Courant-sharp Robin eigenvalues for the square and other planar domains (after K. Gittins and B. Helffer)

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Main goals

We would like to determine the cases where there is equality in Courant's nodal domain theorem in the case of the realization of the Laplacian in a square with a Robin boundary condition.

The initial motivation was the analysis of the problem of minimal partitions, see Helffer–Hoffmann-0stenhof–Terracini (2009) who prove that minimal partitions which are nodal correspond to Courant sharp eigenvalues, but beyond this motivation this is a natural question in spectral theory involving the analysis of the nodal structure of eigenfunctions in case of multiplicity.

One of the new points here is to try to understand the transition between the Dirichlet case and the Neumann case by analysing the deformation of the nodal structure when the Robin parameter varies.

For the square, we partially extend the results that were obtained by Pleijel (1956), Bérard–Helffer (2015) for the Dirichlet problem and Helffer–Persson–Sundqvist (2015) for the Neumann problem.

After proving some general results that hold for any value of the Robin parameter h, we focus on the case when h is large.

We also obtain some semi-stability results for the number of nodal domains of a Robin eigenfunction of a domain with $C^{2,+}$ boundary as h large varies.

This is joint work with Katie Gittins (Neuchâtel University).

The Robin problem

Let $\Omega \subset \mathbb{R}^2$, be a bounded, connected, open set with Lipschitz boundary and let $h \in \mathbb{R}^+$.

The Robin eigenvalues of the Laplacian on Ω with parameter h are $\lambda_{k,h}(\Omega) \in \mathbb{R}$, $k \in \mathbb{N}$, s.t. there exists a function $u_k \in H^1(\Omega)$ which satisfies

$$
\begin{aligned}
-\Delta u_k(x) = \lambda_{k,h}(\Omega) u_k(x)\,, \quad x \in \Omega\,,\\
\frac{\partial}{\partial \nu} u_k(x) + h\, u_k(x) = 0\,, \quad x \in \partial\Omega\,,
\end{aligned}
$$

where ν is the outward-pointing unit normal to $\partial\Omega$.

The Robin problem is associated with the quadratic form:

$$H^1(\Omega) \ni u \mapsto \int_\Omega |\nabla u|^2 + h \int_{\partial\Omega} |u_{\partial\Omega}|^2 d\sigma\,,$$

where $u_{\partial\Omega}$ is the trace of u.
So the spectrum is monotonically increasing with respect to h for $h \in [0, +\infty)$.
Hence the Robin eigenvalues with $h > 0$ interpolate between the Neumann eigenvalues ($h = 0$) and the Dirichlet eigenvalues ($h = +\infty$).

The Robin eigenvalues satisfy the Courant nodal domain theorem (1923) stating that any eigenfunction corresponding to $\lambda_{k,h}(\Omega)$ has at most k nodal domains. We consider the Courant-sharp Robin eigenvalues of Ω.

We call a Robin eigenvalue $\lambda_{k,h}(\Omega)$ Courant-sharp if it has a corresponding eigenfunction that has exactly k nodal domains.

As for the Dirichlet and Neumann eigenvalues, $\lambda_{1,h}(\Omega)$ and $\lambda_{2,h}(\Omega)$ are Courant-sharp for all $h \geq 0$.

It is not too difficult to verify that $\lambda_{4,h}(\Omega)$ is also Courant sharp for any $h \in [0, +\infty]$.

Another interesting question is whether it is possible to follow the Courant-sharp (Neumann) eigenvalues with $h = 0$ to Courant-sharp (Dirichlet) eigenvalues as $h \to +\infty$.

In other words, we can ask whether there are some critical values $h^*(k, \Omega)$ after which the Robin eigenvalues $\lambda_{k,h}(\Omega)$, $h \geq h^*(k, \Omega)$ become Courant-sharp or are no longer Courant-sharp.

With this respect the cases $k = 5$ and $k = 9$ are quite interesting and presumably the only ones.

We denote the Dirichlet eigenvalues by λ_k^D and the Neumann eigenvalues by λ_k^N.
We consider the particular example where Ω is a square in $\mathbb{R}^2$ of side-length $\ell = \pi$ and the main question is:
Is it possible to determine the Courant-sharp Robin eigenvalues of the square $S := (0,\pi) \times (0,\pi)$*?*

It was asserted by Pleijel in [33] (1956) that the only Courant-sharp Dirichlet eigenvalues of the square are for $k = 1, 2, 4$. This was shown rigorously in Bérard-Helffer [4].

On the other hand the only Courant-sharp Neumann eigenvalues of the square are for $k = 1, 2, 4, 5, 9$, as shown in Helffer–Persson-Sundqvist [27] (see the talk in this conference).

The first step is to reduce the number of potential Courant-sharp eigenvalues by invoking an argument inspired by the founding paper of Pleijel [33].

Uniform Reduction Theorem

Let $h \geq 0$. If $k \geq 520$, then $\lambda_{k,h}(S)$ is not Courant-sharp.

In the case of a Dirichlet boundary condition, the equivalent statement in [33] gives $k \geq 48$ and in the case of a Neumann boundary condition, [27], $k \geq 209$.

The strategies of [4, 27] are then either to re-implement the Faber-Krahn inequality, or to use symmetry properties of the corresponding eigenfunctions to further eliminate potential Courant-sharp eigenvalues.

One is then reduced to the analysis of the nodal structure of very few families of eigenfunctions (one for Dirichlet, much more for Neumann) that belong to two-dimensional spectral spaces.

The proof of this theorem is rather close to the proof given for Neumann. Note that the bound is independent of h. In the asymptotic situations $h \to 0$ and $h \to +\infty$ one can improve the theorem. This will be detailed in this talk in the limit h large.

In Gittins-Léna [21], upper bounds (related to the geometry of the domain) are obtained for the Courant-sharp Neumann and Robin eigenvalues with $h > 0$ of a bounded, connected, open set $\Omega \subset \mathbb{R}^n$ with C^2 boundary, extending previous results by Bérard-Helffer and Gittins-Van den Berg for the Dirichlet case).

Our main result

In this talk we show that for h large enough the only Courant-sharp Robin eigenvalues are for $k = 1, 2, 4$.

h large Theorem

There exists $h_1 > 0$ such that for $h \geq h_1$, the Courant-sharp cases for the Robin problem are the same as those for $h = +\infty$ (i.e. the Dirichlet case).

In order to prove this theorem, it is necessary to estimate the number of nodal domains whose boundaries intersect the boundary of the square on at least a non-trivial interval.
For such nodal domains, we cannot use the Faber-Krahn inequality for the Dirichlet problem.
Nevertheless, there is a Faber-Krahn inequality for the Robin problem when $h > 0$ (see Bossel (1988), Daners (2006), Bucur-Giacomini ((2010) and (2015)) [8, 10, 13]).

On the way for the proof of this theorem, we obtain some semi-stability results for the number of nodal domains as the Robin parameter (h large) varies for general domains.

One can also look at the situation when the Robin parameter h tends to 0 and study the following conjecture.

h small Conjecture

There exists $h_0 > 0$ such that for $0 < h \leq h_0$, the Courant-sharp cases for the Robin problem are the same, except the fifth one, as those for $h = 0$ (i.e. the Neumann case for which $k = 1, 2, 4, 5, 9$ were shown as the only Courant Sharp eigenvalues) .

As will be seen in Mikael Persson Sundqvist's talk for Neumann, there are many cases to consider after the first reduction. We have to control the stability by perturbation. At the moment, we have only partial results with K. Gittins.

Formulas for the eigenvalues and eigenfunctions of the Robin Laplacian for a rectangle

For rectangles $\Omega = (0, \ell_1) \times (0, \ell_2) \subset \mathbb{R}^2$ and $(x, y) \in \Omega$, an orthonormal basis for the Robin problem is given by

$$u_{p,q}(x, y) = u_p(x) u_q(y), \tag{1}$$

where, for $p, q \in \mathbb{N}$, u_p is the $(p+1)$-st eigenfunction of the Robin problem in $(0, \ell_1)$.

We will mainly consider the case of the square with $\ell_1 = \ell_2 = \pi$.

The Robin eigenvalues are given by

$$\left(\frac{\alpha_p}{\ell_1}\right)^2 + \left(\frac{\alpha_q}{\ell_2}\right)^2 .$$

So in two dimensions, the Robin eigenvalues correspond to pairs $\lambda_{p,q}(h)$ of non-negative integers (p, q).

If we consider the symmetric case (p even), we have

$$\alpha_p(h) \tan\left(\frac{\alpha_p(h)}{2}\right) = h\ell_1 . \quad (2)$$

Similarly, if we consider the antisymmetric case (p odd) we get

$$\frac{\alpha_p(h)}{h\ell_1} = -\tan\left(\frac{\alpha_p(h)}{2}\right) . \quad (3)$$

With these formulas in mind, we get in the first case

$$u_p(x) = \frac{1}{\sin \frac{\alpha_p}{2}} \cos(\frac{\alpha_p}{x}\ell - \frac{\alpha_p}{2})\,,$$

and in the second case

$$u_p(x) = \frac{1}{\cos \frac{\alpha_p}{2}} \sin(\frac{\alpha_p}{x}\ell - \frac{\alpha_p}{2})\,.$$

In this way, we clearly see the symmetry properties of the (1D)-eigenfunctions :

$$u_p(\ell - x) = (-1)^p u_p(x)\,.$$

Graphs of $\alpha_p(h)$

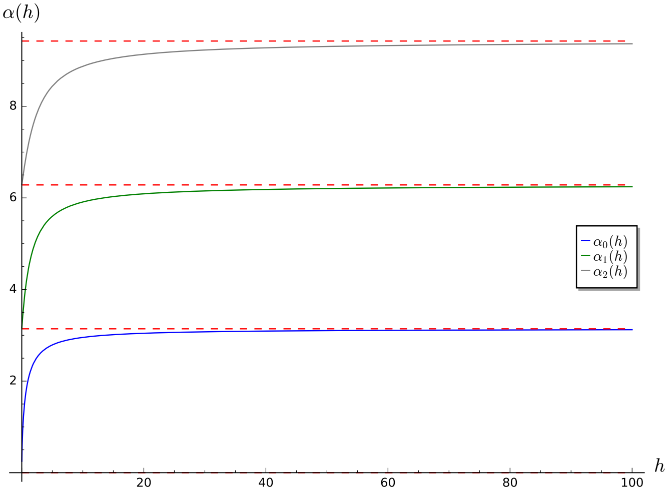

Figure: Solutions $\alpha_0(h)$, $\alpha_1(h)$, $\alpha_2(h)$ for $h \leq 100$.

Symmetry properties

The use of symmetries was quite powerful in the context of the Neumann case, [27], via an argument due to Leydold, [32]. That is, a Courant nodal theorem for eigenfunctions that satisfy certain symmetry properties. In addition, the number of nodal domains inherits some particular properties from these symmetries.
This invariance by symmetry is actually common to all the Robin problems.
See the talk by Mikael P. Sundqvist for details in the Neumann case.

Upper bounds for the number of Courant-sharp Robin eigenvalues of a square

This was the first step proposed by Pleijel [33] in the Dirichlet case to reduce the analysis of the Courant-sharp cases to the analysis of finitely many eigenvalues.
His proof was a combination of the Faber-Krahn inequality and the Weyl formula.
In the Neumann case considered in [27], a new difficulty arises as it is not possible to apply the Faber-Krahn inequality to the elements of the nodal partition whose boundaries touch the boundary of the square at more than isolated points.

We can extend the analysis to the Robin case.

Lower bound for the Robin counting function

Recall that for $\lambda > 0$, the Robin counting function for the corresponding eigenvalues of Ω is defined as

$$N_\Omega^{R,h}(\lambda) := \#\{k \in \mathbb{N} : \lambda_{k,h}(\Omega) < \lambda\}. \tag{4}$$

Similarly we have the Dirichlet counting function

$$N_\Omega^{D}(\lambda) := \#\{k \in \mathbb{N} : \lambda_k^D(\Omega) < \lambda\}, \tag{5}$$

and the Neumann counting function

$$N_\Omega^{Ne}(\lambda) := \#\{k \in \mathbb{N} : \lambda_k^N(\Omega) < \lambda\}. \tag{6}$$

Due to the monotonicity of the Robin eigenvalues with respect to $h \in [0, +\infty)$, it is rather easy to have a lower bound for the $N_\Omega^{R,h}(\lambda)$. In particular, we have

$$N_\Omega^{R,h}(\lambda) \geq N_\Omega^{R,+\infty}(\lambda) = N_\Omega^D(\lambda)\,.$$

For the Neumann counting function of S, we have

$$\frac{\pi}{4}\lambda + 2\lfloor\sqrt{\lambda}\rfloor + 1 \geq N_S^{Ne}(\lambda) > \frac{\pi}{4}\lambda, \tag{7}$$

and for the Dirichlet counting function of S, if $\lambda \geq 2$, we have by [33],

$$N_S^D(\lambda) > \frac{\pi}{4}\lambda - 2\sqrt{\lambda} + 1\,. \tag{8}$$

Assume that $\lambda \geq 2$ (this is true for $k \geq 4$). Then, by (8) and monotonicity of the Robin eigenvalues with respect to h,

$$N_S^{R,h}(\lambda) \geq N_S^D(\lambda) > \frac{\pi}{4}\lambda - 2\sqrt{\lambda} + 1. \quad (9)$$

We now work analogously to the proof of the Neumann case. Denote by Ω^{inn} the union of nodal domains of the eigenfunction Ψ whose boundaries do not touch the boundary of Ω (except at isolated points), and $\mu^{\text{inn}}(\Psi)$ the number of nodal domains of Ψ in Ω^{inn}. We denote by Ω^{out} the nodal domains in $\Omega \setminus \Omega^{\text{inn}}$, and $\mu^{\text{out}}(\Psi)$ the number of nodal domains in Ω^{out}. We have

$$\mu^{\text{inn}}(\Psi) = \mu(\Psi) - \mu^{\text{out}}(\Psi)$$

and we require an upper bound for $\mu^{\text{out}}(\Psi)$.

Counting the number of nodal domains touching the boundary for the Robin problem

We observe (Pleijel remark) that he restriction to one side of the square, say $x = 0$, of the eigenfunction

$$u(x, y) = \sum_{i,j:\lambda_{n,h}(S)=\pi^{-2}(\alpha_i^2+\alpha_j^2)} a_{ij}\, u_i(x)u_j(y)\,.$$

is a linear combination of eigenfunctions on the segment $(0, \pi)$:

$$u(0, y) = \sum_{i,j:\lambda_{n,h}(S)=\pi^{-2}(\alpha_i^2+\alpha_j^2)} a_{ij}\, u_i(0)u_j(y)\,.$$

We can then use Sturm's theorem which gives bounds on the number of zeros of $u(0, y)$ in $(0, \pi)$ by

$$i_n(h) := \min(i : \lambda_{n,h}(S) = \pi^{-2}(\alpha_i^2 + \alpha_j^2))\,,$$

and

$$j_n(h) := \max(j : \lambda_{n,h}(S) = \pi^{-2}(\alpha_i^2 + \alpha_j^2))\,. \qquad (10)$$

We have

$$\lambda_{n,h}(S) = (\alpha^2_{i_n(h)} + \alpha^2_{j_n(h)})/\pi^2 \geq i_n(h)^2 + j_n(h)^2 \geq j_n(h)^2,$$

which gives

$$j_n(h) \leq \sqrt{\lambda_{n,h}(S)}\,.$$

We can argue in the same way for the other sides of the square and get

Lemma A

Let λ be a Robin eigenvalue of S with $h < +\infty$. If ψ is a Robin eigenfunction associated to λ, then

$$\mu^{out}(\psi) \leq 4\sqrt{\lambda}\,. \tag{11}$$

So, following the proof given for Neumann, we obtain our uniform reduction theorem.

Analysis as $h \to +\infty$.

We have to show that for h sufficiently large, the Courant-sharp Robin eigenvalues of the square are the same as those in the Dirichlet case, [33, 4].

Let us first revisit the Pleijel's argument in the Dirichlet case. We recall from (8) that if λ_n is Courant-sharp, then

$$n > \frac{\pi}{4}\lambda_n - 2\sqrt{\lambda_n} + 2\,. \tag{12}$$

On the other hand, if λ_n is Courant-sharp, the Faber-Krahn inequality gives

$$\frac{n}{\lambda_n} \leq \pi \mathbf{j}^{-2} < 0.54323\,. \tag{13}$$

Recall that $\pi \mathbf{j}^2$ is the ground state energy of the disc of area 1. Combining (12) and (13), leads to

$$\pi \mathbf{j}^{-2} > \frac{\pi}{4} - 2\lambda_n^{-\frac{1}{2}} + 2\lambda_n^{-1}\,, \tag{14}$$

and to

$$\lambda_n \leq 50\,. \tag{15}$$

Then the proof is achieved in the following steps (see Bérard-Helffer [4] for the full details).

- By a direct computation of the quotient of $\frac{n}{\lambda_n}$, it is possible to eliminate all the eigenvalues except for $n = 1, 2, 4, 5, 7$ and 9 .
- The eigenvalues for $n = 7$ and $n = 9$ are eliminated by symmetry arguments.
- The final step is to analyse the fifth eigenvalue for which a specific analysis of the nodal structure can be done (see [4]).

We now follow these steps and investigate the extension for h large.

Faber-Krahn for the Robin case

We recall the result of Bossel-Daners which asserts that the Robin eigenvalues of the Laplacian satisfy the following Faber-Krahn inequality. For a Lipschitz domain $\omega \subset \mathbb{R}^2$ and $h > 0$,

$$\lambda_{1,h}(\omega) \geq \lambda_{1,h}(D_\omega), \tag{16}$$

where $D_\omega \subset \mathbb{R}^2$ is a disc such that $A(D_\omega) = A(\omega)$.

For the interior nodal domains, the best approach is to use the standard Faber-Krahn inequality.

For the boundary domains, we have mixed boundary conditions with Robin on some arcs and Dirichlet on the remaining arcs. But for a lower bound, by monotonicity, it is enough to use the Robin Faber-Krahn inequality.

Scaling

The Robin eigenvalues satisfy the following scaling property.

$$\lambda_{n,h}(\omega) = t^2 \lambda_{n,h/t}(t\,\omega), \tag{17}$$

where $t\,\omega := \{tx \in \mathbb{R}^2 : x \in \omega\}$.
Hence the scaling also affects the Robin parameter. So, in particular, replacing D by D_1, the disc of area 1, we have

$$\lambda_{1,h}(D_\omega) = \lambda_{1,hA(\omega)^{\frac{1}{2}}}(D_1)/A(\omega)\,. \tag{18}$$

When $h = +\infty$, the reference is $\lambda_{1,+\infty}(D_1)$.

In the Robin case, if we start from h large, we will not necessarily have $hA(\omega)^{\frac{1}{2}}$ large if we use this inequality with ω a "boundary" nodal domain.
Hence we have to be careful in the application of the Faber-Krahn argument. This is actually the main difficulty.

Asymptotics in the case of the disk

In the case of the disk, there exists $c > 0$ such that, as $\tilde{h}$ tends to $+\infty$,

$$\lambda_{1,\tilde{h}}(D_1) = \lambda_{1,+\infty}(D_1) - \frac{c}{\tilde{h}} + \mathcal{O}\left(\frac{1}{\tilde{h}^2}\right) . \tag{19}$$

As $\tilde{h}$ tends to 0, there exists $d > 0$ such that,

$$\lambda_{1,\tilde{h}}(D_1) = d\,\tilde{h} + \mathcal{O}(\tilde{h}^2) . \tag{20}$$

We will apply the Faber-Krahn inequality to a nodal domain of a Robin eigenfunction $u = u_{n,h}$ associated with $\lambda_{n,h}$.
We have no time to discuss the question of the regularity of the nodal domains and the corresponding regularity needed for Faber-Krahn (see in addition Bucur-Giacomini for a version with very weak assumptions).

Pleijel's approach as $h \to +\infty$.

In light of what was done for $h = +\infty$, we now consider the different steps in the limit $h \to +\infty$.
The eigenvalues depend continuously on h until $+\infty$, in particular

$$\forall n \in \mathbb{N}, \lim_{h \to +\infty} \lambda_{n,h} = \lambda_n^D \,. \tag{21}$$

If we are in the Courant-sharp situation, then $\mu(u) = n$.

If there exists ω_i^{inn} such that $A(\omega_i^{\text{inn}}) \leq A(S)/n$, we are done like in the Dirichlet case.

If not, there exists ω_j^{out} such that

$$A(\omega_j^{\text{out}}) \leq A(S)/n\,. \qquad (22)$$

Combining the previous estimates, we find that

$$\frac{A(S)}{\lambda_{1,hA(\omega_j^{\text{out}})^{1/2}}(D_1)} > \frac{\pi}{4} - \frac{2}{\sqrt{\lambda_{n,h}}} + \frac{2}{\lambda_{n,h}}\,. \qquad (23)$$

Here, comparing with (14), we need to have $\tilde{h} := hA(\omega_j^{\mathrm{out}})^{1/2}$ large enough if we want to arrive at the same conclusion as for the Dirichlet case.

So we have to find a lower bound for $A(\omega_j^{\mathrm{out}})^{1/2}$. This is difficult, at least with explicit lower bounds and we use for this proof our initial h-independent upper bound. Hence, we can assume in this Courant-sharp situation, that

$$n \leq 520\,. \tag{24}$$

and under this assumption we get a uniform lower bound for $A(\omega_j^{\mathrm{out}})^{1/2}$.
At the end, we get, that for h large enough, $\lambda_{n,h} \leq 50$.

We can now follow essentially the proof of Pleijel for the Dirichlet case, modulo some relatively easy perturbation arguments and symmetry arguments.
Hence at this stage, we have proved the following.

Proposition

There exists $h_1 > 0$ such that for $h \geq h_1$, the Courant-sharp cases for the Robin problem are the same, except possibly for the fifth eigenvalue, as those for $h = +\infty$.

So, having in mind what was done for the Dirichlet case [4], it remains for h large enough to count the number of nodal domains of any eigenfunction corresponding to the fifth eigenvalue.

A general perturbation argument

We analyse a θ-dependent family $\Phi_{h,\theta}$ of eigenfunctions, more explicitly

$$\Phi_{h,\theta,p,q}(x,y) = \cos\theta\, u_{p,h}(x)u_{q,h}(y) + \sin\theta\, u_{p,h}(y)u_{q,h}(x)\,.$$

For most of the arguments we will not use the explicit expression of the family of eigenfunctions. Hence the arguments extend to more general bounded, planar domains with piecewise $C^{2,+}$ boundary.

Hence the question is to transfer an information that we have for $h = +\infty$ (or $h = h_0 > 0$) and $\theta = \theta_0$, to close values of the parameters.

The proof involves various general statements which are interesting in a more general context, hence not restricted to the case of the square.

An important point, which is a consequence of the Robin Faber-Krahn inequality, is the following lemma

Lemma on nodal loops

Let $h_0 > 0$ and $M > 0$. Then there exists $\epsilon_0 > 0$ such that no nodal domain of an eigenfunction Φ_h associated with $\lambda(h)$ for the Robin problem with parameter $h \geq h_0$ in some open set Ω (this includes Dirichlet) and $\lambda(h) \leq M$ can have area less than ϵ_0.

Note that our condition excludes the Neumann case which is more difficult.

Proof

This follows directly from the h-Faber Krahn inequality. If ω is a nodal domain of Φ_h satisfying the assumptions of the lemma, we have

$$\begin{aligned} M & \geq \lambda(h) \\ & \geq \lambda(h_0) \\ & \geq \lambda_{1,h_0}(D_\omega) \\ & = \lambda_{1,h_0 A(\omega)^{\frac{1}{2}}}(D_1)/A(\omega) \\ & \sim d\, h_0/A(\omega)^{\frac{1}{2}}\,. \end{aligned} \tag{25}$$

This shows that as soon as we avoid the Neumann situation, the ground state energy in a domain ω tends to $+\infty$ as the area of the domain tends to 0.

The proof is delicate for domains with corners, but a direct proof can be obtained for the square using that eigenfunctions have an extension to $\mathbb{R}^2$.

On the variation of the cardinality of the nodal domains by perturbation.

Our main result is the following proposition.

Proposition B

Let $\rho(h, \theta)$ denote the cardinality of the nodal domains of $\Phi_{h,\theta}$.
For any θ_0, $h_0 \in (0, +\infty]$, there exists $\eta_0 > 0$ such that if $|\frac{1}{h} - \frac{1}{h_0}| + |\theta - \theta_0| < \eta_0$, then

$$\rho(h, \theta) \leq \rho(h_0, \theta_0) .$$

We prove this proposition by analysing what is going on at the interior critical points and at the boundary points of the zero set.

Analysis in a neighbourhood of an interior point.

We treat, following a suggestion of T. Hoffman-Ostenhof what is going on at an interior point z_0. We assume that z_0 is a critical point of Φ_{h_0,θ_0} associated with an eigenvalue $\lambda(h_0)$. We choose $\epsilon_0 > 0$ small enough such that

- $D(z_0, \epsilon_0) \subset \Omega$;
- Lemma on nodal loops applies with $M > \lambda(h_0)$;
- the circle $\mathcal{C}(z_0, \epsilon_0)$ crosses the 2ℓ half-lines emanating from z_0 transversally at 2ℓ points $z_j(h_0, \theta_0)$.

Here we have used the general results on the local structure of an eigenfunction of the Laplacian (see Bers).

Lemma on local semi-stability

There exists $\eta_0 > 0$ such that if $|\frac{1}{h} - \frac{1}{h_0}| + |\theta - \theta_0| < \eta_0$, then the number of nodal domains of $\Phi_{h,\theta}$ intersecting the disk $D(z_0, \epsilon_0)$ cannot increase.

Proof

If we look at the nodal structure inside $D(z_0, \epsilon_0)$, we have 2ℓ local nodal domains (i.e nodal domains of the restriction of $\Phi_{h,\theta}$ to $D(z_0, \epsilon_0)$).
Starting from (h_0, θ_0) we now look at a small perturbation. By considering the restriction of $\Phi_{h,\theta}$ to the circle $\partial D(z_0, \epsilon_0)$, we observe that the 2ℓ zeros $z_j(h, \theta)$. of $\Phi_{h,\theta}$ in $\partial D(z_0, \epsilon_0)$ move very smoothly. Hence the restriction of $\Phi_{h,\theta}$ changes sign at each point $z_j(h, \theta)$.
Moreover, there are 2ℓ local domains $\omega_j(h, \theta)$ of $\Phi_{h,\theta}$ with the property that $\partial\omega_j(h, \theta)$ intersects $\partial D(z_0, \epsilon_0)$ along the arc $(z_j(h, \theta), z_{j+1}(h, \theta))$ (with the convention that $j + 1$ is 1 for $j = 2\ell$).

We now observe that if $\omega_j(h_0, \theta_0)$ and $\omega_{j'}(h_0, \theta_0)$ belong to the same nodal domain ($j \neq j'$), the property remains true for (h, θ) sufficiently close to (h_0, θ_0) (i.e. for η_0 in the lemma sufficiently small).
If, for (θ_0, h_0), $\omega_j(h_0, \theta_0)$ and $\omega_{j'}(h_0, \theta_0)$ do not belong to the same nodal domain, then there are two cases

- either the situation is unchanged by perturbation;
- or they belong after perturbation to the same nodal domain via a new path in $D(z_0, \epsilon_0)$.

In the second case, the number of nodal domains touching $\partial D(z_0, \epsilon_0)$ is decreasing.
On the other hand, by Lemma on nodal loops, any nodal domain that intersects $D(z_0, \epsilon_0)$ crosses $\partial D(z_0, \epsilon_0)$. This achieves the proof. □

Analysis at the boundary.

More difficult but no time to detail.

Application to the square

We come back to the case of the square and achieve the proof of our main Theorem. To this end, it is sufficient to obtain the following.

Proposition

There exists $h_0 > 0$ such that for any $h > h_0$, any eigenfunction corresponding to $\frac{1}{\pi^2}(\alpha_0(h)^2 + \alpha_2(h))^2$ has 2, 3, or 4 nodal domains (as in the Dirichlet case). Hence for $h > h_0$, $\lambda_{5,h}$ is not Courant-sharp.

The property is indeed true for $h = +\infty$ and, by the preceding results, the number of nodal domains cannot increase and is necessarily > 1.
One can carry out a deeper analysis for the eigenfunction associated with the fifth eigenvalue, where we count the nodal domains case by case.

Analysis of crossings

We analyse the possible crossings of two curves $h \mapsto \lambda_{p,q,h}(S)$ and $h \mapsto \lambda_{p',q',h}(S)$ defined in an interval of $[0,+\infty)$. This is indeed quite important as we want to follow the labelling of these eigenvalues when h varies. For this we have the general following result

Proposition C

For distinct pairs (p,q) and (p',q'), with $p \leq q$ and $p' \leq q'$, there is at most one value of h in $[0,+\infty)$ such that $\lambda_{p,q,h}(S) = \lambda_{p',q',h}(S)$.

The proof is based on a Wronskian argument.

$k = 5$, the Dirichlet case

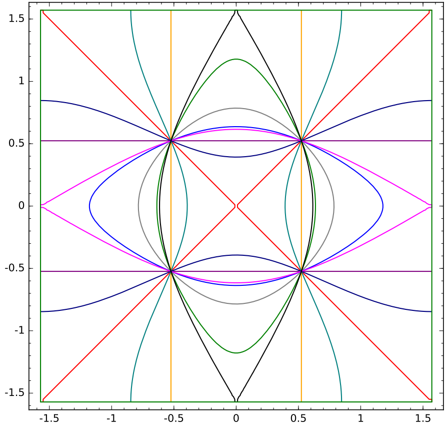

Figure: The fifth Dirichlet eigenfunction for various values of θ. The values $\theta = 0$, $\theta_1^* = \arctan(1/3)$, $\frac{\pi}{8}$, $\frac{\pi}{4}$, $\frac{3\pi}{8}$, $\theta_2^* = \frac{\pi}{2} - \arctan(1/3)$, $\frac{\pi}{2}$, $\frac{5\pi}{8}$, $\theta_3^* = \frac{3\pi}{4}$, $\frac{7\pi}{8}$ correspond to the purple, magenta, blue, grey, green, black, orange, teal, red, navy curves respectively.

Numerical study for $h = 1$

In Figure 3, we plot the fifth Robin eigenfunction for $h = 1$ for various values of θ.

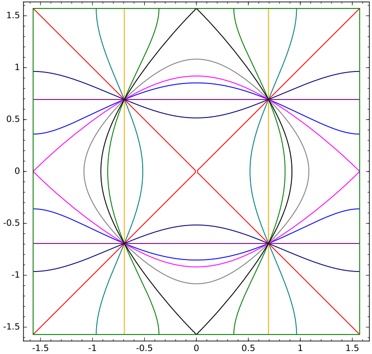

Figure: The fifth Robin eigenfunction with $h = 1$ for various values of θ. The values $\theta = 0$, $\frac{\pi}{8}$, $\arctan(-1/q_2(1))$, $\frac{\pi}{4}$, $\frac{\pi}{2} - \arctan(-1/q_2(1))$, $\frac{3\pi}{8}$, $\frac{\pi}{2}$, $\frac{5\pi}{8}$, $\frac{3\pi}{4}$, $\frac{7\pi}{4}$ correspond to the purple, blue, magenta, grey, black, green, orange, teal, red, navy curves respectively.

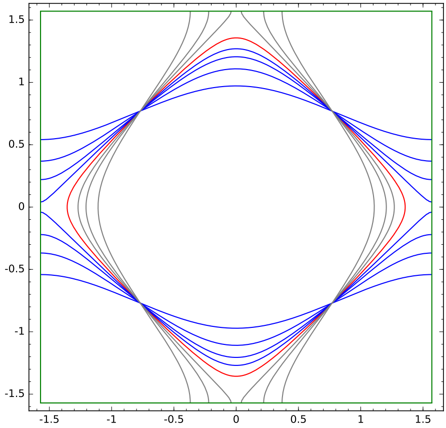

Figure: The fifth eigenfunction when $h\pi = 0.4$ for $\theta = \frac{\pi}{8}, \frac{3\pi}{16}, \frac{7\pi}{32}, \frac{15\pi}{64}, \frac{\pi}{4}, \frac{5\pi}{16}, \frac{9\pi}{32}, \frac{17\pi}{64}$.

The maximal number of nodal domains is three.

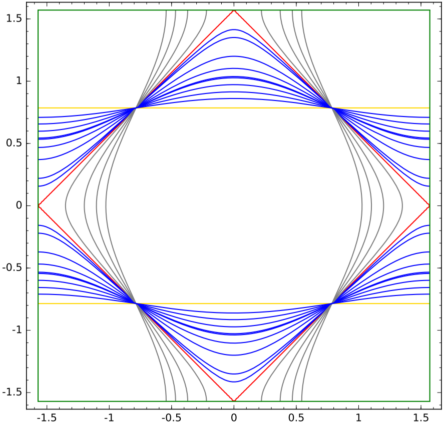

Figure: The fifth eigenfunction when $h = 0$ for different values of $0 \leq \theta < \pi$. When $\theta = \frac{\pi}{4}$, there are five nodal domains (red curve) and for some $\theta < \frac{\pi}{4}$ there are three nodal domains (blue curves). We also see a transition to 3 nodal domains for some $\theta > \frac{\pi}{4}$ (grey curves). The gold lines are for $\theta = 0$.

$k = 9$

For $k = 9$ and $h = 0$, the nine-th eigenvalue corresponds to the eigenvalue $2^2 + 2^2 = 8$. This eigenvalue is simple and corresponds to the labelling $(2, 2)$.
The eigenfunction reads (after translation)
$(-\frac{\pi}{2}, \frac{\pi}{2})^2 \ni (x, y) \mapsto \cos 2x \, \cos 2y$.
It is easy to see that the Courant-sharp property is still true for h small enough.

By deformation, the eigenfunction is

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^2 \ni (x, y) \mapsto \cos(\alpha_2(h)x/\pi) \, \cos(\alpha_2(h)y/\pi)$$

with corresponding eigenvalue $\frac{2}{\pi^2}(\alpha_2(h))^2$.

For $h \in [0, +\infty)$, we have nine nodal domains.

The issue is to follow its labelling and we observe that when $h = +\infty$ the eigenvalue is 18 and has minimal labelling 11. This eigenfunction is NOT Courant-sharp for h sufficiently large.

On the other hand the eigenvalue $\frac{1}{\pi^2}(\alpha_0(h)^2 + \alpha_3(h)^2)$ which has minimal labelling 10 for $h = 0$ arrives with labelling 9 at $h = +\infty$. Hence some transition occurs for at least one $h_9^* > 0$ which satisfies

$$\alpha_0(h)^2 + \alpha_3(h)^2 = 2\alpha_2(h)^2 .$$

Using the mathematics software system “SageMath”, we now plot the Robin eigenvalues of the square

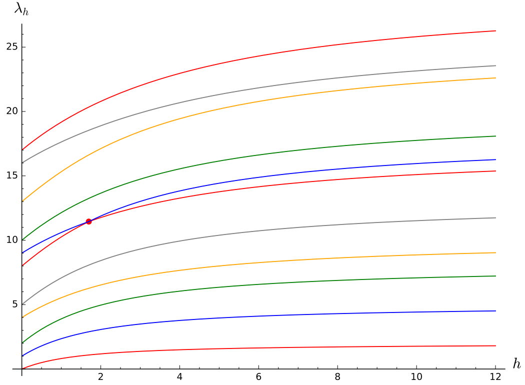

Figure: The Robin eigenvalues of the square $(\alpha_m(h)^2 + \alpha_n(h)^2)/\pi^2$ for $h \leq 12$ corresponding to the pairs $(0,0)$, $(1,0)$, $(1,1)$, $(2,0)$, $(2,1)$, $(2,2)$, $(3,0)$, $(3,1)$, $(3,2)$, $(4,0)$, $(4,1)$. The intersection between the curves corresponding to $(2,2)$ and $(3,0)$ occurs at $(1.6970, 11.4498)$.

Finally, we can prove:

Last proposition

There exists $h_9^* > 0$ such that $\lambda_{9,h}$ is Courant-sharp for $0 \leq h \leq h_9^*$ and not Courant-sharp for $h > h_9^*$.

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