Computing nodal deficiency with a refined spectral flow

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Abstract

Recent work of the authors and their collaborators has uncovered fundamental connections between the Dirichlet-to-Neumann map, the spectral flow of a certain family of self-adjoint operators, and the nodal deficiency of a Laplacian eigenfunction (or an analogous deficiency associated to a non-bipartite equipartition). Using a more refined construction of the Dirichlet-to-Neumann map, we strengthen all of these results, in particular getting improved bounds on the nodal deficiency of degenerate eigenfunctions. Our framework is very general, allowing for non-bipartite partitions, non-simple eigenvalues, and non-smooth nodal sets. Consequently, our results can be used in the general study of spectral minimal partitions, not just nodal partitions of generic Laplacian eigenfunctions.

Main goals

We consider the Dirichlet Laplacian $-\Delta_\Omega = -\Delta$ in a bounded domain $\Omega \subset \mathbb{R}^2$, where $\partial\Omega$ is piecewise C^1.

Our goal is to analyze the relations between spectral properties of this Laplacian and partitions $\mathcal{D}$ of Ω by k open sets $\{D_i\}_{i=1}^k$, which are spectral equipartitions in the sense that:
In each D_i's the ground state energy $\lambda_1(D_i)$ of the Dirichlet realization of the Laplacian in D_i is the same;
In addition they satisfy a pair compatibility condition (PCC):
For any pair of neighbors D_i, D_j , there is a linear combination of the ground states in D_i and D_j which is an eigenfunction of the Dirichlet problem in $\mathrm{Int}(\overline{D_i \cup D_j})$.

Nodal partitions and minimal partitions are typical examples of these PCC-equipartitions.
A difficult question is to recognize which PCC-equipartitions are minimal. This problem has been solved by Helffer–Hoffmann-Ostenhof–Terracini in the bipartite case (which corresponds to the Courant sharp situation) but the problem remains open in the general case.

Our main goal is to extend the construction and analysis of the spectral flow and Dirichlet-to-Neumann operators, which was done for nodal partitions in Berkolaiko-Cox-Marzuola [BCM], to spectral equipartitions that satisfy PCC.
In this talk, I refer mainly to two papers Helffer-(P)Sundqvist (CPDE) and Berkolaiko-Cox-Helffer-(P)Sundqvist.

The construction of [BCM]

Let $\Omega \subset \mathbb{R}^2$ and λ_* be some eigenvalue of the Dirichlet Laplacian $-\Delta_\Omega$, with corresponding eigenfunction ϕ_*.
We let

$$\Gamma = \{x \in \Omega \ : \ \phi_*(x) = 0\} ,$$

and

$$\mathcal{D} = (D_1, \ldots, D_\nu)$$

the components of $\Omega \setminus \Gamma$, where $\nu = \nu(\phi_*)$ is the number of these components. Finally let k_* be the the minimal label of λ_*.

The Dirichlet-to-Neumann operator

Assume that $E \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary (nodal domains and later our more general partitions have this property), and that λ is not in the spectrum of $-\Delta_E$. Given g on ∂E, let u be the unique solution to

$$\begin{cases} -\Delta u = \lambda u & \text{in } E, \\ u = g & \text{on } \partial E. \end{cases}$$

Then the Dirichlet-to-Neumann operator $\mathrm{DN}_E(\lambda)$ is defined as an unbounded operator on $L^2(\partial E)$

$$\mathrm{DN}_E(\lambda) g := \frac{\partial u}{\partial \nu},$$

where ν is a unit normal vector pointing out of E.

The theorem[1] of Berkolaiko-Cox-Marzuola can be reformulated as follows:

Theorem BCM

If $\epsilon > 0$ is sufficiently small, then

$$k_* - \nu(\phi_*) = 1 - \dim \ker(-\Delta_\Omega - \lambda_*) + \mathrm{Mor}\,\mathrm{DN}(\Gamma, \lambda^* + \epsilon) \quad (1)$$

[1] This theorem was initially obtained with another proof based oon the Maslov index by G. Cox, C. K. R. T. Jones, and J. L. Marzuola in [CJM2015].

Here

- Mor counts the number of negative eigenvalues of an operator (the so-called Morse index of the operator),
- $\mathrm{DN}(\Gamma,\lambda)$ is for $\lambda \notin \sigma(-\Delta_\Omega)$ defined by

$$\mathrm{DN}(\Gamma,\lambda) = \sum_{i=1}^{k} R_{\Gamma,\partial D_i} \mathrm{DN}_{D_i}(\lambda) E_{\partial D_i,\Gamma}$$

- $E_{\partial D_i,\Gamma}$ is the operator from $L^2(\Gamma)$ to $L^2(\partial D_i)$ that first extends by 0 on $\partial\Omega$ to get a function on $\Gamma \cup \partial\Omega$ and then restricts to ∂D_i
- $R_{\Gamma,\partial D_i}$ the extension by 0 operator to $\partial\Omega \cup \Gamma$ composed by the restriction operator from $L^2(\Gamma \cup \partial\Omega)$ to $L^2(\Gamma)$.

Spectral flow for a family with delta potentials on Γ

To characterize the negative eigenvalues of $\mathrm{DN}(\Gamma, \lambda_* + \epsilon)$ it is fruitful to study the family of operators $-\Delta_{\Omega,\sigma}$, $0 \leq \sigma < +\infty$, induced by the bilinear form

$$\mathfrak{B}_\sigma(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \sigma \int_\Gamma u \, v \, ds, \quad u, v \in H_0^1(\Omega).$$

Indeed, if we denote by $\{\lambda_k(\sigma)\}_{k=1}^{+\infty}$ the set of eigenvalues of $-\Delta_{\Omega,\sigma}$, in increasing order, then [BCM] shows that if $\epsilon > 0$ is sufficiently small, then $-\sigma$ is an eigenvalue of $\mathrm{DN}(\Gamma, \lambda_* + \epsilon)$ if, and only if, $\lambda_* + \epsilon = \lambda_k(\sigma)$ for some $k \in \mathbb{N}$.

They also show that each analytic branch of the eigenvalues is increasing with σ. Moreover, as $\sigma \to +\infty$, the eigenvalues $\lambda_k(\sigma)$ converges to the eigenvalues of $-\Delta_{\Omega,+\infty}$ which is the Laplacian in Ω with Dirichlet boundary conditions imposed on $\partial\Omega \cup \Gamma$.

Due to the construction, the eigenvalue λ_* is in fact the lowest eigenvalue of $-\Delta_{\Omega,+\infty}$, with multiplicity $\nu(\phi_*)$.

Thus,

$$\lim_{\sigma\to+\infty} \lambda_k(\sigma) \begin{cases} = \lambda_*, & \text{if } 1 \leq k \leq \nu(\phi_*), \\ > \lambda_*, & \text{if } k > \nu(\phi_*). \end{cases}$$

By the definition of k_*, the operator $-\Delta_{\Omega,0} = -\Delta_\Omega$ has exactly $\leq k_* - 1 + \dim\ker(-\Delta_\Omega - \lambda_*)$ eigenvalues $\leq \lambda_*$, and so exactly $k_* - 1 + \dim\ker(-\Delta_\Omega - \lambda_*) - \nu(\phi_*)$ of them will pass $\lambda_* + \epsilon$, for $\epsilon > 0$ sufficiently small.

Equipartitions: Notation and definitions

We consider a bounded connected open set Ω in $\mathbb{R}^2$. A *k-partition* of Ω is a family $\mathcal{D} = \{D_i\}_{i=1}^k$ of mutually disjoint, connected, open sets in Ω such that $\overline{\Omega} = \overline{\cup_{i=1}^k D_i}$.

If $\mathcal{D} = \{D_i\}_{i=1}^k$ is a k-partition and the eigenvalues $\lambda_1(D_i)$ of the Dirichlet Laplacian in D_i are equal for $1 \leq i \leq k$, we say that the partition $\mathcal{D}$ is a *spectral equipartition*.

Nodal partitions

Since an eigenfunction u_j, restricted to each nodal domain D_i satisfy the eigenvalue equation $-\Delta u_j = \lambda_j u_j$, with the Dirichlet boundary condition on ∂D_i, each nodal partition is indeed a spectral equipartition.

By the Courant nodal theorem, $\mu(u_j) \leq j$. We say that the pair (λ_j, u_j) is *Courant sharp* if $\mu(u_j) = j$, i.e. has nodal deficiency 0.

Minimal partitions (after [HHOT2009])

For any integer $k \geq 1$, and for $\mathcal{D}$ in $\mathfrak{O}_k(\Omega)$, we introduce the *energy of the partition*,

$$\mathcal{E}(\mathcal{D}) = \max_i \lambda_1(D_i).$$

Then we define

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D}\in\mathfrak{O}_k} \mathcal{E}(\mathcal{D})\,.$$

We call $\mathcal{D} \in \mathfrak{O}_k$ a *minimal spectral k-partition* if $\mathfrak{L}_k(\Omega) = \mathcal{E}(\mathcal{D})$.

In general, every minimal spectral partition is an equipartition (see Helffer-Hoffmann-Ostenhof-Terracini [HHOT2009]).

Nodal sets and minimal partitions are regular (Bers [Be1955], H.-Hoffmann-Ostenhof-Terracini [HHOT2009]).

Odd and even points

To simplify we assume in this talk that Ω is simply connected.
Given a partition $\mathcal{D} = \{D_i\}$ of Ω, we denote by $X^{\mathrm{odd}}(\mathcal{D})$ the set of odd critical points, i.e. points x_ℓ for which ν_ℓ is odd.

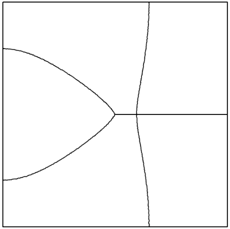

Figure: Example with one odd point and one even point

Nodal partitions have no odd points. Minimal partitions can have odd points.

We say that D_i and D_j are *neighbors*, which we write $D_i \sim D_j$, if the set $D_{ij} := \mathrm{Int}(\overline{D_i \cup D_j}) \setminus \partial\Omega$ is connected.

We say that $\mathcal{D}$ is *admissible* (or bipartite) if we can color the partition with two colors with the property that two neighbors have a different color.

Nodal partitions are always admissible, since the eigenfunction changes sign when going from one nodal domain to a neighboring nodal domain.

Weak Compatibility condition between neighbors

Let $\mathcal{D} = \{D_i\}_{i=1}^k$ be a regular equipartition of energy $\lambda := \mathcal{E}(\mathcal{D})$.

Definition of WPCC

A regular equipartition $\mathcal{D} = \{D_i\}_{i=1}^k$ satisfies the *weak pair compatibility condition*, (for short WPCC), if, for any pair (i, j) s.t. $D_i \sim D_j$, there is an eigenfunction $u_{ij} \not\equiv 0$ of $-\Delta_{D_{ij}}$ s. t.

- $-\Delta_{D_{ij}} u_{ij} = \lambda u_{ij}$,
- the nodal set of u_{ij} is given by $\partial D_i \cap \partial D_j$.

Strong Compatibility condition between neighbors

Definition of SPCC

A regular equipartition $\mathcal{D} = \{D_i\}_{i=1}^k$ satisfies the *strong pair compatibility condition*, (for short SPCC), if there exist positive ground states u_i of $-\Delta_{D_i}$ s. t. for any pair (i,j) such that $D_i \sim D_j$, $u_{ij} = u_i - u_j$ is an eigenfunction of $-\Delta_{D_{ij}}$.

Nodal partitions and spectral minimal partitions satisfy the SPCC.

When Ω is simply connected (WPCC) implies (SPCC). (see [HHO2007]).

We also refer to Berkolaiko-Kuchment-Smilansky [BKS2012] for conditions implying (SPCC).

Avoiding the "ϵ"

The improved theorem obtained by Berkolaiko-Cox-Helffer-Sunqvist reads

Theorem [BCHS] (nodal case)

The eigenfunction ϕ_* has nodal deficiency

$$\delta(\phi_*) = \mathcal{M}\mathrm{orDN}^{new}(\Gamma, \lambda_*), \tag{2}$$

and the corresponding eigenvalue λ_* has multiplicity

$$\dim\ker(\Delta + \lambda_*) = \dim\ker \mathrm{DN}^{new}(\Gamma, \lambda_*) + 1. \tag{3}$$

The version of the Dirichlet-to-Neumann map $\mathrm{DN}^{new}(\Gamma, \lambda_*)$ appearing in the above theorem has some more involved form than the one used in [BCM, CJM2015], but consequently gives us a stronger result.

Construction of $\mathrm{DN}^{new}(\Gamma, \lambda_*)$

We denote the nodal domains of ϕ_* by $D_1, \dots, D_k$.
When defining the Dirichlet-to-Neumann map, one must take into account that λ_* is a Dirichlet eigenvalue on each D_i. Introducing the notation $\Gamma_i = \overline{\partial D_i \cap \Omega}$, we define the closed subspace

$$S = \left\{ g \in L^2(\Gamma) : \int_{\Gamma_i} g_i \frac{\partial \phi_{*,i}}{\partial \nu_i} = 0, \ i = 1, \dots, k \right\} \tag{4}$$

of $L^2(\Gamma)$, where g_i denotes the restriction of g to Γ_i, $\phi_{*,i}$ is the restriction of ϕ_* to D_i, and ν_i is the outward unit normal to D_i.

For sufficiently smooth $g \in S$, each boundary value problem

$$\begin{cases} -\Delta u_i = \lambda_* u_i & \text{in } D_i, \\ u_i = g_i & \text{on } \partial D_i \cap \Omega, \\ u_i = 0 & \text{on } \partial D_i \cap \partial\Omega, \end{cases} \tag{5}$$

has a solution u_i^g.

Defining a function $\gamma_N u^g$ on Γ by

$$\gamma_N u^g\big|_{\Gamma_i \cap \Gamma_j} = \frac{\partial u_i^g}{\partial \nu_i} + \frac{\partial u_j^g}{\partial \nu_j} \tag{6}$$

for all $i \neq j$, we let

$$\mathrm{DN}^{new}(\Gamma, \lambda_*)g = \Pi_S\big(\gamma_N u^g\big), \tag{7}$$

where Π_S denotes the $L^2(\Gamma)$-orthogonal projection onto S.

The solution to the problem (5) is non-unique, but the choice of particular solution u_i^g is irrelevant for the definition on account of the projection in (7).
We can use this freedom to give a more explicit definition of the Dirichlet-to-Neumann map that does not involve Π_S.

Extension to equipartitions

The analysis of this case has been considered in two papers Helffer-Sundqvist and Berkolaiko-Cox-Helffer-Sundqvist. More recently other approachs have been explored by Berkolaiko-Canzani-Cox-Marzuola in the paper : Stability of spectral partitions and the Dirichlet-to-Neumann map (ArXiv July 2022).

Details if time permits

For a two-sided, weakly regular partition each D_i is a Lipschitz domain, so we can define trace operators and solve boundary value problems in a standard way.
To extend the notion of a "nodal partition" to partitions that are not necessarily bipartite, it is convenient to introduce signed weight functions — which will also be used to define a generalized two-sided Dirichlet-to-Neumann map on the partition boundary set.

Definition of valid weights

Given a two-sided, weakly regular partition $\mathcal{D} = \{D_i\}$, with

$$\Gamma_i := \overline{\partial D_i \cap \Omega}.$$

We call *valid weight* a family of functions χ_i defined on ∂D_i which can be obtained by the following construction:
Given an orientation of each ∂D_i, and an orientation of each smooth component of Γ, we define χ_i on each smooth component of ∂D_i to be $+1$ if the orientation of ∂D_i agrees with the orientation of the corresponding smooth component of Γ, and equal to -1 otherwise.

One example

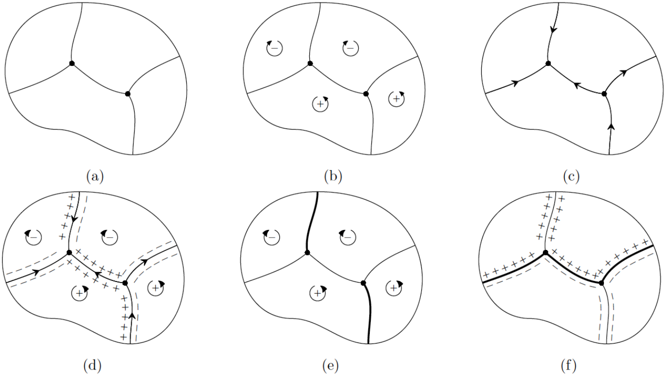

FIGURE 1.2. A partition (a), with a choice of orientation for the boundary ∂D_i of each subdomain (b), and an orientation of each smooth part of the boundary set Γ (c). In (d) we show the corresponding valid weights χ_i, and in (e) we show the resulting cut, as described in Appendix A. In (f) we display a non-valid choice of weights, i.e. functions $\chi_i \colon \Gamma_i \to \{\pm 1\}$ that are not induced by any choice of orientations.

Note that χ_i is constant on each smooth segment of Γ_i.
There are two ways χ_i can change sign on Γ_i:

- it can change sign at a corner.
- it can take different signs on different connected components.

It is easily shown that a partition is bipartite if and only if the weights $\chi_i \equiv 1$ are valid and so non-constant weights are essential for the study of non-bipartite partitions.
Valid weights have a natural geometric interpretation in terms of the cutting construction in [HPS] where one removes a portion Γ^* of the nodal set from the domain Ω in such a way that the resulting partition of $\Omega \setminus \Gamma^*$ is bipartite.

We now introduce a weighted version of the Laplacian, $-\Delta^{\chi}$, corresponding to the bilinear form defined on the domain $\mathrm{Dom}(t^{\chi})$ consisting of $u \in L^2(\Omega)$ such that

$$u_i := u\big|_{D_i} \in H^1(D_i),$$
$$u_i = 0 \text{ on } \partial D_i \cap \partial\Omega$$
$$\chi_i u_i = \chi_j u_j \text{ on } \Gamma_i \cap \Gamma_j \text{ for all } i, j = 1, \ldots, k,$$

and given by

$$t^{\chi}(u, v) = \sum_{i=1}^{k} \int_{D_i} \nabla u_i \cdot \nabla v_i.$$

The Laplacians Δ^{χ} for different valid weights can be shown to be unitarily equivalent. Hence, if the partition is bipartite, Δ^{χ} is unitarily equivalent to the Dirichlet Laplacian on Ω. Furthermore, the nodal sets of the eigenfunctions of Δ^{χ} are independent of χ, justifying the following definition.

Definition

A two-sided, weakly regular partition $\mathcal{D}$ is said to be χ-*nodal* if it is the nodal partition for some eigenfunction of Δ^{χ}. The *defect* of a χ-nodal k-partition is defined to be

$$\delta(\mathcal{D}) = \ell(\mathcal{D}) - k, \tag{8}$$

where $\ell(\mathcal{D})$ denotes the minimal label of λ_* in the spectrum of $-\Delta^{\chi}$.

We can show that a partition is χ-nodal if and only if it satisfies the strong pair compatibility condition.

Finally, we will define a χ-weighted version of the two-sided Dirichlet-to-Neumann map, denoted $\mathrm{DN}(\Gamma, \lambda_*, \chi)$. The full definition is rather delicate because λ_* is a Dirichlet eigenvalue and Γ has corners. We just mention here that (similarly to the Laplacian Δ^χ), the Dirichlet-to-Neumann maps defined with different valid $\{\chi_i\}$ are in some sense unitarily equivalent.

The main result is the following.

Theorem

A two-sided, weakly regular partition $\mathcal{D}$ satisfies the SPCC if and only if it is χ-nodal.
In this case it has defect

$$\delta(\mathcal{D}) = \mathrm{MorDN}(\Gamma, \lambda_*, \chi),$$

and the corresponding eigenvalue λ_* of $-\Delta^\chi$ has multiplicity

$$\dim\ker(\Delta^\chi + \lambda_*) = \dim\ker \mathrm{DN}(\Gamma, \lambda_*, \chi) + 1.$$

THANK YOU.

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