Introduction to spectral minimal partitions, Aharonov-Bohm's operators and Pleijel's theorem

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Talk in Bâton Rouge (March 2014)

Abstract

Given a bounded open set Ω in $\mathbb{R}^n$ (or in a Riemannian manifold) and a partition $\mathcal{D}$ of by k open sets D_j, we can consider the quantity $\Lambda(\mathcal{D}) := max_j \lambda(D_j)$ where $\lambda(D_j)$ is the ground state energy of the Dirichlet realization of the Laplacian in D_j. If we denote by $\mathfrak{L}_k(\Omega)$ the infimum over all the k-partitions of $\Lambda(\mathcal{D})$ a minimal k-partition is then a partition which realizes the infimum. Although the analysis is rather standard when $k = 2$ (we find the nodal domains of a second eigenfunction), the analysis of higher k's becomes non trivial and quite interesting.

In this talk, we consider the two-dimensional case and discuss the properties of minimal spectral partitions, illustrate the difficulties by considering a simple case like the rectangle and then give a "magnetic" characterization of these minimal partitions. We also discuss the large k problem in connexion with recent papers by Bourgain and Steinerberger on the Pleijel theorem and with I. Polterovich's conjecture. This work has started in collaboration with T. Hoffmann-Ostenhof (with a preliminary work with M. and T. Hoffmann-Ostenhof and M. Owen) and has been continued with him and other coauthors : V. Bonnaillie-Noël, S. Terracini, G. Vial or my PHD student: C. Lena.

Section 1: Introduction to the mathematical problem

We consider mainly two-dimensional Laplacians operators in bounded domains. We would like to analyze the relations between the nodal domains of the eigenfunctions of the Dirichlet Laplacians and the partitions by k open sets D_i which are minimal in the sense that the maximum over the D_i's of the ground state energy of the Dirichlet realization of the Laplacian in D_i is minimal.

Let Ω be a regular bounded domain. Let $H(\Omega)$ be the Laplacian $-\Delta$ on $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary condition ($u_{\partial\Omega} = 0$). In the case of a Riemannian manifold we will consider the Laplace Beltrami operator.

We could also consider other operators like the harmonic oscillator and $\Omega = \mathbb{R}^m$. This problem appears in the Bose-Einstein condensation theory. We will not continue in this direction in this talk.

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For any $u \in C_0^0(\overline{\Omega})$, we introduce the nodal set of u by:

$$N(u) = \overline{\{x \in \Omega \,\big|\, u(x) = 0\}} \tag{1}$$

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$$N(u) = \overline{\{x \in \Omega \mid u(x) = 0\}} \tag{1}$$

and call the components of $\Omega \setminus N(u)$ the nodal domains of u.

The $k = \mu(u)$ nodal domains define a partition of Ω.

We keep in mind the Courant nodal theorem and the Pleijel theorem. The main points in the proof of the Pleijel theorem are the Faber-Krahn inequality :

$$\lambda(\omega) \geq \frac{\pi j^2}{A(\omega)} . \tag{2}$$

(where $A(\omega)$ is the area of ω) and the Weyl law for the counting function.

Partitions

We first introduce the notion of partition.

Definition 1

Let $1 \leq k \in \mathbb{N}$. We call **partition** (or k-partition for indicating the cardinal of the partition) of Ω a family $\mathcal{D} = \{D_i\}_{i=1}^{k}$ of mutually disjoint sets such that

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We call it **open** if the D_i are open sets of Ω, **connected** if the D_i are connected.

We denote by $\mathfrak{O}_k$ the set of open connected partitions.

Spectral minimal partitions

We now introduce the notion of spectral minimal partition sequence.

Definition 2

For any integer $k \geq 1$, and for $\mathcal{D}$ in $\mathfrak{O}_k$, we introduce the "energy" of $\mathcal{D}$:

$$\Lambda(\mathcal{D}) = \max_i \lambda(D_i). \tag{4}$$

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Then we define

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D} \in \mathfrak{D}_k} \Lambda(\mathcal{D}). \tag{5}$$

and call $\mathcal{D} \in \mathfrak{D}_k$ minimal if $\mathfrak{L}_k = \Lambda(\mathcal{D})$.

Remark A

If $k = 2$, it is rather well known (see [HH1] or [CTV3]) that $\mathfrak{L}_2 = \lambda_2$ and that the associated minimal 2-partition is a nodal partition.

We discuss briefly the notion of regular and strong partition.

Definition 3: strong partition

A partition $\mathcal{D} = \{D_i\}_{i=1}^k$ of Ω in $\mathfrak{O}_k$ is called strong if

$$\mathrm{Int}\,(\overline{\cup_i D_i}) \setminus \partial\Omega = \Omega \text{ and } \mathrm{Int}\,(\overline{D_i}) \setminus \partial\Omega = D_i\ . \tag{6}$$

Attached to a strong partition, we associate a closed set in $\overline{\Omega}$:

Definition 4: Boundary set

$$N(\mathcal{D}) = \overline{\cup_i (\partial D_i \cap \Omega)}\ . \tag{7}$$

$N(\mathcal{D})$ plays the role of the nodal set (in the case of a nodal partition).

Regular partitions

We now introduce the set $\mathcal{R}(\Omega)$ of regular partitions (or nodal like) through the properties of its associated boundary set N, which should satisfy :

Definition 5: Regular boundary set

(i) Except finitely many distinct $x_i \in \Omega \cap N$ in the nbhd of which N is the union of $\nu_i = \nu(x_i)$ smooth curves ($\nu_i \geq 2$) with one end at x_i, N is locally diffeomorphic to a regular curve.
(ii) $\partial\Omega \cap N$ consists of a (possibly empty) finite set of points z_i. Moreover N is near z_i the union of ρ_i distinct smooth half-curves which hit z_i.
(iii) N has the equal angle meeting property

By equal angle meeting property, we mean that the half curves cross with equal angle at each critical point of N and also at the boundary together with the tangent to the boundary.

Partitions and bipartite property.

We say that D_i, D_j are neighbors or $D_i \sim D_j$, if $D_{i,j} := \mathrm{Int}\,(\overline{D_i \cup D_j}) \setminus \partial\Omega$ is connected.

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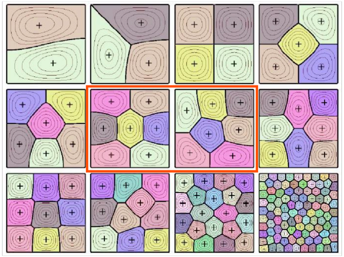
We recall that a collection of nodal domains of an eigenfunction is always bipartite.

Here are examples of regular partitions. These examples are supposed (unproved and not clearly stated) to correspond to minimal partitions of the square. Some of these pictures have been recently (2013) recognized numerically as minimal spectral partitions by V. Bonnaillie and C. Lena.

Multiple populations



“Minimization of the Renyi entropy production in the space-partitioning process”
Cybulski, Babin, and Holyst, Phys. Rev. E 71, 046130 (2005)

Section 2: Main results in the $2D$ case

It has been proved by Conti-Terracini-Verzini [CTV1, CTV2, CTV3] and Helffer–Hoffmann-Ostenhof–Terracini [HHOT1] that

Theorem 1

$\forall k \in \mathbb{N} \setminus \{0\}$, $\exists$ a minimal regular k-partition. Moreover any minimal k-partition has a regular representative.

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Theorem 1

$\forall k \in \mathbb{N} \setminus \{0\}$, $\exists$ a minimal regular k-partition. Moreover any minimal k-partition has a regular representative.

Other proofs of a somewhat weaker version of this statement have been given by Bucur-Buttazzo-Henrot [BBH], Caffarelli- F.H. Lin [CL].

Note that spectral minimal partitions are equi-partitions:

$$\lambda(D_i) = \mathfrak{L}_k(\Omega)\,.$$

Note also that for any pair of neighbours D_i, D_j

$$\lambda_2(D_{ij}) = \mathfrak{L}_k(\Omega)\,.$$

Hence minimal partitions satisfy the pair compatibility condition introduced in [HH1].

A natural question is whether a minimal partition of Ω is a nodal partition, i.e. the family of nodal domains of an eigenfunction of $H(\Omega)$.

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We have first the following converse theorem ([HH1], [HHOT1]):

Theorem 2

If the minimal partition is bipartite this is a nodal partition.

A natural question is now to determine how general this previous situation is.

Surprisingly this only occurs in the so called Courant-sharp situation. We say that:

Definition 6: Courant-sharp

A pair (u, λ_k) is Courant-sharp if
$u \in E(\lambda_k) \setminus \{0\}$ and $\mu(u) = k$.

An eigenvalue is
called Courant-sharp if there exists an associated Courant-sharp pair.

For any integer $k \geq 1$, we denote by $L_k(\Omega)$ the smallest eigenvalue whose eigenspace contains an eigenfunction of $H(\Omega)$ with k nodal domains. We set $L_k(\Omega) = \infty$, if there are no eigenfunctions with k nodal domains.

In general, one can show, that

$$\lambda_k(\Omega) \leq \mathfrak{L}_k(\Omega) \leq L_k(\Omega) \,. \tag{8}$$

The last result gives the full picture of the equality cases :

Theorem 3

Suppose $\Omega \subset \mathbb{R}^2$ is regular.

If $\mathfrak{L}_k = L_k$ or $\mathfrak{L}_k = \lambda_k$ then

$$\lambda_k = \mathfrak{L}_k = L_k \,.$$

In addition, one can find a Courant-sharp pair (u, λ_k).

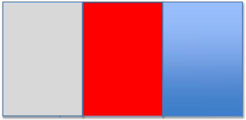
Section 3: Examples of k-minimal partitions for special domains

Numerics (V. Bonnaillie-Noel, G. Vial, C. Lena and coauthors) has been done for many natural examples like the disk, the circular sector and the torus. Except the sphere (3-partitions) and thin tori, the rigorous results correspond to Courant-sharp situations. We only discuss below the case of the rectangle.

The case of a rectangle

Using Theorem 3, it is now easier to analyze the situation for rectangles (at least in the irrational case), since we have just to look for Courant-sharp pairs.

In the long rectangle $]0, a[\times]0, 1[$ the eigenfunction $\sin(k\pi x/a)\sin \pi y$ is Courant-sharp for $a \geq \sqrt{(k^2-1)/3}$. See the nodal domain for $k = 3$.



The case of the square

We verify that $\mathfrak{L}_2 = \lambda_2$.
It is not to difficult to see that $\mathfrak{L}_3$ is strictly less than L_3. We observe indeed that there is no eigenfunction corresponding to $\lambda_2 = \lambda_3$ with three nodal domains (by Courant's Theorem).
Finally λ_4 is Courant-sharp, so $\mathfrak{L}_4 = \lambda_4$.
One can prove that these are the only Courant sharp cases ([Pl], [BeHe1]).

Section 4: The Aharonov-Bohm Operator

Let us recall some definitions and results about the Aharonov-Bohm Hamiltonian (for short **AB**X-Hamiltonian) with a singularity at X introduced in [BHHO, HHOO] and motivated by the work of Berger-Rubinstein.
We denote by $X = (x_0, y_0)$ the coordinates of the pole and consider the magnetic potential with flux at X

$$\Phi = \pi$$

$$\mathbf{A}^X(x,y) = (A_1^X(x,y), A_2^X(x,y)) = \frac{1}{2}\left(-\frac{y-y_0}{r^2}, \frac{x-x_0}{r^2}\right) . \quad (9)$$

We know that the magnetic field vanishes identically in $\dot{\Omega}_X$. The $\mathbf{AB}X$-Hamiltonian is defined by considering the Friedrichs extension starting from $C_0^\infty(\dot{\Omega}_X)$ and the associated differential operator is

$$-\Delta_{\mathbf{A}^X} := (D_x - A_1^X)^2 + (D_y - A_2^X)^2 \text{ with } D_x = -i\partial_x \text{ and } D_y = -i\partial_y. \tag{10}$$

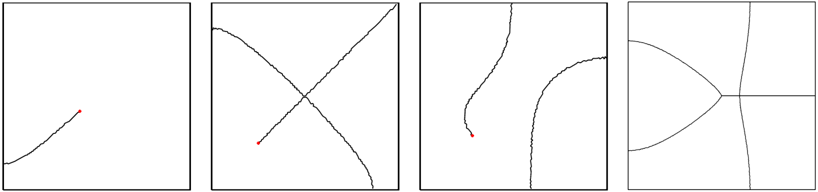
Let K_X be the antilinear operator

$$K_X = e^{i\theta_X}\,\Gamma\,,$$

with $(x - x_0) + i(y - y_0) = \sqrt{|x - x_0|^2 + |y - y_0|^2}\, e^{i\theta_X}$, and where Γ is the complex conjugation operator $\Gamma u = \bar{u}$.
A function u is called K_X-real, if $K_X u = u$.
The operator $-\Delta_{\mathbf{A}^X}$ is preserving the K_X-real functions and we can consider a basis of K_X-real eigenfunctions.
Hence we only analyze the restriction of the $\mathbf{AB}X$-Hamiltonian to the K_X-real space $L^2_{K_X}$ where

$$L^2_{K_X}(\dot{\Omega}_X) = \{u \in L^2(\dot{\Omega}_X)\,,\ K_X\, u = u\,\}\,.$$

It was shown that the nodal set of such a K_X real eigenfunction has the same structure as the nodal set of an eigenfunction of the Laplacian except that an odd number of half-lines meet at X.



For a "real" groundstate (one pole), one can prove [HHOO] that the nodal set consists of one line joining the pole and the boundary.

Extension to many poles

First we can extend our construction of an Aharonov-Bohm Hamiltonian in the case of a configuration with ℓ distinct points $X_1, \dots, X_\ell$ (putting a flux π at each of these points). We can just take as magnetic potential

$$\mathbf{A}^{\mathbf{X}} = \sum_{j=1}^{\ell} \mathbf{A}^{X_j} \,,$$

where $\mathbf{X} = (X_1, \dots, X_\ell)$.

We can also construct (see [HHOO]) the antilinear operator $K_{\mathbf{X}}$, where θ_X is replaced by a multivalued-function $\phi_{\mathbf{X}}$ such that $d\phi_X = 2\mathbf{A}^{\mathbf{X}}$ and $e^{i\phi_{\mathbf{X}}}$ is univalued and C^∞.
We can then consider the real subspace of the $K_{\mathbf{X}}$-real functions in $L^2_{K_{\mathbf{X}}}(\dot{\Omega}_{\mathbf{X}})$. It has been shown in [HHOO] (see in addition [1]) that the $K_{\mathbf{X}}$-real eigenfunctions have a regular nodal set (like the eigenfunctions of the Dirichlet Laplacian) with the exception that at each singular point X_j ($j = 1, \dots, \ell$) an odd number of half-lines should meet.
We denote by $L_k(\dot{\Omega}_{\mathbf{X}})$ the lowest eigenvalue (if any) such that there exists a $K_{\mathbf{X}}$-real eigenfunction with k nodal domains.

Remark

Another equivalent version is to consider double coverings. This point of view was the initial one in [HHOO]. See also the talk of L. Hillairet ([HiKo], who prefers to speak of flat surfaces with conical singularities. The K-real eigenfunctions become on the double covering real eigenfunctions on the double covering which are antisymmetric with respect to the Deck-map.

Section 5: A magnetic characterization of a minimal partition

We only discuss the following theorem which is the most interesting part of this magnetic characterization

Theorem 4

Let Ω be simply connected. Then if $\mathcal{D}$ is a k-minimal partition, then there exists $(X_1, \ldots, X_\ell)$ such that $\mathcal{D}$ is the nodal partition of some k-th $K_{\mathbf{X}}$-real eigenfunction of the Aharonov-Bohm Laplacian associated with $\dot{\Omega}_{\mathbf{X}}$.

When $k = 2$, there is no need to consider punctured Ω's. We have $\ell = 0$.

When $k = 3$, it is possible to show that $\ell \leq 2$.

In the case of the disk and the square, it is proven that the infimum cannot be for $\ell = 0$ and we conjecture that the infimum is for $\ell = 1$ and attained for the punctured domain at the center. We do not know about examples with $\ell = 2$.

Section 6: Asymptotics of the energy for minimal k-partitions for k large.

We recall results of [HHOT]. Faber-Krahn implies :

$$\mathfrak{L}_k(\Omega) \geq k\lambda(\mathrm{Disk}_1)A(\Omega)^{-1}\,.$$

Recent improvements have been obtained this year by Bourgain and Steinerberger separately.
Using the hexagonal tiling, it is easy to see that:

$$\limsup_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k} \leq \lambda(\mathrm{Hexa}_1)A(\Omega)^{-1}\,.$$

The hexagonal conjecture (Van den Berg, Caffarelli-Lin [CL], Bourdin- Bucur-Oudet [BBO], Bonnaillie-Helffer-Vial [BHV]) is

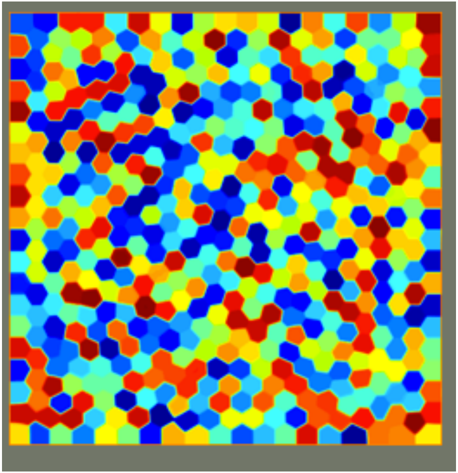
Hexagonal Conjecture

$$\lim_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k} = \lambda(\mathrm{Hexa}_1) A(\Omega)^{-1}\,.$$

Remarks

- There are various controls of the validity of the conjecture using numerics directly or indirectly on theoretical consequences of this conjecture [BHV]. For example, taking as Ω a connected union of k hexagons of area 1 and putting poles at all the vertices of these hexagones in Ω, the Aharonov-Bohm operator should have $\lambda(\mathrm{Hexa1})$ as k-th eigenvalue.
- There is a corresponding (proved by Hales [Ha]) conjecture for k- partitions of equal area and minimal length called the honeycomb conjecture.
- There is a stronger conjecture (see [CL]) corresponding to the averaged sum (in the definition of $\mathfrak{L}_k(\Omega)$) instead of the max. The lower bound by Faber-Krahn is OK. Steinerberger's proof mentioned below will give an improvement like for $\mathfrak{L}_k(\Omega)$.

Hexagonal conjecture.



This was computed for the torus by Bourdin-Bucur-Oudet [BBO] (for the sum).

Section 7: Pleijel's theorem revisited

Pleijel's theorem (see below) is a quantitative version of Courant's theorem.

Generally, the known proofs are going through the non made explicite proposition (deduced from Weyl's formula)

Proposition P1

$$\limsup_{n\to+\infty} \frac{N(\phi_n)}{n} \leq \frac{4\pi}{A(\Omega)\liminf_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k}}\,, \tag{11}$$

and then through a lower bound for $A(\Omega)\liminf_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k}$. using Faber-Krahn's inequality as mentioned in the previous section.

Observing that in Pleijel's proof we only meet Nodal partitions we can hope an improvement in writing

Proposition P2

$$\limsup_{n\to+\infty} \frac{N(\phi_n)}{n} \leq \frac{4\pi}{A(\Omega)\liminf_{k\to+\infty}\frac{L_k(\Omega)}{k}}\,, \tag{12}$$

where we recall that $L_k(\Omega)$ is the smallest eigenvalue (if any) such that in the corresponding eigenspace, one can find an eigenfunction with k nodal domains.
This does not seem to lead to universal results.

May be it is more enlighting to write Equation (11) in the form

$$\limsup_{n\to+\infty} \frac{N(\phi_n)}{n} \leq \frac{\lim_{k\to+\infty} \frac{\lambda_k(\Omega)}{k}}{\liminf_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k}}, \tag{13}$$

Classical Pleijel's Theorem is the immediate consequence of Proposition P1 and of Faber-Krahn's inequality

$$A(\Omega) \liminf_{k \to +\infty} \frac{\mathfrak{L}_k(\Omega)}{k} \geq \lambda(Disk_1)\,. \tag{14}$$

This leads to

Theorem (Pleijel)

$$A(\Omega) \limsup_{n \to +\infty} \frac{N(\phi_n)}{n} \leq \nu_{Pl}\,, \tag{15}$$

with $\nu_{Pl} = \frac{4\pi}{\lambda(Disk_1)} \sim 0.691\,.$

Note that the same result is true in the Neumann case (Polterovich [3]) under some analyticity assumption on the boundary (the case of the square was treated by Pleijel.

It is rather easy to prove that:

$$A(\Omega)\liminf_{k\to+\infty}\frac{\mathfrak{L}_k(\Omega)}{k}\leq A(\Omega)\limsup_{k\to+\infty}\frac{\mathfrak{L}_k(\Omega)}{k}\leq\lambda(Hexa_1)\,. \tag{16}$$

Having in mind the Hexagonal Conjecture:

$$A(\Omega)\liminf_{k\to+\infty}\frac{\mathfrak{L}_k(\Omega)}{k}=A(\Omega)\limsup_{k\to+\infty}\frac{\mathfrak{L}_k(\Omega)}{k}=\lambda(Hexa_1) \tag{17}$$

this would lead to the conjecture that in Pleijel's estimate we have actually:

Hexagonal Conjecture for Pleijel

$$A(\Omega) \limsup_{n \to +\infty} \frac{N(\phi_n)}{n} \leq \nu_{Hex} \,, \tag{18}$$

with $\nu_{Hex} = \frac{4\pi}{\lambda(Hexa_1)} \sim 0.677$.

We note indeed that

$$\frac{\nu_{Hex}}{\nu_{Pl}} = \frac{\lambda(Disk_1)}{\lambda(Hexa_1)} \sim 0.98 \,.$$

Bourgain and Steinerberger separately improve the lower bound of $\liminf_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k}$. Bourgain gives an estimate of his improvement on the size of order 10^{-9} and Steinerberger does not measure his improvment.

In any case, it is clear that

$$\nu_{Hex} \leq \nu_{Bo} < \nu_{Pl} \,,$$

and

$$\nu_{Hex} \leq \nu_{St} < \nu_{Pl} \,,$$

where ν_{Bo} and ν_{St} are the constants of Bourgain [Bo] and Steinerberger [St].

Surely more difficult (than the Hexagonal Conjecture) and more important will be to prove that

Square conjecture

$$A(\Omega) \limsup_{n\to+\infty} \frac{N(\phi_n)}{n} \leq \nu_{square} , \quad (19)$$

with $\nu_{square} = \frac{4\pi}{\lambda(Sq_1)} = \frac{2}{\pi}$.

The philosophy is clear : the hexagonal conjecture for k-partitions should be replaced by the square conjecture when bipartite k-partitions are involved because square tilings are bipartite.

This conjecture is due to Iosif Polterovich on the basis of computations of Blum-Gutzman-Smilanski [2]. Due to the computations on the square, this would be the optimal result.

Analysis of the critical sets in the large limit case (after [HH7])

We first consider the case of one pole X. We look at a sequence of K_X-real eigenfunctions and follow the proof of Pleijel on the number of nodal domains. We observe that the part devoted to the lower bound works along the same lines and the way we shall meet $\mathfrak{L}_k(\Omega)$ is unchanged. When using the Weyl formula, we observe that only a lower bound of the counting function is used. If the distance of X to the boundary is larger than ϵ, we introduce a disk $D(X,\epsilon)$ of radius ϵ around X ($\epsilon > 0$) and consider the Dirichlet magnetic Laplacian in $\Omega \setminus \bar{D}(X,\epsilon)$. For the X at the distance less than ϵ of the boundary, we look at the magnetic Laplacian on Ω minus a (2ϵ)-tubular neighborhood of the boundary.

In the two cases, we get an elliptic operator where the main term is the Laplacian $-\Delta$. Hence we can combine the monotonicity of the Dirichlet problem with respect to the variation of the domain to the use of the standard Weyl formula to get (uniformly for X in Ω, an estimate for the counting function $N_X(\lambda)$ of $-\Delta_{\mathbf{A}^\mathbf{X}}$ in the following way:

There exists a constant $C > 0$ such that, for any $\epsilon > 0$, as $\lambda \to +\infty$,

$$N_X(\lambda) \geq \frac{1}{4\pi}(1 - C\epsilon)A(\Omega)\,\lambda + o(\lambda)\,.$$

Hence, for any $\epsilon > 0$, any $X \in \Omega$,

$$\limsup_{n\to+\infty} \mu(\phi_n^X)/n \leq (1 + C\epsilon)\frac{4\pi}{A(\Omega)\liminf_{k\to+\infty}\frac{\mathfrak{L}_k(\Omega)}{k}}\,.$$

Taking the limit $\epsilon \to 0$, we get:

$$\limsup_{n\to+\infty} \mu(\phi_n^X)/n \leq \frac{4\pi}{A(\Omega)\liminf_{k\to+\infty}\frac{\mathfrak{L}_k(\Omega)}{k}}\,. \tag{20}$$

Everything being uniform with respect to X, we can also consider a sequence $\phi_n^{X_n}$ corresponding to the n-th eigenvalue of $-\Delta_{\mathbf{A}_{X_n}}$. Suppose that for a subsequence k_j, we have a k_j-minimal partition with only one pole X_j in Ω. Let $\phi_{k_j}^{X_j}$ the corresponding eigenfunction. Hence, we are in a Courant sharp situation. The inequality above leads to

$$1 \leq \frac{4\pi}{A(\Omega)\liminf_{k\to+\infty}\frac{\mathfrak{L}_k(\Omega)}{k}} \leq \nu_{Pl} \sim 0.691\,.$$

Hence a contradiction.

We can play the same game with more than one pole and get in [HH7] as consequence:

Proposition

If for $k \in \mathbb{N}$, $\mathcal{D}_k$ denotes a minimal k-partition, then

$$\lim_{k \to +\infty} \# X^{odd}(N(\mathcal{D}_k)) = +\infty \,. \tag{21}$$

Remarks

- We recall that an upper bound for $\#X(N(\mathcal{D}_k))$ is given in [HHOT1] (case with no holes) by using Euler's formula:

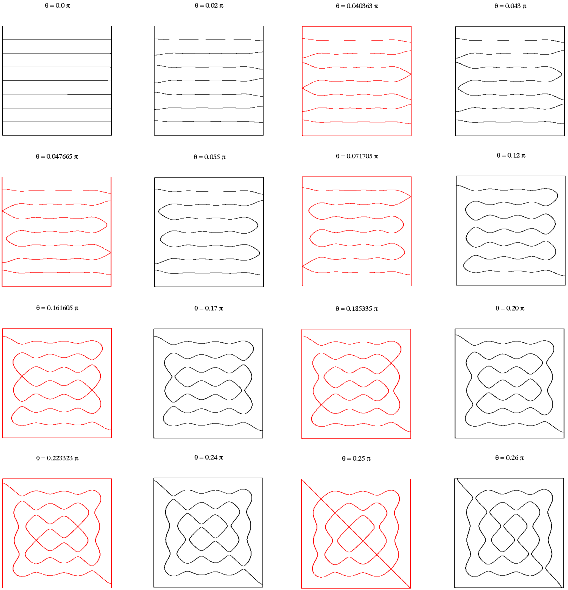
$$\#X^{odd}(N(\mathcal{D}_k)) \leq 2k - 4\,. \tag{22}$$

  On the other hand, the hexagonal conjecture suggests:

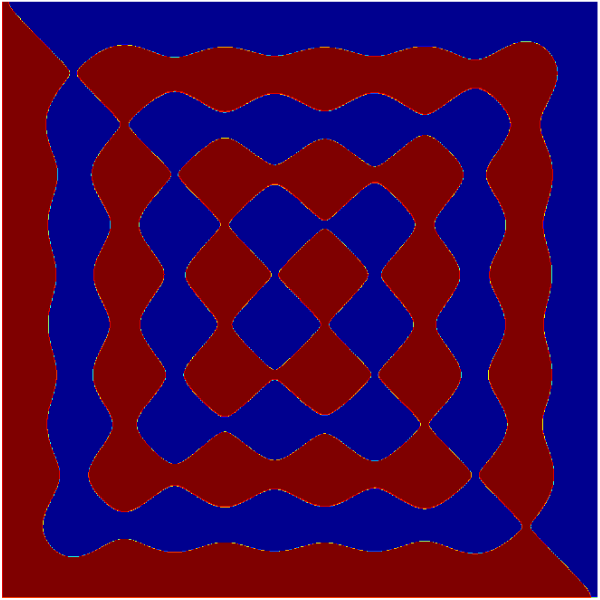
$$\lim_{k \to +\infty} \frac{\#X^{odd}(N(\mathcal{D}_k))}{k} = 2\,. \tag{23}$$

  Hence there are good reasons to believe that upper bound (22) is asymptotically optimal.
- This is not true that the number of critical points tends to ∞ as $\lambda_k \to +\infty$ for the nodal partitions on the square. There is (A. Stern (1925), Bérard-Helffer (2014)) an infinite sequence of eigenvalues such that the corresponding eigenfunctions have two nodal sets and no critical points. The same is true for the sphere (A. Stern (1925), H. Lewy (1977), Bérard-Helffer (2014) (work in progress).

Here is a family of examples corresponding to the eigenvalue $1 + 8^2$:



Here is another example (recomputed by V. Bonnaillie) corresponding to the eigenvalue $1 + 12^2$:



B. Alziary, J. Fleckinger-Pellé, P. Takáč.
Eigenfunctions and Hardy inequalities for a magnetic Schrödinger operator in $\mathbb{R}^2$.
Math. Methods Appl. Sci. **26**(13), 1093–1136 (2003).

P. Bérard and B. Helffer.
Dirichlet eigenfunctions of the square membrane: Courant's property, and A.Stern's and A.Pleijel's analyses
Prépublication hal-00951531, version 2.

P. Bérard and B. Helffer.
Eigenfunctions of the sphere: Courant's property, and A.Stern's and H. Lewy's analyses
Prépublication hal-00951531, version 2

J. Berger, J. Rubinstein.
On the zero set of the wave function in superconductivity.
Comm. Math. Phys. **202**(3), 621–628 (1999).

L. Brasco, G. de Philippis and B. Velichkov.
Faber-Krahn inequaqlities in sharp quantitative form.

ArXiv 13060392.

G. Blind,
Über Unterdeckungen der Ebene durch Kreise.
J. Reine Angew. Math. 236 1969 145173.

V. Bonnaillie, and B. Helffer.
Numerical analysis of nodal sets for eigenvalues of Aharonov-Bohm Hamiltonians on the square and application to minimal partitions.
Experimental Mathematics 20 (3), p. 931-945 (2010).

V. Bonnaillie-Noël, B. Helffer and T. Hoffmann-Ostenhof.
Spectral minimal partitions, Aharonov-Bohm hamiltonians and application the case of the rectangle.
Journal of Physics A : Math. Theor. 42 (18) (2009) 185203.

G. Blum, S. Gnutzmann, and U. Smilansky.
Nodal domain statistics: A criterion for quantum chaos,
Phys. Rev. Lett. 88 (2002), 114101-114104.

V. Bonnaillie-Noël, B. Helffer and G. Vial.

Numerical simulations for nodal domains and spectral minimal partitions.
ESAIM Control Optim. Calc.Var. DOI:10.1051/cocv:2008074 (2009).

V. Bonnaillie and C. Lena.
Personal communication.

B. Bourdin, D. Bucur, and E. Oudet.
Optimal partitions for eigenvalues.
SIAM J.Sci. Comp. 31(6) 4100-4114, (2009).

J. Bourgain.
On Pleijel's Nodal Domain Theorem,
arXiv:1308.4422v1 [math.SP] 20 Aug 2013 (correcting a previous version circulating and then retired beginning of 2013).

D. Bucur, G. Buttazzo, and A. Henrot.
Existence results for some optimal partition problems.
Adv. Math. Sci. Appl. 8 (1998), 571-579.

K. Burdzy, R. Holyst, D. Ingerman, and P. March.
Configurational transition in a Fleming-Viot-type model and probabilistic interpretation of Laplacian eigenfunctions.
J. Phys.A: Math. Gen. 29 (1996), 2633-2642.

L.A. Caffarelli and F.H. Lin.
An optimal partition problem for eigenvalues.
Journal of scientific Computing 31 (1/2) DOI: 10.1007/s10915-006-9114.8 (2007)

M. Conti, S. Terracini, and G. Verzini.
An optimal partition problem related to nonlinear eigenvalues.
Journal of Functional Analysis 198, p. 160-196 (2003).

M. Conti, S. Terracini, and G. Verzini.
A variational problem for the spatial segregation of reaction-diffusion systems.
Indiana Univ. Math. J. 54, p. 779-815 (2005).

M. Conti, S. Terracini, and G. Verzini.

On a class of optimal partition problems related to the Fučik spectrum and to the monotonicity formula.
Calc. Var. 22, p. 45-72 (2005).

O. Cybulski, V. Babin, and R. Holyst.
Minimization of the Renyi entropy production in the space-partitioning process.
Physical Review E71 046130 (2005).

R.T. Dong.
Nodal sets of eigenfunctions on Riemann surfaces.
Journal of differential Geometry 36, p. 493-506 (1992).

N. Fusco, F. Maggi and A. Pratelli,
Stability estimates for certain Faber-Krahn, isocapacitary and Cheeger inequalities.
Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 8 (2009), no. 1, 5171.

N. Fusco, F. Maggi and A. Pratelli,
The sharp quantitative isoperimetric inequality.
Ann. of Math. (2) 168 (2008), no. 3, 941980.

T. C. Hales.
The honeycomb conjecture.
Disc. Comp. Geom. 25 (2001), p. 1-22.

B. Helffer, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, M. P. Owen.
Nodal sets for groundstates of Schrödinger operators with zero magnetic field in non-simply connected domains.
Comm. Math. Phys. **202**(3) (1999) 629–649.

B. Helffer, T. Hoffmann-Ostenhof.
Converse spectral problems for nodal domains.
Mosc. Math. J. **7**(1) (2007) 67–84.

B. Helffer, T. Hoffmann-Ostenhof.
On spectral minimal partitions : the case of the disk.
CRM proceedings **52**, 119–136 (2010).

B. Helffer, T. Hoffmann-Ostenhof.
On two notions of minimal spectral partitions.
Adv. Math. Sci. Appl. 20 (2010), no. 1, 249263.

B. Helffer, T. Hoffmann-Ostenhof.
On a magnetic characterization of spectral minimal partitions.
To appear in JEMS (2012).

B. Helffer, T. Hoffmann-Ostenhof.
Spectral minimal partitions for a thin strip on a cylinder or a thin annulus like domain with Neumann condition
OTAMP proceedings 2010.

B. Helffer, T. Hoffmann-Ostenhof.
On spectral minimal partitions : the case of the torus.
To appear in Journal of Spectral Theory (2013).

B. Helffer, T. Hoffmann-Ostenhof.
On large k minimal spectral partitions and Pleijel's theorem.
Submitted.

B. Helffer, T. Hoffmann-Ostenhof, S. Terracini.
Nodal domains and spectral minimal partitions.
Ann. Inst. H. Poincaré Anal. Non Linéaire (2009), p. 101-138.

B. Helffer, T. Hoffmann-Ostenhof, S. Terracini.

On spectral minimal partitions : the case of the sphere.
Around the Research of Vladimir Maz'ya .III, International Math. Series, Springer, Vol. 13, p. 153-178 (2010).

B. Helffer, T. Hoffmann-Ostenhof, S. Terracini.
On minimal spectral partition in $3D$.
DCDS-A (Vol. 28, No. 2, 2010) special issues Part I, dedicated to Professor Louis Nirenberg on the occasion of his 85th birthday.

L. Hillairet and A. Kokotov.
Variational formulas for the spectrum of translation surfaces.
Work in progress (September 2013).

C. Lena.
On the eigenvalues of Aharonov-Bohm operators with varying poles
Preprint 2013. Submitted.

C. Lena.
Minimal partitions on the torus.
In preparation.

H. Lewy.
On the minimum number of domains in which the nodal lines of spherical harmonics divide the sphere.
Comm. Partial Differential Equations 2 (1977), no. 12, 1233–1244.

B. Noris and S. Terracini.
Nodal sets of magnetic Schrödinger operators of Aharonov-Bohm type and energy minimizing partitions.
Indiana Univ. Math. J. **58**(2), 617–676 (2009).

V. Bonnaillie, B. Noris, M. Nys and S. Terracini.
On the eigenvalues of Aharonov-Bohm operators with varying poles
Preprint (2013)

J. Peetre.
A generalization of Courant nodal theorem.
Math. Scandinavica 5, p. 15-20 (1957).

A. Pleijel.

Remarks on Courant's nodal theorem.
Comm. Pure. Appl. Math., 9: 543–550, 1956.

I. Polterovich.
Pleijel's nodal domain theorem for free membranes.
Proceeding of the AMS, Volume 137, Number 3, March 2009, Pages 10211024

S. Steinerberger.
A geometric uncertainty principle with an application to Pleijel's estimate.
arXiv:1306.3103v2 [math.MG] 14 June 2013, v3 August 2013.

A. Stern.
Bemerkungen über asymptotisches Verhalten von Eigenwerten und Eigenfunctionen.
Thesis in Göttingen (1925).

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