# Helical magnetic fields and semi-classical asymptotics of the first eigenvalue

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# Abstract

We study the $3D$ Neumann magnetic Laplacian in the presence of a semi-classical parameter and a non-uniform magnetic field with constant intensity. We determine a sharp two terms asymptotics for the lowest eigenvalue, where the second term involves a quantity related to the magnetic field and the geometry of the domain. In the special case of the unit ball and a helical magnetic field, the concentration takes place on two symmetric points of the unit sphere. This is continuation of earlier work with (or by) A. Morame and X. Pan.

# The problem

Let $\Omega \subset \mathbb{R}^3$ be an open and bounded set with a smooth boundary $\partial\Omega$. Let us consider a smooth magnetic $\mathbf{B} : \overline{\Omega} \to \mathbb{R}^3$ such that

$$\forall\, x \in \Omega, \quad |\mathbf{B}(x)| = b$$

where $b > 0$ is a constant. W.l.o.g, we assume that $b = 1$. Let $\mathbf{A}(x)$ be a magnetic field such that

$$\mathrm{curl}\, \mathbf{A} = \mathbf{B} \; .$$

We are interested in the analysis of the first eigenvalue $\lambda_1(\mathbf{A}, h)$ of the Neumann realization of

$$P_{\mathbf{A}}^h := \Delta_{h,\mathbf{A}} = \sum_{j=1}^{3} (hD_{x_j} + A_j(x))^2 \; .$$

We introduce the following assumptions.

## Assumption [C1]

- $$\Gamma := \{x \in \partial\Omega \,\big|\, \mathbf{B}\cdot\mathbf{N}(x) = 0\},$$
  is a regular submanifold of $\partial\Omega$ .
- The "magnetic curvature" along $\Gamma$ satisfies
  $$\kappa_{n,\mathbf{B}}(x) := |d^T(\mathbf{B}\cdot\mathbf{N})(x)| \neq 0\,,\ \forall x \in \Gamma\,.$$
  Here $d^T$ is the differential defined on functions on $\partial\Omega$ and $\mathbf{N}(x)$ is the unit inward normal of $\Omega$.

### Assumption[C2]

The set of points where $\mathbf{B}$ is tangent to $\Gamma$ is finite.

These assumptions are rather generic and for instance satisfied for ellipsoids, when $\mathbf{B}$ is constant. When $|\mathbf{B}|$ is constant, the above assumptions hold for the sphere with a helical magnetic field.

Let us introduce the constant $\widehat{\gamma}_{0,\mathbf{B}}$ involving the "magnetic curvature",

$$\widehat{\gamma}_{0,\mathbf{B}} := \inf_{x\in\Gamma} \widetilde{\gamma}_{0,\mathbf{B}}(x),$$

where

$$\widetilde{\gamma}_{0,\mathbf{B}}(x) := 2^{-2/3}\widehat{\nu}_0\delta_0^{1/3}|\kappa_{n,\mathbf{B}}(x)|^{2/3}\Big(1-(1-\delta_0)|\mathbf{T}(x)\cdot\mathbf{B}(x)|^2\Big)^{1/3}.$$

Here $\mathbf{T}(x)$ is the oriented, unit tangent vector to $\Gamma$ at $x$, $\delta_0 \in ]0,1[$ and $\widehat{\nu}_0 > 0$ are spectral quantities relative to the De Gennes and Montgomery operators which will be introduced later.

# The case **B** constant

When **B** is constant, the following two-term asymptotics of $\lambda_1(B)$ has been established by Helffer-Morame and Pan (2002-2004).

## HMP Theorem (2004)

When **B** is constant and satisfies $(\mathrm{C1})$-$(\mathrm{C2})$, there exists $\eta > 0$ such that $\lambda_1^N(\mathbf{A}, h)$ satisfies as $h \to 0$

$$(*)\ \lambda_1^N(\mathbf{A}, h) = \Theta_0 h + \widehat{\gamma}_{0,\mathbf{B}} h^{\frac{4}{3}} + \mathcal{O}(h^{\frac{4}{3}+\eta})\,.$$

Since the proof in 2003-2004 of the theorem, let us mention three contributions:

- With variable $B(x)$, various contributions by N. Raymond.
- Fournais-Persson Sundqvist gave a finer analysis in the case of the sphere (2009).
- F. Hérau, N. Raymond (2022) gave the analysis of the gap between the first and second eigenvalues.

# (2D)-case. Old and new

We recall that the $(2D)$-case was solved earlier by Lu-Pan (1999-2000), Helffer-Morame (2001) (after preliminary works by Bernoff-Sternberg (1998)).

Since these works, many improvements have been obtained by

- Fournais-Helffer (2006) (complete expansion of the ground state),
- Bonnaillie-Hérau-Raymond (2022) (analysis of the tunneling in the case of a symmetric domain).

The aim of this new paper with A. Kachmar is to extend the proof of the [HMP] Theorem under the weaker assumption that $|\mathbf{B}(x)|$ is constant.

### HKP Theorem

Under the assumptions $(\mathrm{C1})$-$(\mathrm{C2})$, if $|\mathbf{B}|$ is constant, then the asymptotics (*) holds for $\lambda_1^N(\mathbf{A}, h)$.

An upper bound was given by X. Pan in [Pan6] (2009).

An interesting example of a non-constant magnetic field but with a constant intensity is the helical magnetic field occurring in the theory of liquid crystals. Up to the action of an orthogonal matrix (see Pan [Pan4]), it is given for some $\tau > 0$

$$\mathbf{B} = \mathrm{curl}\, \mathbf{n}_\tau = -\tau \mathbf{n}_\tau, \quad \mathbf{n}_\tau(x_1, x_2, x_3) = \left(\frac{1}{\tau}\cos(\tau x_3), \frac{1}{\tau}\sin(\tau x_3), 0\right).$$

We now recall the now standard properties of the two important $(1D)$-models.

# The de Gennes model

We refer to [DaHe, HelMo2] for the proof of these now standard properties which are presented below. For $\xi \in \mathbb{R}$, we consider the harmonic oscillator on $\mathbb{R}_+$:

$$H(\xi) := D_t^2 + (t - \xi)^2 \,,$$

with Neumann boundary condition at $0$. We denote by $\mu(\xi)$ its first eigenvalue. $\xi \mapsto \mu(\xi)$ admits a unique and non degenerate minimum at $\xi_0$. This leads to introduce

$$\Theta_0 = \inf_{\xi \in \mathbb{R}} \mu(\xi) = \mu(\xi_0), \quad \delta_0 = \mu''(\xi_0)\,, \text{ where } \xi_0 = \sqrt{\Theta_0}\,.$$

Moreover $\frac{1}{2} < \Theta_0 < 1$ and $0 < \delta_0 < 1$. $\Theta_0$ is called the de Gennes constant.

# The Montgomery model

Here we refer to Helffer-Morame [HelMo1] and Pan-Kwek [PanKw]. In [HMP]-Theorem, the constant $\widehat{\nu}_0 > 0$ is related to the Montgomery model whose spectral analysis has a long story before the problem in superconductivity and after.
For $\rho \in \mathbb{R}$, we introduce, in $L^2(\mathbb{R})$, the operator

$$S(\rho) = D_r^2 + (r^2 - \rho)^2 \,,$$

and denote its first eigenvalue by $\mu^{\mathrm{Mon}}(\rho)$. Then

$$\widehat{\nu}_0 := \inf_{\rho \in \mathbb{R}} \mu^{\mathrm{Mon}}(\rho) = \mu^{\mathrm{Mon}}(\rho_0) \,,$$

where $\rho_0 \in \mathbb{R}$ is the unique minimum non degenerate of $\mu^{\mathrm{Mon}}$.

# Model operator for non-uniform magnetic fields

Given real parameters $\eta, \zeta, \gamma$ and $\theta$, we consider the operator

$$\begin{aligned} P_{0;\gamma,\theta}^{h,\eta,\zeta} \quad := & \,(hD_r - \sin\theta\, t - \cos\theta\,(\eta s + \zeta r)t)^2 \\ & +(hD_s + \cos\theta\, t - \sin\theta\,(\eta s + \zeta r)\, t + \gamma \tfrac{r^2}{2})^2 \\ & +h^2 D_t^2\ , \end{aligned}$$

on $\mathbb{R}^2 \times \mathbb{R}^+$ (actually in a neighborhood of $(0,0,0)$).
Roughly speaking

- $r$ corresponds to a signed distance to $\Gamma$
- $s$ parametrizes $\Gamma$.
- $t$ corresponds to the distance to the boundary $\partial\Omega$,

When $\eta = \zeta = 0$, we recover the model studied in [HelMo4]. Hence our aim is to compare this situation with that when $\eta = \zeta = 0$. Our main result on this model (see below), is an important new step in our derivation of the lower bound matching with the asymptotics (*).

We start like in the case $\eta = \zeta = 0$ and consider the following scaling

$$r = h^{\frac{1}{3}}\hat{r}\ ,\ s = h^{\frac{1}{3}}\hat{s}\ ,\ t = h^{\frac{1}{2}}\hat{t}\ .$$

After division by $h$, this leads to (forgetting the hats)

$$P^{h,\eta,\zeta}_{1;\gamma,\theta} := \left(h^{\frac{1}{6}} D_r - \sin\theta\, t - h^{\frac{1}{3}} \cos\theta\, t(\eta s + \zeta r)\right)^2 + \left(h^{\frac{1}{6}} D_s + \cos\theta\, t + h^{\frac{1}{6}} \gamma \frac{r^2}{2} - h^{\frac{1}{3}} \sin\theta\, t(\eta s + \zeta r)\right)^2 + D_t^2$$

on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$.
Hence we have

$$\sigma(P^{h,\eta,\zeta}_{0;\gamma,\theta}) = h\, \sigma(P^{h,\eta,\zeta}_{1;\gamma,\theta}).$$

Unlike the case where $\eta = \zeta = 0$, we can no more perform a partial Fourier transform in the $s$-variable.

But we can rewrite this operator as in the following form semi-abstract form.

## New form

$$P^{h,\eta,\zeta}_{1;\gamma,\theta} = D_t^2 + (t - h^{\frac{1}{6}} L_{1,\gamma,\theta})^2 + h^{\frac{1}{3}} (L^{h,\eta,\zeta}_{2,\gamma,\theta})^2 \ ,$$

where

$$\begin{aligned} L_{1;\gamma,\theta} &= \sin\theta D_r - \cos\theta \left(\tfrac{\gamma}{2} r^2 + D_s\right) \ , \\ L^{h,\eta,\zeta}_{2;\gamma,\theta} &:= \cos\theta D_r + \sin\theta \left(\tfrac{\gamma}{2} r^2 + D_s\right) - h^{\frac{1}{6}} (\zeta r + \eta s) t \, . \end{aligned}$$

Note that to compare with the case considered $\eta = \zeta = 0$ we have

$$L^{h,\eta,\zeta}_{2;\gamma,\theta} = L_{2;\gamma,\theta} - h^{\frac{1}{6}} (\zeta r + \eta s) t \, ,$$

where $L_{2;\gamma,\theta} := L^{0,0,0}_{2;\gamma,\theta}$.
When $\eta = \zeta = 0$, this is the operator studied in [HelMo4] modulo a Fourier transformation with respect to the $s$ variable.

## Proposition by Helffer-Morame [HelMo4]

For any $\delta \in ]0, \frac{1}{3}[$ and $M > 0$, there exist positive constants $C$ and $h_0$ such that, for all $\theta \in \mathbb{R}$, $|\gamma| \leq M$, and $h \in ]0, h_0]$, any $u$ with suitable support[1], we have,

$$\langle P^{h,0,0}_{1;\gamma,\theta} u\,,\, u\rangle \geq \left(\Theta_0 + h^{\frac{1}{3}} c^{\mathrm{conj}}(\gamma,\theta) - C\big(h^{\frac{3}{8}} + h^{\delta+\frac{1}{12}}\big)\right) \|u\|^2\,,$$

where

$$c^{\mathrm{conj}}(\gamma,\theta) := \left(\frac{1}{2}\right)^{\frac{2}{3}} \delta_0^{\frac{1}{3}} |\gamma|^{\frac{2}{3}} (\delta_0 \sin^2\theta + \cos^2\theta)^{\frac{1}{3}}\,\hat{\nu}_0.$$

[1] $u \in C_0^\infty(] - C_0 h^\delta, C_0 h^\delta[\times \mathbb{R} \times \overline{\mathbb{R}_+})$, where $C_0 > 0$ is given.

We can not directly compare $P^{h,\eta,\zeta}_{1;\gamma,\theta}$ and $P^{h,0,0}_{1;\gamma,\theta}$ but this can be done by introducing a small perturbation of $P^{h,0,0}_{1;\gamma,\theta}$ whose spectrum is just lifted. To achieve this goal we introduce for $\tau > 0$

$$P^h_{1;\gamma,\theta,\tau} := D_t^2 + (t - h^{\frac{1}{6}} L_{1,\gamma,\theta})^2 + (1 - h^\tau) h^{\frac{1}{3}} (L_{2,\gamma,\theta})^2 ,$$

where we have modified the coefficient of $(L_{2,\gamma,\theta})^2$ by $h^{1/3+\tau}$. Heuristically this leads to a maximal shift of the bottom of the spectrum by $\mathcal{O}(h^{1/3+\tau})$.
In particular we show

$$\langle P^{h,\eta,\zeta}_{1;\gamma,\theta} u, u\rangle \geq \langle P^h_{1;\gamma,\theta,\tau} u, u\rangle - C(\eta^2 + \zeta^2) h^{2\delta-\tau} ||tu||^2 ,$$

What remains to prove is too technical to be exposed here.

# Upper bounds

Fortunately, the same quasi-mode constructed in [HelMo4] (see also Pan for a different formulation) yields an upper bound of the first eigenvalue $\lambda_1(\mathbf{A}, h)$ matching with the asymptotics (*). More precisely, under Assumptions (C1)-(C2), we can prove that:

$$\lambda_1^N(\mathbf{A}, h) \leq \Theta_0 h + \widehat{\gamma}_{0,\mathbf{B}} h^{\frac{4}{3}} + \mathcal{O}(h^{\frac{4}{3}+\eta^*}),$$

for some constant $\eta^* > 0$.

However, while computing the energy of the quasi-mode, we observe additional terms (not present in [HelMo4]) due to the non-homogeneity of the magnetic field which should be treated carefully.

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