2, [Y_1, Y_2]] = 0\,.$
In exponential coordinates

$$Y_1 = \partial_{u_1}\,,\; Y_2 = \partial_{u_2} + u_1 \partial_{u_3} + \frac{1}{2} u_1^2 \partial_{u_4}\,,\; Z = \partial_{u_3} + u_1 \partial_{u_4}\,,\; W = \partial_{u_4}\,.$$

Here $p = 2$ and $r = 3$.

Induced representations

Let $\mathcal{H}$ a subalgebra in $\mathcal{G}$ (respecting the stratification) of codimension k and $\ell : \mathcal{H} \mapsto \mathbb{R}$ a linear form such that

$$\ell([X,Y]) = 0\,,\ \forall X, Y \in \mathcal{H}\,.$$

One can show that one can find a basis e_j ($j = 1, \dots, k$) of a supplementary space to $\mathcal{H}$ (each e_j being homogeneous with respect to δ_t) such that for any $a \in \mathcal{G}$, we can write

$$\exp a = \exp h \cdot \exp t_k e_k \cdots \exp t_1 e_1 := \exp h \cdot \exp \gamma(t)\,,$$

where the map $a \mapsto (h, t)$ is a global diffeomorphism of $\mathcal{G}$ onto $\mathcal{H} \times \mathbb{R}^k$.

We then introduce $h(t,a)$ and $\sigma(t,a)$ by the relation

$$\gamma(t) \circ a = h(t,a) \circ \sigma(t,a)\,.$$

We can now define the induced representation $\pi_{\ell,\mathcal{H}}$ of the group G in $L^2(\mathbb{R}^k)$ by

$$(\pi_{\ell,\mathcal{H}}(\exp a) f)(t) = e^{i<\ell, h(t,a)>} f(\sigma(t,a))\,,\ \forall t \in \mathbb{R}^k, \forall a \in \mathcal{G}\,.$$

Note that for $k = 0$, $L^2(\mathbb{R}^k) = \mathbb{C}$.
When $\ell = 0$, $\pi_{0,\mathcal{H}}$ is the natural representation (on the right) of G in $L^2(H\backslash G)$.

Induced representation of the Lie algebra

For $f \in \mathcal{S}(\mathbb{R}^k)$ and $a \in \mathcal{G}$ we define

$$\pi_{\ell,\mathcal{H}}(a)f = \frac{d}{ds}(\pi_{\ell,\mathcal{H}}(\exp sa)f)_{/s=0}$$

which after computation gives

$$\pi_{\ell,\mathcal{H}}(a) = i < \ell, h'(t,a) > + \sum_{j=1}^{k} \sigma_j'(t,a)\frac{\partial}{\partial t_j}\,,$$

where

$$h'(t,a) := \frac{d}{ds}h(t,sa)_{/s=0}\,,\ \sigma'(t,a) := \frac{d}{ds}\sigma(t,sa)_{/s=0}\,.$$

We can then naturally extend $\pi_{\ell,\mathcal{H}}$ to $\mathcal{U}(\mathcal{G})$.

Examples

- For G = Heisenberg, $\mathcal{H} = \mathbb{R} Y_2$, $\ell = 0$, we get with $k = 2$,

$$X_1 := \pi_{0,\mathcal{H}}(Y_1) = \partial_{t_1}\,,\; X_2 := \pi_{0,\mathcal{H}}(Y_2) = t_1 \partial_{t_2}\,.$$

  $X_1^2 + X_2^2$ is a Baouendi-Grushin operator. The analysis of the hypoellipticity of $X_1^2 + X_2^2 + \lambda [X_1, X_2]$ is due to V. Grushin.

- For G = Engel, $\mathcal{H} = \mathbb{R}\, Y_2$, $\ell = 0$, we get with $k = 3$,

$$\begin{aligned} X_1 &:= \pi_{0,\mathcal{H}}(Y_1) = \partial_{t_1} \\ X_2 &:= \pi_{0,\mathcal{H}}(Y_2) = \tfrac{1}{2} t_1^2 \partial_{t_3} + t_1 \partial_{t_2} \\ [X_1, X_2] &= \pi_{0,\mathcal{H}}(Z) = t_1 \partial_{t_3} + \partial_{t_2}\,. \\ [X_1 [X_1, X_2]] &= \pi_{0,\mathcal{H}}(W) = \partial_{t_3}\,. \end{aligned}$$

- For $G =$ Engel, $\mathcal{H} = \mathbb{R}\, Y_2 + \mathbb{R} Z$, $\ell = 0$, we get with $k = 2$,

$$\begin{array}{rl} X_1 & := \pi_{0,\mathcal{H}}(Y_1) = \partial_{t_1} \\ X_2 & := \pi_{0,\mathcal{H}}(Y_2) = \frac{1}{2} t_1^2 \partial_{t_2} \\ [X_1, X_2] & = \pi_{0,\mathcal{H}}(Z) = t_1 \partial_{t_2}\,. \\ [X_1[X_1, X_2]] & = \pi_{0,\mathcal{H}}(W) = \partial_{t_2}\,. \end{array}$$

$X_1^2 + X_2^2$ is a (more degenerate) Baouendi-Grushin operator:

$$X_1^2 + X_2^2 = \partial_{t_1^2} + \frac{1}{4} t_1^4 \partial_{t_2}^2\,.$$

Kirillov's theory

For $\ell \in \mathcal{G}^*$, we consider the bilinear form on $\mathcal{G} \times \mathcal{G}$

$$B_\ell(x,y) = \langle \ell \,,\, [x,y] \rangle \,.$$

We now consider a subalgebra $\mathcal{H}$ which is isotropic for B_ℓ and look at the induced representation $\pi_{\ell,\mathcal{H}}$. One can show that $\pi_{\ell,\mathcal{H}}$ is irreducible iff $\mathrm{Codim}\mathcal{H} = \frac{1}{2}\mathrm{rank} B_\ell$.
Moreover for any $\ell \in \mathcal{G}^*$, there exists a (non unique) maximal $\mathcal{H}$.
Hence we can associate to each ℓ an irreducible unitary representation of G $\pi_{\ell,\mathcal{H}}$ which is unique up to unitary conjugation, hence defining a map κ of $\mathcal{G}^*$ to $\hat{G}$ the set of the irreducible representations of G.

This map is not injective. To understand this non injectivity we have to explain how G naturally acts on $\mathcal{G}^*$.
If $g = \exp a$ and $\ell \in \mathcal{G}^*$, we define (coadjoint action)

$$g \cdot \ell = \sum_{k=0}^{r} \frac{1}{k!} \mathrm{ad}^*(-a)^k \ell \,,$$

where

$$((\mathrm{ad}^*(b))\ell))(c) = \ell([b, c]) \,.$$

Kirillov's theory says that the (equivalent class of the) representation π_ℓ depends only on the orbit of ℓ and that in this way we recover all the irreducible unitary representations of G.

Exercise 1. Irreducible representation of Heisenberg (by hand)

This presentation follows the way we use for the proof of Rockland's conjecture.
We will look at the representation of the corresponding Lie algebra: Y_1, Y_2, Z, $[Y_1, Y_2] = Z$ starting of the regular representation (in exponential coordinates)

$$Y_1 = \partial_{u_1} - \frac{1}{2} u_2 \partial_{u_3}\,,\; Y_2 = \partial_{u_2} + \frac{1}{2} u_1 \partial_{u_3}\,,\; Z = \partial_{u_3}\,.$$

A partial Fourier transform with respect to u_3 gives the family (parametrized by ℓ_3)

$$\pi_{\ell_3}(Y_1) = \partial_{u_1} - \frac{i}{2} \ell_3 u_2\,,\; Y_2 = \partial_{u_2} + \frac{i}{2} \ell_3 u_1\,,\; \pi_{\ell_3}(Z) = i\ell_3\,.$$

After a gauge transformation, we get the family

$$\tilde{\pi}_{\ell_3}(Y_1) = \partial_{u_1}\,,\ \tilde{\pi}_{\ell_3}(Y_2) = \partial_{u_2} + i\ell_3 u_1\,,\ \tilde{\pi}_{\ell_3}(Z) = i\ell_3\,.$$

This is clearly not irreducible. A partial Fourier transform in u_2 gives the family (parametrized by ℓ_2, ℓ_3)

$$\tilde{\pi}_{\ell_2,\ell_3}(Y_1) = \partial_{u_1}\,,\ \tilde{\pi}_{\ell_2,\ell_3}(Y_2) = i(\ell_2 + \ell_3 u_1)\,,\ \tilde{\pi}_{\ell_2,\ell_3}(Z) = i\ell_3\,.$$

This is irreducible if $\ell_3 \neq 0$. In this case, a translation in u_1 shows that it is enough to consider $\ell_2 = 0$. The orbit of $(0,0,\ell_3)$ is

$$\mathcal{O}((0,0,\ell_3)) = \{(\ell_1,\ell_2,\ell_3)\,,\ (\ell_1,\ell_2) \in \mathbb{R}^2\}\,.$$

If $\ell_3 = 0$, $\tilde{\pi}_{\ell_2,0}$ is not irreducible. A partial Fourier transform in u_1 gives

$$\pi_{\ell_1,\ell_2,0}(Y_1) = i\ell_1\,,\ \pi_{\ell_1,\ell_2,0}(Y_2) = i\ell_2\,,\ \pi_{\ell_1,\ell_2,0}(Z) = 0\,.$$

Rockland calls these representations the "degenerate" representations (corresponding with the ℓ vanishing on $\mathcal{G}_2 = \mathbb{R}Z$). The orbits are reduced to points.

Exercise 2. Engel.

We start from

$$Y_1 = \partial_{u_1}\,,\; Y_2 = \partial_{u_2} + u_1\partial_{u_3} + \frac{1}{2}u_1^2\partial_{u_4}\,,\; Z = \partial_{u_3} + u_1\partial_{u_4}\,, W = \partial_{u_4}\,.$$

A Fourier transform in (u_2, u_3, u_4) leads to

$$\pi_{\ell_2,\ell_3,\ell_4}(Y_1) = \partial_{u_1}\,,\; \pi_{\ell_2,\ell_3,\ell_4}(Y_2) = i(\ell_2 + u_1\ell_3 + \frac{\ell_4}{2}u_1^2)\,,\dots\,.$$

This is irreducible if $\ell_4 \neq 0$.
The orbit of $(0, \ell_2, \ell_3, \ell_4)$ is parametrized by (ℓ_1, β)

$$\mathcal{O}((0, \ell_2, \ell_3, \ell_4)) = \{(\ell_1, \ell_2 + \beta\ell_3 + \frac{1}{2}\beta^2\ell_4, \ell_3 + \beta\ell_4, \ell_4), (\ell_1, \beta) \in \mathbb{R}^2\}$$

If $\ell_4 = 0$,

$$\begin{aligned} \pi_{\ell_2,\ell_3,0}(Y_1) &= \partial_{u_1}\,, \quad & \pi_{\ell_2,\ell_3,0}(Y_2) &= i(\ell_2 + u_1\ell_3)\,,\\ \pi_{\ell_2,\ell_3,0}(Z) &= i\ell_3\,, \quad & \pi_{\ell_2,\ell_3,0}(W) &= 0\,. \end{aligned}$$

We can continue like for Heisenberg.

Rockland's conjecture (1976)

Theorem of Helffer-Nourrigat (1979)

Let $\mathcal{G} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_r$ a graded stratified Lie algebra and let $P \in \mathcal{U}_m(\mathcal{G})$, then the three following conditions are equivalent

1. P is hypoelliptic in G ($G = \exp \mathcal{G}$ is the associated Lie group and P is identified with a left invariant operator on G).
2. For any non trivial irreducible representation in $\widehat{G}$, $\pi(P)$ is injective in $\mathcal{S}_\pi$, the space of C^∞ vectors of the representation.
3. For any $Q \in \mathcal{U}_m(\mathcal{G})$, there exists C_Q s.t. for any $\pi \in \widehat{G}$, any $u \in \mathcal{S}_\pi$ we have

$$||\pi(Q)u||^2 \leq C_Q ||\pi(P)u||^2$$

Historics

- The formulation of the conjecture is due to Charles Rockland (1976) (published in (1978) [32]) who proves the conjecture in the case of the Heisenberg group.
- B. Helffer and R. Beals observed independently that when $r = 2$ the theorem, modulo the establishment of a dictionary, was a consequence of general theorems about the hypoellipticity of operators with multiple characteristics (J. Sjöstrand (1974), L. Boutet de Monvel (1974), Boutet de Monvel-Grigis-Helffer (1976)). The proof (for the two last papers) was based on a very nice class of pseudo-differential operators introduced by L. Boutet de Monvel and adapted with operators with multiple characteristics.
- Extension to nondifferential convolution operators is considered by P. Glowacki in [9] and Hebisch (1998).

- R. Beals (1977) also proves in full generality "(1) implies (2)". Helffer and Nourrigat prove that "(2) implies (3)" in two steps: first $r = 3$ (1978) and one year later the general case. Kirillov's theory [24] plays an important role but cannot be used as a black box.
  The feeling at this time was that one cannot use a standard class of pseudo-differential operators and that $r = 2$ was in some sense the limit for this kind of approach.
- Since this proof, only A. Melin (1981) gives a partially alternative proof using a group adapted pseudo-differential calculus but he can not avoid to use an important step of Helffer-Nourrigat's proof to complete his proof. See also later [5], P. Glowacki [10] and references therein, Hebisch (1998).
- More properties of the so-called positive Rockland's operators are presented in the book of V. Fischer and M. Ruzhansky [7].

The case of Homogeneous groups $H\backslash G$

Once Rockland's conjecture was proven, we (with J. Nourrigat) wanted to extend the theorem to $H\backslash G$ where H is the sub-group associated with the subalgebra $\mathcal{H}$. If $\mathcal{H}$ is an ideal, we can apply Rockland's criterion to the nilpotent group $H\backslash G$, hence we do not make this assumption and we look for criteria for analyzing the maximal hypoellipticity $\pi_{0,\mathcal{H}}(P)$.

Here we use the notion of Spectrum of an induced representation (a notion due to Brown). If $\ell \in \mathcal{G}^*$ and $\mathcal{H}$ is a sub-algebra such that $\ell([\mathcal{H},\mathcal{H}]) = 0$, we consider

$$\Omega_{\ell,\mathcal{H}} = G \cdot (\ell + \mathcal{H}^\perp)\,.$$

We consider its closure $\overline{\Omega_{\ell,\mathcal{H}}}$ and since it is G-invariant and closed in $\mathcal{G}^*$ we get via κ a closed set

$$\widehat{\Omega}_{\ell,\mathcal{H}} := \kappa(\overline{\Omega_{\ell,\mathcal{H}}}) \text{ in } \hat{G}$$

(for a suitable topology).

This can be identified to what is called the spectrum of $\pi_{\ell,\mathcal{H}}$.

Note that if $\ell = 0$, $\overline{\Omega_{0,\mathcal{H}}}$ is a cone for δ_t.

Helffer-Nourrigat for $\pi_{0,\mathcal{H}}(P)$

Theorem Helffer-Nourrigat (book) + Nourrigat (1987)

Let $\mathcal{G}$ be a graded stratified Lie algebra, $\mathcal{H}$ a graded subalgebra of $\mathcal{G}$, and let $P \in \mathcal{U}_m(\mathcal{G})$, then the following conditions are equivalent

1. For any non trivial irreducible representation in $\widehat{\Omega}_{0,\mathcal{H}}$, $\pi(P)$ is injective in $\mathcal{S}_\pi$, the space of C^∞ vectors of the representation.
2. For any $Q \in \mathcal{U}_m(\mathcal{G})$, there exists C_Q s.t. for any $\pi \in \widehat{\Omega}_{0,\mathcal{H}}$, any $u \in \mathcal{S}_\pi$ we have

$$||\pi(Q)u||^2 \leq C_Q||\pi(P)u||^2$$

3. $\pi_{0,\mathcal{H}}(P)$ is maximally hypoelliptic in the sense of above but with π replaced by $\pi_{0,\mathcal{H}}$.

Notice that in the statement we have replaced (in comparison with Rockland's statement) "hypoelliptic" by "maximally hypoelliptic".

About the proof

The proof given in our book (extended by J. Nourrigat [28]) is analytic, but we discuss in one chapter an alternative "algebraic" proof when $\overline{\Omega_{0,\mathcal{H}}}$ is closed for the Zariski topology. In this case, we can find operators P_j in $\mathcal{U}_{m_j}(\mathcal{G})$ ($j = 1, \cdots, q$) such that $\pi_{0,\mathcal{H}}(P_j) = 0$ and the system $(P, P_1, \cdots, P_q)$ satisfies Rockland condition.

We fail with this approach when considering cases when $\overline{\Omega_{0,\mathcal{H}}}$ is strictly included in the Zariski closure of $\Omega_{0,\mathcal{H}}$.

But if we extend Rockland's theorem to a suitable class of pseudo-differential operators, the previous strategy work in full generality by replacing $(P_1, \cdots, P_q)$ by a suitable pseudo-differential operator (Melin,Glowacki, . . .).

This approach was only completed in 1998 by Hebisch (see below).

Hebisch theorem

For a function $\phi \in \mathcal{S}(G)$ and a unitary representation π, we can define

$$\pi(\phi) = \int_G \phi(g)\pi(g^{-1})dg \,.$$

Formally, we recover the elements of the enveloping algebra by considering linear combinations of Dirac distributions at the neutral element of G.

Hebisch Theorem

Let F be a closed set of $\mathcal{G}^*$ stable by the coadjoint action and the dilation. Then there exists a function ϕ in $\mathcal{S}(G)$ such that for all $\ell \in F$, $\pi_\ell(\phi) = 0$ and for all $\ell \notin F$, the operator $\pi_\ell(\phi)$ is positive definite and injective.

One can then apply a Rockland like theorem to the system $(P, \pi_{reg}(\phi))$ where π_{reg} is the (right)- regular representation of G in $L^2(G)$.

Exercises

We come back to previous exercises related to Heisenberg and Engel.

- For G = Heisenberg, $\mathcal{H} = \mathbb{R} Y_2$,

$$\overline{\Omega_{0,\mathcal{H}}} = \mathcal{G}^* .$$

  Hence we need all the non trivial representations.

- For G = Engel, $\mathcal{H} = \mathbb{R}\, Y_2$, we get

$$\overline{\Omega_{0,\mathcal{H}}} = \{(\eta_1, \eta_2, \xi, \tau) | 2\tau\eta_2 - \xi^2 \leq 0\} .$$

  This set is NOT Zariski closed.

- For G = Engel, $\mathcal{H} = \mathbb{R}\, Y_2 + \mathbb{R} Z$, (Grushin operators) we get

$$\overline{\Omega_{0,\mathcal{H}}} = \{(\eta_1, \eta_2, \xi, \tau) | \xi^2 = 2\eta_2\tau\} .$$

  We are in a Zariski closed situation.

THIS IS THE END of the SHORT preliminary talk.

Towards Helffer-Nourrigat's conjecture

At about the same time appears the fundamental paper of Rothschild-Stein (1976) (C.Rockland is citing the paper which was submitted to Acta in June 1975) which gives a new light on the paper of Lars Hörmander (1967) on the operator $\sum X_j^2 + X_0$, where the X_j's are vector fields satisfying the celebrated

Hörmander condition $(CH)_r$

The X_j and all their brackets up to rank r generate at each point the whole tangent space.

We write $(CH)_r(x)$ if the condition is satisfied at x.
One important step was that this condition implies

$$||u||^2_{1/r} \leq C\left(\sum_j ||X_j u||^2 + ||u||_2^2\right).$$

Except Kohn's paper (1973) giving an alternative easier proof of the hypoellipticity (but with weaker estimates), no progress was done except in the case $r = 2$ (see above).

From the PDE point of view, the interest of the paper by Rotschild-Stein was that they get maximal estimates for an operator in the form

$$P := \sum_{|\alpha|\leq m} a_\alpha(x) X^\alpha$$

i.e. it holds

$$\sum_{|\alpha|\leq m} ||X^\alpha u||^2 \leq C\left(||Pu||^2 + ||u||^2\right), \forall u \in C_0^\infty ,$$

as a consequence of construction of a nice calculus modelled on nilpotent groups.
Note that the two inequalities imply hypoellipticity but maximal hypoellipticity is much stronger.

Without to enter in the details, I would like to mention the following points

- The Lifting theorem (see also Folland, Hörmander–Melin, Helffer-Nourrigat). This lifting (addition of variable) permits to associate with a polynomial of vector fields $\sum_{|\alpha|\leq m} a_\alpha(x) X^\alpha$ an operator $\sum_{|\alpha|\leq m} a_\alpha(\lambda(x))\tilde{X}^\alpha$ where the $\tilde{X}_j$ are this time well approximated by corresponding Y_j generating a free nilpotent, stratified, Lie Algebra of rank r with p generators.
- Assuming that

$$\mathcal{P}_{x_0} := \sum_{|\alpha|=m} a_\alpha(x_0) Y^\alpha$$

is hypoelliptic for any x_0, a singular integral calculus for hypoelliptic operators which are polynomial of these vector fields.

If this approach worked perfectly well for $\sum_j X_j^2$ or more generally for $\sum_j X_j^{2k}$ (the lifted operator is hypoelliptic), this does not work in general. Hence the assumption that $\mathcal{P}_{x_0} := \sum_{|\alpha|=m} a_\alpha(x_0) Y^\alpha$ is hypoelliptic is too strong.

The first idea was to consider the case when the lifting can be done with a smaller Lie Algebra. This case was for example considered by L.P. Rothschild (1979) (see also G. Métivier for the corresponding theory) and combined with the proved Rockland's Conjecture.

Thinking of Rockland's conjecture and many particular cases (Grushin's like results) one is led to the formulation of our conjecture.

Conjecture of Helffer-Nourrigat (1979)

Conjecture locale

We assume that at some point x_0 the vector fields X_i satisfy $(CH)_r(x_0)$. Then there exists a closed subset $\widehat{\Gamma}_{x_0}$ in $\widehat{G}$ such that the following conditions are equivalent

1. P is maximally hypoelliptic in x_0
2. For any non trivial representation π in $\widehat{\Gamma}_{x_0}$, $\pi(\mathcal{P}_{x_0})$ is injective $\mathcal{S}_\pi$.

The conjecture gives in addition the candidate !

If λ is the lifting map, i.e. the unique linear application of $\mathcal{G}$ into the algebra of the vector fiels defined on Ω such that

$$\lambda(Y_i) = X_i$$

which is a partial homomorphism of rank r, we define λ_x by $\lambda_x(a) = \lambda(a)(x)$ and denote by λ_x^* the transposed map.

Definition of Γ_x

Assuming $(CH)_r(x_0)$ we introduce $\Gamma_{x_0} \subset \mathcal{G}^*$ as the set of the ℓ such that there exists a sequence (t_n, x_n, ξ_n) in $\mathbb{R}^+ \times T^*\Omega \setminus \{0\}$ such that

$$\begin{cases} t_n \to 0\,,\, x_n \to x_0, |\xi_n| \to +\infty \\ t_n^r|\xi_n| \text{ is bounded} \\ \ell = \lim_{n\to+\infty} \delta_{t_n}^* \lambda_{x_n}^* \xi_n\,. \end{cases}$$

One can prove that Γ_{x_0} is a closed set in $\mathcal{G}^*$ which is invariant by dilation and by the coadjoint action of G on $\mathcal{G}^*$. By definition $\widehat{\Gamma}_{x_0}$ is the corresponding set (via Kirillov's theory) in $\widehat{G}$.

The book of 1985 by B. Helffer and J. Nourrigat [19]

The book is the result of five years of investigations around this conjecture by the two authors separately or together. It presents the proof of Rockland's conjecture in a self contained way.

Then it explores particular cases where the conjecture of Helffer-Nourrigat can be proved.

The book is also exploring cases where one can make Rockland's conditions more explicit, in particular for the analysis of problems connected with complex analysis $\bar{\partial}_b$, $\Box_b$.

The following result obtained by J. Nourrigat in 1987 ([28]) is enlightning for some of the techniques appearing in the proof of Rockland's Conjecture and other results in the book

Nourrigat's Theorem

Let F be a closed subset of $\mathcal{G}^*$ stable by dilation and the coadjoint action of G. Let $P \in \mathcal{U}_m(\mathcal{G})$.
Then if $\pi_\ell(P)$ is injective for any $\ell \in F \setminus \{0\}$, then there exists a constant $C > 0$ such that for any $Q \in \mathcal{U}_m(\mathcal{G})$, there exists C_Q s.t. for any $\pi \in \widehat{F}$, any $u \in \mathcal{S}_\pi$ we have

$$||\pi(Q)u|| \leq C_Q ||\pi(P)u||$$

Note that this result can have many other applications than for Hypoellipticity. The case when $F = \mathcal{G}^*$ corresponds to Rockland's conjecture. The case when $F = \overline{G.\mathcal{H}^{\perp}}$ where $\mathcal{H}$ is a graded subalgebra of $\mathcal{G}$ appears also naturally and was analyzed in the book.

In 1998, W. Hebisch [13] (Theorem 2) gives a nice simple proof of this theorem, modulo the extension of Rockland's conjecture and some adapted pseudo-differential calculus due to [5].

Microlocal questions

In order to present the problem "microlocally", one has

- first to mention a microlocalized version of Hörmander-Kohn inequality (due to Bolley-Camus-Nourrigat [3]),
- then to give a microlocalized definition of maximal estimates,
- and finally to give the microlocal definition of Γ_x.

To the vector field $X_j = \sum_k a_{jk}(x)\partial_{x_k}$, we can attach its symbol

$$U_j(x,\xi) = i \sum_k a_{jk}(x)\xi_k \,.$$

The symbol of $[X_j, X_k]$ is the Poisson bracket $-i\{U_j, U_k\}$.
In other words, X_j can be considered as a pseudo-differential operator of symbol U_j.

Microlocalized Hörmander condition $(CH)_r(x_0,\xi_0)$

Let $(x_0,\xi_0) \in T^*\Omega \setminus \{0\}$. We say that $(CH)_r(x_0,\xi_0)$ holds if the system of the U_j and all their Poisson brackets up to rank r is elliptic at (x_0,ξ_0).

This definition immediately extends to pseudo-differential operators $U_j(x,D_x)$ of degree one with purely imaginary symbols.

Bolley-Camus-Nourrigat have shown

BoCaNo theorem

If $(CH)_r(x_0, \xi_0)$ holds, then there exists a pseudo-differential operator of degree 0 $\psi(x, D_x)$, elliptic at (x_0, ξ_0) such that

$$||\psi(x, D_x)u||^2_{1/r} \leq C\Big(\sum_j ||U_j(x, D_x)u||^2 + ||u||^2_2\Big)\,.$$

Definition of Γ_{x_0,ξ_0}

Let $(x_0, \xi_0) \in T^*\Omega \setminus \{0\}$ and assume $(CH)_r(x_0, \xi_0)$.
$\Gamma_{x_0,\xi_0} \subset \mathcal{G}^*$ is the set of the ℓ such that there exists a sequence (t_n, x_n, ξ_n) in $\mathbb{R}^+ \times T^*\Omega \setminus \{0\}$ such that

$$\begin{cases} t_n \to 0\,,\, x_n \to x_0, |\xi_n| \to +\infty, \xi_n/|\xi_n| \to \xi_0/|\xi_0| \\ t_n^r|\xi_n| \text{ is bounded} \\ \ell = \lim_{n\to+\infty} \delta_{t_n}^* \lambda_{x_n}^* \xi_n\,. \end{cases}$$

The last condition can also be written in the following way:
For any bracket Y_I of length $|I|$ of the generators of $\mathcal{G}$ we have

$$\ell(Y_I) = (-i)^{|I|} \lim_{n\to+\infty} t_n^{|I|} U_I(x_n, \xi_n)\,,$$

where U_I denotes the iterated Poisson bracket of the symbols of the pseudo-differential operators U_i. In this way we can define Γ_{x_0,ξ_0} for a family of pseudo-differential operators of degree one U_i satisfying $(CH)_r(x_0, \xi_0)$. Note that Γ_{x_0,ξ_0} is a closed G-invariant cone.

Maximal Microhypoellipticity

We consider an operator in the form

$$P := \sum_{|\alpha| \leq m} a_\alpha(x) U^\alpha$$

More generally one can replace $a_\alpha(x)$ by pseudo-differential operators of order 0 $a_\alpha(x, D_x)$.
We say that it is maximally microhypoelliptic at (x_0, ξ_0) if there exists a pseudo-differential operator of degree 0 $\psi(x, D_x)$, elliptic at (x_0, ξ_0) such that

$$\sum_{|\alpha| \leq m} ||\psi(x, D_x) U^\alpha u||^2 \leq C \left(||Pu||^2 + ||u||^2 \right), \forall u \in C_0^\infty ,$$

Together with $(CH)_r(x_0, \xi_0)$ this implies micro-hypoellipticity at (x_0, ξ_0).

Microlocal conjecture

Conjecture

We assume that at some point (x_0, ξ_0) the operators U_i satisfy $(CH)_r(x_0, \xi_0)$. Then the following conditions are equivalent

1. P is microlocally maximally hypoelliptic at (x_0, ξ_0)
2. For any non trivial representation π in $\widehat{\Gamma}_{x_0,\xi_0}$, $\pi(\mathcal{P}_{x_0,\xi_0})$ is injective $\mathcal{S}_\pi$.

J. Nourrigat has shown the necessary part. The sufficient part is rather well understood as $r = 2$ since the end of the seventies. J. Nourrigat has shown in [29] the sufficiency part for a class of systems of order 1. The proof is extremely technical, and inspired by Fefferman Phong techniques.

Bibliography.

Iakovos Androulidakis, Omar Mohsen, Robert Yuncken.
A pseudodifferential calculus for maximally hypoelliptic operators and the Helffer-Nourrigat conjecture.
arXiv:2201.12060 v2. 7 Dec 2022

R. Beals.
Opérateurs invariants hypoelliptiques sur un groupe de Lie nilpotent.
In: Séminaire Goulaouic-Schwartz 1976/1977: Équations aux dérivées partielles et analyse fonctionnelle, Exp. No. 19. April 1977.

P. Bolley, J. Camus, and J. Nourrigat.
La condition de Hörmander-Kohn pour les opérateurs pseudo-différentiels.
Comm. Partial Differential Equations 7 (1982), no. 2, 197–221.

L. Boutet de Monvel, A. Grigis, and B. Helffer.

Parametrixes d'opérateurs pseudo-différentiels à caractéristiques multiples. (French)
Journées: Équations aux Dérivées Partielles de Rennes (1975), pp. 93–121. Astérisque, No. 34–35, Soc. Math. France, Paris, 1976.

M. Christ, D. Geller, P. Glowacki, and L. Polin.
Pseudodifferential operators on groups with dilations.
Vol. 68, No. 1 Duke Math. Journal (C) October 1992

Y.V. Egorov.
Subelliptic operators.
Russian Math. Survey 30 (2), 59–118 and 30(3), 55–105 (1975).

V. Fischer, M. Ruzhansky.
Quantization of nilpotent Lie groups.
Progress in Mathematics 314 (2015).

G.B. Folland.
On the Rothschild-Stein lifting theorem.

Comm. in PDE 212 (1977), pp. 165-191.

- P. Glowacki.
  The Rockland condition for non-differential convolution operators.
  Duke Math. Journal, Vol. 58, n 2, 371–395 (1989).

- P. Glowacki.
  The Melin calculus for general homogeneous groups.
  Ark. Mat., 45 (1): 31-48, 2007.

- V. Grushin.
  On a class of hypoelliptic operators,
  Math. USSR Sbornik 3, NO. 3 (1970), 458-475.

- D. Guibourg.
  Inégalités maximales pour l'opérateur de Schrödinger.
  CRAS 316 (1993), 249–252.

- W. Hebisch.
  On operators satisfying the Rockland condition.

Studia Math. 131 (1998), no. 1, 63–71.

- B. Helffer
  Hypoellipticité pour des opérateurs différentiels sur les groupes nilpotents.
  in Pseudodifferential operators with applications lectures given at the Centro internazionale matematico estivo (C.I.M.E.) held in Bressanone (Bolzano), Italy, June 16-24, 1977.

- B. Helffer and J. Nourrigat.
  Hypoellipticité pour des groupes nilpotents de rang de nilpotence 3.
  Comm. Partial Differential Equations 3 (1978), no. 8, 643–743.

- B. Helffer and J. Nourrigat.
  Caractérisation des opérateurs hypoelliptiques homogènes invariants à gauche sur un groupe de Lie nilpotent gradué.
  Comm. Partial Differen- tial Equations 4.8 (1979), pp. 899–958.

- B. Helffer and J. Nourrigat.

Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs.
C. R. Acad. Sci. Paris Sér. A-B 289.16 (1979), A775–778. .

B. Helffer and J. Nourrigat.
Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs.
Conference on linear partial and pseudodifferential operators (Torino, 1982). Rend. Sem. Mat. Univ. Politec. Torino 1983, Special Issue, 115–134 (1984).

B. Helffer and J. Nourrigat.
Hypoellipticité Maximale pour des Opérateurs Polynômes de Champs de Vecteurs.
Progress in Mathematics, Birkhäuser, Vol. 58 (1985).

L. Hörmander.
Hypoelliptic second order differential equations.
Acta Math. 119 (1967), pp. 147-171.

L. Hörmander.

Subelliptic operators.
Ann. Math. Studies 91, Princeton (1979).

L. Hörmander.
The Analysis of Linear Partial Differential Operators.
Volume III and Vol IV, Springer, (1985).

L. Hörmander, A. Melin.
Free systems of vector fields.
Ark. Mat. 16 (1978), no. 1, 83–88.

A.A. Kirillov.
Unitary representations of nilpotent Lie groups,
Russian Mathematical Surveys,17(1962),53-100.

J. Kohn.
Lectures on degenerate elliptic problems.
Pseudodifferential operators with applications, C.I.M.E., Bressanone 1977, p. 89-151 (1978).

A. Melin.

Parametrix constructions for right invariant differential operators on nilpotent groups.
Ann. Glob. Analysis and Geometry Vol. 1, No. 1 (1983), 79-130

O. Mohsen.
Work in progress.

J. Nourrigat.
Subelliptic estimates for systems of pseudo-differential operators.
Course in Recife (1982). University of Recife.

J. Nourrigat.
Réduction microlocale des systèmes d'opérateurs pseudodifférentiels.
Ann. Inst. Fourier 26 (3), 83–108 (1986).

J. Nourrigat.
Inégalités L^2 et représentations de groupes nilpotents.
Journal of Functional Analysis 74. 300–327 October 1987.

J. Nourrigat.
L^2 inequalities and representations of nilpotent groups.
CIMPA School of Harmonic Analysis. Wuhan (China), April-May 1991.

J. Nourrigat,
Systèmes sous-elliptiques. I
Comm in PDE 15(3), 341–405 (1990).

J. Nourrigat,
Systèmes sous-elliptiques. II.
Invent. Math. 104 (1991), no. 2, 377-400.

C. Rockland.
Hypoellipticity on the Heisenberg group: representation-theoretic criteria.
Preprint (1976) and Trans. Amer. Math. Soc., 240:1. (1978).

C. Rockland.
Intrinsic Nilpotent Approximation

Acta Applicandae Mathematicae 8 (1987), 213-270.

L.P. Rothschild.
A criterion for hypoellipticity of operators constructed from vector fields.
Communications in Partial Differential Equations, 4(6):645–699, 1979.

L.P. Rothschild and E. Stein.
Hypoelliptic operators and nilpotent groups.
Acta Mathematica 137 (1976), pp. 248-315.