The memoir "Hofstadter butterfly revisited" of Helffer-Kerdelhué-Sjöstrand revisited.

Lecture at the University of Nanjing

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Abstract

The aim of these notes[1] is to give a presentation of extracts (with some corrections or modifications) of Helffer-Kerdelhué-Sjöstrand [18] translated from the french into english devoted to a better understanding of the structure of the Hofstadter butterfly for any $\lambda > 0$. This was at the time based on a conjecture which is now proven [3] for $\lambda \in (0,1)$ due to its equivalence with the Dry Ten Martini conjecture.

1 Introduction

The aim of these notes is to give extracts (with some corrections or modifications) of the Memoir of Helffer-Kerdelhué-Sjöstrand [18] which appears in french in 1990. The goal was to get a better understanding of the structure of the Hofstadter butterfly for any $\lambda > 0$ and to justify formal arguments proposed by the Physicists (starting from D.G. Hofstadter) or semi-rigourous arguments proposed by Wilkinson (for the link with Chern classes). At the time the authors learn a lot from discussions with J. Bellissard.
The four first parts of the memoir were based on a conjecture which is now proven [3] for $\lambda \in (0,1)$ due to its equivalence with the Dry Ten Martini conjecture. Of course since 1990 most of the conjectures have been proved [2] and references therein but may be the content of our memoir remains interesting, if one wants to understand more deeply the topological structure of the "Wings" of the butterfly. In these notes, we have added a few references to more recent contributions transmitted by J. You and Q. Zhou and some comments coming from discussions with them in Nanjing (September 2016). We would like to thank Professor J. You for his kind invitation and would not to forget our collaborators for the Memoir.
Remark *Note that in the original Memoir, there was a confusion done by the commercial printer which has printed 8 pages of the Memoir in reverse order. One should read starting of p. 29 in the following order: 29, 37, 36, 35, 34, 33, 32, 31, 30, 38. The arXiv system NUMDAM permits to have access to a corrected version of the mémoire, where these pages have been reordered.*

[1] This is an extended version of the last lecture of a course given at the University of Nanjing in September 2016

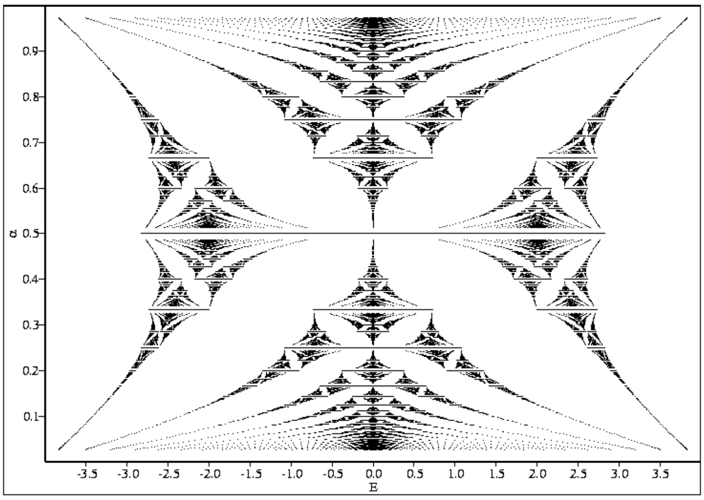


Figure 1: The Hofstadter butterfly

2 Reminder on the density of states

We consider the operator defined on $\ell^2(\mathbb{Z})$ by

$$\ell^2(\mathbb{Z}) \ni u \mapsto \left(P^\lambda_{\alpha,\theta} u\right)(n) := \frac{\lambda}{1+\lambda}\left((\Delta^{dis} u)(n) + \cos 2\pi(\alpha n + \theta) u(n)\right) , \tag{2.1}$$

On $\widetilde{Q} = [0,1] \times [0,1]$ we introduce the set

$$\Sigma^\lambda = \cup_\alpha (\alpha, \Sigma^\lambda_\alpha) , \tag{2.2}$$

where

$$\Sigma^\lambda_\alpha = \cup_\theta \Sigma^\lambda_{\alpha,\theta} , \tag{2.3}$$

with $\Sigma^\lambda_{\alpha,\theta} = \sigma(P^\lambda_{\alpha,\theta})$ denoting the spectrum of $P_\alpha(\lambda,\theta)$.
The integrated density of states (IDS) was defined a long time ago by Physicists (see [13] and references therein). If χ_ℓ is the characteristic function of $\{-\ell, -\ell-1, \dots, , \ell-1, \ell\}$ seen as a multiplicative operator on $\ell^2(\mathbb{Z})$, it can be defined as the limit

$$k^\lambda_{\alpha,\theta}(E) = \lim_{\ell \to +\infty} \frac{1}{2\ell+1} \mathrm{Tr}\left(\chi_\ell P_{-\infty,E)}(P^\lambda_{\alpha,\theta})\right) \tag{2.4}$$

where P_Ω denotes the spectral projector of $P^\lambda_{\alpha,\theta}$ on the interval Ω.
The IDS has the following properties:

$$\left\{ \begin{array}{l} E \mapsto k^\lambda_{\alpha,\theta}(E) \text{ is independent of } \theta \text{ in } \mathbb{R} \setminus \Sigma^\lambda_\alpha . \\ \text{ It is independent of } \theta \text{ if } \alpha \text{ is irrational.} \end{array} \right. \tag{2.5}$$

$$E \mapsto k^\lambda_{\alpha,\theta}(E) \text{ is continuous and constant in each gap of the spectrum.} \tag{2.6}$$

According to (2.4) and (2.5), we delete the reference to θ when α is irrational or when $E \in \complement\Sigma^\lambda_\alpha$.

$$\begin{array}{c} \text{For } E \in \complement\Sigma^\lambda_\alpha \,, \text{ there exist two integers } m \text{ and } n \text{ in } \mathbb{Z} \text{ such that} \\ k^\lambda_\alpha(E) = m\alpha + n\,. \end{array} \tag{2.7}$$

Moreover, in a connected component W^λ of $\widetilde{Q} \setminus \Sigma^\lambda$, we can find a pair (m,n) such that (2.7) is satisfied for all $(\alpha, E) \in W^\lambda$.

$$\text{For } E \in \complement\Sigma^\lambda_\alpha \text{ and } \alpha = \frac{p}{q}\,,\ k^\lambda_\alpha(E) \text{ is the number of bands on the left of } E \text{ divided by } q\,. \tag{2.8}$$

This results indeed (by Floquet theory) of

$$k^\lambda_\alpha(E) = \frac{1}{q}\left(\sum_n \int_{\lambda_n(\theta)\leq E} d\theta\right)\,, \tag{2.9}$$

where $\lambda_n(\theta)$ is the n-th Floquet eigenvalue.
Let $\alpha \in (0,1)$, I an interval in $(0,1)$ and $I \ni \lambda \mapsto E(\lambda)$ a continuous path such that $(\alpha, E(\lambda)) \in \widetilde{Q} \setminus \Sigma^\lambda$, then

$$k^\lambda_\alpha(E(\lambda)) \text{ does not depend on } \lambda\,. \tag{2.10}$$

$$\text{If } E \in \Sigma^\lambda_{\alpha,\theta}\,,\ \text{ then } k^\lambda_{\alpha,\theta}(E+\epsilon) - k^\lambda_{\alpha,\theta}(E-\epsilon) > 0\,,\ \forall \epsilon > 0\,. \tag{2.11}$$

For α irrational

$$\int_0^1 k^\lambda_{\alpha,\theta}(E)d\theta = k^\lambda_{\alpha,\theta}(E)\,. \tag{2.12}$$

We will some time take the notation

$$k(\alpha, E) = k_\alpha(E)\,.$$

Finally, we can complete (2.10) by

$$\text{If } E \in \Sigma^\lambda_{\alpha}\,,\ \text{ then } k^\lambda_{\alpha}(E+\epsilon) - k^\lambda_{\alpha}(E-\epsilon) > 0\,,\ \forall \epsilon > 0\,. \tag{2.13}$$

3 A perturbative theorem near $\lambda = 0$.

To follow [18] we consider instead

$$\ell^2(\mathbb{Z}) \ni u \mapsto \left(H^\lambda_{\alpha,\theta} u\right)(n) := \lambda(\Delta^{dis}u)(n) + \cos 2\pi(\alpha n + \theta)u(n)\,, \tag{3.1}$$

where Δ^{dis} is the discrete Laplacian.
For $\lambda = 0$ the spectrum of $H^\lambda_{\alpha,\theta}$ is of course the closure of the set of eigenvalues $(\cos 2\pi(\alpha n + \theta))$ with $n \in \mathbb{Z}$. For $\lambda \neq 0$, we would like to show the existence of gaps in the spectrum.
Given some positive integer ℓ, we assume that the following condition

$$\alpha \in NR(\ell) := (0,1) \setminus \cup_{j=1}^{2\ell} \mathbb{N}\frac{1}{j}\,. \tag{3.2}$$

The first observation is that under this condition, if $n \in \mathbb{Z}$, $\theta \in \mathbb{R}$ satisfy $\cos 2\pi(\alpha n + \theta) = \cos 2\pi(\alpha(n+\ell) + \theta)$, then there exists $k \in \mathbb{Z}$ such that

$$2\pi\left(\alpha(n+\ell) + \theta\right) = 2\pi k - 2\pi(\alpha n + \theta)\,. \tag{3.3}$$

This implies

$$\alpha n + \theta = \frac{k}{2} - \frac{\alpha\ell}{2}\,, \tag{3.4}$$

hence

$$\cos 2\pi(\alpha n + \theta) = (-1)^k \cos \pi\alpha\ell\,. \tag{3.5}$$

We choose k even and now introduce

$$E_{\alpha,\ell} = \cos(\pi\alpha\ell) \tag{3.6}$$

and observe that under condition (3.2) we have

$$E_{\alpha,\ell} \notin \{0,1,-1\}\,, \tag{3.7}$$

$$\text{for } \ell' \in \{1,2,\dots,\ell-1\}\,,\ |E_{\alpha,\ell'}| \neq |E_{\alpha,\ell}|. \tag{3.8}$$

and

$$\cos 2\pi(\alpha n+\theta) \neq \cos 2\pi(\alpha(n-\ell)+\theta)\,. \tag{3.9}$$

All the results below will hold for fixed ℓ and α in a compact subset $\mathcal{A}$ of $NR(\ell)$ uniformly for λ small enough and $\theta \in \mathbb{R}$. . The aim is to show the existence of a gap in the spectrum of $\hat{H}^\lambda$ tending to $E_{\alpha,\ell}$ as $\lambda \to 0$.
For $\epsilon > 0$, we introduce

$$A_{\epsilon,\theta,\alpha} = \{n \in \mathbb{Z}\,;\, \alpha n + \theta \pm \frac{\alpha\ell}{2} \in \mathbb{Z} + [-\epsilon,+\epsilon] \text{ for one sign }\} \tag{3.10}$$

and we observe that $A_{\epsilon,\theta,\alpha}$ is a union of pairs $(n_j, n_j+\ell)$ with $n_j \in \mathbb{Z}$.

Lemma 3.1. *There exists $\epsilon(\ell,\mathcal{A}) > 0$ and $C = C(\ell,\mathcal{A}) > 0$ such that, for $\epsilon \in (0,\epsilon(\ell,\mathcal{A}))$, either $A_{\epsilon,\theta,\alpha} = \emptyset$ or $A_{\epsilon,\theta,\alpha} = \cup_{j\in\mathbb{Z}}(n_j, n_j+\ell)$ where $n_{j+1} \geq n_j + 2\ell + 1$.*
Moreover,

$$|\cos(2\pi(\alpha(n+\theta)) - E_{\alpha,\ell}| \left\{ \begin{array}{ll} \geq \frac{\epsilon}{C}\,, & \text{if } n \in \mathbb{Z}\setminus A_{\epsilon,\theta,\alpha} \\ \leq \epsilon C\,, & \text{if } n \in A_{\epsilon,\theta,\alpha} \end{array}\right. \tag{3.11}$$

We now introduce what is called a Grushin problem (which is a variant of a Schur complements method). For $\epsilon > 0$ small enough, we introduce $\mathcal{P}^\lambda(z)$ in $\mathcal{L}(\ell^2(\mathbb{Z}) \times \ell^2(A_\epsilon))$ by

$$\mathcal{P}^\lambda(z) = \begin{pmatrix} \hat{H}^\lambda - z & i \\ \pi & 0 \end{pmatrix}, \tag{3.12}$$

where $\pi : \ell^2(\mathbb{Z}) \mapsto \ell^2(A_\epsilon)$ is the restriction operator and $i = \pi^* : \ell^2(A_\epsilon) \mapsto \ell^2(\mathbb{Z})$ is the natural injection given by

$$(iu)(n) = \left\{ \begin{array}{ll} u(n) & \text{if } n \in A_\epsilon \\ 0 & \text{if } n \notin A_\epsilon \end{array}\right.$$

For $\lambda = 0$, one can see $\mathcal{P}^0(z)$ as a direct sum parametrized by $\mathbb{Z}$ of scalars (when $n \in \mathbb{Z}\setminus A_\epsilon$) or 2×2 matrices corresponding to the pairs $n, n+\ell \in A_\epsilon$. In the first case the scalar is $\cos 2\pi(\alpha n+\theta) - z$ and in the second case the matrix is

$$\begin{pmatrix} \cos 2\pi(\alpha n+\theta) - z & 1 \\ 1 & \cos 2\pi(\alpha(n+\ell)+\theta) - z \end{pmatrix}.$$

If $|z - E_{\alpha,\ell}| \leq \frac{1}{2C}\epsilon$ (with the C as in the previous lemma), $\mathcal{P}^0$ is invertible and its inverse reads

$$\begin{aligned} \mathcal{P}^0(z)^{-1} &:= \begin{pmatrix} E_0(z) & E_0^+ \\ E_0^- & E_0^{-+} \end{pmatrix} \\ &= \begin{pmatrix} (1-\pi)\left((H^0-z)_{\mathbb{Z}\setminus A_\epsilon}\right)^{-1}(1-\pi) & i \\ \pi & (z-H^0)_{/A_\epsilon} \end{pmatrix}. \end{aligned} \tag{3.13}$$

We immediately get for a new constant $\hat{C} = \hat{C}(\ell, \mathcal{A}) \geq C$ that

$$||\mathcal{P}^0(z)^{-1}|| \leq \hat{C}\epsilon^{-1} . \tag{3.14}$$

With a new constant $\check{C} \geq \hat{C}$, we immediately deduce that $\left(\mathcal{P}^\lambda - z\right)$ is invertible for $|\lambda| \leq \frac{\epsilon}{\check{C}}$ and $|z - E_{\alpha,\ell}| \leq \frac{1}{\check{C}}\epsilon$, with in addition the control

$$||\mathcal{P}^\lambda(z)^{-1}|| \leq \check{C}\epsilon^{-1} . \tag{3.15}$$

This inverse is indeed given by the Neumann series

$$\mathcal{E}_\lambda(z) = \mathcal{P}^\lambda(z)^{-1} = \sum_{j\geq 0} (-\lambda)^j \mathcal{P}^0(z)^{-1} \left(\begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} \mathcal{P}^0(z)^{-1} \right)^j . \tag{3.16}$$

Writing $\mathcal{E}_\lambda(z)$ in the form

$$\mathcal{E}_\lambda(z) = \begin{pmatrix} E_\lambda(z) & E_\lambda^+(z) \\ E_\lambda^-(z) & E_\lambda^{-+}(z) \end{pmatrix}$$

it is important to note (this is the interest of the Grushin method) that for z and λ satisfying the above assumptions

$$z \in \sigma(\hat{H}^\lambda) \text{ iff } 0 \in \sigma(E_\lambda^{-+}(z)) . \tag{3.17}$$

We have for $E_\lambda^{-+}(z)$ the following expression

$$\begin{aligned} &E_\lambda^{-+}(z) \\ &\quad = (z - \hat{H}^0)_{/A_\epsilon} \\ &\qquad + \textstyle\sum_{j\geq 1} (-\lambda)^j \pi \left(\Delta(1-\pi) \left((z - \hat{H}^0)_{/\mathbb{Z}\setminus A_\epsilon} \right)^{-1} (1-\pi) \right)^{j-1} \Delta i . \end{aligned} \tag{3.18}$$

If $(n,m) \in (A_\epsilon)^2$ the element of the matrix of $E_\lambda^{-+}(z)$ is denoted by $E_\lambda^{-+}(z)(n,m)$. We observe from the above expression that

$$|\partial_z^k E_\lambda^{-+}(z)(n,m)| \leq C_{\epsilon,k} \lambda^{|n-m|} . \tag{3.19}$$

Let $n, n+\ell \in A_\epsilon$ and consider the 2×2 block matrix

$$\begin{pmatrix} E_\lambda^{-+}(z)(n,n) & E_\lambda^{-+}(z)(n,n+\ell) \\ E_\lambda^{-+}(z)(n+\ell,n) & E_\lambda^{-+}(z)(n+\ell,n+\ell) \end{pmatrix} .$$

It has the form

$$\begin{pmatrix} \hat{\lambda}_{n,\lambda,z} & \mu_{n,\lambda,z} \\ \mu_{n,\lambda,z} & \hat{\lambda}_{n+\ell,\lambda,z} \end{pmatrix} + \mathcal{O}_\epsilon(\lambda^{\ell+1}) , \tag{3.20}$$

where the remainder corresponds to the contribution of

$$+ \sum_{j\geq \ell+1} (-\lambda)^j \pi \left(\Delta(1-\pi) \left((z - \hat{H}^0)_{/\mathbb{Z}\setminus A_\epsilon} \right)^{-1} (1-\pi) \right)^{j-1} \Delta i$$

in the above formula.
We first look at the two terms on the diagonal that we now write $\hat{\lambda}_{n,\lambda,z}(\theta)$ and $\hat{\lambda}_{n+\ell,\lambda,z}(\theta)$ to recall now the dependence on θ which will now play a role.
These elements are still well defined when we replace θ by a variable $\tilde{\theta}$ varying in the

largest integral $\mathcal{J}_n$ containing θ with the property that $n, n+\ell \in A_{\epsilon_0,\tilde{\theta}}$, where ϵ_0 is fixed, small but satisfying $\epsilon_0 >> \epsilon$. More explicitly, if $k \in \mathbb{Z}$ is such that

$$n\alpha + \theta + \frac{\alpha\ell}{2} \in k \in [-\epsilon, +\epsilon]\,,$$

then $\hat{\lambda}_{n,\lambda,z}(\tilde{\theta})$ and $\hat{\lambda}_{n+\ell,\lambda,z}(\tilde{\theta})$ are well defined for $|\tilde{\theta} - \theta_0| \leq \epsilon_0$, where θ_0 is defined by

$$n\alpha + \theta_0 + \frac{\alpha\ell}{2} = k\,.$$

Let $\delta(\tilde{\theta}) = \tilde{\theta} - \theta_0$. Then $|\delta(\theta)| \leq \epsilon$. Note that θ_0 depends on n but all the estimates below will be uniform with respect to n. For $\tilde{\theta} = \theta_0$, n and $n+\ell$ are in a symmetric situation for the map $\tilde{n} \mapsto \cos 2\pi(\alpha\tilde{n} + \tilde{\theta})$:

$$\cos 2\pi(\alpha(n+\nu) + \theta_0) = \cos 2\pi\alpha(\nu - \frac{\ell}{2})\,,\ \forall \nu \in \mathbb{R}\,, \tag{3.21}$$

which implies

$$\cos 2\pi(\alpha n + \theta_0) = \cos 2\pi(\alpha(n+\ell) + \theta_0)\,. \tag{3.22}$$

By symmetry arguments we get

$$\hat{\lambda}_{n,\lambda,z}(\theta_0) = \hat{\lambda}_{n+\ell,\lambda,z}(\theta_0) := z - E_{\alpha,\ell,\lambda,z}\,, \tag{3.23}$$

where

$$\begin{cases} E_{\alpha,\ell,\lambda,z} & = E_{\alpha,\ell} + \mathcal{O}(\lambda^2)\,, \\ \partial_z^p E_{\alpha,\ell,\lambda,z} & = \mathcal{O}_p(\lambda^2)\,, \end{cases} \tag{3.24}$$

and $E_{\alpha,\ell,\lambda,z}$ is independent of n.
Let us also observe that the dependence on n in the expression of $\hat{\lambda}_{n,\lambda,z}(\tilde{\theta})$ and $\hat{\lambda}_{n+\ell,\lambda,z}(\tilde{\theta})$ appears only through $\tilde{\theta} - \theta_0$.
From (3.20), we also get the information

$$\partial_{\tilde{\theta}}\hat{\lambda}_{n,\theta,z}(\theta_0) = q + \mathcal{O}(\lambda^2)\,,\ \partial_{\tilde{\theta}}\hat{\lambda}_{n+\ell,\theta,z}(\theta_0) = -q + \mathcal{O}(\lambda^2)\,, \tag{3.25}$$

where

$$q = 2\pi\sin(\pi\alpha\ell) \neq 0\,. \tag{3.26}$$

With $\delta = \theta - \theta_0$, we obtain by a Taylor expansion

$$\begin{cases} \hat{\lambda}_{n,\lambda,z}(\theta) & = z - E_{\alpha,\ell,\lambda,z} + q\delta + \mathcal{O}(\lambda^2|\delta| + |\delta|^2)\,, \\ \hat{\lambda}_{n+\ell,\lambda,z}(\theta) & = z - E_{\alpha,\ell,\lambda,z} - q\delta + \mathcal{O}(\lambda^2|\delta| + |\delta|^2)\,. \end{cases} \tag{3.27}$$

We now introduce

$$w(z,\lambda) = z - E_{\alpha,\ell,\lambda,z}\,, \tag{3.28}$$

and we observe that

$$|w(z,\lambda)| \sim |z - z(\lambda)|\,, \tag{3.29}$$

where $z(\lambda) = E_{\alpha,\ell} + \mathcal{O}(\lambda^2)$ is independent of n.
For $\mu_{n,\lambda}$, we have the explicit formula

$$\mu_{n,\lambda} = (-\lambda/2)^\ell / \left(\prod_{j=1}^{\ell-1}((\cos(2\pi((n+j)\alpha+\theta)) - z))\right)\,, \tag{3.30}$$

but we will only use

$$|\mu_{n,\ell}| \sim \lambda^\ell\,. \tag{3.31}$$

The eigenvalues of the first term of (3.20) are given by

$$\hat{\lambda}_\pm = \left(\hat{\lambda}_n + \hat{\lambda}_{n+\ell}\right)/2 \pm \sqrt{\mu^2 + \left((\hat{\lambda}_n - \hat{\lambda}_{n+\ell})/2\right)^2}\,. \tag{3.32}$$

Hence we get

$$(\hat{\lambda}_+ + \hat{\lambda}_-)/2 = w(z,\lambda) + \mathcal{O}(\lambda^2|\delta| + |\delta|^2)\,, \tag{3.33}$$

and

$$\begin{aligned}(\hat{\lambda}_+ - \hat{\lambda}_-)/2 \quad &= \sqrt{\mu^2 + \left((\hat{\lambda}_n - \hat{\lambda}_{n+\ell})/2\right)^2} \\ &\sim \lambda^\ell + |\delta| + \mathcal{O}(\lambda^2|\delta| + |\delta|^2) \\ &\sim \lambda^\ell + |\delta| \text{ if } \epsilon \text{ and } \lambda \text{ are small enough.}\end{aligned} \tag{3.34}$$

If we impose the condition

$$|z - z(\lambda)| \leq \lambda^\ell / D \tag{3.35}$$

with $D > 0$ large enough, we get

$$|w(z,\lambda)| \leq \lambda^\ell / \tilde{D}\,,$$

with $\tilde{D}$ as large as we want (through the choice of D).
Comparing (3.33) and (3.34), we see that, we can choose $\tilde{D}$ such that

$$\hat{\lambda}_+ - \hat{\lambda}_- >> (\hat{\lambda}_+ + \hat{\lambda}_-)/2\,,$$

hence we get

$$\inf(|\hat{\lambda}_+|, |\hat{\lambda}_-|) \geq \frac{1}{C}(\lambda^\ell + |\delta|)\,,\ \forall z \in (z(\lambda) - \frac{1}{D}\lambda^\ell, z(\lambda) + \frac{1}{D}\lambda^\ell)\,.$$

The first block in (3.20) therefore admits an inverse of norm $\mathcal{O}(1))\frac{1}{\lambda^\ell + |\delta|}$. The interval $(z(\lambda) - \frac{1}{D}\lambda^\ell, z(\lambda) + \frac{1}{D}\lambda^\ell)$ is independent of n and the result is obtained for fixed ϵ small enough. The perturbation term in (3.20) is controlled in $\mathcal{L}(\ell^2(A_\epsilon))$ as $\mathcal{O}(\lambda^{\ell+1})$. Hence we obtain the invertibility of $E_\lambda^{-+}(z)$ for $z \in (z(\lambda) - \frac{1}{D}\lambda^\ell, z(\lambda) + \frac{1}{D}\lambda^\ell)$ and λ small enough depending only on ℓ and $\mathcal{A}$.
We have finally obtained

Theorem 3.2. *(Helffer-Kerdelhué-Sjöstrand) There exists $\lambda_0 = \lambda_0(\ell, \mathcal{A})$ and $C = C(\ell, \alpha)$, s. t. for all $\lambda \in (0, \lambda_0)$, there exists*

$$z_\ell(\lambda, \alpha) = E_{\alpha,\ell} + O(\lambda^2)$$

such that for all $\theta \in \mathbb{R}$, we have

$$(z_\ell(\lambda,\alpha) - \frac{\lambda^\ell}{C}, z_\ell(\lambda) + \frac{\lambda^\ell}{C}) \cap \sigma(H^\lambda_{\alpha,\theta}) = \emptyset\,, \tag{3.36}$$

and the same result holds near $-z_\ell(\lambda,\alpha)$.

Remark 3.3. *This gives a lower bound for the gap which appears to be optimal in the rational case due to some explicit computations given by P. Van Mouche in his analysis of the spectrum of Harper in the rational case [25].Moreover we observe that for α rational we have obtained all the gaps in the spectrum.*

It is also proven under the same conditions as in the theorem:

Proposition 3.4. *The value of the integrated density of states in the gap around $z_\ell(\lambda, \alpha)$ is given by*

$$\rho^\lambda((-\infty, z_\ell(\lambda,\alpha)]) = 1 - \mathrm{dist}(\ell\alpha, 2\mathbb{Z}) = |2\{\frac{\ell\alpha}{2}\} - 1|\,. \tag{3.37}$$

4 Some conjectural analysis of the wings

We denote by Σ_α^λ the spectrum of $\frac{1}{1+\lambda}\hat{H}^\lambda$. We then consider in $Q=[0,1]\times[-1,+1]$ the set

$$\Sigma^\lambda := \cup_\alpha(\alpha, \Sigma_\alpha^\lambda)\,.$$

It is well known that Σ^λ is closed and we call wing (in [HKS] this is called "fuseau") a connected component of the complementary $\complement\Sigma^\lambda$ of Σ^λ in Q. Hence by definition a wing is open.

4.1 Preliminary discussion

The aim is to discuss the structure of the wings. We then observe the following properties:

Property 4.1.
(P1) For any $\alpha_0 \in (0,1)$, the line $\alpha=\alpha_0$ cuts a wing $\mathcal{W}^\lambda$ on an (possibly empty) open interval $I(\alpha_0,\lambda)$.

Property 4.2.
(P2) For any wing $\mathcal{W}$, there exists $\alpha_\pm(\mathcal{W})$ such that

$$\mathcal{W}\subset(\alpha_-,\alpha_+)\times[-1,+1]\,,\,0\leq\alpha_-<\alpha_+\leq 1\,,\,\alpha_\pm\in\pi(\overline{\mathcal{W}})\,,$$

where π denotes the projection $\mathbb{R}^2\ni(\alpha,E)\mapsto\alpha$.

This is obtained immediately by connectedness.
For any interval $J\subset\mathbb{R}^+$, we introduce

Conjecture 4.3. $C3(J)$
For any $\lambda\in J$ and for any wing $\mathcal{W}^\lambda$, the points $\alpha_\pm$ are rational.

This conjecture was open in the 90's. It would be interesting to know if it can be proved by more recent results (see below Avila-You-Zhou [3]).

Property 4.4. *Continuity*
(P4) If, for $J=(0,\lambda_1)$, Conjecture $C3(J)$ holds, then, for any $\lambda\in J$, the boundary of the wing $\mathcal{W}^\lambda$ is continuous.

By continuity, we mean that if we write $I(\alpha,\lambda)=(f_-(\alpha,\lambda),f_+(\alpha,\lambda))$ the maps $(\alpha_-,\alpha_+)\ni\alpha\mapsto f_\pm(\alpha,\lambda)$ belong to $C^0([\alpha_-,\alpha_+])$ with $f_-(\alpha_\pm,\lambda)=f_+(\alpha_\pm,\lambda)$. The continuity property is attributed to Elliott [15] in [8]. The Hölder character results from an estimate given by [12]. Note that Elliott gets $C^{\frac{1}{3}}_{loc}(\alpha_-,\alpha_+)$.
The continuity at α_- and α_+ is less clear. In [18], the authors give two proofs. The first one is based on $C_3(\{\lambda\}$ and involves the semi-classical analysis of [20]. This proof is complete when $\lambda=1$ but seems to have a gap (in view of the published literature in semi-classical analysis) when $\lambda<1$.
A second proof is based on Assumption $C3((0,\lambda])$ and a deformation argument in λ (this is explained in Remark 3.5 in [18] which gives in addition that $f_-(\alpha_\pm,\lambda)=f_+(\alpha_\pm,\lambda)$ is at the end of a band).

Semi-classical approach

Let see however, what we get from this first approach. We freely use the result of [20]. We assume that $\lambda > 0$ is fixed and
$$\alpha_- = \frac{p}{q}\,.$$
The difficult point is to prove the continuity of $\alpha \mapsto f_\pm(\alpha,\lambda)$ at the ends of the interval $(\alpha_-(W^\lambda), \alpha_+(W^\lambda))$. Let us look at α_- and consider the case of a single band. Let
$$\ell_-(\lambda) = \liminf_{\alpha\to\alpha_-} f_-(\alpha,\lambda)\,,\ \ell_+(\lambda) = \limsup_{\alpha\to\alpha_-} f_+(\alpha,\lambda)\,.$$
It is clear that
$$[\ell_-(\lambda), \ell_+(\lambda)] \subset [a_\lambda, b_\lambda]\,, \tag{4.1}$$
where $[a_\lambda, b_\lambda]$ is a band of the spectrum. By Chambers' formula [11], we know that this is one connected component of
$$z \in (f^\lambda_{p,q})^{-1}([-1-\lambda^q, +1+\lambda^q])\,, \tag{4.2}$$
where $f^\lambda_{p,q}$ is a polynomial.

The semi-classical analysis leads to three zones in the band:
$$\begin{array}{ll} Z_1 & := B_\lambda \cap (f^\lambda_{p,q})^{-1}([-1-\lambda^q, -1+\lambda^q))\,,\\ Z_2 & := B_\lambda \cap (f^\lambda_{p,q})^{-1}([-1+\lambda^q, 1-\lambda^q])\,,\\ Z_3 & := B_\lambda \cap (f^\lambda_{p,q})^{-1}([1-\lambda^q, 1+\lambda^q))\,. \end{array} \tag{4.3}$$
We observe that for $\lambda = 1$ Z_2 is reduced to one point. The semi-classical analysis developped in [19, 20] works outside an arbitrarily small neighborhood of Z_2. Near these two regions the semi-classical analysis gives that the gaps tend necessarily to 0. One could probably do a semi-classical analysis in the interior of Z_2 but no reference[2] is available. From this analysis, we get that

- either $a_\lambda = \ell_-(\lambda) = \ell_+(\lambda)$
- either $[\ell_-(\lambda), \ell_+(\lambda)] \subset Z_2$
- or $b_\lambda = \ell_-(\lambda) = \ell_+(\lambda)\,.$

When $\lambda = 1$, observing that $\lim_{\alpha\mapsto\alpha_-}(f_-(\alpha,\lambda) - f_+(\alpha,\lambda)) = 0$, we get the continuity to $\ell_+(\lambda) = \ell_-(\lambda)$ as claimed in [18] with three possibilities.

We will see later how to exclude the case corresponding to Z_2 and this will permit to treat all the λ's and at the same time to show that the limit is either a_λ or b_λ.

Remark 4.5. *Another proof of (P4) is proposed in [18] (Remark 3.5) which avoids any semi-classical analysis.*

Property 4.6.
(P5) If, for some $J = (0, \lambda_1)$, Conjecture C3(J) holds, then for all $\alpha \in (0,1)$, $I(\alpha,\lambda)$ is either empty for all $\lambda \in (0,+\infty)$ or never closes for all $\lambda \in J$.

For α rational, this was the object of the results of Van Mouche [25] and Choi-Elliott-Yui [12]. We can label each interval between two bounds by some $r \in \{1, \dots q-1\}$ if q is odd or in $\{1, \dots, \frac{q}{2}-1, \frac{q}{2}+1, \dots, q-1\}$ if q is even.

[2]The vague explanation given in [18] is questionnable and seems only correct when $\lambda = 1$.

We have first to clarify the notion of deformation of a gap. The deformation for $\alpha \in \mathbb{Q}$ as λ varies is clearly defined by counting the number $r \in \{0, \cdots, q-1\}$ of bands on the left of the gap. We simply fix r in the deformation. This will permit us to define a notion of continuous variation of a wing W^λ for λ in an interval J. Let $\lambda_0 \in J$ and W^{λ_0} a non empty wing. Then we choose α_0 rational such that $I(\alpha_0, \lambda_0) \subset W^{\lambda_0}$ and non empty. Then we define the variation of the wing by considering for any $\lambda \in J$ the wing containing $I(\alpha_0, \lambda)$. The problem is to verify that there are necessarily uniqueness of the defomed wing, if we take another α_0. If we consider indeed α_0 and α_1 with this property, we could obtain for some $\lambda \in J$ the existence of two distinct wings W_0^λ and W_1^λ. But this will imply the existence of $\alpha \in (\alpha_0, \alpha_1)$ and $\tilde{\lambda} \in (\lambda_0, \lambda_1)$ (or (λ_1, λ_0) if $\lambda_1 < \lambda_0$) such that $I(\alpha, \lambda)$ becomes empty. But this is impossible for α rational and excluded by $C_3(J)$ for α irrational.
As a consequence, the interval $I(\alpha, \lambda)$ never closes and the same argument gives also the

Property 4.7.
(P6) If, for some $J = (0, \lambda_1)$, Conjecture C3(J) holds, then $\alpha_\pm(\mathcal{W}_\lambda)$ is independent of λ for $\lambda \in (0, \lambda_1)$.

4.2 Main statement

The discussion can be summarized and extended in the following statement:

Theorem 4.8. *If for some $J = (0, \lambda_1)$, Conjecture $C3(J)$ holds, then for any "continuous" family $(\mathcal{W}_\lambda)_{\lambda \in J}$, there exist two rationals $\alpha_\pm$ (with $0 \leq \alpha_- < \alpha_+ \leq 1$) such that*

$$\forall \lambda \in (0, \lambda_1), \pi(\overline{\mathcal{W}_\lambda}) = [\alpha_-, \alpha_+] \,,$$

and two functions $f_\pm$ defined on $(\alpha_-, \alpha_+) \times (0, \lambda_1)$ such that:

$$f_\pm(\alpha, \lambda) \in C^0([\alpha_-, \alpha_+] \times [0, \lambda_1)) \,, \tag{4.4}$$

$$|f_\pm(\alpha, \lambda) - f_\pm(\alpha, \lambda')| \leq 2|\lambda - \lambda'| \tag{4.5}$$

$$f_-(\alpha, \lambda) < E < f_+(\alpha, \lambda), \alpha \in (\alpha_-, \alpha_+) \text{ iff } (\alpha, E) \in \mathcal{W}_\lambda \,, \tag{4.6}$$

and

$$f_-(\alpha_\pm, \lambda) = f_+(\alpha_\pm, \lambda) \text{ belongs to the end of a band of } \Sigma^\lambda_{\alpha_\pm} \,. \tag{4.7}$$

Moreover, if the integrated density of states in $\mathcal{W}^\lambda$ is given by $k(\alpha, \mu) = m\alpha + n$, then

$$\alpha_\pm \in \cup_{j \leq 2|m|} \mathbb{N}\{\frac{1}{j}\} \cap (0, 1) \tag{4.8}$$

and

$$(\alpha_-, \alpha_+) \cap \left(\cup_{j \leq 2|m|} \mathbb{N}\{\frac{1}{j}\} \cap (0, 1) \right) = \emptyset \,. \tag{4.9}$$

Remark 4.9. *Conjecture $C3(J)$ implies the "dry" or "strong" form of the ten Martinis Conjecture $(DTM)(\lambda)$ for $\lambda \in J$ which was formulated by B. Simon in the form, for a given $\lambda \neq 0$,*

Conjecture 4.10 ($DTM(\lambda)$). *For all $\alpha \notin \mathbb{Q}$ and any integers m, n with $0 < m\alpha + n < 1$, there exists a gap for which the IDS in the gap satisfies*

$$k(\alpha, \cdot) = m\alpha + n \,.$$

Note that this conjecture is now proved for $\lambda \in (0, 1)$ by Avila-You-Zhou [3]. Hence Conjecture $C3(0, +\infty)$ appears to be a stronger form of the "dry ten Martinis conjecture" (and probably equivalent).

4.3 Proof of Theorem 4.8

The existence of $\alpha_\pm$, $f_\pm$ has been already proved in the previous subsection.

4.3.1 Proof of (4.11)

For fixed $\alpha \in (\alpha_-, \alpha_+)$ and $\lambda \in J$, we show by an immediate perturbation argument that if $|\lambda-\lambda'| < \frac{1}{4}|f_+(\alpha,\lambda')-f_-(\alpha,\lambda')|$ then (4.11) is satisfied. For obtaining (4.11) for general λ_1, λ_2, we construct a finite increasing sequence $\lambda^{(j)}$ in $[\lambda_1, \lambda_2]$ such that $\lambda^{(1)} = \lambda_1$ and $\lambda^{(N)} = \lambda_2$ and such that

$$\forall j\,,\ |\lambda^{(j+1)} - \lambda^{(j)}| < \frac{1}{4}|f_+(\alpha,\lambda^{(j)}) - f_-(\alpha,\lambda^{(j)})|\,.$$

We get then (4.11) by summing over j the inequalities

$$|f_\pm(\alpha,\lambda^{(j)}) - f_\pm(\alpha,\lambda^{(j+1)})| \leq 2\,|\lambda^{(j+1)} - \lambda^{(j)}|\,.$$

4.3.2 Proof of (4.4)–weak form

In $(\alpha_-, \alpha_+) \times J$ we have seen the partial continuity with respect to α in the discussion of (P4) and we conclude through (4.11) that $f_\pm(\alpha,\lambda) \in C^0((\alpha_-,\alpha_+)\times[0,\lambda_1))$, by observing that the Lipschitzian property is uniform with respect to α. What is missing at this stage is the control at α_- and α_+.

4.3.3 Proof of (4.8)

As $\lambda \to 0$, each of the wings W^λ is contracted on an arc supported by the curve $E = \pm\cos(\pi\alpha\ell)$. More precisely, there exist $\ell \in \mathbb{N}$ and $\epsilon = \pm 1$ such that, for any compact set in (α_-, α_+), there exists C such that

$$\sup_{x\in I(\alpha,\lambda)} |x - \epsilon\cos(\pi\alpha\ell)| \leq C\lambda\,,\ \forall \alpha \in K\,. \tag{4.10}$$

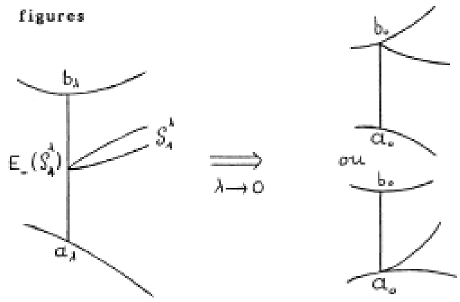
This property is an immediate consequence of Theorem 3.2 and of (4.11) (we have $f_+(\alpha,0) - f_-(\alpha,0) = 0$). Theorem 3.2 gives first the result for compact sets in $(\alpha_-,\alpha_+) \setminus NR(\ell)$ for some ℓ determined by the analysis of one rational number $\alpha^\#$ in (α_-,α_+) with odd denominator. ϵ is then determined by writing $\epsilon\cos\pi\alpha^\#\ell > 0$.
This is determined by the value of the density of states in this wing. For this $\alpha^\# = \frac{p}{q}$, the density of state $k^\lambda_{\alpha^\#}$ is constant $\frac{r}{q}$ for some $r \in \{1,\cdots,q-1\}$, in the wing and should choose $\ell > 0$ minimal in $(0,2q)$ such that $\frac{r}{q} = 1 - dist(\frac{p\ell}{q}, 2\mathbb{Z})$. The exceptional points β' in $(\alpha_-,\alpha_+)\setminus NR(\ell)$ are rational points, for which one can make the same construction. Hence[3] there exists ℓ' such that $\beta' \in NR(\ell')$ and such that W^λ is now contracted in $(\beta'-\eta,\beta'+\eta)\setminus NR(\ell')$ on $\epsilon'\cos(\alpha\pi\ell')$ (with η small enough). We get immediately $\epsilon = \epsilon'$ $\ell' = \ell$ and consequently $(\alpha_-,\alpha_+)\setminus NR(\ell) = \emptyset$. Theorem 3.2 actually proves the wings are open for $\lambda \neq 0$ except possibly in $\complement NR(\ell)$. This achieves consequently the proof of (4.8).

4.3.4 Proof of (4.7)

Suppose for a while that $(\alpha, E) \mapsto k^\lambda(\alpha, E)$ is continuous and that for some $\lambda \in J$ we have a wing W_λ such that $E_-(\lambda)$ (or $E_+(\lambda)$) belongs to the interior of a band $B(\lambda) = [a_\lambda, b_\lambda]$

[3] see also Remark 3.3

(keep in mind the discussion of (P4)). As $\lambda \to 0$, W^λ necessarily tends to an arc having for some ℓ having as end $\cos \pi\alpha_-\ell$ for some ℓ. This end corresponds to one of the ends of



the limiting band a_0 or b_0.
To fix the ideas, suppose that it is a_0. The IDS for $(\alpha, E) \in W^\lambda$ satisfies

$$k^\lambda(\alpha, E) = m\alpha + n\,,$$

for some integers m, n independent of λ.
We treat[4] the case when the band does not touch a neighboring band at a_0. Let W_0^λ the wing whose section by $\alpha = \alpha_-$ is an interval with end a_λ. This wing cannot be W^λ. The IDS, for $(\alpha, E) \in W_0^\lambda$, has the form

$$k^\lambda(\alpha, E) = m_0\alpha + n_0\,,$$

for some integers m_0, n_0 independent of λ.
We have seen previously that the expressions of the IDS remain true for the limiting wings as $\lambda \to 0$. At the point (α_-, a_0) where the two limiting wings cross, we get

$$m_0\alpha_- + n_0 = m\alpha_- + n\,. \tag{4.11}$$

On the other hand, if we admit the continuity of $k^\lambda(\alpha, E)$ as (α, E) tends to (α_-, E_-^λ) we get

$$m\alpha_- + n = k^\lambda(\alpha_-, E_-^\lambda) \neq k^\lambda(\alpha_-, a_\lambda) = m_0\alpha_- + n_0\,. \tag{4.12}$$

The inequality in the middle of (4.12) is due to our assumption that $E_-^\lambda \neq a_\lambda$ and to the strict monotonicity of $E \mapsto k_{\alpha_-}^\lambda(E)$ on the band $B(\lambda)$. Hence we have obtained the contradiction.
We have actually cheated because we admit the existence of E_-^λ which is not proven at this stage in the case Z_2 (see the discussion in (P4)) which remains to be treated. But we can redo the proof above by considering instead of E_-^λ any

$$\hat{E}^\lambda = \lim_{\alpha_n \to \alpha_-} E_n\,,$$

where $(\alpha_n, E_n) \in W_\lambda$.
We indeed observe that in the case Z_2 $a_\lambda < \hat{E}_\lambda < b_\lambda$. Hence this case should be excluded and we get simultaneously the existence of E^λ and the fact that E^λ should be a_λ or b_λ.

We did not know the continuity of $(\alpha, E) \mapsto k^\lambda(\alpha_\pm, E)$ when we wrote [18] and an alternative way was proposed there. But combining separate continuity [5] in E and E and the monotonicity of the IDS function gives the continuity.

Remark 4.11. *Another proof of* (4.7) *is proposed in [18] (p. 31) consisting in considering all the wings whose boundary touches the ends of the band.*

[4]the other case is treated in [18] p. 32 or p. 35 in the NUMDAM version.

4.3.5 Proof of (4.9)

If W_λ is a wing, we have already seen how by continuous deformation W^λ is contracted on the limiting wing W^0 $W^0 = \{(\alpha, E)\,,\, E = \pm\cos\pi\alpha\ell\,,\, \alpha\in(\alpha_-,\alpha_+)\}$ for some ℓ.
Using the homeomorphism ψ defined by

$$(\alpha, E) \mapsto \psi(\alpha, E) = (\alpha, 1 - \frac{1}{\pi}\arccos E)\,, \tag{4.13}$$

Σ^0 can be send on a closed set $\hat\Sigma^0$ in $[0,1]\times[0,1]$ and each limiting wing is send on a segment defined as

$$\hat W^0 = \{(x, y)\,,\, x = \alpha \in (\alpha_-,\alpha_+)\,,\, y = \pm x\ell + n\} \tag{4.14}$$

for some pair (ℓ, n).
It remains to show how the different values of α_- and α_+, which have been already localized in $\cup_{j\leq 2\ell}\mathbb{N}\{\frac{1}{j}\}$. The proof given in [18] is by recursion on ℓ and at the same time gives a practical way for the construction of these segments (see [16] for pictures) and the

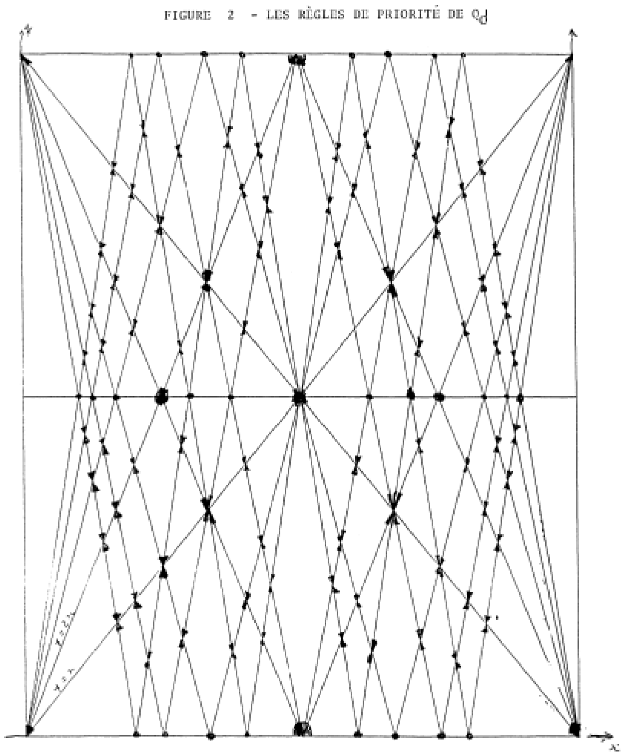


figure from [18] reproduced here.

Remark 4.12. *In the picture, for given $x = \alpha$, the IDS is determined by the value y on the segment $\hat W^0$.*

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